<u>Note</u> For 1*) and 2*) above, see ex. 8 (Section 7.6), ex. 25 (Section 10.3)
10 26

1)* Let R be a comm. ring $(0_R \neq 1_R)$ and $S \subseteq R$ a multiplicatively closed set. Let $\{M_i\}_{i\in I}$ be a directed system of R-modules with respect to R-module morphisms $\varphi_{ij} : M_i \longrightarrow M_j$, $\forall i,j \in I$ such that $i \leq j$.

a) Show that $\{S^{-1}M_i\}_{i\in I}$ is a directed system of $S^{-1}R$-modules with respect to $\{S^{-1}\varphi_{ij}\}_{\substack{i,j\in I\\ i\leq j}}$.

b) Show that there is a canonical $S^{-1}R$-module isomorphism

$$\varinjlim_{i\in I} S^{-1}M_i \xrightarrow{\sim} S^{-1}\varinjlim_{i\in I} M_i$$

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2)* Give an example of an inverse system $\{M_i\}_{i\in I}$ of R-modules, such that

$$\varprojlim_{i\in I} S^{-1}M_i \not\xrightarrow{\sim} S^{-1}\varprojlim_{i\in I} M_i$$

as $S^{-1}R$-modules, for some multiplicatively closed subset $S \subseteq R$.

3*) Let $n \in \mathbb{N}$, p prime number, $S_p := \mathbb{Z} \setminus p\mathbb{Z}$.

a) Show that there is a canonical $\mathbb{Z}_{(p)}$-module isomorphism

$$S_p^{-1}\left(\mathbb{Z}/n\mathbb{Z}\right) \simeq \begin{cases} \mathbb{Z}/p^\alpha\mathbb{Z}, & \text{if } p \mid n \\ 0, & \text{otherwise} \end{cases}$$

where p^α is the largest power of p dividing n.

(Note: First show that $\mathbb{Z}/p^\alpha\mathbb{Z} \simeq \mathbb{Z}_{(p)}/p^\alpha\mathbb{Z}_{(p)}$, and therefore $\mathbb{Z}/p^\alpha\mathbb{Z}$ has a canonical $\mathbb{Z}_{(p)}$-module structure.)

b) Generalize the result in a) by replacing $\mathbb{Z}$ with an arbitrary PID R and $\mathbb{Z}/n\mathbb{Z}$ with an arbitrary cyclic, torsion R-module M. (Note: A cyclic R-module is an R-module generated by one element.)

4*) Let $f \in \mathbb{Z}[x] \setminus \mathbb{Z}$, such that $\gcd(f, f') = 1$. (f' is the formal derivative $\frac{d}{dx} f$ viewed in $\mathbb{Z}[x]$.)

Let S be the set of non-zero divisors in $\mathbb{Z}[x]/(f)$.

a) Show that $S^{-1}\left(\mathbb{Z}[x]/(f)\right)$ is isomorphic (as a ring) to a finite direct sum of fields.

b) Is the condition "$\gcd(f, f') = 1$" imposed in a) necessary? Justify with an example.

5*) a) Use the result in 4*) to show that ~~the~~ if G is a finite cyclic group, then the total ring of fractions of $\mathbb{Z}[G]$ is isomorphic to a direct sum of fields.

b) What can you ~~say~~ say about the total ring of fractions of $\mathbb{Z}[G]$, if G is a finite ~~pr~~ direct product of finite cyclic groups?