MATH 21C, MIDTERM 2 SOLUTIONS, WINTER 2003, OKIKIOLU.

1. A curve is given by the equation

$$\mathbf{r}(t) = \left\langle \frac{5t^2}{2}, \frac{4t^3}{3}, -t^3 \right\rangle, \qquad 0 \le t \le \sqrt{3}.$$

(a). Calculate the equation of the tangent line to the curve at the point $\left(\frac{5}{2}, \frac{4}{3}, -1\right)$.

(b). Calculate the length of the curve.

Solution. (a). $\mathbf{r}'(t) = \langle 5t, 4t^2, -3t^2 \rangle$. At the point given,

$$\frac{5t^2}{2} = \frac{5}{2}, \qquad \frac{4t^3}{3} = \frac{4}{3}, \qquad -t^3 = -1.$$

The unique solution to these three equations is $t = 1$. Now $\mathbf{r}'(1) = \langle 5, 4, -3 \rangle$ and so the equation of the tangent line in vector form is $\mathbf{r}_0(t) = \langle \frac{5}{2} + 5t, \frac{4}{3} + 4t, -1 - 3t \rangle$.

(b). $|\mathbf{r}'(t)| = \sqrt{(5t)^2 + (4t^2)^2 + (3t^2)^2} = 5t\sqrt{1+t^2}$. Then the length is

$$\int_0^{\sqrt{3}} 5t\sqrt{1+t^2}\,dt \;=\; \frac{5}{3}(1+t^2)^{3/2}\Big|_0^{\sqrt{3}} \;=\; \frac{35}{3}.$$

2. Let $f(x, y) = e^x(1 + \sin y)$.
(a). Calculate the equation of the tangent plane to the graph $z = f(x, y)$ at the point $(x, y, z) = (0, \pi, 1)$.

(b). By making a linear approximation of f at the point $(x, y) = (0, \pi)$, estimate $f(0.2, \pi - 0.1)$.

Solution. (a).

$$f_x = e^x(1 + \sin y), \qquad f^y = e^x \cos y, \qquad f_x(0, \pi) = 1, \qquad f_y(0, \pi) = -1.$$

Tangent plane at $(0, \pi, 1)$ is $z - 1 = (x - 0) - (y - \pi)$.

(b). The value on the graph $z = f(x, y)$ is approximated by the value of z on the tangent plane: $f(0.2, \pi - 0.1) \approx 1 + (0.2 - 0) - (\pi - 0.1 - \pi) = 1 + 0.2 + 0.1 = 1.3$. (Alternatively use differentials to get the same answer.)

3. The temperature of a heated plate is given by $f(x, y) = x^2/y$.
(a). At the point $(1, 1)$, in which direction is the rate of change of temperature greatest?
(b). An ant travels on the plane at unit speed and passes through the point $(1, 1)$ in the direction $\langle 3, 4\rangle$. What rate of change of temperature does the ant experience as it passes through $(1, 1)$?

Solution. (a). Rate is greatest in the direction $\nabla f(1, 1)$. Now $\nabla f = \big\langle 2x/y\,,\, -x^2/y^2\big\rangle$, and so $\nabla f(1, 1) = \langle 2, -1\rangle$.

(b). Unit vector in the direction $\langle 3, 4\rangle$ is $\big\langle \frac{3}{5}, \frac{4}{5}\big\rangle$. Rate of change of temperature is

$$D_{\mathbf{u}}f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = \langle 2, -1\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{2}{5}.$$

4. Find all the critical points of the function $f(x, y) = x^2 - 2xy + \dfrac{y^3}{3}$ and determine whether each critical point is a local maximum, a local minimum or a saddle point.

Solution. Solve for critical points

$$\begin{aligned} 0 = f_x &= 2x - 2y \quad \Rightarrow \quad x = y. \\ 0 = f_y &= -2x + y^2 \quad \Rightarrow \quad 0 = -2y + y^2 \quad \Rightarrow \quad y(y - 2) = 0. \end{aligned}$$

We get the points $(0, 0)$ and $(2, 2)$. Now

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 2y \end{vmatrix} = 4y - 4.$$

At $(0, 0)$ we have $D = -4 < 0$ so $(0, 0)$ is a saddle. At $(2, 2)$ we have $D = 4 > 0$ and $f_{xx} = 2 > 0$ so $(2, 2)$ is a local minimum.

5. Use **Lagrange multipliers** to calculate the maximum and minimum values of $x^4 + y^4$ on the curve $x^2 + y^4 = 1$.

Solution. $f = x^4 + y^4$, $g = x^2 + y^4$. Solve $\nabla f = \lambda \nabla g$ and $g = 4$:

$$\begin{cases} 4x^3 = \lambda 2x \\ 4y^3 = \lambda 4y^3 \\ x^2 + y^4 = 1 \end{cases}$$

The second equation implies $\lambda = 1$ or $y = 0$. If $\lambda = 1$ the the first equation implies $2x(2x^2 - 1) = 0$ so $x = 0$ or $x = \pm 2^{-1/2}$. The constraint gives the points $(x, y) = (0, \pm 1)$ and the four points $(x, y) = (\pm 2^{-1/2}, \pm 2^{-1/4})$. If $y = 0$ then the constraint gives the points $(\pm 1, 0)$. Evaluating f at these points,

$$f(0, \pm 1) = 1, \qquad f(\pm 2^{-1/2}, \pm 2^{-1/4}) = 3/4, \qquad f(\pm 1, 0) = 1.$$

The max is 1 and the min is 3/4.