Solutions to Math 21C Final, Winter 02.

1. Two vectors for two of the sides of the triangle are $\mathbf{a} = \langle 1, 2, 1\rangle$ and $\mathbf{b} = \langle 2, -2, 3\rangle$. The area of the triangle is $|\mathbf{a} \times \mathbf{b}|/2$. We have $\mathbf{a} \times \mathbf{b} = ... = 8\,\mathbf{i} - \mathbf{j} - 6\,\mathbf{k}$ and $|\mathbf{a} \times \mathbf{b}| = \sqrt{8^2 + 1 + 6^2} = \sqrt{101}$.

2. The curves intersect when $\langle t, t^2, t^3\rangle = \langle 1 + s, 4s, 8s^2\rangle$ for some t and s. This means that $t = 1+s$ and $t^2 = 4s = 4(t-1)$ so $t^2-4t+4 = 0$ and hence $(t-2)^2 = 0$, i.e. $t = 2$ and hence $s = 1$. Its is easy to check that also the last equation is satisfied for these values. We have $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2\rangle$ so $\mathbf{r}_1'(2) = \langle 1, 4, 12\rangle$ and $\mathbf{r}_2'(s) = \langle 1, 4, 16s\rangle$ so $\mathbf{r}_2'(1) = \langle 1, 4, 16\rangle$. The angle is given by $\cos\theta = \mathbf{r}_1'(2) \cdot \mathbf{r}_2'(1)/(|\mathbf{r}_1'(2)|\,|\mathbf{r}_2'(1)|) = 209/(\sqrt{1 + 4^2 + 12^2} \cdot \sqrt{1 + 4^2 + 16^2}) = 209/\sqrt{161 \cdot 273}$.

3. The vector $\mathbf{n} = \langle 2, 1, 4\rangle$ between the points is normal to the plane and the point in the middle between the points $P = (0, 3/2, 1)$ is on the plane. The equation of the plane is therefore $2(x - 0) + 1(y - 3/2) + 4(z - 1) = 0$.

4. $\mathbf{r}'(t) = 3t^{1/2}\,\mathbf{i} - 2\sin 2t\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$. The initial speed is $|\mathbf{r}'(0)| = |2\mathbf{k}| = 2$. The arc length is $\int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 \sqrt{9t + 4}\,dt = (9t+4)^{3/2}2/27\big|_0^1 = 2(13^{3/2} - 4^{3/2})/27$.

5. The tangent plane to $F(x, y, z) = x^2 + y^2/4 + z^2/9 = 1$ at a point (x_0, y_0, z_0) has normal $\nabla F(x_0, y_0, z_0) = \langle 2x_0, y_0/2, 2z_0/9\rangle$. This vector is parallel to the normal of the plane $x+y-z=0$, which is $\langle 1, 1, -1\rangle$, if $2x_0 = \lambda$, $y_0/2 = \lambda$ and $2z_0/9 = -\lambda$ Since the point also must lie on the surface we must have $F(\lambda/2, 2\lambda, -9\lambda/2) = \lambda^2/4 + \lambda^2 + 9\lambda^2/4 = 7\lambda^2/2 = 1$ so $\lambda = \pm\sqrt{2/7}$. The point is $(x_0, y_0, z_0) = \pm\sqrt{2/7}(1/2, 2, -9/2)$.

6. $f_x(x, y) = 4x^3 - 8y = 0$ and $f_y(x, y) = -8x + 4y = 0$ gives $y = 2x$ and $4x(x^2 - 4) = 0$. Hence $x = 0$ or $x = \pm 2$ so the critical points are $(0, 0), (2, 4), (-2, -4)$. $f_{xx} = 12x^2$, $f_{xy} = -8$, $f_{yy} = 4$ so $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 48x^2 - 64$. Then $D(0, 0) < 0$ so $(0, 0)$ is a saddle point, $D(2, 4) = 128 > 0$ and $f_{xx}(2, 4) = 48 > 0$ so $(2, 4)$ is local min, $D(-2, -4) = 128 > 0$ and $f_{xx}(-2, -4) = 48 > 0$ so $(-2, -4)$ is local min.

7. Minimize the area $A = xy + 2xz + 2yz$, subject to the constraint that the volume is $V = xyz = 32{,}000$. Lagrange multipliers: $\nabla A(x, y, z) = \langle y + 2z, x + 2z, 2x + 2y\rangle$ and $\nabla V(x, y, z) = \langle yz, xz, xy\rangle$ so we must find all (x, y, z) and λ such that $y + 2z = \lambda yz$, $x + 2z = \lambda xz$, $2x + 2y = \lambda xy$ and $V(x, y, z) = 32{,}000$. Multiplying the first equation by x, the second by y and the third by z we get $x(y + 2z) = y(x + 2z) = z(2x + 2y)$. Subtracting the first two equations gives $2z(x - y) = 0$ and subtracting the first and third gives $(x - 2z)y = 0$. If $z = 0$ then $y = 0$ or $x = 0$. If $z \neq 0$ then $x = y = 0$ or $x = y = 2z$. Hence we have the points $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$ and $(2z, 2z, z)$. Only the last one gives $V \neq 0$ and we must have $V(2z, 2z, z) = 4z^3 = 32{,}000$ which is equivalent to $z = 20$ so $(x, y, z) = (40, 40, 20)$.

8. Let $D = \{(x, y);\ 10 - 3x^2 - 3y^2 \geq 4\} = \{(x, y);\ x^2 + y^2 \leq 2\}$.
The volume is $\iint_D z\,dA = \iint_D (10 - 3x^2 - 3y^2)\,dA = \int_0^{2\pi}\int_0^{\sqrt{2}} (10 - 3r^2)\,r dr\,d\theta = \int_0^{2\pi} (5r^2 - 3r^4/4)\big|_0^{\sqrt{2}}\,d\theta = \int_0^{2\pi} 7\,d\theta = 14\pi$.

9. $\iint_T \sqrt{1 + 1 + 4y^2}\,dA = \int_0^1\int_0^y \sqrt{2 + 4y^2}\,dx\,dy = \int_0^1 \sqrt{2 + 4y^2}x\big|_{x=0}^y\,dy = \int_0^1 \sqrt{2 + 4y^2}y\,dy (2 + 4y^2)^{3/2}/12\big|_0^1 = (6^{3/2} - 2^{3/2})/12$.

10. $E = \{(x, y, z);\ x \geq 0, y \geq 0, z \geq 0, x + y + z/2 \leq 1\} = \{(x, y, z);\ 0 \leq z \leq 2,\ 0 \leq y \leq 1 - z/2,\ 0 \leq x \leq 1 - y - z/2\}$.
$\iiint_E y\ \ dV = \int_0^2\int_0^{1-z/2}\int_0^{1-y-z/2} y\,dx\,dy\,dz = \int_0^2\int_0^{1-z/2} yx\big|_{x=0}^{1-y-z/2}\,dy\,dz = \int_0^2\int_0^{1-z/2} y(1 - y - z/2)\,dy\,dz = \int_0^2 (y^2/2 - y^3/3 - zy^2/4)\big|_{y=0}^{1-z/2}\,dz = \int_0^2 (1 - z/2)^3/6\,dz = \int_{1/2}^1 t^3/3\,dt = t^4/12\big|_0^1 = 1/12$.