Chapter 7

Notation: For a group $G$, $S \leq G$, $a \in G$ define

- $aS = \{as : s \in S\}$ in additive notation
- $Sa = \{sa : s \in S\}$ in additive notation
- $aS_{a^{-1}} = \{asa^{-1} : s \in S\}$ in additive notation
- $|S| = \text{number of elements in } S$

Defn: Let $G$ group, $H \leq G$ subgroup, $a \in G$.

- $aH$ is the left-coset of $H$ containing $a$
  - $a$ is called a coset representative of $aH$
- $Ha$ is the right-coset of $H$ containing $a$
  - $a$ is called a coset representative of $Ha$. 

HW 6 due Wed.

OH today: 11:00-11:30, 4:00-4:30
Lem. 7. A: Let $G$ be a group, $H \leq G$, $a, b \in G$.

Then $aH \subseteq bH \iff a \subseteq b^{-1}a \subseteq H$.

Furthermore either $aH = bH$ or $aH \cap bH = \emptyset$.
(Similarly $Ha = Hb \iff ae \subseteq H \iff a^{-1}e \subseteq H$ and either $Ha = Hb$ or $Ha \cap Hb = \emptyset$)

Ref: (1) $\Rightarrow$ (2) Assume $aH = bH$. Since $e \in H$ we have $a = ae \subseteq aH = bH$.

(2) $\Rightarrow$ (3) Assume $a \subseteq bH$. Then there is $h \in H$ with $a = bh$. So $b^{-1}a = h \in H$.

(3) $\Rightarrow$ (1) Assume $b^{-1}a \subseteq H$. Set $h_0 = b^{-1}a \in H$.

Notice $h_0^{-1} = a^{-1}b$.

(aH $\subseteq$ bH) For any $h \in H$ we have $ah = (bh^{-1})ah = b(b^{-1}a)h = bh$, $h \in bH$.

(bH $\subseteq$ aH) For any $h \in H$ we have $bh = (aa^{-1})bh = a(a^{-1}b)h = a h_0^{-1}h \in aH$.

Now we prove the "Furthermore".

Case 1: $aH \cap bH = \emptyset$ Done.

Case 2: $aH \cap bH \neq \emptyset$.

Pick any $c \in aH \cap bH$.
Then $c \in aH$ and $c \in bH$.

By (2) $\Rightarrow$ (1) $cH = aH$ and $cH = bH$.

Therefore $aH = bH$. \[\square\]
Lem. 7.B: The collection of left-cosets \( \{ aH : a \in G \} \) partitions \( G \). Also \( |aH| = |H| \) for all \( a \in G \). (Similarly \( \{ Ha : a \in G \} \) partitions \( G \) and \( |Ha| = |H| \) for all \( a \in G \).)

**Pf:** Since \( e \in H \), we have \( a = ae \in aH \). So, the union of the \( aH \) (\( a \in G \)) is equal to \( G \). By Lem 7.A the sets \( aH \) (\( a \in G \)) are disjoint when they are not equal. This shows that \( \{ aH : a \in G \} \) is a partition of \( G \).

Lastly, \( |aH| = |H| \) since the map from \( H \) to \( aH \) sending \( hh \in H \) to \( ah \in aH \) is one-to-one and onto. \( \Box \)

**Warning:** Generally, \( aH \neq Ha \). However...

Lem 7.C: \( aH = Ha \iff aHa^{-1} = H \)

**Pf:** Multiplication on the right by \( a^{-1} \) is a one-to-one operation that sends \( aH \) to \( aHa^{-1} \) and sends \( Ha \) to \( H \). \( \Box \)
Ex: Set $H = \{ \alpha \in S_3 : \alpha(1) = 1^3 = \varepsilon, (23)^3 \}$
$H$ subgroup of $S_3$.

The left cosets of $H$ are
$\varepsilon H = H = \{ \varepsilon \}$, $(23)^3 = (23)H = \{ \alpha \in S_3 : \alpha(1) = 1^3 \}$
$(12)H = \{ \varepsilon (12) \}$, $(123)^3 = (123)H = \{ \alpha \in S_3 : \alpha(1) = 2^3 \}$
$(13)H = \{ \varepsilon (13) \}$, $(132)^3 = (132)H = \{ \alpha \in S_3 : \alpha(1) = 3^3 \}$

The right cosets of $H$ are
$H \varepsilon = H = \{ \varepsilon \}$, $(23)^3 = H(23) = \{ \alpha \in S_3 : \alpha(1) = 1^3 \}$
$H(12) = \{ \varepsilon (12) \}$, $(132)^3 = H(132) = \{ \alpha \in S_3 : \alpha(2) = 1^3 \}$
$H(13) = \{ \varepsilon (13) \}$, $(123)^3 = H(123) = \{ \alpha \in S_3 : \alpha(3) = 1^3 \}$

Lagrange's Thm 7.1:
If $G$ is a finite group and $H \leq G$ is a subgroup then $|H| \text{ divides } |G|$. Moreover, the number of distinct left (or right) cosets of $H$ in $G$ is $\frac{|G|}{|H|}$

Called the index of $H$ in $G$ and is denoted $|G:H|$. 
\textbf{Pf.} Let \( r = |G:H| = \# \text{ of distinct left cosets of } H. \)

Let \( a_1 H, a_2 H, \ldots, a_r H \) be the distinct left cosets of \( H. \) Then by Lem \( 7.8, \)
\( a_1 H, a_2 H, \ldots, a_r H \) partition \( G \) so

\[
|G| = |a_1 H| + |a_2 H| + \cdots + |a_r H|
\]

\[
\text{Lem 7.8}\quad = \left( |H| + |H| + \cdots + |H| \right) \quad \underbrace{}_{r}
\]

\[
= r \cdot |H|.
\]

Therefore \( |H| \mid |G| \) and

\[
|G:H| = r = \frac{|G|}{|H|}.
\]

\( \square \)