Final Exam, Mathematics 20D
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Name:
SSN:
Section Number:

Note: There are 8 problems on this exam, worth 25 points each. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

(25 pts.) I. (1) Solve the following initial value problem.

$$y'' + 4y' + 4y = t^{-2} \cdot e^{-2t}, \qquad y(0) = 0,\ y'(0) = 0.$$

(2) Indicate the interval of definition for the solution you found in (1).

(1) Inhomogeneous, 2^{nd} order, Corresponding Homogeneous Equation: $y''+4y'+4y=0$ has constant coefficients.

Solve Homogeneous Eq.:

Corresponding Polynomial: $r^2+4r+4=(r+2)^2=0$

Repeated real root $r=-2$, so general solution is

$$y=C_1e^{-2t}+C_2te^{-2t}$$

Because of t^{-2} in the inhomogeneous term, cannot use undetermined coefficients.

Using Theorem 3.7.1 in Boyce and DiPrima, where

$p(t)=4, q(t)=4, g(t)=t^{-2}e^{-2t}$,

$y_1(t)=e^{-2t}$, $y_2(t)=te^{-2t}$ and $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$=\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}-2te^{-2t} \end{vmatrix} = e^{-4t}-2te^{-4t}+2te^{-4t}$$
$$=e^{-4t}$$

we have the particular solution

$$Y(t)=-e^{-2t}\int_{t_0}^{t}\frac{se^{-2s}\cdot s^{-2}e^{-2s}}{e^{-4s}}\,ds+te^{-2t}\int_{t_0}^{t}\frac{e^{-2s}s^{-2}e^{-2s}}{e^{-4s}}\,ds$$

$$= -e^{-2t}\int_{t_0}^{t} \frac{s^{-1}e^{-4s}}{e^{-4s}}\,ds + te^{-2t}\int_{t_0}^{t} \frac{s^{-2}e^{-4s}}{e^{-4s}}\,ds$$

$$= -e^{-2t}\int_{t_0}^{t} s^{-1}\,ds + te^{-2t}\int_{t_0}^{t} s^{-2}\,ds$$

Note: t_0 can be any convenient value we choose.

Often, $t_0=0$ is a good choice, but here, s^{-1} and s^{-2} are not defined at 0, so setting $t_0=0$ would make these improper integrals. Choose $t_0=1$ instead.

Then $Y(t) = -e^{-2t}\int_1^t s^{-1}\,ds + te^{-2t}\int_1^t s^{-2}\,ds$

$$= -e^{-2t}\left(\ln|s|\Big|_1^t\right) + te^{-2t}\left(-s^{-1}\Big|_1^t\right)$$

$$= -e^{-2t}\ln|t| + e^{-2t}\ln(1) + te^{-2t}(-t^{-1}) - te^{-2t}(-1)$$

$$= -e^{-2t}\ln|t| - e^{-2t} + te^{-2t}$$

Note: $-e^{-2t}+te^{-2t}$ is already a solution to the corresponding homogeneous equation, so can choose the particular solution to be $-e^{-2t}\ln|t|$

So the general solution to the inhomogeneous equation is

$$y(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln|t|$$

The initial conditions are bad, since $g(t)=t^{-2}e^{-t}$ is undefined at the initial time $t=0$.

If the initial conditions were instead: $y(1)=0$, $y'(1)=0$,

then $y(1) = c_1e^{-2} + c_2e^{-2} - e^{-2}\ln(1) = c_1e^{-2}+c_2e^{-2} = 0 \Rightarrow c_1+c_2=0$

$y'(t) = -2c_1e^{-2t} - 2c_2te^{-2t} + c_2e^{-2t} + 2e^{-2t}\ln|t| - t^{-1}e^{-2t}$

so $y'(1) = -2c_1e^{-2} - 2c_2e^{-2} + c_2e^{-2} + 2e^{-2}\ln(1) - e^{-2} = -2c_1e^{-2} - c_2e^{-2} - e^{-2} = 0$

$\Rightarrow -2c_1 - c_2 - 1 = 0$

Adding the previous equation $c_1+c_2=0$ to this gives $-c_1-1=0 \Rightarrow c_1=-1$, and then $c_2=1$.

