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A coloring property for countable groups

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Abstract

Motivated by research on hyperfinite equivalence relations we define a coloring property for countable groups. We prove that every countable group has the coloring property. This implies a compactness theorem for closed complete sections of the free part of the shift action of G on 2^G. Our theorems generalize known results about $\mathbb{Z}$.

1. *Introduction*

The coloring property we will establish for all countable groups in this paper was motivated by the study of hyperfinite equivalence relations. One of the most well-known results in this area is the hyperfiniteness of the orbit equivalence relation of the shift action of the group $\mathbb{Z}$ on $2^{\mathbb{Z}}$ (c.f. [**2**]). In the proof of this result a marker lemma of Slaman and Steel ([**5**]) played an important role.

LEMMA 1·1 (Slaman–Steel). *Let $F(\mathbb{Z})$ be the free part of $2^{\mathbb{Z}}$ under the shift action of $\mathbb{Z}$. Then there is an infinite decreasing sequence of Borel complete sections of $F(\mathbb{Z})$*

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq \cdots$$

such that $\bigcap_n S_n = \varnothing$.

This Slaman–Steel marker lemma is true when $\mathbb{Z}$ is replaced by an arbitrary countable group G. In [**3**] the first two authors studied, among other things, the existence of decreasing sequences of complete sections that are relatively closed in the free part of $2^{\mathbb{Z}}$. If such sequences existed then the hyperfiniteness of $2^{\mathbb{Z}}$ could be strengthened easily (to a continuous embedding of E_0). However, in [**3**] it was noted that, while it is possible to have decreasing sequences of clopen complete sections such that their intersection contains at most one point of each orbit, requiring the intersection to be empty is impossible. (An earlier, weaker version of the following theorem was joint work with Ben Miller.)

THEOREM 1·2 (Gao–Jackson). *There is no infinite sequence of closed complete sections of $F(\mathbb{Z})$*

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq \cdots$$

such that $\bigcap_n S_n = \varnothing$.

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In this paper the above theorem is generalized to an arbitrary countable group G.

THEOREM 1·3. *Let $F(G)$ be the free part of the shift action on 2^G by G. Then there is no infinite sequence of closed complete sections of $F(G)$*

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n \supseteq \cdots$$

such that $\bigcap_n S_n = \varnothing$.

This can be interpreted as a compactness theorem for closed complete sections of $F(G)$. The proof of the theorem, however, turns out to rely on a combinatorial analysis of the group G, and ultimately boils down to the following coloring property.

Definition 1·4. Let G be a countable group. A 2-*coloring* on G is a function $c : G \to \{0, 1\}$ such that for any $s \in G$ with $s \neq 1_G$ there is a finite set $T \subseteq G$ such that

$$\forall g \in G\ \exists t \in T\ c(gt) \neq c(gst).$$

We say that G has *the coloring property* if there is a 2-coloring on G.

Our main theorem of the paper is the following.

THEOREM 1·5. *Every countable group has the coloring property.*

In fact we demonstrate a stronger theorem which asserts the existence of continuum many distinct 2-colorings on any countably infinite group G. The proof of our main theorem is entirely algebraic.

After an earlier version of this paper was completed we learned that Glasner and Uspenskij [**4**] have asked the general question whether every countable group has the coloring property and obtained partial results on this problem.

It is worth noting that although our results imply that the Slaman–Steel lemma cannot be improved to closed complete sections, it is true that the shift equivalence relation on $2^{\mathbb{Z}}$ is continuously embeddable into E_0 ([**1**]). Moreover, the same is true even when $\mathbb{Z}$ is replaced by $\mathbb{Z}^n$ for any $n \geqslant 1$ or by $\mathbb{Z}^{<\omega}$, the direct sum of infinitely many copies of $\mathbb{Z}$ ([**3**]). The proof of these results are much more sophisticated, and the necessity of the sophistication is suggested by our results here.

The rest of the paper is organized as follows. In Section 2 we give the connection between the coloring property and the compactness theorem for closed complete sections. We also introduce a concept of orthogonality and characterize it topologically. In Section 3 we prove our main theorems that every countably infinite group has the coloring property, and moreover on any countably infinite group there is a perfect set of pairwise orthogonal 2-colorings.

2. *Definitions and connections*

We reformulate the coloring property in a slightly broader context.

Definition 2·1. Let G be a countable group and $k \geqslant 2$ an integer. A k-*coloring* on G is a function $c : G \to k$ such that for any $s \in G$ with $s \neq 1_G$ there is a finite set $T \subseteq G$ such that

$$\forall g \in G\ \exists t \in T\ c(gt) \neq c(gst).$$

We also consider the following concept of orthogonality.

Definition 2·2. Let G be a countable group, $k \geqslant 2$ an integer and c_0, c_1 k-colorings on G. We say that c_0 and c_1 are *orthogonal*, denoted $c_0 \perp c_1$, if there is a finite set $T \subseteq G$ such that

$$\forall g_0, g_1 \in G\ \exists t \in T\ c_0(g_0 t) \neq c_1(g_1 t).$$

If two k-colorings are orthogonal we regard them to be different in an effective way. We will see below that the orthogonality corresponds to a nice topological characterization. We note that this concept was used in essential ways in some of our earlier partial results. In the current proof the concept is not explicitly used but a local version of it is still instrumental in the proof of our main theorem.

We next give some topological characterizations for these concepts.

Fix a countable group G and an integer $k \geqslant 2$. Let G be enumerated without repetition as $1_G = g_0, g_1, g_2, \ldots$. Define a metric on $k^G = \{0, \ldots, k-1\}^G$ by

$$d_k(x, y) = \begin{cases} 2^{-n}, & \text{if } x \neq y \text{ and } n \in \omega \text{ is the least such that } x(g_n) \neq y(g_n), \\ 0, & \text{if } x = y. \end{cases}$$

Then d_k is an ultrametric on k^G compatible with the compact product topology on k^G, where $k = \{0, \ldots, k-1\}$ is endowed with the discrete topology.

The shift action of G on k^G is given by

$$(g \cdot x)(h) = x(g^{-1}h).$$

This action is continuous. Let F_k be the free part of this action, i.e., $x \in F_k$ iff $\forall g \in G - \{1_G\}\ g \cdot x \neq x$. Then F_k is an invariant dense G_δ subset of k^G.

For each $x \in k^G$ let $[x]$ denote the orbit of x, i.e., the set of elements $g \cdot x$ for $g \in G$. Then we have the following characterization.

LEMMA 2·1. *For any $x \in k^G$, x is a k-coloring on G iff $\overline{[x]} \subseteq F_k$.*

Proof. ($\Rightarrow$) Assume that x is a k-coloring on G. Suppose $z \in \overline{[x]}$, that is, there are $h_m \in G$ with $h_m \cdot x \to z$ as $m \to \infty$. We show that $z \in F_k$. Assume not and suppose $s \cdot z = z$ for $s \neq 1_G$. Then by the continuity of the action we have that $s^{-1}h_m \cdot x \to s^{-1} \cdot z = z$. Let $T \subseteq G$ be a finite set such that for any $g \in G$ there is $t \in T$ with $x(gt) \neq x(gst)$. Let n be large enough so that $T \subseteq \{g_0, \ldots, g_n\}$ and let $m \geqslant n$ be such that $d(h_m \cdot x, z), d(s^{-1}h_m \cdot x, z) < 2^{-n}$. Now fix $t \in T$ with $(h_m \cdot x)(t) = x(h_m^{-1}t) \neq x(h_m^{-1}st) = (s^{-1}h_m \cdot x)(t)$. Then $z(t) = (h_m \cdot x)(t) \neq (s^{-1}h_m \cdot x)(t) = z(t)$, a contradiction.

($\Leftarrow$) Assume $\overline{[x]} \subseteq F_k$. Denote $C = \overline{[x]}$. Fix any $s \in G$ with $s \neq 1_G$. Then for any $y \in C$, $s^{-1} \cdot y \neq y$, and hence there is $t \in G$ with $(s^{-1} \cdot y)(t) \neq y(t)$. Define a function $\tau : C \to G$ by letting $\tau(y) = g_n$ where n is the least so that $(s^{-1} \cdot y)(g_n) \neq y(g_n)$. Then τ is a continuous function. Since C is compact we get that $\tau(C) \subseteq G$ is finite. Let $T = \tau(C)$. Then for any $g \in G$, there is a $t \in T$ such that $x(gt) = (g^{-1} \cdot x)(t) \neq (s^{-1}g^{-1} \cdot x)(t) = x(gst)$. This proves that x is a k-coloring.

