Lecture 18, Friday, May 6, 2002

Today's and next lecture:

1. Lifting properties of X to \((P_2(X), W_2)\), Polish, compact, but not proper.
2. Characterization of convergence in \((P_2(X), W_2)\).
3. Some examples.

Recall:

**Theorem.** If X is a Polish space, then \((P_2(X), W_2)\) is a metric space.

The \(\beta\)-metric of \(P(X)\):
\[
\beta(u, v) = \sup \{ \| \int_X f d\mu - \int_X f d\nu \| : f \in BL(X), \|f\|_1 \leq 1 \}
\]

\(\circ\) If X is separable, then \(\mu_n \to \mu\) narrowly \iff \int_X f d\mu_n \to \int_X f d\mu \ \forall f \in BL(X) \iff \beta(\mu_n, \mu) \to 0.

\(\circ\) If X is a Polish space, then \((P(X), \beta)\) is complete.

**Theorem.** Let X be a Polish space. Then \((P_2(X), W_2)\) is a complete metric space.

**Proof.** Let \(\{\mu_n\}\) be a Cauchy sequence in \((P_2(X), W_2)\).

Since \(\beta(\mu, \nu) \leq W_2(\mu, \nu) \ \forall \mu, \nu \in P_2(X)\), \(\{\mu_n\}\) is also a Cauchy sequence in \((P(X), \beta)\), a complete metric space. Hence, \(\beta(\mu_n, \mu) \to 0\) for some \(\mu \in P(X)\). We show that \(\mu \in P_2(X)\) and \(W_2(\mu_n, \mu) \to 0\).

Let \(\varepsilon > 0\). Let \(N \geq 1\) be such that \(W_2(\mu_n, \mu_m) \leq \varepsilon\) for \(m, n \geq N\).

We claim for a given \(x \in X\) and \(N \geq N\) that
\[
\int_X d(x, \bar{x}) d\mu(x) \leq 2\varepsilon^2 + 2\int_X d^*(y, \bar{x}) d\mu_n(y) < \infty, \quad (\star)
\]

which would imply \(\mu \in P_2(X)\), and
which would imply $W_2(\mu, \nu) \to 0$.

Let $\gamma_{m, n} \in \mathcal{A}(\mu_m, \nu_n)$ be such that

$$W_2^b(\mu_m, \nu_n) = \int_{X \times X} d^2(x, y) d\gamma_{m, n}(x, y).$$

Since $\beta(\mu_m, \nu) \to 0$, $\mu_m \to \mu$ narrowly, and $\{\mu_m\}$ is precompact narrowly, hence, by Prokhorov's Theorem, $\{\mu_m\}$ is equi-tight. \(\forall \delta > 0 \exists \text{ compact } K \subseteq X \text{ such that } \mu_m(X \setminus K) < \delta \text{ for all } m \geq 1\). Thus, for any $n \geq 1$,

$$\gamma_{m, n}(XX \setminus (X \times K)) \leq \gamma_{m, n}(X \setminus K) \times X = m_m(X \setminus K) < \delta \text{ (} m = 1, 2, \ldots\).$$

Hence, $\{\gamma_{m, n}\}_{m=1}^\infty$ is equi-tight. By Prokhorov's Theorem again, $\forall n \geq 1, \exists \gamma_n \in \mathcal{P}(X \times X)$ and a subseq. $\{\gamma_{m, n}\}$ of $\{\gamma_{m, n}\}$ such that $\gamma_{m, n} \to \gamma_n$ narrowly in $\mathcal{P}(X \times X)$. But $\gamma_n^X$ is continuous w.r.t. the narrow convergence, thus $\gamma_n \in \mathcal{A}(\mu, \nu)$.

