§3.2

1 (a) Yes
(b) No
(c) Yes
(d) No
(e) No

Lets do a yes and a no. First we show (a) is a subspace.

(a) $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1 + x_2 = 0 \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1 = -x_2 \right\} = \left\{ \begin{pmatrix} -s \\ s \end{pmatrix} \middle| s \text{ any } \# \right\}$.

If $\vec{x}, \vec{y} \in S$ and $\vec{x} = \begin{pmatrix} -s \\ s \end{pmatrix}$, $\vec{y} = \begin{pmatrix} -t \\ t \end{pmatrix}$, then $\vec{x} + \vec{y} = \begin{pmatrix} -(s+t) \\ s+t \end{pmatrix}$ which satisfies the property of S $(x_1 = -x_2)$. Thus, $\vec{x} + \vec{y} \in S$.

If $\vec{x} \in S$, α is a # and $\vec{x} = \begin{pmatrix} s \\ -s \end{pmatrix}$, Then $\alpha x = \begin{pmatrix} \alpha s \\ -(\alpha s) \end{pmatrix}$ which satisfies the property $x_1 = -x_2$ and so is in S.

Since S satisfies the definition on page 134 for subspaces, S is a subspace.

(b) We will show $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1 x_2 = 0 \right\}$ is not a subspace. We do this by showing That S does not satisfy The property: If $\vec{x}, \vec{y} \in S$, Then $\vec{x} + \vec{y} \in S$.

We do this by example. If $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, Then $\vec{x}, \vec{y}$ are in S. But, $\vec{x} + \vec{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in S, since $1 \cdot 1 = 1 \neq 0$.

The other parts are proven or disproven the same way.

3 (a) Yes
(b) No
(c) No
(d) Yes
(e) Yes
(f) No

We prove a couple Yes's and disprove a couple No's.

(a) $S = \left\{ \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \middle| \; d_1, d_2 \text{ are any \#s} \right\} =$ set of 2x2 diagonal matrices.

We show S is a subspace in the usual way.

If $\vec{x} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\vec{y} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, then $\vec{x} + \vec{y} = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix}$

which is diagonal and so is in S.

If $\vec{x} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and α is any scalar, then

$\alpha \vec{x} = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix}$ which is diagonal and so is in S.

Thus, S is a subspace.

(f) $S = \{\text{singular } 2\times2 \text{ matrices}\}$.

S is <u>not</u> a subspace. Let $\vec{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$\vec{x}, \vec{y}$ are both singular $(\det(\vec{x}) = \det(\vec{y}) = 0)$, but

$\vec{x} + \vec{y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is not singular.

So, S is not closed under addition and hence

it is not a subspace.

(c) $S = \left\{ \begin{pmatrix} a_{11} & 1 \\ a_{21} & a_{22} \end{pmatrix} \right\}$.

$\vec{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are both in S, but

$\vec{x} + \vec{y} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ which is not in S.

So, S is not a subspace.

(d) $S = \left\{ \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\}$.

If $\vec{x} = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 0 & d \\ e & f \end{pmatrix}$ are in S, Then $\vec{x} + \vec{y} =$

$= \begin{pmatrix} 0 & a+d \\ b+e & c+f \end{pmatrix}$ which is in S. And if α is a

scalar and $\vec{x} = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \in S$, then $\alpha\vec{x} = \begin{pmatrix} 0 & \alpha a \\ \alpha b & \alpha c \end{pmatrix}$

which is in S. So, S is a subspace.

④ (a) ($\text{Nullspace}(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$)

If $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, we want the $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where $A\vec{x} = \vec{0}$.

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \qquad \left. \begin{aligned} x_1 + \tfrac{1}{2}x_2 &= 0 \\ \tfrac{1}{2}x_2 &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} x_2 &= 0 \\ x_1 &= 0. \end{aligned}$$

$\text{Nullspace}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

(d) $$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Get : $$\left. \begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 0 \\ -x_3 - 3x_4 &= 0 \end{aligned} \right\} \quad \begin{aligned} x_1 &= -x_2 + x_3 - 2x_4 = -t - 3s - 2s \\ x_2 &= t \\ x_3 &= -3x_4 = -3s \\ x_4 &= s \end{aligned}$$

$$\text{Nullspace}(A) = \left\{ \begin{pmatrix} -t-5s \\ t \\ -3s \\ s \end{pmatrix} \middle| \ t, s \text{ any } \# \right\} = \left\{ t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix} \middle| \ t, s \text{ any } \# \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}$$ ← Note: This is 2 dimensional

$P_4 = \{a+bx+cx^2+dx^3 \mid a,b,c,d \text{ any } \#\}$

5 (a) $S = \{$polys in P_4 of even degree$\}$

Let $p(x) = x+x^2$, $q(x) = x-x^2$. Then p,q are in S. But, $p(x)+q(x) = 2x$ which does not have even degree and so is not in S. Thus, S is not a subspace of P_4.