Thus we have the following proposition (also due independently to Pestov and can be found in [**4**]).

PROPOSITION 2·2. *Let G be a countable group. Then the following are equivalent*:

(i) *G has the coloring property;*
(ii) *$\overline{[x]} \subseteq F_2$ for some $x \in 2^G$;*
(iii) *F_2 contains a compact invariant subset.*

The compactness theorem for complete sections is now a corollary of the coloring property. Recall that a complete section of F_2 is a subset $A \subseteq F_2$ so that $A \cap [x] \neq \varnothing$ for every $x \in F_2$.

THEOREM 2·5. *Let G be a countably infinite group with the coloring property. Suppose $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$ is a decreasing sequence of closed complete sections in F_2. Then $\bigcap_n S_n \neq \varnothing$.*

Proof. Let $x \in 2^G$ be a 2-coloring on G. Then $\overline{[x]} \subseteq F_2$. Now each S_n is a complete section and therefore $S_n \cap [x] \neq \varnothing$. Thus the sequence $S_n \cap \overline{[x]}$ is a decreasing sequence of nonempty closed subsets of a compact space $\overline{[x]}$ and therefore $\bigcap_n (S_n \cap \overline{[x]}) \neq \varnothing$. In particular, $\bigcap_n S_n \neq \varnothing$.

We also give the promised topological characterization for orthogonality.

LEMMA 2·6. *Let G be a countable group, $k \geqslant 2$ an integer and c_0, c_1 k-colorings on G. Then $c_0 \perp c_1$ iff $\overline{[c_0]} \cap \overline{[c_1]} = \varnothing$.*

Proof. ($\Rightarrow$) Let n be large enough such that $T \subseteq \{g_0, \ldots, g_n\}$. Then for any $x_0 \in [c_0]$ and $x_1 \in [c_1]$, there is $t \in T$ such that $x_0(t) \neq x_1(t)$, and thus $d(x_0, x_1) \geqslant 2^{-n}$. It follows that $d(y_0, y_1) \geqslant 2^{-n}$ for any $y_0 \in \overline{[c_0]}$ and $y_1 \in \overline{[c_1]}$, and therefore $\overline{[c_0]} \cap \overline{[c_1]} = \varnothing$.

($\Leftarrow$) Conversely, suppose $\overline{[c_0]} \cap \overline{[c_1]} = \varnothing$. Since they are both compact it follows that there is some $\delta > 0$ such that for any $y_0 \in \overline{[c_0]}$ and $y_1 \in \overline{[c_1]}$, $d(y_0, y_1) \geqslant \delta$. Let n be large enough such that $\delta \geqslant 2^{-n}$. Then in particular for any $x_0 \in [c_0]$ and $x_1 \in [c_1]$, $d(x_0, x_1) \geqslant 2^{-n}$. This implies that there is $t \in \{g_0, \ldots, g_n\}$ such that $x_0(t) \neq x_1(t)$.

We briefly turn our attention to finite groups. It is easy to see that every finite group has the coloring property. In fact if G is finite we may let $c(1_G) = 0$ and $c(g) = 1$ for all $g \neq 1_G$; then c is a 2-coloring on G. It is not clear, however, how many pairwise orthogonal 2-colorings a general finite group G can have. The group $\mathbb{Z}_2$ has only two 2-colorings, but they are in the same orbit, and therefore not orthogonal.

3. *The proof of the main theorem*

In this section we prove our main result that every countably infinite group has the coloring property. The proof is technical but elementary. Before we give the presentation of the coloring we will prove some preparatory propositions and lemmas about the combinatorial structure of the group. The first major step is Proposition 3·2 below. We will use the following concept in its proof.

Definition 3·1. Let G be a group and let $A, B, \Delta \subseteq G$. We say that the Δ-translates of A are *maximally disjoint within B* if the following properties hold:

(i) for all $\gamma, \psi \in \Delta$, if $\gamma \neq \psi$ then $\gamma A \bigcap \psi A = \varnothing$;

(ii) for every $g \in G$, if $gA \subseteq B$ then there exists $\gamma \in \Delta$ with $gA \bigcap \gamma A \neq \varnothing$.

When property (i) holds we say that the Δ-translates of A are *disjoint*. Furthermore, we say that the Δ-translates of A are *contained and maximally disjoint within B* if the Δ-translates of A are maximally disjoint within B and $\Delta A \subseteq B$.

Notice that in the definition above we were referring to the left translates of A by Δ but never explicitly used the term left translates. Throughout this section when we use the word translate(s) it will be understood that we are always referring to left translate(s).

PROPOSITION 3·2. *Let G be a countably infinite group. Given a sequence $(H_n)_{n\in\mathbb{N}}$ of finite subsets of G such that $1_G \in H_0$, $\bigcup_{n\in\mathbb{N}} H_n = G$, and such that for all $n \geqslant 1$*

$$H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n$$

there exists an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G and a sequence $(\Delta_n)_{n\in\mathbb{N}}$ of subsets of G such that:

(i) $F_0 = H_0$;

(ii) $F_n \subseteq H_n$ *for all* $n \geqslant 1$;

(iii) $1_G \in \Delta_n$ *for all* $n \in \mathbb{N}$;

(iv) *for all $n \in \mathbb{N}$ the Δ_n-translates of F_n are maximally disjoint within G*;

(v) *for all $n \geqslant 1$ the $\Delta_{n-1} \bigcap F_n$-translates of F_{n-1} are contained and maximally disjoint within H_n*;

(vi) $\gamma(\Delta_k \bigcap F_n) = \Delta_k \bigcap \gamma F_n$ *for all* $n \geqslant k$ *and* $\gamma \in \Delta_n$;

(vii) $(\Delta_k \bigcap F_n)F_k \subseteq F_n$ *for all* $n \geqslant k$.

Proof. Set $F_0 = H_0$ so (i) is satisfied. We will construct $(F_n)_{n\in\mathbb{N}}$. Let δ_0^1 be such that $1_G \in \delta_0^1$ and such that the δ_0^1-translates of F_0 are contained and maximally disjoint within H_1. Then define $F_1 = \bigcup_{\gamma\in\delta_0^1} \gamma F_0$ and note $F_1 \subseteq H_1$.

We will continue the construction inductively. Assume F_0 through F_{n-1} have been defined with $n > 1$ and with $F_m \subseteq H_m$ for $m < n$. Let δ_{n-1}^n be such that $1_G \in \delta_{n-1}^n$ and such that the δ_{n-1}^n-translates of F_{n-1} are contained and maximally disjoint within H_n. Once δ_{n-1}^n through δ_{n-k+1}^n have been defined with $1 < k \leqslant n$, choose δ_{n-k}^n such that the δ_{n-k}^n-translates of F_{n-k} are contained and maximally disjoint within

$$\beta(n, n-k) - \bigcup_{1\leqslant m<k} \bigcup_{\gamma\in\delta_{n-m}^n} \gamma F_{n-m} = \beta(n, n-k) - \bigcup_{1\leqslant m<k} \delta_{n-m}^n F_{n-m}$$

where for $r, s \in \mathbb{N}$ with $r < s$

$$\beta(s, r) = \{g \in G \mid \{g\}(F_{r+1}^{-1}F_{r+1})(F_{r+2}^{-1}F_{r+2})\cdots(F_{s-1}^{-1}F_{s-1}) \subseteq H_s\}.$$

We placed the requirement that $H_{n-1}(H_0^{-1}H_0)(H_1^{-1}H_1)\cdots(H_{n-1}^{-1}H_{n-1}) \subseteq H_n$ in order to ensure that $H_{n-1} \subseteq \beta(n, n-k)$ for all $k \leqslant n$ so that $\beta(n, n-k) \neq \varnothing$ and more importantly (later on) $\bigcup_{n>k} \beta(n, k) = G$ for fixed $k \in \mathbb{N}$. Intuitively we want to control how translates of F_{n-k} are placed in order for this collection of translates to eventually become maximally disjoint within G (in fact eventually become Δ_{n-k}). Requiring the δ_{n-k}^n-translates of F_{n-k} to be contained in $\beta(n, n-k)$ currently seems like an obscurity but will later be shown to give us what we desire.

