Realizations of equivalence relations and subshifts
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- A basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space.
- The continuum hypothesis (is there some $A \subseteq \mathbb{R}$ with $|\mathbb{N}| < |A| < |\mathbb{R}|$??)

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Fact: Any two uncountable Polish spaces are Borel isomorphic.

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- Non-measurable sets? Every Borel set is measurable.
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- The continuum hypothesis? Every Borel set satisfies the continuum hypothesis, i.e. if $A \subseteq \mathbb{R}$ is a Borel subset, then either A is countable, or $|A| = |\mathbb{R}|$.

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For instance, the classification of 5×5 unitary matrices up to similarity (aka conjugacy) is smooth, where the concrete invariants are the 5 eigenvalues.

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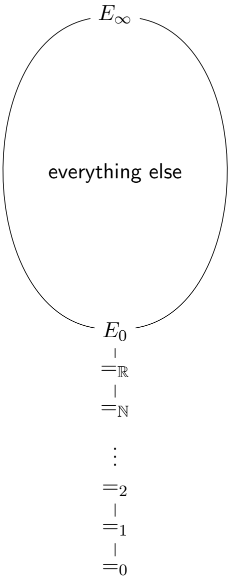

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If E has a minimal action realization, then E is not smooth. Converse?

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This implies a stronger version of the marker lemma (purely Borel fact about every CBER).

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Question

Does every non-smooth aperiodic CBER have a minimal action realization?

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- The universal compressible CBER.

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Theorem ([FKSV21])

Every aperiodic CBER E has a K_σ realization.
That is, E is Borel isomorphic to a K_σ CBER.

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In general, we know many groups Γ for which 2^Γ has a minimal subshift with universal CBER (certain wreath products).

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Andy Zucker has informed me that this also follows from facts about strongly proximal actions

Side remark on amenability

A countable group Γ is **amenable** if every continuous action $\Gamma \curvearrowright X$ on a compact space has an invariant measure.
We show that it suffices to check subshifts of 2^Γ.

Theorem ([FKSV21])

A group Γ is amenable iff every subshift of 2^Γ has an invariant measure.

Andy Zucker has informed me that this also follows from facts about strongly proximal actions (which I am not very good at, sorry Josh).

The space of subshifts

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$\mathrm{Sh}([0,1]^{\mathbb{N}})$ is a **universal space for compact actions**.

Similarly, $\mathrm{Sh}(\mathbb{R}^{\mathbb{N}})$ is a **universal space for arbitrary actions**.

Topological and descriptive complexity

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Is this set comeager?

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Argument is indirect, we can't show G_δ directly.

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This implies that even if every CBER has a compact realization, there is no **effective** way to obtain this realization.
Precisely:

Theorem ([FKSV21])

There is a non-smooth aperiodic subshift $F \in \mathrm{Sh}(\mathbb{R}^{\mathbb{N}})$*, such that for every* $K \in \mathrm{Sh}([0,1]^{\mathbb{N}})$*, there is no* $\Delta^1_1(F)$ *isomorphism of* E_F *with* E_K*.*

Thank you!

Joshua Frisch, Alexander S. Kechris, Forte Shinko, and Zoltán Vidnyánszky.
Realizations of countable Borel equivalence relations.
arXiv:2109.12486, 2021.