Exam 1, Mathematics 109
Prof. Cristian D. Popescu
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SOLUTIONS
Name:
Student ID:

Note: There are 3 questions on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (30 points)

Use truth tables to prove the following logical equivalences:

(1)

$$(P \Longrightarrow (Q \vee R)) \quad \Longleftrightarrow \quad ((P \wedge \overline{Q}) \Longrightarrow R)$$

for any three propositions P, Q and R;

(2)

$$\overline{(P \Longrightarrow Q)} \quad \Longleftrightarrow (P \wedge \overline{Q})$$

for any two propositions P and Q.

(1)

P	Q	R	$P \Rightarrow Q \vee R$	$P \wedge \overline{Q} \Rightarrow R$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

(2)

P	Q	$\overline{P \Rightarrow Q}$	$P \wedge \overline{Q}$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

II. (40 points)

(1) Use **I**(2) to prove that for all $a \in \mathbb{R}$, $a \neq 0$, we have an equivalence

$$a > 0 \Longleftrightarrow a^{-1} > 0.$$

In the process, you may use only the axioms for $(\mathbb{R}, +, \cdot, <)$ and the fact that $1 > 0$.

(2) Use **I**(1) and **II**(1) above to prove that for all $x \in \mathbb{R}$, we have

$$(x^2 - 3x + 2 > 0) \Longrightarrow (x \leq 1) \vee (x > 2).$$

~~Pro~~ Let $a \in \mathbb{R}, a \neq 0$.

(1) "$\Rightarrow$" We prove $(a > 0) \Rightarrow (a^{-1} > 0)$ by contradiction. Assume that $(a > 0) \Rightarrow (a^{-1} > 0)$ is false. According to $\underline{I}(2)$ this is equivalent to

$$(a > 0) \wedge (a^{-1} \leq 0)$$

However, via the mult. axiom, this implies that

$$a \cdot a^{-1} \leq a \cdot 0$$

Therefore $1 \leq 0$, which is false. Consequently the implication $(a > 0) \Rightarrow (a^{-1} > 0)$ is true for all $a \in \mathbb{R}, a \neq 0$

In the above proof, we have used the equality

$$a \cdot 0 = 0$$

which will be proved in the solution of $\underline{\overline{III}}(1)$ below.

"$\Leftarrow$" Let $a \in \mathbb{R}$, $a \neq 0$. We will prove the implication $(a^{-1} > 0) \Rightarrow (a > 0)$ via direct proof.

Assume that $a^{-1} > 0$. According to ~~part~~ the first part, this implies that

$$(a^{-1})^{-1} > 0.$$

However, we claim that $(a^{-1})^{-1} = a$. Indeed, we have $(a^{-1})^{-1} \cdot a^{-1} = 1 = a \cdot a^{-1}$, which by multiplication with a gives $(a^{-1})^{-1} = a$. Consequently $a > 0$ $\square$

(2) According to $\underline{I}$ (1), we have an equivalence

$$\Big(\big((x^2 - 3x + 2) > 0 \implies (x < 1) \vee (x > 2)\big)\Big)$$

$$\Updownarrow$$

$$\Big(\big((x^2 - 3x + 2) > 0\big) \wedge (x > 1) \implies (x > 2)\Big)$$

We prove the second implication above by direct proof. So, assume that

$$(x^2 - 3x + 2) > 0 \quad \wedge \quad (x > 1)$$

This implies that :

$$(x - 1)(x - 2) > 0 \quad \wedge \quad x - 1 > 0 .$$

However, according to $\underline{II}$ (1) applied to $a := x - 1$, this implies that

$$(x-1)(x-2) > 0 \quad \wedge \quad (x-1)^{-1} > 0 .$$

The multiplicativity axiom implies that

$$(x-1)^{-1}\big((x-1)(x-2)\big) > (x-1)^{-1} \cdot 0$$

Distributivity, combined with $\underline{III}$ (2) imply that

$$x - 2 > 0 .$$

$\square$

III. (30 points)

Prove based only on the axioms for $(\mathbb{R}, +, \cdot, <)$ that

(1) $a \cdot 0 = 0$, for all real numbers a;

(2) $(a \cdot b = 0) \Longleftrightarrow ((a = 0) \vee (b = 0))$, for all real numbers a and b.

(1) Let $a \in \mathbb{R}$. Since $0 + 0 = 0$ (neutral element axiom), ~~this implies that~~ we have

$$a \cdot (0+0) = a \cdot 0.$$

Distributivity implies that

$$a \cdot 0 + a \cdot 0 = a \cdot 0.$$

Since $a \cdot 0 \in \mathbb{R}$, the opposite $-(a \cdot 0)$ exists. Add $-(a\cdot 0)$ to both sides of the last displayed equality to obtain

$$-(a \cdot 0) + (a \cdot 0 + a \cdot 0) = -(a \cdot 0) + (a \cdot 0)$$

Associativity implies that

$$a \cdot 0 = 0. \qquad \square$$

(2) "$\Longleftarrow$" follows from part (1).

"$\Longrightarrow$" We prove this implication by contradiction. Assume $\neg\,(a \cdot b = 0 \Rightarrow (a=0) \vee (b=0))$

According to I (2) this is equivalent to

$$(a \cdot b = 0) \wedge (a \neq 0) \wedge (b \neq 0)$$

(we have also used one of the De Morgan rules.)

However, since $a \neq 0$, a^{-1} exists. Therefore

$$a^{-1} (a \cdot b) = a^{-1} \cdot 0 = 0 \quad \text{(according to (1))}$$

Therefore $(a^{-1} a) \cdot b = 0 \Rightarrow 1 \cdot b = 0 \Rightarrow b = 0$, which is false. Consequently the "$\Rightarrow$" implication in the statement holds.