(b) $S = \{$polys in P_4 of degree $=3\}$.

Let $p(x) = x^3+x$, $q(x) = -x^3+x$. Then $p(x)+q(x) = 2x$ which is not in S, but $p,q \in S$. Hence, S is not a subspace of P_4.

(c) $S = \{p(x) \in P_4 \mid p(0)=0\}$.

Let $p(x), q(x) \in S$, ie $\deg(p), \deg(q) \leq 3$ and $p(0) = 0 = q(0)$. Then $(p+q)(0) = p(0)+q(0) = 0+0=0$. Hence $(p+q)(x) \in S$. If α is a scalar, Then $(\alpha p)(0) = \alpha p(0) = \alpha \cdot 0 = 0$. So, $(\alpha p)(x)$ is in S. Thus, S is a subspace of P_4.

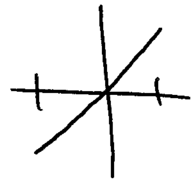
(d) $S = \{p \in P_4 \mid p(x)$ has at least one real root$\}$.

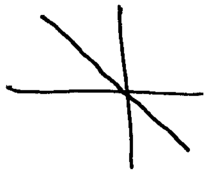
Let $p(x) = x^2-x$, $q(x) = x+1$. The roots of $p(x)$ are $0,1$ and The roots of $q(x)$ are -1. But, $(p+q)(x) = x^2+1$ which has roots $\pm i$. Thus, $p,q \in S$, but $p+q \notin S$. So, S is not a subspace of P_4.

6 (a) $S = \{ f \in C[-1,1] \mid f(-1) = f(1) \}$.

If $f, g \in S$, then $f(-1) = f(1)$ and $g(-1) = g(1)$. And, $(f+g)(-1) = f(-1) + g(-1) = f(1) + g(1) = (f+g)(1)$. If α is a scalar, then $(\alpha f)(-1) = \alpha f(-1) = \alpha f(1) = (\alpha f)(1)$. So, if $f \in S$, then $\alpha f \in S$. So, S is a subspace of $C[-1,1]$.

(c) $S = \{$ non decreasing functions in $C[-1,1] \}$.

$f(x) = x$ is not decreasing. Look at picture 

But, $(-1) \cdot x = -x$ is not .

So, S is not closed under scalar multiplication and so is not a subspace.

(e) $S = \{ f \in C[-1,1] \mid f(-1) = 0$ or $f(1) = 0 \}$.

Let $f(x) = x+1$, $g(x) = -x+1$. Then $f(-1) = 0$ and $g(1) = 0$. So, $f, g \in S$. But, $f+g = 2$. And $(f+g)(-1) = 2 \neq 0$ and $(f+g)(1) = 2 \neq 0$. Thus, $f+g \notin S$. So, S is not a subspace.

9 (a) Since we know about rowspace now, we can do this the easy way. $\text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} =$ $= \text{rowspace} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Find rowspace by reducing matrix to row echelon form.

$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$. Basis for rowspace is $\left\{ \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right\}$ = basis for $\text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$.
So, span is 2 dimensional and is $= \mathbb{R}^2$.

(d) Do same thing. $\text{span}\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}2\\-4\end{pmatrix}\right\}=$

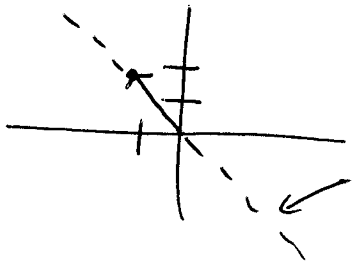
$=\text{rowspace}\begin{pmatrix}-1 & 2\\ 1 & -2\\ 2 & -4\end{pmatrix}.$

$\begin{pmatrix}-1 & 2\\ 1 & -2\\ 2 & -4\end{pmatrix}\rightarrow\begin{pmatrix}-1 & 2\\ 0 & 0\\ 0 & 0\end{pmatrix}$. So, basis for $\text{rowspace}\begin{pmatrix}-1 & 2\\ 1 & -2\\ 2 & -4\end{pmatrix}=$

$=\text{span}\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}2\\-4\end{pmatrix}\right\}$ is $\left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}$. Thus,

$\text{span}\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right\}=\text{span}\left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}$

Heres a picture



this line is $\text{span}\left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}$

So, the 3 vectors do not span $\mathbb{R}^2$. They span a line. They all lie on the line $y=-2x$.