Finally, define

$$F_n = \bigcup_{0\leqslant m<n} \bigcup_{\gamma\in\delta_m^n} \gamma F_m = \bigcup_{0\leqslant m<n} \delta_m^n F_m$$

and note $F_n \subseteq H_n$ since $\beta(n, k) \subseteq H_n$ for all $0 \leqslant k < n-1$.

The construction of $(F_n)_{n\in\mathbb{N}}$ is now complete and satisfies (i) and (ii). The collection $(\delta_k^n)_{k<n}$ was useful in constructing $(F_n)_{n\in\mathbb{N}}$ but is inadequate for our further needs. For $k \leqslant n$ we wish to recognize exactly how translates of F_k were both explicitly and implicitly used in constructing F_n and all of the parts of F_n. For example, for $k < m < n$ $\delta_k^m F_k \subseteq F_m$ and $\delta_m^n F_m \subseteq F_n$ so $\delta_m^n \delta_k^m F_k \subseteq F_n$. Thus informally we would say the $\delta_m^n \delta_k^m$-translates of F_k were implicitly used in constructing F_n. However if for $g \in F_n$ we only have $gF_k \subseteq F_n$ we would not necessarily want to say the g-translate of F_k was used in constructing F_n. Hopefully we

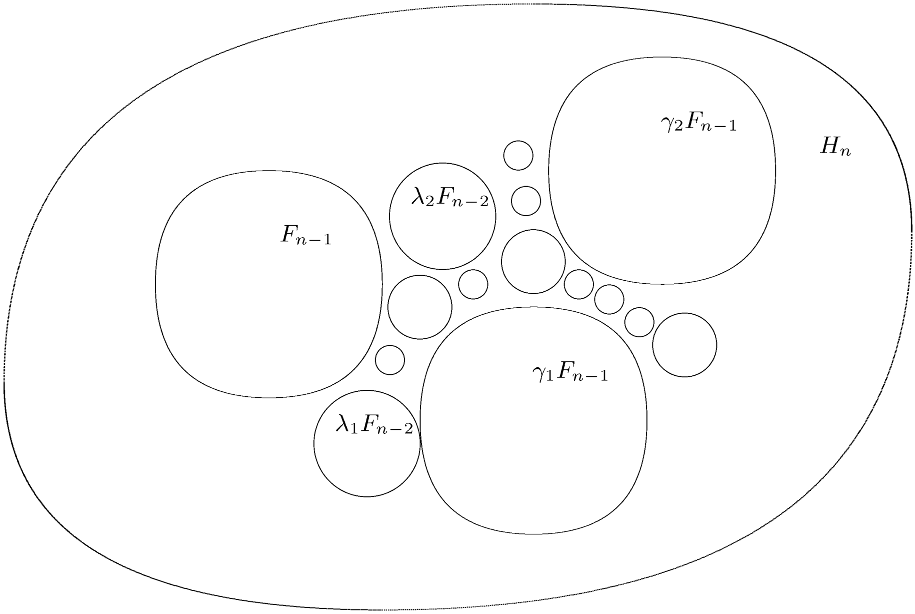

Fig. 1. The composition of F_n.

have made the point that we only wish to consider translates which, in some sense, were either explicitly or implicitly used. Informally, we wish to define D_k^n to be the set of all γ's in F_n (recall $1_G \in F_k$) such that the γ-translate of F_k was used in constructing F_n. For $k \in \mathbb{N}$ define $D_k^k = \{1_G\}$, $D_k^{k+1} = \delta_k^{k+1}$, and in general for $n > k$

$$D_k^n = \delta_{n-1}^n D_k^{n-1} \bigcup \delta_{n-2}^n D_k^{n-2} \bigcup \cdots \bigcup \delta_{k+1}^n D_k^{k+1} \bigcup \delta_k^n = \bigcup_{k \leqslant m < n} \delta_m^n D_k^m.$$

Note that $D_k^n F_k \subseteq F_n$ for all $k, n \in \mathbb{N}$ with $k \leqslant n$. This follows from the fact that $D_k^k F_k = F_k$ and assuming $D_k^m F_k \subseteq F_m$ for all $k \leqslant m < n$ we have

$$D_k^n F_k = \bigcup_{k \leqslant m < n} \delta_m^n D_k^m F_k \subseteq \bigcup_{k \leqslant m < n} \delta_m^n F_m \subseteq F_n.$$

Additionally we have that for all $s, n, k \in \mathbb{N}$ with $s \geqslant n \geqslant k$ $D_n^s D_k^n \subseteq D_k^s$. Clearly when $s = n$ $D_n^s D_k^n = D_n^n D_k^n = D_k^n$ and if we assume $D_n^r D_k^n \subseteq D_k^r$ for all $n \leqslant r < s$ then

$$D_n^s D_k^n = \bigcup_{n \leqslant r < s} \delta_r^s D_n^r D_k^n \subseteq \bigcup_{n \leqslant r < s} \delta_r^s D_k^r \subseteq \bigcup_{k \leqslant r < s} \delta_r^s D_k^r = D_k^s.$$

We wish to show that for all $k, n \in \mathbb{N}$ with $k \leqslant n$ the D_k^n-translates of F_k are disjoint. This is clear when $n = k$ and $n = k + 1$ since $D_k^k = \{1_G\}$ and $D_k^{k+1} = \delta_k^{k+1}$. Fix k and $n > k + 1$ and assume the D_k^m-translates of F_k are disjoint for all $k \leqslant m < n$. Recall $D_k^n = \bigcup_{k \leqslant m < n} \delta_m^n D_k^m$. If $k \leqslant r, s < n$ and $r \neq s$ then $\delta_r^n D_k^r F_k \bigcap \delta_s^n D_k^s F_k = \varnothing$ since $\delta_r^n D_k^r F_k \subseteq \delta_r^n F_r$, similarly $\delta_s^n D_k^s F_k \subseteq \delta_s^n F_s$, and $\delta_r^n F_r \bigcap \delta_s^n F_s = \varnothing$ by construction. Also for $k \leqslant m < n$ and $\gamma, \psi \in \delta_m^n$ with $\gamma \neq \psi$, $\gamma D_k^m F_k \bigcap \psi D_k^m F_k = \varnothing$ since again $D_k^m F_k \subseteq F_m$ and the δ_m^n-translates of F_m are disjoint. Finally by the induction hypothesis, for $k \leqslant m < n$ the D_k^m-translates of F_k are disjoint. It then clearly follows that the D_k^n-translates of F_k are disjoint as well and our claim follows by induction.

The D_k^n's we have constructed are a discrete version of the Δ_k's which we will soon construct to fulfill (iii) through (vii). However there is one more thing we must establish first. We claim that for all $n, k \in \mathbb{N}$ with $k < n$ the D_k^n-translates of F_k are maximally disjoint within $\beta(n, k)$. This is clearly true whenever $n = k + 1$ (we take $\beta(n, n-1) = H_n$). Fix $k \in \mathbb{N}$ and towards a contradiction suppose $n > k + 1$ is minimal such that the D_k^n-translates of F_k are not maximally disjoint within $\beta(n, k)$. Fix $g \in \beta(n, k)$ such that $gF_k \subseteq \beta(n, k)$ and $gF_k \bigcap D_k^n F_k = \varnothing$. Our argument will rely on inductively creating a finite sequence of natural numbers. We first detail how the starting number v_0 is determined. Recall that in the construction of F_n δ_{n-1}^n through δ_{k+1}^n are defined first and then δ_k^n is chosen maximally disjoint within $\beta(n, k) - \bigcup_{k<m<n} \delta_m^n F_m$. We cannot have $gF_k \subseteq \beta(n, k) - \bigcup_{k<m<n} \delta_m^n F_m$ since the δ_k^n-translates of F_k are maximally disjoint within this region and $gF_k \bigcap \delta_k^n F_k = gF_k \bigcap \delta_k^n D_k^k F_k \subseteq gF_k \bigcap D_k^n F_k = \varnothing$. As $gF_k \subseteq \beta(n, k)$ we must have $gF_k \bigcap (\bigcup_{k<m<n} \delta_m^n F_m) \neq \varnothing$. Therefore there exists $v_0 \in \mathbb{N}$ with $k < v_0 < n$ and $\alpha_0 \in \delta_{v_0}^n$ such that $gF_k \bigcap \alpha_0 F_{v_0} \neq \varnothing$. Note that $\alpha_0 D_k^{v_0} \subseteq \delta_{v_0}^n D_k^{v_0} \subseteq D_k^n$ so $\alpha_0^{-1} gF_k \bigcap D_k^{v_0} F_k = \alpha_0^{-1}(gF_k \bigcap \alpha_0 D_k^{v_0} F_k) = \varnothing$.

