SOLUTIONS

Exam 1, Mathematics 20D
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Name:
SSN:
Section Number:

Note: There are 4 problems on this exam. Each of them is worth 25 points. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

(25 pts.) I. Given the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{n}\cdot x^n,$$

determine

(1) its radius of convergence;
(2) its interval of convergence;
(3) the values of x for which it converges absolutely respectively conditionally.

(1) ratio test

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{4^{n+1}/n+1 \cdot |x|^{n+1}}{4^n/n \cdot |x|^n} = 4\cdot|x|\cdot\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 4\cdot|x| < 1$$

$|x| < 1/4$ $\quad R = 1/4.$

(2) $x = -\frac{1}{4}$

$\sum_{n=1}^{\infty}\frac{1}{n}$ divergent p-series $p=1$

$x = \frac{1}{4}$

$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ — abs. conv No! — Cond. conv. yes! (alternating series test)

interval of convergence $= (-\frac{1}{4}, \frac{1}{4}]$

(3) Abs. conv. $(-\frac{1}{4}, \frac{1}{4})$

Cond. convergence $1/4$

divergence $\mathbb{R}\setminus(-\frac{1}{4}, \frac{1}{4}]$.

(25 pts.) II. (1) Write down the MacLaurin power series for

$$f(x) = x^2 \cdot e^{-x^2},$$

and determine the values of x for which $f(x)$ equals the sum of its MacLaurin series.

(2) Use the answer to (1) to approximate the integral

$$\int_0^1 x^2 \cdot e^{-x^2}\, dx,$$

with an error smaller than 0.1.

(1) $e^{x} = \sum_{n\geq 0} \frac{1}{n!} x^n, \quad \forall x \in \mathbb{R}$

$e^{-x^2} = \sum_{n\geq 0} \frac{1}{n!} (-x^2)^n = \sum_{n\geq 0} (-1)^n \frac{1}{n!} x^{2n}, \quad \forall x \in \mathbb{R}$

$x^2 \cdot e^{-x^2} = \sum_{n\geq 0} (-1)^n \frac{1}{n!} x^{2n+2}, \quad \forall x \in \mathbb{R}$

(2) $\int_0^1 x^2 \cdot e^{-x^2} dx = \sum_{n\geq 0} (-1)^n \frac{1}{n!} \int_0^1 x^{2n+2} dx =$

$= \sum_{n\geq 0} (-1)^n \frac{1}{n!} \cdot \frac{x^{2n+3}}{2n+3}\Big|_0^1 =$

$= \underbrace{\sum_{n\geq 0} (-1)^n \frac{1}{n!(2n+3)}}_{(*)}$

(*) is an alternating series. Let S be its sum and S_n its nth partial sum, then

$|S - S_n| \leq \frac{1}{(n+1)!(2n+5)} < 0.1$ for $n \geq 3$.

Take $n=3$: $S \sim S_3 = \frac{1}{0! \cdot 3} - \frac{1}{1! \cdot 5} + \frac{1}{2! \cdot 7} - \frac{1}{3! \cdot 9} =$

$= 1/3 - 1/5 + 1/14 - 1/54 = \ldots$

(25 pts.) III. For each of the following series determine whether it is absolutely convergent, conditionally convergent, or divergent. Compute the sums (i.e. limits) of those series which are absolutely convergent.

(a) $\sum_{n=1}^{\infty}(-1)^n \ln\left(\frac{n}{2n+5}\right)$; (b) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$; (c) $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$.

(a) $\sum_{n=1}^{\infty}(-1)^n \ln\left(\frac{n}{2n+5}\right)$ fails the basic convergence test

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{2+5/n}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2 \neq 0$$

therefore $\lim_{n\to\infty} (-1)^n \ln\left(\frac{n}{2n+5}\right)$ does not exist.

Consequently, (a) is divergent.

(b) $\sum_{n\geq 1} \frac{1}{n(n+1)}$ is a series with positive terms. For these convergence $\Leftrightarrow$ absolute convergence.

Let S_n be its nth partial sum. Then, since

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \quad \forall k,$$

we have

$$S_n = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right) =$$

$$= 1 - \frac{1}{n+1}$$

$\sum_{n\geq 1}\frac{1}{n(n+1)}$ converges to $S := \lim_{n\to\infty} S_n = 1$.

see verso.

(c) $\sum_{n\geq 1} \frac{2n}{\sqrt{n^5+1}}$

series with positive terms.

Basic comparison test

$$\frac{2n}{\sqrt{n^5+1}} \leq \frac{2n}{\sqrt{n^5}} = \frac{2n}{n^{5/2}} = \frac{2}{n^{3/2}}$$

$$\sum_{n\geq 1} \frac{2n}{\sqrt{n^5+1}} \leq 2 \underbrace{\sum_{n\geq 1} \frac{1}{n^{3/2}}}_{\text{Convergent}}$$

Convergent $p = 3/2 > 1$

Therefore $\sum_{n\geq 1} \frac{2n}{\sqrt{n^5+1}}$ is **convergent**

(25 pts.) IV. Determine whether the following series are convergent or not.

(a) $\sum_{n=1}^{\infty} \sin(1/n)$; (b) $\sum_{n=1}^{\infty} \tan(1/n^2)$; (c) $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+1}}$.

(a). $\sum_{n\geq 1} \sin\left(\frac{1}{n}\right)$ is a series with positive terms

$\left(n \geq 1,\ \frac{1}{n} \in \left(0, \frac{\pi}{4}\right) \Rightarrow \sin\left(\frac{1}{n}\right) > 0\right)$.

Apply the limit comparison test. Compare with

$\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent, p-series, $p=1$)

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{1/n} \underset{1/n = x}{=} \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \neq 0 \Rightarrow$$

$\Rightarrow \sum_{n\geq 1} \sin 1/n$ divergent

(b) $\sum_{n\geq 1} \tan(1/n^2)$ — series with positive terms

limit comparison test. — Compare with

$\sum_{n\geq 1} \frac{1}{n^2}$ (convergent, p-series, $p=2>1$)

$$\lim_{n\to\infty} \frac{\tan(1/n^2)}{1/n^2} = \lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} \cdot \frac{1}{\cos(1/n^2)} \underset{1/n^2 = x}{=}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1 \neq 0.$$

$\Rightarrow$ $\sum_{n\geq 1} \tan(1/n^2)$ convergent

see verso

(c) $\sum_{n\geq 1} \frac{2n}{\sqrt{n^5+1}}$

series with positive terms.

<u>Basic comparison test</u>

$$\frac{2n}{\sqrt{n^5+1}} \leq \frac{2n}{\sqrt{n^5}} = \frac{2n}{n^{5/2}} = \frac{2}{n^{3/2}}$$

$$\sum_{n\geq 1} \frac{2n}{\sqrt{n^5+1}} \leq 2 \underbrace{\sum_{n\geq 1} \frac{1}{n^{3/2}}}_{\text{Convergent}}$$

Convergent $p = 3/2 > 1$

Therefore $\sum_{n\geq 1} \frac{2n}{\sqrt{n^5+1}}$ is $\boxed{\text{convergent}}$