10 (a) $\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}=\text{rowspace}\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}\right\}$

$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$. So, $\text{rowspace}\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}\right\}=$

$=\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}=\mathbb{R}^3.$

(d) $\text{span}\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix},\begin{pmatrix}-2\\-1\\2\end{pmatrix},\begin{pmatrix}4\\2\\-4\end{pmatrix}\right\}=\text{rowspace}\left\{\begin{pmatrix}2 & 1 & -2\\ -2 & -1 & 2\\ 4 & 2 & -4\end{pmatrix}\right\}$

$\begin{pmatrix}2 & 1 & -2\\ -2 & 1 & 2\\ 4 & 2 & -4\end{pmatrix}\rightarrow\begin{pmatrix}2 & 1 & -2\\ 0 & 2 & 0\\ 0 & 0 & 0\end{pmatrix}$. So, $\text{span}\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix},\begin{pmatrix}-2\\-1\\2\end{pmatrix},\begin{pmatrix}4\\2\\-4\end{pmatrix}\right\}=$

$=\text{span}\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right\}\leftarrow$ 2 dimensional. Its a plane through the origin.

So, doesn't span $\mathbb{R}^3$.

11 (a) Is $\vec{x} = \alpha\vec{x}_1 + \beta\vec{x}_2$ for some α, β ?

$$\begin{pmatrix}2\\6\\6\end{pmatrix} = \alpha\begin{pmatrix}-1\\2\\3\end{pmatrix} + \beta\begin{pmatrix}3\\4\\2\end{pmatrix} = \begin{pmatrix}-\alpha+3\beta\\2\alpha+4\beta\\3\alpha+2\beta\end{pmatrix}$$

Are there solutions to

$$\begin{aligned}2 &= -\alpha + 3\beta\\ 6 &= 2\alpha + 4\beta\\ 6 &= 3\alpha + 2\beta\end{aligned} \quad ??$$

$$\left(\begin{array}{cc|c}-1 & 3 & 2\\ 2 & 4 & 6\\ 3 & 2 & 6\end{array}\right) \to \left(\begin{array}{cc|c}1 & -3 & -2\\ 2 & 4 & 6\\ 3 & 2 & 6\end{array}\right) \to \left(\begin{array}{cc|c}1 & -3 & -2\\ 0 & 10 & 10\\ 0 & 11 & 12\end{array}\right) \to$$

$$\to \left(\begin{array}{cc|c}1 & -3 & -2\\ 0 & 1 & 1\\ 0 & 11 & 12\end{array}\right) \to \left(\begin{array}{cc|c}1 & -3 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{array}\right) \leftarrow \text{No solutions.}$$

$\vec{x} \notin \text{span}\{\vec{x}_1, \vec{x}_2\}$.

(b) Is $\vec{y} = \alpha\vec{x}_1 + \beta\vec{x}_2$?

$$\begin{pmatrix}-9\\-2\\5\end{pmatrix} = \alpha\begin{pmatrix}-1\\2\\3\end{pmatrix} + \beta\begin{pmatrix}3\\4\\2\end{pmatrix} = \begin{pmatrix}-\alpha+3\beta\\2\alpha+4\beta\\3\alpha+2\beta\end{pmatrix}$$

$$\left(\begin{array}{cc|c}-1 & 3 & -9\\ 2 & 4 & -2\\ 3 & 2 & 5\end{array}\right) \to \left(\begin{array}{cc|c}1 & -3 & 9\\ 2 & 4 & -2\\ 3 & 2 & 5\end{array}\right) \to \left(\begin{array}{cc|c}1 & -3 & 9\\ 0 & 10 & -20\\ 0 & 11 & -22\end{array}\right) \to \left(\begin{array}{cc|c}1 & -3 & 9\\ 0 & 1 & -2\\ 0 & 0 & 0\end{array}\right)$$

$$\left.\begin{aligned}\alpha - 3\beta &= 9\\ \beta &= -2\end{aligned}\right\} \quad \begin{aligned}\alpha &= 9 + 3\beta = 9 - 6 = 3\\ \beta &= -2.\end{aligned}$$

So, $\vec{y} = \begin{pmatrix}-9\\-2\\5\end{pmatrix} = \alpha\vec{x}_1 + \beta\vec{x}_2 = 3\begin{pmatrix}-1\\2\\3\end{pmatrix} + (-2)\begin{pmatrix}3\\4\\2\end{pmatrix}$

So, <u>Yes.</u> $\vec{y} \in \text{span}\{\vec{x}_1, \vec{x}_2\}$.

