SOLUTIONS

Exam 1, Mathematics 20F
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January 30, 2004

Name:
SSN:
Section Number:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

(40 pts.) I. Consider the following system of linear equations.

$$\begin{cases} x_1 + x_2 + x_3 + \quad x_4 + \quad x_5 = 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2 \end{cases}$$

(1) Find the reduced echelon form of the augmented matrix associated to the system above.
(2) Determine if the system above is consistent or not. If it is consistent, describe its full solution set.
(3) Use the answer you obtained in (2) to write the vector

$$\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

as an explicit linear combination of the following vectors

$$\vec{a_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{a_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{a_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{a_4} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \vec{a_5} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(1) $$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \xrightarrow[r_3 - r_1]{r_2 - r_1} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right) \to$$

$$\xrightarrow{r_3 - r_2} \left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{array}\right) \xrightarrow[r_1 - r_2]{} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right) \to$$

$$\xrightarrow[r_2 - r_3]{} \boxed{\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{array}\right)}$$

(2) The system is consistent, as the reduced echelon form of its augmented matrix does not contain any row of type

$$(0\ 0\ 0\ 0\ 0 \mid \alpha)$$

with $\alpha \neq 0$.

Free variables : x_2, x_3.
Lead variables : x_1, x_4, x_5.

$$\begin{cases} x_1 = 1 - x_2 - x_3. \\ x_4 = 2 \\ x_5 = -1. \end{cases}$$

Solution set :

(*) $\{(1-\alpha-\beta, \alpha, \beta, 2, -1) \mid \alpha, \beta \in \mathbb{R}\}$.

(3). $\vec{b}$ is the vector of free terms and $\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{a_4}, \vec{a_5}$ are the column vectors of the matrix of coefficients in the linear system under consideration.

Therefore, if $(x_1, x_2, x_3, x_4, x_5)$ is a solution to the linear system, $\vec{b} = x_1\vec{a_1} + x_2\vec{a_2} + x_3\vec{a_3} + x_4\vec{a_4} + x_5\vec{a_5}$.

Let $\alpha = 0$, $\beta = 0$ in (*) above. This leads to the following solution of our linear system $(x_1, x_2, x_3, x_4, x_5) = (1, 0, 0, 2, -1)$.

Therefore $\vec{b} = \cancel{1\cdot x_1}\ 1\cdot\vec{a_1} + 0\cdot\vec{a_2} + 0\cdot\vec{a_3} + 2\vec{a_4} + (-1)\vec{a_5}$.

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(40 pts.) II. Consider the following 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

(1) Compute $\det(A)$ and show that A is non–singular.
(2) Compute the inverse A^{-1} of A.
(3) Use the answer you obtained in (2) to find the unique solution

$$\vec{x} = (\, x_1 \quad x_2 \quad x_3\,)^T$$

of the linear system $A \cdot \vec{x} = \vec{b}$, where

$$\vec{b} = (1 \quad 2 \quad 3)^T \,.$$

(1) Expand $\det(A)$ with respect to row 1.

$$\det(A) = 1 \cdot (-1)^{1+1} \cdot \det\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} + 0 \cdot A_{12} + 1 \cdot (-1)^{1+3} \det\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} =$$

$$= 1 \cdot (-1)^{1+1} \cdot 1 + 0 + 1 \cdot (-1)^{1+3} \cdot 0 = \boxed{1}$$

Since $\det(A) = 1 \neq 0$, A is nonsingular.

(2) $$(A \mid I_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right) \rightarrow$$

$$\xrightarrow[r_2 - r_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & -1 & 1 & -1 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right) \xrightarrow[r_3 - 2r_2]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right) \longrightarrow$$

$$\xrightarrow[r_1 - r_3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right) = (I_3 \mid A^{-1}) \Longrightarrow$$

$$\Longrightarrow \boxed{A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}}$$

(3) $A^{-1} \,|\, A\vec{x} = \vec{b}$

$$\Downarrow$$

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$\Downarrow$$

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b} \Rightarrow I_3 \cdot \vec{x} = A^{-1}\vec{b} \Rightarrow$$

$$\Rightarrow \vec{x} = A^{-1}\vec{b} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix}$$

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(20 pts.) III.

(1) Give an example of two square matrices A and B of the same dimension, such that

$$A \cdot B \neq B \cdot A .$$

(2) Give an example of a square matrix A which is not the 0–matrix (i.e. not all the entries of A are equal to 0) whose square is the 0–matrix (i.e. $A^2 = 0$).

(3) Let A and B be two $n \times n$ non–singular matrices. Prove that

$$\det(B^{-1} \cdot (AB)^T) = \det(A) .$$

(1) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$A \cdot B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\therefore$ $\underline{\underline{AB \neq BA}}$

(2) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(3) $\det\left(B^{-1} \cdot (AB)^T\right) \overset{\text{Thm.}}{=} \det\left(B^{-1} \cdot (B^T \cdot A^T)\right) =$

$\underset{\text{Thm}}{=} \det(B^{-1}) \cdot \det(B^T \cdot A^T) \underset{\text{Thm.}}{=} \det(B^{-1}) \cdot \det(B^T) \cdot \det(A^T) =$

$\underset{\text{Thm.}}{=} \left(\det(B^{-1}) \cdot \det(B)\right) \cdot \det(A) \underset{\text{Thm.}}{=} \det(B^{-1}B) \cdot \det(A) =$

$\underset{\text{Thm.}}{=} \det(I_n) \cdot \det(A) = 1 \cdot \det(A) = \det(A)$ $\blacksquare$