So the solution to the initial value problem (with $y(1)=0$, $y'(1)=0$) is

$$y(t) = -e^{-2t} + te^{-2t} - e^{-2t}\ln|t|$$

(2) Using the new initial time $t=1$, we see that $y(t)$ is defined if $t \neq 0$ (so $|t|>0$).
Since the initial time $t=1>0$, the interval of definition is $(0,\infty)$

(Remark: Thus, we can drop the absolute value and

$$y(t) = -e^{-2t} + te^{-2t} - e^{-2t}\ln t, \quad t>0.)$$

(25 pts.) II. **A.** For the differential equation

$$y'' - y = 0\,,$$

at the regular point $x_0 = 0$

(1) determine the recurrence relation satisfied by the coefficients $\{a_n\}_{n\geq 0}$ of its general power series solution $\sum_{n\geq 0} a_n \cdot x^n$.

(2) find the first four terms in each of the two linearly independent series solutions.

B. Determine a lower bound for the radius of convergence of series solutions to the differential equation

$$(x^2 - 2x + 5)y'' + xy' + 4y = 0\,,$$

about the regular point $x_0 = 4$.

A

1) Look for a solution of the form $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n x^n$.

Assume this converges in some interval $|x| < \rho$.

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ (note the $n=0$ term of y is a_0 which is constant, so it disappears upon differentiation.)

and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting into $y'' - y = 0$, we find

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

In the first sum, replace n by $n+2$ everywhere and begin the sum at $n=0$:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or $\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n = 0.$

If this is satisfied for all x, must have each coefficient is 0, so

$$(n+2)(n+1) a_{n+2} - a_n = 0, \quad n = 0, 1, 2, \ldots$$

2) Choosing $a_0 = 1$ and $a_1 = 1$, we find from setting $n=0$ above

$2 \cdot 1 \cdot a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2} = \frac{1}{2}$. Setting $n=1$ above,

$3 \cdot 2 \cdot a_3 - a_1 = 0 \Rightarrow a_3 = \frac{a_1}{6} = \frac{1}{6}$.

So a solution begins $y_1 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ (note: this solution is $y = e^x$)

Setting $a_0 = 1$, $a_1 = -1$, and $n = 0$ in the recurrence relation:

$$2 \cdot 1 \cdot a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2} = \frac{1}{2}$$

Setting $n = 1$ in the recurrence relation,

$$3 \cdot 2 \cdot a_3 - a_1 = 0 \Rightarrow a_3 = \frac{a_1}{6} = -\frac{1}{6}$$

So another solution begins $y_2 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$ (note: this solution is $y = e^{-x}$)

Note: $y_1(0) = 1$, $y_2(0) = 1$, $y_1' = 1 + x + \frac{x^2}{2} + \cdots$ so $y_1'(0) = 1$

$y_2' = -1 + x - \frac{x^2}{2} + \cdots$ so $y_2'(0) = -1$

$$\text{So } W(y_1, y_2)(0) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

Thus, y_1 and y_2 are linearly independent by Theorem 3.3.3.

Remark: Setting $a_0 = 1$, $a_1 = 0$ yields the solution $y = \cosh(x)$ and setting $a_0 = 0$, $a_1 = 1$ yields the solution $y = \sinh(x)$.

Using these solutions would be an alternative correct answer.

B

First, if $P(x) = x^2 - 2x + 5$, $Q(x) = x$, and $R(x) = 4$, then the zeros of $P(x)$ are $\frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$.

The distance in the complex plane from $x = 4$ to the zero of $P(x)$ $1+2i$ is $d = \sqrt{(4-1)^2 + (0-2)^2} = \sqrt{9+4} = \sqrt{13}$ and similarly for $1 - 2i$.

Thus, since $Q(x)$ and $R(x)$ have no zeros in common with $P(x)$, Theorem 5.3.1 shows that the radius of convergence of series solutions is at least $\sqrt{13}$.

(25 pts.) III. (1) Find the radius of convergence and interval of convergence for the following power series.

$$\sum_{n\geq 1} \frac{2^n \cdot (x-2)^n}{n+2}.$$

(2) Specify for which points in its interval of convergence the power series in (1) converges absolutely, respectively conditionally.

(1) Ratio Test: $a_n = \frac{2^n(x-2)^n}{n+2}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x-2)^{n+1}}{n+3}}{\frac{2^n(x-2)^n}{n+2}} \right| = \lim_{n\to\infty} 2\frac{n+2}{n+3}|x-2|$$

$$= 2|x-2| \lim_{n\to\infty} \frac{1+\frac{2}{n}\to 0}{1+\frac{3}{n}\to 0} = 2|x-2|$$

The series is absolutely convergent if $2|x-2|<1$

$\Rightarrow |x-2|<\frac{1}{2}$

$\Rightarrow 0\leq x-2<\frac{1}{2}$ or $-\frac{1}{2}<x-2<0$

i.e., for $-\frac{1}{2}<x-2<\frac{1}{2}$

so for $\frac{3}{2}<x<\frac{5}{2}$

This is an interval centered at $x=2$ of radius $\frac{1}{2}$, so

Radius of Convergence $=\frac{1}{2}$.