(14) (a) $\{1, x^2, x^2-2\}$

x is not in span $\{1, x^2, x^2-2\}$ since

$x = a\cdot 1 + b\cdot x^2 + c\cdot(x^2-2)$ has no solutions.

i.e. $x = (a-2c)\cdot 1 + 0\cdot x + (b+c)x^2$

cannot be solved for a, b, c since $1 \neq 0$.

Thus, $\{1, x^2, x^2-2\}$ ~~do not span P_3.~~

are not a spanning set for P_3.

(b) $\{2, x^2, x, 2x+3\}$.

Span $\{2, x^2, x, 2x+3\} = P_3$ since

$a + bx + cx^2 = \left(\frac{a}{2}\right)\cdot 2 + (b)x + (c)x^2 + (0)\cdot(2x+3)$.

(17) Use thm 1.4.3 pg. 71.

(a) $\Rightarrow$ (b) If $N(A) = \{0\}$, then $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$. Then thm 1.4.3 says that A is nonsingular.

(b) $\Rightarrow$ (c). If A is nonsingular. Then the equation $A\vec{x} = \vec{b}$ has unique solution $\vec{x} = A^{-1}\vec{b}$ since

if $A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = A^{-1}b \Rightarrow I\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$.

And $A^{-1}\vec{b}$ is a solution.

(c) $\Rightarrow$ (a) Take $\vec{b} = \vec{0}$ in (c). Then by hypothesis $A\vec{x} = \vec{0}$ has only one solution $\vec{x} = \vec{0}$. Then, $N(A) = \{\vec{0}\}$.

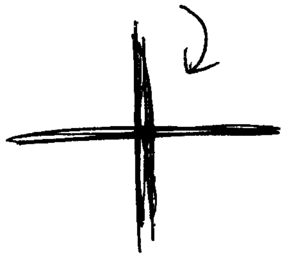
18. **Proof:** Let $\vec{x},\vec{y} \in U \cap V$. Then $\vec{x} \in U$ & $\vec{x} \in V$ and $\vec{y} \in U$ & $\vec{y} \in V$. Since $\vec{x},\vec{y} \in U$ and U is a subspace of W, $\vec{x}+\vec{y} \in U$. Since $\vec{x},\vec{y} \in V$ and V is a subspace of W, $\vec{x}+\vec{y} \in V$. Since, $\vec{x}+\vec{y} \in U$ and $\vec{x}+\vec{y} \in V \Rightarrow \vec{x}+\vec{y} \in U \cap V$.

Let $\vec{x} \in U \cap V$, α a scalar. Since $\vec{x} \in U \cap V$, $\vec{x} \in U$ and $\vec{x} \in V$. Since U is a subspace of W, $\alpha\vec{x} \in U$; Since V is a subspace of W, $\alpha\vec{x} \in V$. Since $\alpha\vec{x} \in U$ & $\alpha\vec{x} \in V \Rightarrow \alpha\vec{x} \in U \cap V$.

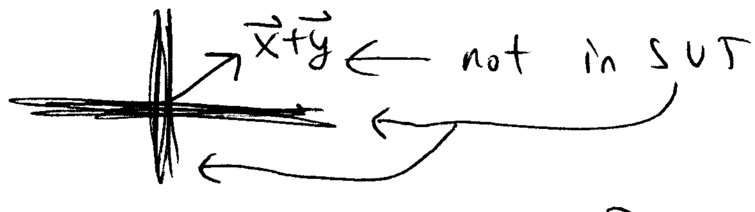
So, $U \cap V$ is a subspace of W.

19. No. $S =$ x-axis and $T =$ y-axis.

$S \cup T$ = x-axis + y-axis = $\left\{\begin{pmatrix} x \\ 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 0 \\ y \end{pmatrix}\right\}$.



