Final Exam, Math. 20C
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Name:
Student ID:
Section Number:

Note: There are 5 problems on this exam. Each of them is worth 40 points. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (40 points) Let $f(x, y)$ be the function given by

$$f(x, y) = x^4 + y^4 - 4xy + 1\,.$$

(1) Determine the critical points of $f(x, y)$.
(2) Classify the critical points of $f(x, y)$ into local maxima, local minima, and saddle points, respectively.
(3) Determine the global maximum and minimum points of $f(x, y)$ on the (closed and bounded) domain

$$D = \{(x, y)\,|\,0 \leq x \leq 3\,, \quad 0 \leq y \leq 2\}\ .$$

(4) Use the method of Lagrange multipliers to find minimum value of $f(x, y)$ along the curve given by the equation $4xy = 1$. Is there a maximum value for f along the given curve ?

II. (40 points)

(1) Compute the integral

$$\iint_D f(x,y)dA\,,$$

where $f(x,y) = y\cos(x^2)$ and D is the plane region situated above the x–axis and bounded by the curves $y = 0$, $x = y^2$, and $x = 9$.

(2) Find the volume of the solid which is inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

III. (40 points) Let $\mathcal{C}$ be the space curve whose vectorial equation is given by

$$\vec{r}(t) = t^2\,\vec{i} + \ln t\,\vec{j} + 2t\,\vec{k}\,, \qquad t \in (0,\,\infty)\,.$$

(1) Find the length of the arc of $\mathcal{C}$ corresponding to $1 \le t \le 2$.
(2) Find the unit tangent vector $\vec{T}$, unit normal vector $\vec{N}$, and unit binormal vector $\vec{B}$ to the curve $\mathcal{C}$ at the point $P(1,0,2)$.
(3) Find the equation of the osculating plane to the curve $\mathcal{C}$ at the point $P(1,0,2)$.
(4) Find the curvature κ of the curve $\mathcal{C}$ at $P(1,0,2)$.
(5) What is the radius of the osculating circle to the curve $\mathcal{C}$ at $P(1,0,2)$?

IV. (40 points) The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2},$$

where T is measured in Celsius degrees and x, y, z are measured in meters.

(1) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $Q(3, -3, 3)$

(2) Find the direction in which the temperature increase the fastest at the point $P(2, -1, 2)$.

(3) Find the maximum rate of increase in temperature at $P(2, -1, 2)$.

(4) Find the equation of the plane tangent to the surface consisting of all those points where the temperature is equal to $200e^{-1}$ Celsius degrees at the point $R(1, 0, 0)$.

V. (40 points) Let $(\pi_1) : 2x + y + z - 1 = 0$ and $(\pi_2) : x + 2y - 2z = 0$ be the equations of two planes (π_1) and (π_2).

(1) Find the angle $\theta \in [0, \pi)$ determined by the two planes above.
(2) Find the vectorial, parametric and symmetric equations of the line of intersection between the two planes above.
(3) Find the distance between the point $P_0(1, 0, 1)$ and the plane (π_1).
(4) Write the cartesian equation of the sphere centered at $P_0(1, 0, 1)$ and tangent to (π_1).
(5) Find the point of intersection between the plane (π_1) and the line which passes through $P_0(1, 0, 1)$ and is perpendicular on (π_1).