For interval of convergence, test convergence at the endpoints $x=\frac{3}{2},\frac{5}{2}$.

$x=\frac{3}{2}$: $\sum_{n\geq 1} \frac{2^n(\frac{3}{2}-2)^n}{n+2} = \sum_{n\geq 1} \frac{2^n(-\frac{1}{2})^n}{n+2} = \sum_{n\geq 1} \frac{(-1)^n\frac{2^n}{2^n}}{n+2} = \sum_{n\geq 1} \frac{(-1)^n}{n+2}$.

This is alternating, and setting $b_n = \left|\frac{(-1)^n}{n+2}\right| = \frac{1}{n+2}$, we have

a) $b_{n+1} = \frac{1}{n+3} < \frac{1}{n+2} = b_n$ for all n since $n+3>n+2$ for all n.

b) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+2} = \lim_{n\to\infty} \frac{\frac{1}{n}\to 0}{1+\frac{2}{n}\to 0} = 0$

so the power series converges at $x=\frac{3}{2}$ by the Alternating Series Test.

$x=\frac{5}{2}$: $\sum_{n\geq 1}\frac{2^n(\frac{5}{2}-2)^n}{n+2} = \sum_{n\geq 1}\frac{2^n\frac{1}{2^n}}{n+2} = \sum_{n\geq 1}\frac{1}{n+2}$.

This is similar to the harmonic series, so expect it diverges.

However, the Comparison Test cannot be used to show this (why not?), so we use the Limit Comparison Test.

$\sum_{n\geq 1}\frac{1}{n+2}$ and $\sum_{n\geq 1}\frac{1}{n}$ both have positive terms.

$\lim_{n\to\infty}\frac{\frac{1}{n+2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+2} = \lim_{n\to\infty}\frac{1}{1+\underbrace{\frac{2}{n}}_{\to 0}} = 1 > 0$ and 1 is finite, so by the Limit Comparison Test, since $\sum_{n\geq 1}\frac{1}{n}$ diverges, so does $\sum_{n\geq 1}\frac{1}{n+2}$.

Thus, the power series diverges at $x=\frac{5}{2}$.

The interval of convergence is thus $[\frac{3}{2},\frac{5}{2})$.

(2): From part (1), the power series converges absolutely for x in $(\frac{3}{2},\frac{5}{2})$.

At $x=\frac{3}{2}$, since $\sum_{n\geq 1}\left|\frac{2^n(\frac{3}{2}-2)^n}{n+2}\right| = \sum_{n\geq 1}\frac{1}{n+2}$ and this series was shown to diverge in part (1), the power series is not absolutely convergent at $x=\frac{3}{2}$.

Since it converges at $x=\frac{3}{2}$, it is conditionally convergent there.

(25 pts.) IV. Determine which of the following infinite series are convergent and which are divergent.

$$(a) \sum_{n\geq 1} n \cdot \sin 1/n; \qquad (b) \sum_{n\geq 1} 1/n \cdot \ln n; \qquad (c) \sum_{n\geq 1} (-1)^n 1/n \cdot \ln n.$$

a) Test for divergence:

$a_n = n\sin(\frac{1}{n})$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin(\frac{1}{n}) \overset{\text{substitute } m=\frac{1}{n}}{=} \lim_{m\to 0} \frac{\sin(m)}{m} = 1 \neq 0$

so it diverges by the test for divergence.

b) Note 1: $\frac{1}{n}\cdot \ln n = \frac{\ln n}{n}$, not $\frac{1}{n\ln n}$.

Note 2: $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, so cannot use test for divergence.

Integral Test:

$f(x) = \frac{\ln x}{x}$ is continuous and positive for $x > 1$.

$f'(x) = \frac{x\cdot\frac{1}{x} - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2} < 0$ when $\ln(x) > 1$, i.e., when $x > e$.

(Since e^x is an increasing function, so $\ln(x) > 1 \Rightarrow e^{\ln(x)} > e^1 \Rightarrow x > e$).

Thus, f is decreasing for $x > e$.

$$\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx \overset{u\text{-sub: } u=\ln(x)}{=} \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t = \lim_{t\to\infty}\frac{(\ln t)^2}{2} - 0$$

$$= \infty$$

Since the integral is divergent, so is $\sum_{n\geq 1} \frac{\ln n}{n}$.