Now assume v_0 through v_{i-1} have been defined and $\alpha_j \in \delta_{v_j}^{v_{j-1}}$ has been fixed for each $0 < j \leqslant i - 1$ such that:

(a) $n > v_0 > v_1 > \cdots > v_{i-1} > k$;

(b) $gF_k \bigcap \alpha_0\alpha_1 \cdots \alpha_{i-1} F_{v_{i-1}} \neq \varnothing$; and

(c) $(\alpha_0\alpha_1 \cdots \alpha_{i-1})^{-1} gF_k \bigcap D_k^{v_{i-1}} F_k = \varnothing$.

We will find a new number v_i and from here the sequence may either terminate or continue further. Since $F_{v_{i-1}} = \bigcup_{0 \leqslant m < v_{i-1}} \delta_m^{v_{i-1}} F_m$, by (b) there exists $0 \leqslant v_i < v_{i-1}$ and $\alpha_i \in \delta_{v_i}^{v_{i-1}}$ such that $gF_k \bigcap \alpha_0\alpha_1 \cdots \alpha_i F_{v_i} \neq \varnothing$. If $v_i = k$ then we would have $\alpha_i F_{v_i} \subseteq \delta_k^{v_{i-1}} F_k = \delta_k^{v_{i-1}} D_k^k F_k \subseteq D_k^{v_{i-1}} F_k$ which would be in contradiction with (c) of the induction hypothesis, so $v_i \neq k$. If $v_i > k$ then $\alpha_i D_k^{v_i} F_k \subseteq \delta_{v_i}^{v_{i-1}} D_k^{v_i} F_k \subseteq D_k^{v_{i-1}} F_k$ which, together with the induction hypothesis (c), shows that (c) is again satisfied. Therefore if $v_i > k$ then (a) and (c) are satisfied and we can continue this construction further. But as we are constructing a strictly decreasing sequence with initial term $k < v_0 < n$ the process will eventually terminate. Note that in the case when $v_i < k$, property (b) is satisfied for i. Also it is important to note that if $k = 0$ then we can never have $v_i < k$ and since the sequence we are constructing is strictly decreasing we eventually have $v_i = k$ which contradicts property (c) of the induction hypothesis as stated earlier. Thus if $k = 0$ we have already arrived at our contradiction.

Assume that $k > 0$ and that the process above terminates at stage j. Since v_0 was explicitly found with $k < v_0 < n$, it must be that $j \geqslant 1$. Set $w = v_j$ and $p = v_{j-1}$ so that $w < k < p < n$. For $\alpha = \alpha_0\alpha_1 \cdots \alpha_{j-1}$ we have $\alpha^{-1} gF_k \bigcap \alpha_j F_w \neq \varnothing$ therefore $\alpha^{-1} gF_k \subseteq \alpha_j F_w F_k^{-1} F_k$ and

$$\alpha^{-1} gF_k(F_{k+1}^{-1} F_{k+1}) \cdots (F_{p-1}^{-1} F_{p-1}) \subseteq \alpha_j F_w F_k^{-1} F_k (F_{k+1}^{-1} F_{k+1}) \cdots (F_{p-1}^{-1} F_{p-1}).$$

Additionally as $\alpha_j \in \delta_w^p$,

$$\alpha_j F_w (F_k^{-1} F_k) \cdots (F_{p-1}^{-1} F_{p-1}) \subseteq \alpha_j F_w (F_{w+1}^{-1} F_{w+1}) \cdots (F_{p-1}^{-1} F_{p-1}) \subseteq H_p.$$

We therefore see that $\alpha^{-1} gF_k(F_{k+1}^{-1} F_{k+1}) \cdots (F_{p-1}^{-1} F_{p-1}) \subseteq H_p$ and hence $\alpha^{-1} gF_k \subseteq \beta(p, k)$. Also by property (c) $\alpha^{-1} gF_k \bigcap D_k^p F_k = \varnothing$. But then the D_k^p-translates of F_k are not maximally disjoint within $\beta(p, k)$ which contradicts the minimality of n.

In particular for all $n > k$ the D_k^n-translates of F_k are maximally disjoint within H_{n-1} since $H_{n-1} \subseteq \beta(n, k)$ by construction. We remark that $D_k^n \subseteq D_k^{n+1}$ since $\delta_n^{n+1} D_k^n \subseteq D_k^{n+1}$ and $1_G \in \delta_n^{n+1}$. As $(H_n)_{n \in \mathbb{N}}$ is an increasing sequence with $\bigcup_{n \in \mathbb{N}} H_n = G$, for each $k \in$

$\mathbb{N}$ $\Delta_k = \bigcup_{n\geqslant k} D^n_k$ is such that the Δ_k-translates of F_k are maximally disjoint within G. Properties (iii) and (iv) are immediately satisfied.

We will now prove that for $n \geqslant k$ and $\gamma \in \Delta_n$ $\Delta_k \bigcap \gamma F_n = \gamma D^n_k$. Properties (vi) and (vii) will clearly follow and since $D^n_{n-1} = \delta^n_{n-1}$ (v) will follow from how we defined δ^n_{n-1}.

Clearly when $n = k$ and $\gamma \in \Delta_k$ we have $\Delta_k \bigcap \gamma F_k = \{\gamma\}$ since $\gamma \in \Delta_k \bigcap \gamma F_k$ and the Δ_k-translates of F_k are disjoint. Note that $\{\gamma\} = \gamma D^k_k$. Now fix $k, n \in \mathbb{N}$ with $k < n$ and assume that for all $k \leqslant m < n$ and $\psi \in \Delta_m$ $\Delta_k \bigcap \psi F_m = \psi D^m_k$. Fix $\gamma \in \Delta_n$. For some $s \geqslant n$ $\gamma \in D^s_n$, so for any $0 \leqslant m < n$ $\gamma\delta^n_m \subseteq D^s_n\delta^n_m D^m_m \subseteq D^s_n D^n_m \subseteq D^s_m \subseteq \Delta_m$. We then have

$$\Delta_k \bigcap \gamma F_n = \Delta_k \bigcap \left(\bigcup_{0\leqslant m<n} \gamma\delta^n_m F_m \right)$$

$$= \left(\Delta_k \bigcap \left(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m \right)\right) \bigcup \left(\bigcup_{k\leqslant m<n} \left(\Delta_k \bigcap \gamma\delta^n_m F_m\right)\right)$$

$$= \left(\Delta_k \bigcap \left(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m \right)\right) \bigcup \left(\bigcup_{k\leqslant m<n} \gamma\delta^n_m D^m_k \right)$$

$$= \left(\Delta_k \bigcap \left(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m \right)\right) \bigcup \gamma D^n_k.$$

By our construction $(\bigcup_{0\leqslant m<k} \delta^n_m F_m) \bigcap (\bigcup_{k\leqslant m<n} \delta^n_m F_m) = \varnothing$. Thus γD^n_k is disjoint with $\Delta_k \bigcap(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m)$. We will show $\Delta_k \bigcap(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m) = \varnothing$. Towards a contradiction suppose $g \in \Delta_k \bigcap(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m)$. Fix $0 \leqslant m < k$ such that $g \in \Delta_k \bigcap \gamma\delta^n_m F_m$. Since $1_G \in F_k$ we have $gF_k \subseteq \gamma\delta^n_m F_m F_k^{-1} F_k$ and

$$gF_k(F^{-1}_{k+1}F_{k+1}) \cdots (F^{-1}_{n-1}F_{n-1}) \subseteq \gamma\delta^n_m F_m(F^{-1}_k F_k)(F^{-1}_{k+1}F_{k+1}) \cdots (F^{-1}_{n-1}F_{n-1}).$$

