SOLUTIONS

Exam 2, Mathematics 20F
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Name:
SSN:
Section Number:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

(40 pts.) I. Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the function defined by

$$L((x_1, x_2, x_3)^T) = (-x_1 + 2x_2, x_2 + x_3, x_2 + x_3)^T,$$

for all vectors $\vec{x} = (x_1, x_2, x_3)^T$ in $\mathbb{R}^3$.

(1) Show that L is a linear transformation.
(2) Show that if $\vec{b}_1 = (1,1,1)^T$, $\vec{b}_2 = (1,1,0)^T$, $\vec{b}_3 = (1,0,0)^T$, then $B = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ is an ordered basis for $\mathbb{R}^3$.
(3) Find the matrix associated to L with respect to the basis B.
(4) Find bases for the kernel $\mathrm{Ker}(L)$ and the image $\mathrm{Im}(L)$ of L.

(1) Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$.

Then
$$L(\vec{x}+\vec{y}) = \begin{pmatrix} -(x_1+y_1) + 2(x_2+y_2) \\ (x_2+y_2) + (x_3+y_3) \\ (x_2+y_2) + (x_3+y_3) \end{pmatrix} = \begin{pmatrix} -x_1+2x_2 \\ x_2+x_3 \\ x_2+x_3 \end{pmatrix} + \begin{pmatrix} -y_1+2y_2 \\ y_2+y_3 \\ y_2+y_3 \end{pmatrix} =$$
$$= L(\vec{x}) + L(\vec{y}). \quad (*)$$

$$L(\alpha\vec{x}) = \begin{pmatrix} -\alpha x_1 + 2\alpha x_2 \\ \alpha x_2 + \alpha x_3 \\ \alpha x_2 + \alpha x_3 \end{pmatrix} = \alpha \cdot \begin{pmatrix} -x_1+2x_2 \\ x_2+x_3 \\ x_2+x_3 \end{pmatrix} = \alpha \cdot L(\vec{x}) \quad (**)$$

$\left.\begin{matrix} (*) \\ (**) \end{matrix}\right\} \Rightarrow L$ is a linear transformation.

(2) $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\det(B) = -1 \neq 0 \Rightarrow [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ is a basis.

③ $\left[\vec{b}_1, \vec{b}_2, \vec{b}_3 \,\middle|\, L(\vec{b}_1), L(\vec{b}_2), L(\vec{b}_3)\right] =$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow[\text{row3} - \text{row1}]{\text{row2} - \text{row1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{row2} \leftrightarrow \text{row3}]{} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[(-1)\,\text{row3}]{(-1)\cdot\text{row2}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right] \longrightarrow$$

$$\xrightarrow[\text{row2} - \text{row3}]{\text{row1} - \text{row2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

4. ~~Ker(L)~~ Let $M := \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. This is the matrix of L in the standard basis $E := [\vec{e}_1, \vec{e}_2, \vec{e}_3]$ of $\mathbb{R}^3$.

Then $\mathrm{Ker}(L) = \mathrm{Nul}(M)$, $\mathrm{Im}(L) = \mathrm{Col}(M)$.

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{row1} \times (-1)]{} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{row3} - \text{row2}]{\text{row1} + 2\text{row2}} \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ 1 \\ 0 \end{matrix} \end{bmatrix}$$

$$\mathrm{Col}(M) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$$\text{Basis for } \mathrm{Col}(M) = \left[\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right].$$

Determine Nul(M)

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

x_1, x_2 lead variables

x_3 free variable

$$\mathrm{Nul}(M) = \left\{ \begin{pmatrix} -2x_3 \\ -x_3 \\ x_3 \end{pmatrix} \middle| \, x_3 \in \mathbb{R} \right\} =$$

$$= \left\{ x_3 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$$

$$\mathrm{Nul}(M) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Basis for $\mathrm{Nul}(M)$ $= \left[\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right].$

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II. (30 pts) Consider the following 3×4 matrix

$$A := \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}.$$

(1) Construct bases for the column space Col(A), row space Row(A), and null space N(A) of A.
(2) What is the rank of A ?
(3) Do the columns of A span $\mathbb{R}^3$?
(4) Are the columns of A linearly independent ?

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \xrightarrow[\text{row echelon form}]{\text{reduced}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) Col (A) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \right\}$

Row (A) $\left\{ (1, 2, 0, 3)^T, (0\ 0\ 1\ 2)^T \right\}$.

Nul (A)

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases}$$

with x_2 and x_4 free variables
x_1 and x_3 lead variables

$$\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = -2x_4 \end{cases}$$

$$\text{Nul}(A) = \left\{ \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{pmatrix} \middle| x_2, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \middle| x_2, x_4 \in \mathbb{R} \right\}$$

Basis for Nul(A) : = $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$.

(2) $\text{rank}(A) = \dim \text{Col}(A) = 2$

(3) No! The columns of A span $\text{Col}(A)$, which is a two dimensional subspace of $\mathbb{R}^3$, therefore smaller than $\mathbb{R}^3$.

(4) No! If they were ~~then~~ linearly independent, then $\dim \text{Col}(A) = 4$, which is false! Therefore, the columns of A are linearly dependent.

III. (30 pts.) As usual, let $\mathbb{P}_n$ denote the space of polynomials in variable x, with real coefficients, of degree at most $n-1$. Let L be the function

$$L : \mathbb{P}_3 \longrightarrow \mathbb{P}_2 ,$$

defined by $L(P) = P'' + P'$, for all polynomials P in $\mathbb{P}_3$.

(1) Show that L is a linear transformation.

(2) Find the matrix of L with respect to the standard bases of $\mathbb{P}^3$ and $\mathbb{P}^2$, respectively

$$E = [1, \quad x, \quad x^2] \qquad F = [1, \quad x] .$$

(3) Is the subset

$$\{P \in \mathbb{P}_3 \,|\, P'' + P' = 0\}$$

a vector subspace of $\mathbb{P}_3$? Justify your answer.

(1) Let $P, Q \in \mathbb{P}_3$. Then $\alpha \in \mathbb{R}$

$L(P+Q) = (P+Q)'' + (P+Q)' = (P''+Q'')+(P'+Q') =$
$= (P''+P') + (Q''+Q') = L(P)+L(Q)$.

$L(\alpha P) = (\alpha P)'' + (\alpha P)' = \alpha P'' + \alpha P' = \alpha(P''+P') =$
$= \alpha L(P)$.

(2) $[L(1)]_F = (1''+1')_F = [0]_F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$[L(x)]_F = (x''+x')_F = [1]_F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$[L(x^2)]_F = (2+2x)_F = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$A = \left([L(1)]_F, [L(x)]_F, [L(x^2)]_F\right) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

(3) Yes! The subset $\{P \in \mathbb{P}_3 \,|\, P''+P'=0\}$ is the kernel of L, $\ker(L)$. The kernel of any linear transformation is a vector subspace of its domain.