(Alternatives: Comparison Test, or see problem 41(b) in section 11.5)

c) $\sum_{n\geq 1} \frac{(-1)^n \ln(n)}{n}$ is alternating. Setting $b_n = \left|\frac{(-1)^n \ln n}{n}\right| = \frac{\ln n}{n}$ for $n \geq 1$,

part b) above showed $\frac{\ln x}{x}$ is decreasing for $x > e$, so

i) $b_{n+1} \leq b_n$ for $n \geq 3$ (note $e = 2.718\ldots$)

ii) $\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{L'Hospitals}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$, so $\sum_{n\geq 1} \frac{(-1)^n \ln n}{n}$ converges by the Alternating Series Test.

(25 pts.) V. (1) Write down the Taylor series expansion of

$$f(x) = e^{-x^2},$$

about $a = 0$ and determine its radius of convergence.

(2) Compute the value $T_2(0.2)$ of the second Taylor polynomial $T_2(x)$ associated to f at $a = 0$ and estimate the error in approximating $e^{-0.04}$ with $T_2(0.2)$. Take into account that $|f^{(3)}(x)| \le 0.6$, for all x in the interval $(-0.2, 0.2)$.

(3) Use the power series in (1) to approximate the integral

$$\int_0^1 e^{-x^2}\, dx$$

within 0.01.

(1) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

substitute $-x^2$ for x above

Since the series for e^x converges to e^x for all $x \in \mathbb{R}$, the series for e^{-x^2} converges to e^{-x^2} for all $x \in \mathbb{R}$. So the radius of convergence is infinite.

(2) $T_2(x) = 1 - x^2$ (sum of the series for e^{-x^2} up to the x^2 term, not always just the first two terms of the series – i.e., $T_2(x)$ for e^x is $1 + x + \frac{x^2}{2}$; 3 terms)

$T_2(0.2) = 1 - (0.2)^2 = 1 - 0.04 = 0.96$

$e^{-0.04}$ is e^{-x^2} evaluated at $x = 0.2$, so is approx. $T(0.2) = 0.96$

The error in this approx. could, in principle, be calculated using the error estimate for alternating series, but this does not take into account $|f^{(3)}(x)| \le 0.6$ for x in $(-0.2, 0.2)$, so we have to use Taylor's Formula (see p.764 in Stewart)

By Taylor's formula, for x in the interval $I = (-0.2, 0.2)$, there is a z between $a = 0$ and x such that $R_2(x) = \frac{f^{3}(z)}{3!} x^3$

see the margin, or just use (9) on p.763 here

Then $|R_2(x)| = \left|\frac{f^{3}(z)}{6} \cdot x^3\right| \le \frac{0.6}{6}|x|^3$ since $|f^{(3)}(x)| \le 0.6$ in I

$< \frac{0.6}{6}(0.2)^3$ since $|x| < 0.2$ for x in I

$= 0.0008$, so the error is at most 0.0008

(3) $\int_0^1 e^{-x^2}dx = \int_0^1 (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \cdots)dx$ answer from part (1)

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

This series is alternating, so $1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}$ approximates the sum of the series with error $< \frac{1}{216} < 0.01$

So within 0.01, $\int_0^1 e^{-x^2}dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{5616}{7560} = \frac{26}{35}$

(25 pts.) VI. Solve the following initial value problem

$$t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$$

and indicate the maximal interval of definition for its solution.

Linear, First Order – use techniques from section 2.1 of Boyce & DiPrima.

Divide out coefficient of y':

$y' + \frac{4}{t} y = \frac{e^{-t}}{t^3}$ is the normal form, $p(t) = \frac{4}{t}$, $g(t) = \frac{e^{-t}}{t^3}$.

Find integrating factor:

$\mu(t) = e^{\int p(t)} = e^{\int \frac{4}{t}} = e^{4\ln|t|} = |t|^4 = t^4$

note: t can be negative – see initial condition

since 4 is even, absolute value doesn't matter.

Multiply normal form of equation by $\mu(t)$:

$t^4 y' + 4t^3 y = e^{-t} \cdot t$

$\Rightarrow \frac{d}{dt}(t^4 y) = e^{-t} \cdot t$

$\Rightarrow t^4 y = \int e^{-t} \cdot t \, dt$ – Integrate by parts:

$u = t \quad dv = e^{-t} dt$

$du = dt \quad v = -e^{-t}$

$\Rightarrow t^4 y = uv - \int v\,du = -te^{-t} - \int -e^{-t} dt$

$= -te^{-t} + \int e^{-t} dt$

$= -te^{-t} - e^{-t} + C$

so $y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$

Evaluate C using initial condition $y(-1) = 0$:

$0 = -\underbrace{\frac{e}{(-1)^3}}_{=e} - \underbrace{\frac{e}{(-1)^4}}_{=e} + \frac{C}{(-1)^4} \Rightarrow C = 0$

so $y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4}$.