Since $\delta^n_m F_m \subseteq \beta(n, m)$ we have

$$\gamma\delta^n_m F_m(F^{-1}_k F_k) \cdots (F^{-1}_{n-1}F_{n-1}) \subseteq \gamma\delta^n_m F_m(F^{-1}_{m+1}F_{m+1}) \cdots (F^{-1}_{n-1}F_{n-1}) \subseteq \gamma H_n.$$

Thus $gF_k \subseteq \gamma\beta(n, k)$. We showed earlier that the D^n_k-translates of F_k are maximally disjoint within $\beta(n, k)$ so we have $gF_k \bigcap \gamma D^n_k F_k \neq \varnothing$. Additionally as $\gamma \in D^s_n$ $\gamma D^n_k \subseteq D^s_n D^n_k \subseteq D^s_k \subseteq \Delta_k$. But the Δ_k-translates of F_k are disjoint and $g \in \Delta_k$ so we must have $g \in \gamma D^n_k$. But this contradicts $\Delta_k \bigcap(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m)$ being disjoint with γD^n_k. We conclude $\Delta_k \bigcap(\bigcup_{0\leqslant m<k} \gamma\delta^n_m F_m) = \varnothing$ and $\Delta_k \bigcap \gamma F_n = \gamma D^n_k$. By induction this establishes (v), (vi), and (vii).

Let G be a group and let $A, B \subseteq G$ be finite with $1_G \in A$. We define $\rho(B; A)$ to be the minimal size of a set $D \subseteq B$ such that for every $g \in B$ with $gA \subseteq B$ there exists $d \in D$ with $gA \bigcap dA \neq \varnothing$. Such a minimum size exists since $D \subseteq B$ which is finite. Note that if $A' \subseteq A$ then $\rho(B; A') \geqslant \rho(B; A)$.

PROPOSITION 3·3. *Let G be a countably infinite group and let $A, B \subseteq G$ be finite with $1_G \in A$. For any $\epsilon > 0$ there exists a finite $C \subseteq G$ such that $C \supseteq B$ and $\rho(C; A) > (|C|/|A|)(1 - \epsilon)$.*

Proof. Let $\Delta \subseteq G$ be countably infinite and such that the Δ-translates of $AA^{-1}A$ are disjoint and $\Delta AA^{-1}A \bigcap B = \varnothing$. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of Δ. For each $n \in \mathbb{N}^+$ define $B_n = B \cup \bigcup_{1 \leqslant k \leqslant n} \lambda_k A$. Fix $n \in \mathbb{N}^+$ and let $D \subseteq B_n$ be such that for every $g \in B_n$ with $gA \subseteq B_n$ $gA \bigcap DA \neq \varnothing$. It follows that for each $1 \leqslant i \leqslant n$ there is $d_i \in D$ with $d_i A \bigcap \lambda_i A \neq \varnothing$. Since the Δ-translates of $AA^{-1}A$ are disjoint and for each $1 \leqslant i \leqslant n$ $d_i A \subseteq \lambda_i AA^{-1}A$, the d_i's are all distinct. Additionally $\Delta AA^{-1}A \bigcap B = \varnothing$ so $\rho(B_n; A) - n \geqslant \rho(B; A)$. Therefore we have

$$\rho(B_n; A)\frac{|A|}{|B_n|} \geqslant \frac{n|A| + \rho(B; A)|A|}{n|A| + |B|}.$$

Clearly as n goes to infinity the fraction on the right goes to 1. So for some $n \in \mathbb{N}^+$ $\rho(B_n; A)(|A|/|B_n|) > 1 - \epsilon$ and $\rho(B_n; A) > (|B_n|/|A|)(1 - \epsilon)$. $B_n \supseteq B$ so we are done.

PROPOSITION 3·4. *If G is a countably infinite group and $A, B \subseteq G$ are finite with $1_G \in A$ then there exists a finite $C \subseteq G$ such that $C \supseteq B$ and $2^{\rho(C;A)} > 32|C|^5$.*

Proof. Clearly there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ $2^{\frac{n}{2|A|}} > 32n^5$. Thus let $B' \subseteq G$ be finite such that $B' \supseteq B$ and $|B'| \geqslant N$. By Proposition 3·3 there exists a finite $C \subseteq G$ with $C \supseteq B'$ and $\rho(C; A) > (1/2)(|C|/|A|)$. Then $C \supseteq B$ and as $n = |C|$ is at least N,

$$2^{\rho(C;A)} > 2^{\frac{1}{2}\frac{|C|}{|A|}} = 2^{\frac{n}{2|A|}} > 32n^5 = 32|C|^5.$$

LEMMA 3·5. *If G is a countably infinite group then there exists a finite $A \subseteq G$ such that $1_G \in A$, $|A| > 1$, and for all $a \in A$ if $a \neq 1_G$ then $aA \neq A$.*

Proof. Choose a finite $A_0 \subseteq G$ with $1_G \in A_0$ and $|A_0| > 1$. Fix $a \in G - A_0A_0 \bigcup A_0A_0^{-1}$. Let $A = A_0 \cup \{a\}$. Immediately we have $aA_0 \bigcap A_0 = \varnothing$ since $a \notin A_0A_0^{-1}$. Thus we must have $aA \neq A$ since $|A_0| > 1$. Now let $g \in A_0$ with $g \neq 1_G$. We have two cases to consider. Case 1: $gA_0 = A_0$. Then $gA \neq A$ since otherwise we would have $ga = a$ contradicting $g \neq 1_G$. Case 2: $gA_0 \neq A_0$. Since $a \notin A_0A_0$ we have $gA \neq A$ as well. We have shown A satisfies the requirements.

THEOREM 3·6. *If G is a countably infinite group then G has the coloring property.*

Proof. Fix an increasing sequence of finite sets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ with $\bigcup_{n \in \mathbb{N}^+} A_n = G$.

We will first construct a sequence $(H_n)_{n \in \mathbb{N}}$ of finite subsets of G. Using Lemma 3·5 we may let $H_0 \subseteq G$ be finite such that $1_G \in H_0$, $|H_0| > 1$, and for all $h \in H_0 - \{1_G\}$ $hH_0 \neq H_0$. Next let $H_1 \subseteq G$ be finite such that $A_1 \bigcup H_0H_0(H_0^{-1}H_0^{-1}H_0H_0) \subseteq H_1$ and such that $2^{\rho(H_1; H_0H_0)} > 32|H_1|^5$.

The construction is continued inductively. Once H_0 through H_{k-1} have been defined for $k > 1$, let $H_k \subseteq G$ be finite such that

$$A_k \bigcup H_{k-1}(H_0^{-1}H_0^{-1}H_0H_0)(H_1^{-1}H_1)(H_2^{-1}H_2)\cdots(H_{k-1}^{-1}H_{k-1}) \subseteq H_k$$

and such that $2^{\rho(H_k; H_{k-1})} > 32|H_k|^5$.

With the exception of F_0, let (F_n) and (Δ_n) be as in Proposition 3·2 with respect to the sequence $H_0H_0, H_1, H_2, H_3, \ldots, H_n, \ldots$. Let $F_0 = H_0$.

For each $n \geqslant 1$ F_n is finite so we may let $\lambda_1^n, \lambda_2^n, \ldots, \lambda_{s(n)+4}^n$ enumerate $\Delta_{n-1} \bigcap F_n$ where $s(n) = |\Delta_{n-1} \bigcap F_n| - 4$. Notice that for $n \geqslant 2$ the $\Delta_{n-1} \bigcap F_n$-translates of F_{n-1} are contained and maximally disjoint within H_n so $s(n) + 4 \geqslant \rho(H_n; F_{n-1}) \geqslant \rho(H_n; H_{n-1})$

as $F_{n-1} \subseteq H_{n-1}$. Also for $n = 1$ the $\Delta_0 \bigcap F_1$-translates of $F_0F_0 = H_0H_0$ are maximally disjoint within H_1 so $s(1) + 4 \geqslant \rho(H_1; H_0H_0)$. Throughout this proof we will frequently invoke properties (vi) and (vii) of Proposition 3·2, usually with respect to Δ_n-translates of the λ_i^n's and without explicit mention of invoking the properties.

