# Homework 3

1. Let $X$ be a set, $\mathcal{C} \subseteq \text{Pow}(X)$, and $Y \subseteq X$ with $Y \neq \emptyset$. Set $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$. Let $\sigma\text{-alg}_Y(\mathcal{C}_Y)$ be the $\sigma$-algebra on $Y$ generated by $\mathcal{C}_Y$ and let $\sigma\text{-alg}_X(\mathcal{C})$ be the $\sigma$-algebra on $X$ generated by $\mathcal{C}$. Prove that
$$\sigma\text{-alg}_Y(\mathcal{C}_Y) = \{E \cap Y : E \in \sigma\text{-alg}_X(\mathcal{C})\}$$

2. Let $(X,d)$ be a metric space and let $Y \subseteq X$ be nonempty. View $Y$ as a metric space by restricting $d$ to $Y \times Y$. Prove the following relation regarding Borel $\sigma$-algebras:
$$\mathcal{B}_Y = \{E \cap Y : E \in \mathcal{B}_X\}$$

3. Equip $\mathbb{R}$ with the usual metric $d$. Let $T = \bigcup_{n \in \mathbb{Z}, n \geq 0} \{0,1\}^n$ be the set of all finite-length tuples of $0$ and $1$. We write $\emptyset$ for the length $0$ tuple, so $\{0,1\}^0 = \{\emptyset\}$. For $s \in T$, say $s = (s_1, s_2, \cdots, s_n) \in \{0,1\}^n$, define
$$s^\frown 0 = (s_1, s_2, \cdots, s_n, 0) \in \{0,1\}^{n+1}$$
$$s^\frown 1 = (s_1, s_2, \cdots, s_n, 1) \in \{0,1\}^{n+1}.$$
For $s \in \{0,1\}^n$ write $|s| = n$ for the length of $s$.
Using $T$ we can construct the standard middle-thirds Cantor set. Specifically, set
$$F_\emptyset = [0,1]$$
and inductively whenever $F_s$ is defined ($s \in T$) set
$$F_{s^\frown 0} = \text{the closed left-third of } F_s$$
$$F_{s^\frown 1} = \text{the closed right-third of } F_s$$
Equivalently, when $s = (s_1, s_2, \cdots, s_n)$
$$F_s = \left[\sum_{k=1}^{n} 2s_k \cdot 3^{-k}, \sum_{k=1}^{n} 2s_k \cdot 3^{-k} + 3^{-n}\right].$$

The Cantor set is then

$$C = \bigcap_{n \in \mathbb{Z}, n \geq 0} \left( \bigcup_{\substack{S \in T \\ |S| = n}} F_S \right).$$

We restrict $d$ to $C$ to view $C$ as a metric space. As a subset of $C$, each set $C_S = F_S \cap C$ is open, closed, and compact.

a. Prove that $\mathcal{B}_C = \sigma\text{-alg}(\{C_S : S \in T\})$. I do not recommend using problems 1 or 2 above — that would require more work.

b. Prove that finite Borel measures on $C$ are in one-to-one correspondence with functions $f : T \to [0, +\infty)$ satisfying the algebraic condition

$$\forall s \in T \quad f(s) = f(s^\frown 0) + f(s^\frown 1).$$

c. Is there a finite Borel measure on $C$ that feels "natural" or "uniform"? (This won't be graded)

4. Ch. 1 # 28 from the book

5. Prove that the Cantor set $C$ has Lebesgue measure 0.