Defined for $t \neq 0$, but initial condition has $t_0 = -1$, so interval of definition is $(-\infty, 0)$.

(25 pts.) VII. Find the explicit solution of the following initial value problem

$$x\,dx + ye^{-x}\,dy = 0\,, \quad y(0) = 1\,,$$

and indicate the maximal interval of definition for its solution.

$x\,dx + ye^{-x}dy = 0$. Hmmm... might be separable.

Try to separate:

$e^{-x} \cdot y\,dy = -x\,dx \Rightarrow y\,dy = -xe^{x}dx$ yes, it is separable!

Integrate: $\frac{y^2}{2} = \int -xe^{x}dx$: Integrate by parts (or, u-substitute $u=-x$ and use integral from problem VI.)

$u = -x \quad dv = e^{x}dx$

$du = -dx \quad v = e^{x}$

$\frac{y^2}{2} = uv - \int v\,du = -xe^{x} - \int e^{x}dx = -xe^{x} + e^{x} + C$

Evaluate C using $y(0)=1$:

$\frac{1^2}{2} = -0e^{0} + e^{0} + C \Rightarrow C = \frac{1}{2} - 1 = -\frac{1}{2}$

So Implicit Solution: $\frac{y^2}{2} = -xe^{x} + e^{x} - \frac{1}{2}$

or $y^2 = -2xe^{x} + 2e^{x} - 1$

For Explicit Solution, take square root:

$y = \pm\sqrt{-2xe^{x} + 2e^{x} - 1}$

To decide if we want $\pm$, notice initial condition $y(0)=1$ y is positive

so the explicit solution is $y = \sqrt{-2xe^{x} + 2e^{x} - 1}$

Maximal interval of definition: need $-2xe^{x} + 2e^{x} - 1 \geq 0$

$-2xe^{x} + 2e^{x} - 1$ cannot be solved for x explicitly, so ignore this part

(25 pts.) VIII. (1) Show that the differential equation

$$ydx + (2x - ye^y)dy = 0$$

is not exact.

(2) Show that the equation in (1) becomes exact when multiplied by the factor of integration $\mu(x, y) = y$.

(3) Solve the equation in (1).

1) $\underbrace{y}_{M}dx + \underbrace{(2x-ye^y)}_{N}dy = 0$

This is exact only if $M_y = N_x$.

$$M_y = \frac{\partial y}{\partial y} = 1, \; N_x = \frac{\partial(2x-ye^y)}{\partial x} = 2$$

$1 \neq 2$, so it is not exact.

2) Multiplying by $\mu(x,y) = y$, we have

$$\underbrace{y^2}_{M}dx + \underbrace{(2xy - y^2e^y)}_{N}dy = 0.$$

$$M_y = \frac{\partial(y^2)}{\partial y} = 2y, \; N_x = \frac{\partial(2xy - y^2e^y)}{\partial x} = 2y$$

Since $M_y = N_x$, it is exact.

3) Need $\Psi(x,y)$ with $\Psi_x = M = y^2$, $\Psi_y = N = 2xy - y^2e^y$

$$\Psi_x = y^2 \Rightarrow \Psi = xy^2 + h(y)$$

Taking derivative w.r.t. y,

$$\Psi_y = 2xy + h'(y) = N = 2xy - y^2e^y$$

$$\Rightarrow h'(y) = -y^2e^y \Rightarrow h(y) = \int -y^2e^y dy$$

Integrate by parts:

$u = -y^2 \quad dv = e^y dy$

$du = -2y dy \quad v = e^y$

$$\Rightarrow h(y) = uv - \int v du = -y^2e^y - \int -2ye^y dy.$$

$$= -y^2e^y + \int 2ye^y dy$$

Integrate by parts again:

$$u = 2y \qquad dv = e^y dy$$
$$du = 2dy \qquad v = e^y \Rightarrow h(y) = y^2e^y + [uv - \int v\,du]$$
$$= y^2e^y + 2ye^y - \int 2e^y dy$$
$$= y^2e^y + 2ye^y - 2e^y + C$$

Substituting back into $\Psi(x,y)$:

$$\Psi(x,y) = xy^2 + h(y) = xy^2 + y^2e^y + 2ye^y - 2e^y + C$$

so $xy^2 + y^2e^y + 2ye^y - 2e^y = C'$ for some constant C'

This gives the solution y implicitly, and since it cannot be solved for y explicitly, we leave it in implicit form.