Define $a_0 = b_0 = 1_G$ and for each $n \geqslant 1$ let $a_n = \lambda^n_{s(n)+2}\lambda^{n-1}_{s(n-1)+2}\cdots\lambda^1_{s(1)+2}$, and $b_n = \lambda^n_{s(n)+3}\lambda^{n-1}_{s(n-1)+3}\cdots\lambda^1_{s(1)+3}$. Clearly $a_0 \in F_0$ and assuming $a_{n-1} \in F_{n-1}$ we have $a_n = \lambda^n_{s(n)+2}a_{n-1} \in \lambda^n_{s(n)+2}F_{n-1} \subseteq F_n$ since $\lambda^n_{s(n)+2} \in \Delta_{n-1}\bigcap F_n$. By induction, and by a similar argument, we have $a_n, b_n \in F_n$ for all $n \in \mathbb{N}$. Additionally for every $n \in \mathbb{N}^+$ and $1 \leqslant i \leqslant s(n)+4$ $\Delta_n\lambda_i^n a_{n-1} \subseteq \Delta_{n-1}a_{n-1}$ and $\Delta_n\lambda_i^n b_{n-1} \subseteq \Delta_{n-1}b_{n-1}$ since $\Delta_n\lambda_i^n \subseteq \Delta_n(\Delta_{n-1}\bigcap F_n) = \Delta_{n-1}\bigcap \Delta_n F_n \subseteq \Delta_{n-1}$ by property (vi) of Proposition 3·2. In particular $\Delta_n\{a_n, b_n\} \subseteq \Delta_{n-1}\{a_{n-1}, b_{n-1}\}$. For each $n \in \mathbb{N}^+$ define $\Omega_n = \bigcup_{1\leqslant m\leqslant n}\Delta_m\lambda^m_{s(m)+4}a_{m-1}$ and note that by our earlier remark $\Omega_n - \Omega_{n-1} \subseteq \Delta_n\lambda^n_{s(n)+4}a_{n-1} \subseteq \Delta_{n-1}a_{n-1}$. These last two statements tell us that in what we are about to introduce properties (i) and (ii) are consistent.

We wish to construct a sequence of functions $(c_n)_{n\in\mathbb{N}}$ satisfying for each $n \in \mathbb{N}^+$:

(i) $\mathrm{dom}(c_n) = G - (\Delta_n\{a_n, b_n\} \bigcup \Omega_n)$;

(ii) $c_{n+1} \supseteq c_n$;

(iii) there exists $V \subseteq F_n \bigcap \mathrm{dom}(c_n)$ such that for any function $c \supseteq c_n$ and $g \in G$, $g \in \Delta_n \Longleftrightarrow \forall a \in V\ \ c(ga) = c(a)$;

(iv) for any $\gamma, \psi \in \Delta_n$, if $\gamma^{-1}\psi \in H_nH_n^{-1}H_n^2H_n^{-1}$ then there exists $a \in F_n$ such that $\gamma a, \psi a \in \mathrm{dom}(c_n)$ and $c_n(\gamma a) \neq c_n(\psi a)$.

After constructing the sequence (c_n) it will be an easy task to extract a 2-coloring on G. The general idea of the construction is as follows. Given c_{k-1} we first define c'_{k-1} to satisfy (iii) using (for the most part) $\Delta_k\lambda^k_{s(k)+1}\{a_{k-1}, b_{k-1}\}$. We then extend c'_{k-1} to c_k, which preserves property (iii), use $\Delta_k\{\lambda^k_1, \lambda^k_2, \ldots, \lambda^k_{s(k)}\}a_{k-1}$ to achieve (iv), and then leave $\Delta_k\{\lambda^k_{s(k)+2}a_{k-1}, \lambda^k_{s(k)+3}b_{k-1}\} = \Delta_k\{a_k, b_k\}$ and $\Delta_k\lambda^k_{s(k)+4}a_{k-1}\bigcup\Omega_{k-1} = \Omega_k$ undefined. We now cover the details.

We first aim to satisfy (iii) and define

$$c_0 : (G - \Delta_1\{\lambda^1_1, \lambda^1_2, \ldots, \lambda^1_{s(1)}, \lambda^1_{s(1)+2}, \lambda^1_{s(1)+3}, \lambda^1_{s(1)+4}\}) \longrightarrow 2$$

by

$$c_0(g) = \begin{cases} 1 & \text{if } g \in \Delta_1\lambda^1_{s(1)+1}F_0 \\ 0 & \text{otherwise} \end{cases}$$

for $g \in \mathrm{dom}(c_0)$.

Let $c\colon G \to 2$ with $c \supseteq c_0$ and let $g \in G$ be arbitrary. Suppose $c(gF_0) = \{1\}$. Since $1_G \in F_0$ $c(g) = 1$ so from how we defined c_0 either $g \in \mathrm{dom}(c_0)$ and therefore $g \in \Delta_1\lambda^1_{s(1)+1}F_0$ or $g \in G - \mathrm{dom}(c_0)$. That is, either $g \in \Delta_1\lambda^1_{s(1)+1}F_0$ or for some $1 \leqslant i \leqslant s(1)+4$, $i \neq s(1)+1$, $g \in \Delta_1\lambda^1_i$. We claim in fact $g \in \Delta_1\lambda^1_{s(1)+1}$, so towards a contradiction suppose $g \notin \Delta_1\lambda^1_{s(1)+1}$. Case 1: $g \in (\Delta_1\lambda^1_{s(1)+1}F_0 - \Delta_1\lambda^1_{s(1)+1})$. Let $\gamma \in \Delta_1$ and $f \in F_0$ be such that $g = \gamma\lambda^1_{s(1)+1}f$. Then $f \neq 1_G$. By construction there is $h \in F_0$ with $fh \notin F_0$ and therefore $\gamma\lambda^1_{s(1)+1}fh \notin \gamma\lambda^1_{s(1)+1}F_0$. But also $\gamma\lambda^1_{s(1)+1}fh \in \gamma\lambda^1_{s(1)+1}F_0F_0 \subseteq \gamma F_1$ and for $1 \leqslant j \leqslant s(1)+4$ with $j \neq s(1)+1$ $\gamma\lambda^1_{s(1)+1}fh \neq \gamma\lambda^1_j$ since $\lambda^1_{s(1)+1}F_0F_0 \bigcap \lambda^1_jF_0F_0 = \varnothing$. Thus $c(\gamma\lambda^1_{s(1)+1}fh) \neq 1$, a contradiction. Case 2: $g \in \Delta_1\{\lambda^1_1, \lambda^1_2, \ldots, \lambda^1_{s(1)}, \lambda^1_{s(1)+2}, \lambda^1_{s(1)+3}, \lambda^1_{s(1)+4}\}$. Let $\gamma \in \Delta_1$ and $1 \leqslant i \leqslant s(1)+4$ with $i \neq s(1)+1$ be such that $g = \gamma\lambda^1_i$. Since $|F_0| > 1$ there is $f \in F_0$, $f \neq 1_G$, with $c(gf) = 1$. Thus $gf \in \Delta_1\lambda^1_{s(1)+1}F_0$ or $gf \in \Delta_1\{\lambda^1_1, \lambda^1_2, \ldots, \lambda^1_{s(1)}, \lambda^1_{s(1)+2}, \lambda^1_{s(1)+3}, \lambda^1_{s(1)+4}\}$. But since $1_G \in F_0$ $gF_0 = \gamma\lambda^1_iF_0 \subseteq \gamma\lambda^1_iF_0^2 \subseteq \gamma F_1$ and by construction for $1 \leqslant j \leqslant s(1)+4$

with $j \neq i$ $\lambda_i^1 F_0^2 \bigcap \lambda_j^1 F_0^2 = \varnothing$. Thus it is impossible for $c(gf)$ to be 1. We conclude $g \in \Delta_1 \lambda^1_{s(1)+1}$.

We therefore have a test for membership of Δ_1 for any function $c \supseteq c_0$. Since $1_G \in \Delta_1$ for any $g \in G$,

$$g \in \Delta_1 \iff c(g\lambda^1_{s(1)+1}F_0) = \{1\} \iff \forall f \in \lambda^1_{s(1)+1}F_0 \;\; c(gf) = c(f).$$

Note that $\lambda^1_{s(1)+1}F_0 \subseteq \lambda^1_{s(1)+1}F_0F_0 \subseteq F_1$ and $\lambda^1_{s(1)+1}F_0 \subseteq \mathrm{dom}(c_0)$.

