Exam 2, Mathematics 109
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Name:
Student ID:

Note: There are 3 questions on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (35 points)

Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be the functions given by

$$f(x) = \begin{cases} x - 1, & \text{if } x \text{ is even;} \\ x + 1, & \text{if } x \text{ is odd.} \end{cases} \qquad g(x) = \begin{cases} x + 1, & \text{if } x \text{ is even;} \\ x - 1, & \text{if } x \text{ is odd.} \end{cases}$$

(1) Is f bijective? Justify your answer.
(2) If the answer to question (1) above is affirmative compute the inverse function $f^{-1} : \mathbb{Z} \to \mathbb{Z}$.
(3) Compute $f \circ g$.

Important note: You may use the fact that if $x \in \mathbb{Z}$, then x is even if and only if there exists $k \in \mathbb{Z}$ such that $x = 2k$ and x is odd if and only if there exists $k \in \mathbb{Z}$ such that $x = 2k + 1$.

II. (35 points)

Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence defined inductively by

$$F_1 = F_2 = 1, \qquad\qquad F_{n+2} = F_{n+1} + F_n, \quad \forall\, n \in \mathbb{N}.$$

(1) Prove that for all natural numbers n we have

$$\sum_{i=1}^{n} F_i^2 = F_n \cdot F_{n+1}.$$

(2) Prove that for all natural numbers n we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Important note: α and β are the roots of the quadratic equation $x^2 = x + 1$.

III. (30 points)

(1) Let A, B, C be three subsets of a universal set X. Prove that

$$(A \cap B = A \cap C) \wedge (A \cup B = A \cup C) \implies B = C.$$

(2) If the following statement is true, prove it; if it is false, negate it and prove its negation.

$$\exists y \in \mathbb{R}, \quad \forall x \in \mathbb{R}, \quad xy = 1.$$