Lollipop Hessenberg Varieties and Garsia–Haiman modules of hook shapes

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Abstract

Regular semisimple Hessenberg varieties have cohomology rings which, as rings, are not generated in degree 2 in general. Masuda–Sato identify the family for which they are, known as the (double) lollipop Hessenberg varieties $X_{a,b}$, and prove that they are fiber bundles over certain toric varieties $Y_{a,b}$.

For the Hessenberg function $h = (k^{k-1}, n^{n-k+1})$, the Garsia–Haiman module D_μ of hook shape $\mu = (n-k+1, 1^{k-1})$ can be explicitly realized as a quotient of a natural subring of $H_T^*(X_{n,k})$. This identification is $\mathfrak{S}_n$-equivariant, intertwining Tymoczko's dot action with the usual diagonal action on D_μ. Adin–Remmel–Roichman give a presentation of D_μ as a quotient of the polynomial ring $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$ by an ideal J_μ, and it turns out the monomial relations in their presentation generate the Stanley–Reisner ideal of the fan of $Y_{n,k}$. This compatibility extends to the level of bases, and we discuss the relationship between Cho–Hong–Lee's module basis of $H_T^*(X_{n,k})$ and Adin–Remmel–Roichman's monomial basis of D_μ.

Time permitting, we will discuss joint work with T. Matsumura which features an explicit identification of $Y_{n,k}$ as the toric variety of the normal fan of a Minkowski sum of hypersimplices. This calculation leads to a Borel presentation of the cohomology rings of all lollipop Hessenberg varieties.