Third Homework Assignment

Homework problems due Friday, April 27:

In text: Sec. 7.3, p. 336: 2; Sec. 7.4, p. 341: 1, 4.

Added Problems (These are also part of the homework.):

1. Let G be a group, and let H and K be subgroups of G. Defined a group action of $H \times K$ on G by

$$(h,k)\,g \;=\; hgk^{-1} \qquad \text{for all } \; h \in H,\ k \in K,\ g \in G.$$

(a) Prove that this is a group action.

It is easy to see that for any $g \in G$, the orbit of g under this group action is

$$\mathcal{O}(g) = HgK = \{hgk \mid h \in H,\ k \in K\}.$$

The set HgK is called an H-K *double coset* of G. Since the H-K double cosets are the orbits of a group action on G, they form a partition of G. The goal for the rest of this problem is to prove a formula for the size of HgK.

(b) For $g \in G$, let $(H \times K)_g = \{(h,k) \in H \times K \mid (h,k)\,g = g\}$, which is the stablilizer of g for this group action. Prove that $(H \times K)_g \;\cong\; H \cap gKg^{-1}$.

(c) Deduce that if H and K are finite groups, then

$$|HgK| \;=\; |H| \cdot |K| \big/ |H \cap gKg^{-1}|.$$

Note: When we take $g = e$, the identity element of G, the formula in problem 1(c) reduces to: $|HK| = |H| \cdot |K| \big/ |H \cap K|$. This formula has another proof using a different group action given in the text p. 336, problem 3. (If H or K is a normal subgroup of G, then the formula for $|HK|$ is easily deducible using the First Isomorphism Theorem and Lagrange's Theorem.)

2. Let G be a p-group, and let M be a maximal proper subgroup of G. That is, M is a subgroup of G with $M \neq G$ and there is no subgroup H of G with $M \subsetneqq H \subsetneqq G$. Prove that M is a normal subgroup of G, and $|G : M| = p$.

Note: The assumption that G is a p-group is needed for problem 2. For example, one can show that S_{n-1} is a maximal proper subgroup of S_n for $n \geq 3$, but S_{n-1} is not a normal subgroup of S_n.

Optional Problem: This is a rather difficult problem which is not part of the required homework, and you may work on it or not, as you like: In text, p. 337: 12.