In constructing c_1, and likewise the sequence (c_n), it is of much use to consider graphs. Let Γ be the graph with vertex set Δ_1 and with edge relation given by

$$(\gamma, \psi) \in E(\Gamma) \iff \gamma^{-1}\psi \in H_1H_1^{-1}H_1^2H_1^{-1} \text{ or } \psi^{-1}\gamma \in H_1H_1^{-1}H_1^2H_1^{-1}.$$

Then for every vertex $v \in V(\Gamma)$, $\deg(v) \leqslant 2|H_1H_1^{-1}H_1^2H_1^{-1}| \leqslant 2|H_1|^5$. It is a simple result in graph theory that Γ is $(2|H_1|^5 + 1)$-colorable. Let $\mu : V(\Gamma) \to (2|H_1|^5 + 1)$ be a $(2|H_1|^5 + 1)$-coloring of Γ. For each $i \in \mathbb{N}^+$ define $B_i : \mathbb{N} \to 2$ to be such that $B_i(k)$ is the i^{th} digit from least to most significant in the binary representation of k when $k \geqslant 2^{i-1}$ and $B_i(k) = 0$ when $k < 2^{i-1}$. Note that since $2^{s(1)} \geqslant 2^{\rho(H_1; H_0H_0)-4} > 2|H_1|^5$ all integers 0 through $2|H_1|^5$ can be written in binary using $s(1)$ digits.

Define $c_1 : G - (\Delta_1\{a_1, b_1\} \bigcup \Omega_1) \to 2$ to be such that $c_1 \supseteq c_0$ and such that for every $\gamma \in \Delta_1$ and $1 \leqslant i \leqslant s(1)$ $c(\gamma\lambda_i^1) = B_i(\mu(\gamma))$. It follows that properties (i) through (iv) are satisfied (property (iii) was satisfied by c_0).

The construction will be continued inductively. Assume c_0 through c_{k-1} have been defined with $k > 1$. We will first construct c'_{k-1} which will satisfy property (iii). Let c'_{k-1} have domain

$$G - \left(\Omega_{k-1} \bigcup \Delta_k\{\lambda_1^k, \lambda_2^k, \ldots, \lambda^k_{s(k)}, \lambda^k_{s(k)+2}, \lambda^k_{s(k)+4}\}a_{k-1} \bigcup \Delta_k\{\lambda_{s(k)+3}b_{k-1}\}\right)$$

and have the following properties:

$$c'_{k-1} \supseteq c_{k-1};$$

$$\forall\psi \in (\Delta_{k-1} - \Delta_k\{\lambda_1^k, \lambda_2^k, \ldots, \lambda^k_{s(k)+4}\}) \;\; c'_{k-1}(\psi a_{k-1}) = c'_{k-1}(\psi b_{k-1}) = 0;$$

$$\forall\psi \in (\Delta_{k-1} - \Delta_k\{\lambda^k_{s(k)+1}, \lambda^k_{s(k)+3}\}) \;\; c'_{k-1}(\psi b_{k-1}) = 0;$$

$$\forall\gamma \in \Delta_k \;\; c'_{k-1}(\gamma\lambda^k_{s(k)+1}a_{k-1}) = c'_{k-1}(\gamma\lambda^k_{s(k)+1}b_{k-1}) = 1; \text{ and}$$

$$\forall\gamma \in \Delta_k \;\; c'_{k-1}(\gamma\lambda^k_{s(k)+3}a_{k-1}) = 0.$$

Note that $\Omega_{k-1} \bigcup \Delta_k\lambda^k_{s(k)+4}a_{k-1} = \Omega_k$ so the domain specified for c'_{k-1} does not contain Ω_k. Also in the properties listed above we specified the values of c'_{k-1} on a subset of $\Delta_{k-1}\{a_{k-1}, b_{k-1}\}$ and $\Delta_{k-1}\{a_{k-1}, b_{k-1}\} \bigcap \Omega_{k-1} = \varnothing$. This is since

$$\Delta_1\{a_1, b_1\} \bigcap \Omega_1 = \Delta_1\{a_1, b_1\} \bigcap \Delta_1\lambda^1_{s(1)+4} = \varnothing$$

and assuming $\Delta_{n-1}\{a_{n-1}, b_{n-1}\} \bigcap \Omega_{n-1} = \varnothing$ we have

$$\Delta_n\{a_n, b_n\} \bigcap \Omega_n = (\Delta_n\{a_n, b_n\} \bigcap \Omega_{n-1}) \bigcup (\Delta_n\{a_n, b_n\} \bigcap \Delta_n\lambda^n_{s(n)+4}a_{n-1}) = \varnothing$$

as $\Delta_n\{a_n, b_n\} \subseteq \Delta_{n-1}\{a_{n-1}, b_{n-1}\}$.

Let $c \supseteq c'_{k-1}$ be any function from G to 2. The function c_{k-1} was not defined on $\Delta_{k-1}b_{k-1}$ and, from how we defined c'_{k-1}, for any $\psi \in \Delta_{k-1}$ $c(\psi b_{k-1}) = 1$ only if $\psi \in \Delta_k\lambda^k_{s(k)+1}$ or $\psi \in \Delta_k\lambda^k_{s(k)+3}$. However $c'_{k-1}(\Delta_k\lambda^k_{s(k)+1}a_{k-1}) = \{1\} \neq \{0\} = c'_{k-1}(\Delta_k\lambda^k_{s(k)+3}a_{k-1})$ so if

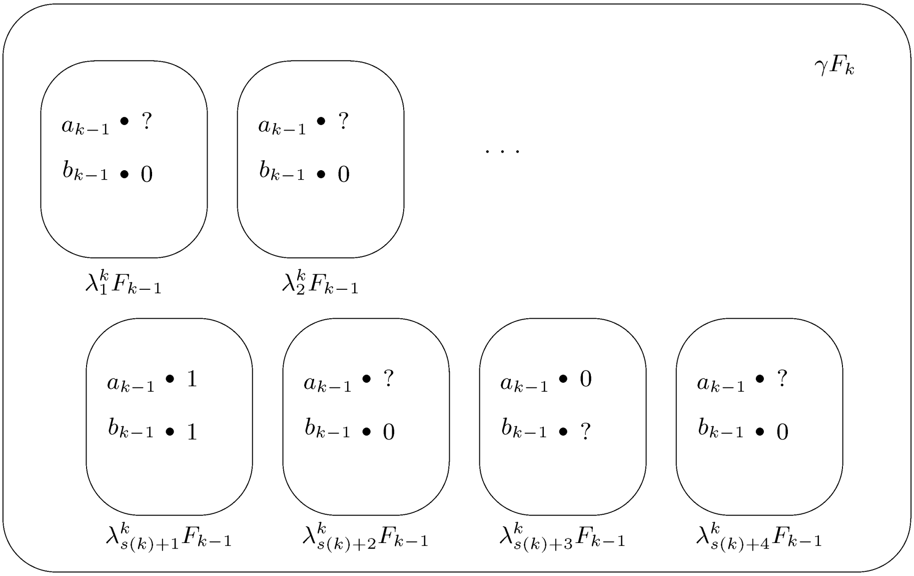

Fig. 2. The coloring c'_{k-1}: ensuring a membership test for Δ_k.

we can recognize membership of Δ_{k-1} then we can recognize membership of Δ_k. But by the induction hypothesis we can do just that. Let $V \subseteq F_{k-1} \bigcap \mathrm{dom}(c_{k-1})$ be such that for $g \in G$, $g \in \Delta_{k-1} \iff \forall a \in V \;\; c(ga) = c(a)$. We clearly have for $g \in G$, $g \in \Delta_k$ if and only if $g\lambda^k_{s(k)+1} \in \Delta_{k-1}$ and $c(g\lambda^k_{s(k)+1}a_{k-1}) = c(g\lambda^k_{s(k)+1}b_{k-1}) = 1$. If we set

$$V' = \lambda^k_{s(k)+1} V \bigcup \{\lambda^k_{s(k)+1}a_{k-1}, \lambda^k_{s(k)+1}b_{k-1}\}$$

then since $1_G \in \Delta_k \;\; V' \subseteq F_k \bigcap \mathrm{dom}(c'_{k-1})$ and for $g \in G$

$$g \in \Delta_k \iff \forall a \in V' \;\; c(ga) = c(a).$$

We will construct c_k to extend c'_{k-1} so that c_k will have property (iii).