But, $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S \cup T$, $\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S \cup T$, and $\vec{x}+\vec{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S \cup T$



So, $S \cup T$ is not a subspace of $\mathbb{R}^2$

3.3 #2 (a) Since $\det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 1 \neq 0$, the vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ are linearly independent.

(b) The vectors are not linearly independent. For example

$$0\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} + 1\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

(c) Not linearly independent, as for example

$$1\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix} + (-1)\begin{pmatrix}3\\2\\-2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(d) Linearly dependent

(e) Linearly independent as $c_1\begin{pmatrix}1\\1\\3\end{pmatrix} + c_2\begin{pmatrix}0\\2\\1\end{pmatrix} = \vec{0} \iff c_1 = 0$ and $c_2 = 0$.

#5. Let $\vec{x}_1, \dots, \vec{x}_k$ be linearly independent vectors in some vector space V.

(a) If we add the vector $\vec{x}_{k+1}$, the new collection may or may not be linearly independent. It will still be linearly independent if $\vec{x}_{k+1} \notin \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$, and will be lin. dependent if $\vec{x}_{k+1} \in \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$.

(b) If we delete $\vec{x}_k$, the collection $\vec{x}_1, \dots, \vec{x}_{k-1}$ is linearly independent. This is because $c_1\vec{x}_1 + \dots + c_{k-1}\vec{x}_{k-1} = \vec{0}$ implies $c_1\vec{x}_1 + \dots + c_{k-1}\vec{x}_{k-1} + 0\cdot\vec{x}_k = \vec{0}$, implies $c_1 = c_2 = \dots = c_{k-1} = 0$

3.3 #7 (a) $W(\cos \pi x, \sin \pi x) = \begin{vmatrix} \cos \pi x & \sin \pi x \\ -\pi \sin \pi x & \pi \cos \pi x \end{vmatrix}$

$= \pi(\cos^2 \pi x + \sin^2 \pi x) = \pi \neq 0.$ So the functions are linearly independent in $C[0,1]$.

(b) $W(x^{3/2}, x^{5/2}) = \begin{vmatrix} x^{3/2} & x^{5/2} \\ \frac{3}{2}x^{1/2} & \frac{5}{2}x^{3/2} \end{vmatrix} = \frac{5}{2}x^3 - \frac{3}{2}x^3 = x^3 \neq 0.$

(c) $W(1, e^x+e^{-x}, e^x-e^{-x}) = \begin{vmatrix} 1 & e^x+e^{-x} & e^x-e^{-x} \\ 0 & e^x-e^{-x} & e^x+e^{-x} \\ 0 & e^x+e^{-x} & e^x-e^{-x} \end{vmatrix}$

$= 1[(e^x-e^{-x})^2 - (e^x+e^{-x})^2] = -4 \neq 0$

(d) $W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} \overset{(-1)}{=} \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ 0 & -2e^{-x} & e^{2x} \\ 0 & 0 & 3e^{2x} \end{vmatrix}$

$= -6e^{2x} \neq 0.$

#11. Consider the collection $\vec{v}_1, \ldots, \vec{v}_{n-1}, \vec{0}$.

Then $0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \ldots + 0 \cdot \vec{v}_{n-1} + 1 \cdot \vec{0} = \vec{0}$

is a nontrivial linear combination of the vectors giving the zero vector.
(not all coefficients are zero.)

So $\vec{v}_1, \ldots, \vec{v}_{n-1}, \vec{0}$ is a linearly dependent set of vectors.

3.3 #14. Let A be an $m \times n$ matrix. Write $A = [\vec{a}_1 | \vec{a}_2 | \cdots | \vec{a}_n]$ and assume $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly independent. Let $\vec{x} \in N(A)$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. By the definition of the nullspace, $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$.

Since $\vec{a}_1, \ldots, \vec{a}_n$ are linearly independent we conclude that $x_1 = x_2 = \cdots = x_n = 0$, so $\vec{x} = \vec{0}$. Therefore $N(A) = \{\vec{0}\}$.

#16. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a spanning set for V, and $\vec{v} \in V$. Then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ for some scalars $c_1, \ldots, c_n$. Rearranging this we get

$$\vec{0} = (-1)\vec{v} + c_1\vec{v}_1 + \cdots + c_n\vec{v}_n,$$

a non-trivial linear combination that equals the zero vector. It follows $\vec{v}, \vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent.

