

SOLUTIONS

Exam 1, Mathematics 109
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Name:
 Student ID:
 Section Number:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck!

I. (40 pts.)

(1) Use truth tables to show that the following propositional expressions are tautologies

(i) $(\neg P \rightarrow (Q \wedge \neg Q)) \rightarrow P$; (ii) $\neg(R \rightarrow S) \leftrightarrow R \wedge \neg S$.

(2) Use the tautology (1)(ii) above and other well-known tautologies (e.g. de Morgan's laws) to write the simplest possible form of the **negation** of the propositional expression (1)(i) above.

(3) Write the simplest possible form of the **contrapositive** of the propositional expression (1)(i).

(1)

P	Q	$\neg P \rightarrow (Q \wedge \neg Q)$	$\neg(\ast)$
T	T	F	T
T	F	F	T
F	T	T	F
F	F	T	F

(ii)

R	S	$\neg(R \rightarrow S)$	$R \wedge \neg S$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

(2)

$$\neg((\neg P \rightarrow (Q \wedge \neg Q)) \rightarrow P) \Leftrightarrow (\neg P \rightarrow (Q \wedge \neg Q)) \wedge \neg P \Leftrightarrow$$

$$\Leftrightarrow \neg(\neg P) \wedge \neg P \Leftrightarrow$$

$$\Leftrightarrow P \wedge \neg P.$$

The second equivalence above is based on the fact that $Q \wedge \neg Q$ is a contradiction.

(3)

$$\neg P \rightarrow \neg(\neg P \rightarrow (Q \wedge \neg Q)) \Leftrightarrow \neg P \rightarrow (\neg P \wedge \neg(Q \wedge \neg Q)) \Leftrightarrow$$

$$\Leftrightarrow \neg P \rightarrow (\neg P \wedge (Q \vee \neg Q)) \Leftrightarrow$$

$$\Leftrightarrow \neg P \rightarrow \neg P.$$

II. (30 pts.) The universe \mathcal{U} for all the variables in the statements below is the set of integers \mathbb{Z} .

(1) Prove or disprove the following statement.

$$(\exists x) \quad x^3 + (x+1)^3 = (x+2)^3.$$

(2) Prove or disprove the following statement.

$$(\forall x)(\exists y)(\exists z) \quad x^2 + y^2 = z^2.$$

(3) Write down the negations of the statements in (1) and (2) above.

(1) We will disprove the statement, in other words show that

$$(\forall x) \quad x^3 + (x+1)^3 \neq (x+2)^3.$$

Assume $(\exists x)$ such that $x^3 + (x+1)^3 = (x+2)^3$. Therefore

$$\cancel{x^3} + \cancel{x^3} + 3x^2 + 3x + 1 = \cancel{x^3} + 6x^2 + 12x + 8$$

Therefore: $x^3 - 3x^2 - 9x = 7.$

Therefore $x(x^2 - 3x - 9) = 7. \quad (*)$

Therefore $x \mid 7$. Consequently, $x \in \{+1, -1, +7, -7\}$.

If one replaces x by $+1, -1, +7, -7$, respectively in $(*)$, one obtains a contradiction. \square

(2) The statement is true. Let $x \in \mathbb{Z}$. Let $y=0$ and $z=x$.

Then
$$\begin{aligned} x^2 + y^2 &= x^2 + 0^2 = \\ &= x^2 = \\ &= z^2 \quad \square \end{aligned}$$

(3) $(\forall x) \quad x^3 + (x+1)^3 \neq (x+2)^3$

$(\exists x)(\forall y)(\forall z) \quad x^2 + y^2 \neq z^2.$

III. (30 pts.)

- (1) Show that for any two subsets A and B of a given universal set \mathcal{U} , we have an equality

$$A \setminus B = A \cap \overline{B},$$

where \overline{B} denotes, as usual, the absolute complement of B .

- (2) Assume that the universal set \mathcal{U} is the set of natural numbers \mathbb{N} . Let

$$A = \{p : p \text{ prime}, p = 4k + 1, \text{ for some } k \in \mathbb{N}\}$$

$$B = \{p : p \text{ prime}, p = 3s + 2, \text{ for some } s \in \mathbb{N}\}$$

Use the set-equality in (1) above to show that

$$A \setminus B = \{p : p = 12l + 1, \text{ for some } l \in \mathbb{N}\}.$$

Proof.

- (1) Let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$.
Therefore $x \in A$ and $x \in \overline{B}$. Consequently
 $x \in A \cap \overline{B}$. Consequently $\#$
 $A \setminus B \subseteq A \cap \overline{B}$ (a)

Let $x \in A \cap \overline{B}$. Therefore $x \in A$ and $x \in \overline{B}$.
Therefore $x \in A$ and $x \notin B$. Therefore $x \in A \setminus B$.
Consequently
 $A \cap \overline{B} \subseteq A \setminus B$ (b)

Inclusions (a) and (b) imply that $A \cap \overline{B} = A \setminus B$ \square

(2)

Please note that

$$\overline{B} = \{p \mid p \text{ not prime or } p \text{ prime and } p = 3s + 1 \text{ or } p = 3\}$$

Therefore, according to (1), we have equalities:

$$(a) \quad A \setminus B = A \cap \overline{B} = \{p \mid p \text{ prime}, p = 3s + 2, p = 4k + 1, \text{ for some } s \in \mathbb{N}, k \in \mathbb{N}\}.$$

See reverse \curvearrowright

However

$$p = 4k + 1 = (3k + 1) + k.$$

Since $p = 3s + 1$ for some $s \in \mathbb{N}$, the last equality implies that $\exists l \in \mathbb{N}$, such that $k = 3l$.

Therefore:

$$p = 4k + 1 = 4(3l) + 1 = 12l + 1.$$

This shows that

$$(b) \quad A \setminus B \subseteq \{p \mid p \text{ prime}, p = 12l + 1, \text{ for some } l \in \mathbb{N}\}.$$

To prove the reverse inclusion, let $p \in \mathbb{N}$, p prime, $p = 12l + 1$, for some $l \in \mathbb{N}$. Then

$$p = 4(3l) + 1 = 3 \cdot (4l) + 1,$$

with $k = 3l$, $s = 4l$ natural numbers.

Therefore

$$(c) \quad \{p \mid p \text{ prime}, p = 12l + 1 \text{ for some } l \in \mathbb{N}\} \subseteq A \setminus B.$$

Equalities (a) combined with inclusions (b) and (c) imply that

$$A \setminus B = \{p \mid p \text{ prime}, p = 12l + 1, \text{ for some } l \in \mathbb{N}\}.$$