Let Γ be the graph with vertex set Δ_k and edge relation given by

$$(\gamma, \psi) \in E(\Gamma) \iff \gamma^{-1}\psi \in H_k H_k^{-1} H_k^2 H_k^{-1} \text{ or } \psi^{-1}\gamma \in H_k H_k^{-1} H_k^2 H_k^{-1}.$$

Clearly for $v \in V(\Gamma) \;\; \deg(v) \leqslant 2|H_k|^5$. So we may let $\mu\colon V(\Gamma) \to (2|H_k|^5 + 1)$ be a $(2|H_k|^5+1)$-coloring of Γ. Since by construction $2^{s(k)} \geqslant 2^{\rho(H_k;H_{k-1})-4} > 2|H_k|^5$ all numbers 0 through $2|H_k|^5$ can be represented in binary with $s(k)$ digits. Recalling the domain of c'_{k-1} we see we only need to extend it by $\Delta_k\{\lambda_1^k, \lambda_2^k, \ldots, \lambda^k_{s(k)}\}a_{k-1}$ to have the domain of c_k as desired. So we let $c_k : G - (\Delta_k\{a_k, b_k\} \bigcup \Omega_k) \to 2$ be such that $c_k \supseteq c'_{k-1}$ and for all $\gamma \in \Delta_k$ and $1 \leqslant i \leqslant s(k) \;\; c_k(\gamma\lambda_i^k a_{k-1}) = B_i(\mu(\gamma))$. Properties (i) through (iv) are then clearly satisfied.

Let $c = \bigcup_{n\in\mathbb{N}} c_n$. Then $\mathrm{dom}(c) \subseteq G - \bigcup_{n\in\mathbb{N}^+} \Omega_n$. We claim that any function $\pi : G \to 2$ with $\pi \supseteq c$ is a 2-coloring on G. Fix such a function π and fix $s \in G$ with $s \neq 1_G$. Since $\bigcup_{n\in\mathbb{N}} H_n = G$ we may let $i \geqslant 1$ be minimal such that $s \in H_i$ and let $T = F_i F_i^{-1} F_i$. Let $g \in G$ be arbitrary. We will find $t \in T$ such that $\pi(gt) \neq \pi(gst)$. Since the Δ_i-translates of F_i are maximally disjoint within $G \;\; gF_i \bigcap \Delta_i F_i \neq \varnothing$ so there exists $f \in F_i F_i^{-1}$ such that $gf \in \Delta_i$. We have two cases to consider. Case 1: $gsf \notin \Delta_i$. Since $\pi \supseteq c_i$ we may let $V \subseteq F_i$ be such that for $h \in G \;\; h \in \Delta_i \iff \forall a \in V \;\; \pi(ha) = \pi(a)$. As $gsf \notin \Delta_i$ there exists $a \in V \subseteq F_i$ with $\pi(gsfa) \neq \pi(a)$. But $gf \in \Delta_i$ so $\pi(gfa) = \pi(a) \neq$

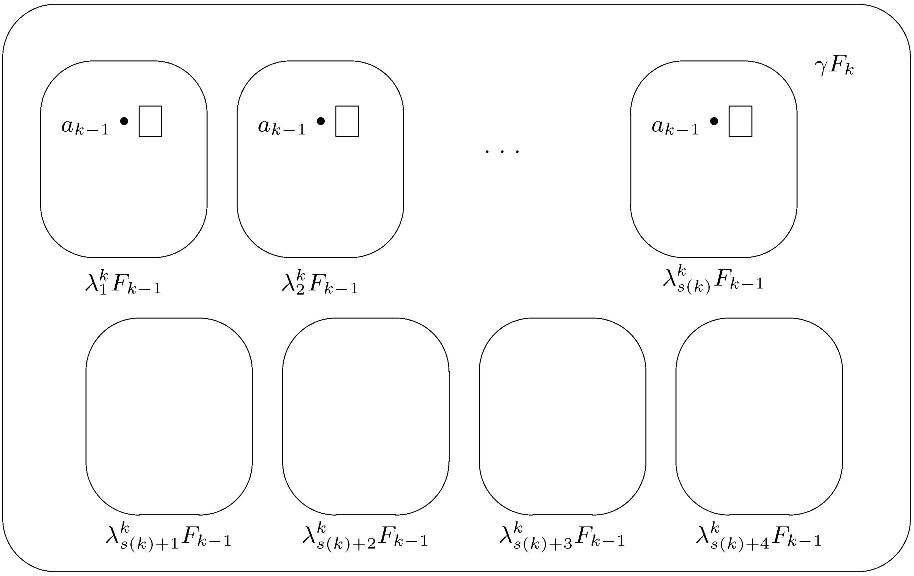

Fig. 3. The coloring c_k: coding a $(2|H_k|^5 + 1)$-coloring of Γ.

$\pi(gsfa)$. In addition we have $fa \in T$ so we are done. Case 2: $gsf \in \Delta_i$. Then we have $(gf)^{-1}(gsf) = f^{-1}sf \in F_i F_i^{-1} H_i F_i F_i^{-1} \subseteq H_i H_i^{-1} H_i^2 H_i^{-1}$. It follows there exists $a \in F_i$ such that $gfa, gsfa \in \mathrm{dom}(c_i)$ and $\pi(gfa) = c_i(gfa) \neq c_i(gsfa) = \pi(gsfa)$. Again $fa \in T$ so we are done.

COROLLARY 3·7. *If G is a countably infinite group then there is a perfect set of pairwise orthogonal 2-colorings on G.*

Proof. The proof of the previous theorem was precisely constructed to allow for a simple proof of this corollary. For this reason it will be understood that we will be using objects from the previous proof as they were defined there. Let c be as in the concluding paragraph of the previous proof. Recall that $\bigcup_{n\in\mathbb{N}^+} \Omega_n = \bigcup_{n\in\mathbb{N}^+} \Delta_n \lambda^n_{s(n)+4} a_{n-1}$. For each $\sigma \in 2^\omega$ define c_σ: $G \to 2$ to be such that $c_\sigma \supseteq c$ and such that for every $i \in \mathbb{N}^+$ and $\gamma \in \Delta_i$ $c_\sigma(\gamma\lambda^i_{s(i)+4} a_{i-1}) = \sigma(i-1)$. By the proof of the previous theorem, for every $\sigma \in 2^\omega$ c_σ is a 2-coloring on G. Now let $\sigma, \tau \in 2^\omega$ with $\sigma \neq \tau$. We will show c_σ and c_τ are orthogonal. Suppose $\sigma(i) \neq \tau(i)$ and let $T = F_{i+1} F_{i+1}^{-1} F_{i+1}$. Fix $h_0, h_1 \in G$. We will find $t \in T$ with $c_\sigma(h_0 t) \neq c_\tau(h_1 t)$. Since the Δ_{i+1}-translates of F_{i+1} are maximally disjoint within G there exists $f \in F_{i+1} F_{i+1}^{-1}$ such that $h_0 f \in \Delta_{i+1}$. We have two cases to consider. Case 1: $h_1 f \notin \Delta_{i+1}$. Let $V \subseteq F_{i+1} \bigcap \mathrm{dom}(c_{i+1})$ be such that for any function $\pi \supseteq c_{i+1}$ and any $g \in G$ $g \in \Delta_{i+1} \iff \forall a \in V \;\; \pi(ga) = \pi(a)$. Since $h_1 f \notin \Delta_{i+1}$ there is $a \in V$ such that $c_\tau(h_1 fa) \neq c_\tau(a) = c_{i+1}(a)$. But $h_0 f \in \Delta_{i+1}$ so $c_\sigma(h_0 fa) = c_\sigma(a) = c_{i+1}(a) = c_\tau(a) \neq c_\tau(h_1 fa)$. This completes this case as $fa \in T$. Case 2: $h_1 f \in \Delta_{i+1}$. Then $c_\sigma(h_0 f \lambda^{i+1}_{s(i+1)+4} a_i) = \sigma(i) \neq \tau(i) = c_\tau(h_1 f \lambda^{i+1}_{s(i+1)+4} a_i)$ and $f\lambda^{i+1}_{s(i+1)+4} a_i \in T$.

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