#17. Let $\vec{v}_1, \ldots, \vec{v}_n$ be linearly independent vectors in V. Suppose (for the sake of contradiction) that $\vec{v}_2, \ldots, \vec{v}_n$ span V. Then, since $\vec{v}_1 \in V$, we'd have $\vec{v}_1 = c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_n\vec{v}_n$, that is $\vec{0} = (-1)\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$. This contradicts the fact that $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent.

(13)

3.4 #2. (a) They do form a basis.

(b) They don't form a basis

(c) Not a basis

(d) not a basis

(e) not a basis.

#4. Since $\vec{x}_2 = -\vec{x}_1$ and $x_3 = -2\vec{x}_1$,

$\operatorname{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = \operatorname{Span}\{\vec{x}_1\}$, and $\dim \operatorname{Span}\{\vec{x}_1, \vec{x}_2, x_3\}$

$= \dim \operatorname{span}\{\vec{x}_1\} = 1.$

#5. Let $\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$.

(a) $\det[\vec{x}_1 | \vec{x}_2 | \vec{x}_3] = 0$, so they are linearly dependent.

b) $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0} \Rightarrow c_1\vec{x}_1 = -c_2\vec{x}_2$. If $c_1 \neq 0$,

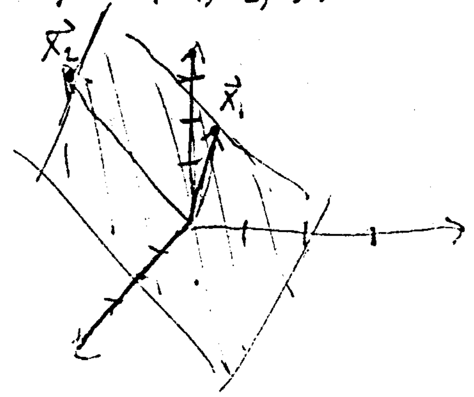
$\vec{x}_1 = -\frac{c_2}{c_1}\vec{x}_2 = \alpha\vec{x}_2$ which is clearly not the case, so $c_1 = 0$.

If $c_2 \neq 0$, we'd have $\vec{x}_2 = -\frac{c_1}{c_2}\vec{x}_1$, again not so, so $c_2 = 0$.

Thus $\vec{x}_1, \vec{x}_2$ are linearly independent

(c) $\dim \operatorname{Span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \overset{\text{part (a)}}{=} \dim \operatorname{Span}\{\vec{x}_1, \vec{x}_2\} \overset{\text{part b)}}{=} 2.$

(d) $\operatorname{Span}\{x_1, x_2, x_3\}$ is the plane spanned by $\vec{x}_1$ and $\vec{x}_2$



3.4 # 9 Let $\vec{a}_1, \vec{a}_2$ be linearly independent vectors in $\mathbb{R}^3$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

(a) Span $\{\vec{a}_1, \vec{a}_2\}$ is a plane in $\mathbb{R}^3$ that passes through the origin.

(b) If $A = (\vec{a}_1 | \vec{a}_2)$ and $b = A\vec{x}$, then $b = x_1 \vec{a}_1 + x_2 \vec{a}_2 \Rightarrow b \in \text{Span}\{\vec{a}_1, \vec{a}_2\}$, so that Span $\{\vec{a}_1, \vec{a}_2, b\} = \text{Span}\{\vec{a}_1, \vec{a}_2\}$.

#13. In $C[-\pi, \pi]$, $0 = (-1)\cos^2 x + \frac{1}{2} \cdot 1 + \frac{1}{2}\cos 2x$ (a trig identity) tells us $1, \cos 2x, \cos^2 x$ are linearly dependent. (One could also use the Wronskian to show this.) But $W(1, \cos 2x) = \begin{vmatrix} 1 & \cos 2x \\ 0 & -2\sin 2x \end{vmatrix} = -2\sin 2x \neq$ shows $1, \cos 2x$ are linearly independent. Hence dim span $\{1, \cos 2x, \cos^2 x\} = 2$.

#14 (a) Span $\{x, x-1, x^2-1\} = P_3$ (since they're lin. ind.)
Thus dim span $\{x, x-1, x^2-1\} = 3$.

(b) Again, the span is P_3.

(c) Since $x+1 = x^2 - (x^2 - x - 1)$, span $\{x^2, x^2-x-1, x+1\}$ $= \text{span}\{x^2, x^2-x-1\}$ so the dimension is 2.

(d) dim span $\{2x, x-2\} = 2$.