Let $\bar{x} \in X$. For $m, n \geq 1$, we have

$$\int_{X \times X} d^2(x, \bar{x}) d\mu_m(x) = \int_{X \times X} d^2(x, \bar{x}) d\gamma_{m, n}(x, y)$$

$$\leq 2 \int_{X \times X} d^2(x, y) d\gamma_{m, n}(x, y) + 2 \int_{X \times X} d^2(y, \bar{x}) d\gamma_{m, n}(x, y)$$

$$= 2 W_2^b(\mu_m, \mu_n) + 2 \int_{X \times X} d^2(y, \bar{x}) d\mu_n(y).$$

Sending $k \to \infty$, we get (**). Since $\gamma_{m, n} \to \gamma_n$ narrowly,

$$\int_{X \times X} d^2(x, y) d\gamma_n(x, y) = \lim_{k \to \infty} \int_{X \times X} d^2(x, y) d\gamma_{m, n}(x, y)$$

$$= \lim_{k \to \infty} W_2^b(\mu_m, \mu_n) \leq \varepsilon^2 \quad \text{for any } n \geq N.$$
Thus since $u_n \in (L^1_u, U_n)$,
\[ W_2^* (U, U_n) \leq \sum_{x \times X} d(x, y) dU(x, y) \leq l \quad \forall U \in N. \]
This is Q.E.D.

**Theorem (Characterization of $W_2$-convergence):** Let $X$ be a Polish space, and $U_n, U \in \mathcal{P}_2(X)$ ($n=1, 2 \ldots$).

1. If $U_n \overset{W_2}{\to} U$ then $U_n \overset{W_2}{\to} U$ narrowly and for any $x_0 \in X$
\[ \int d^2(x, x_0) dU_n(x) \to \int d^2(x, x_0) dU(x) \quad (*) \]

2. If $U_n \overset{W_2}{\to} U$ narrowly and $(*)$ holds true for some $x_0 \in X$, then $U_n \overset{W_2}{\to} U$.

**Remark:** The condition $(*)$ is necessary, cf. an example below.

**Corollary:** If $X$ is a compact metric space then $(\mathcal{P}_2(X), W_2)$ is a compact metric space.

**Proof:** Fix $x_0 \in X$. Then $x \mapsto d(x, x_0)$ is bounded and continuous on $X$, where $d$ is the metric of $X$. Thus, $U \in \mathcal{P}(X) \Rightarrow U \in \mathcal{P}_2(X)$. Hence, $\mathcal{P}_2(X) = \mathcal{P}(X)$.

Since $X$ is compact, $[C_b(X)]^* = [C(X)]^*$ and $\mathcal{M}_e(X)$ is the compact v.r.t. the weak-* topology, same as the narrow topology. Namely, for any sequence $\{U_n \in \mathcal{P}_2(X) \overset{W_2}{\to} \mathcal{P}(X) (n=1, \ldots)$, there exists a subseq. $\{U_{nj}\}$ and some $f \in [C(X)]^*$ such $U_{nj} \to f$ weak-*.

By Riesz’s Theorem $f \in \mathcal{M}_e(X)$. Moreover, $f(X) = 1$ as all $U_{nj}(X) = 1$. Thus, $f \in \mathcal{P}(X)$. Hence, any $\{U_n\}$, a sequence in $(\mathcal{P}_2(X), W_2)$ has a convergent subseq. So, $(\mathcal{P}_2(X), W_2)$ is compact. Q.E.D.
The following observation is interesting:

**Proposition** If \( X \) is a Polish space but unbounded, then \((\mathcal{P}_2(X), W_2)\) is not locally compact.

**Proof** We show that for any \( \mu \in \mathcal{P}_2(X) \) and any \( R > 0 \), the closed ball \( B(\mu, R) \) in \((\mathcal{P}_2(X), W_2)\) is not compact.

Pick up some \( x_0 \in X \). Let \( x_n \in X \) be such that \( n < d(x_n, x_0) \uparrow \infty \). Set for \( n \in \mathbb{N} \)

\[
\Sigma_n = R^2 \left[ 2 \int_X d^2(x, x_0) \, d\mu(x) + 2 d^2(x_n, x_0) \right]^{-1} \uparrow 0.
\]

\[
\underline{\mu}_n = (1 - \Sigma_n) \mu + \Sigma_n \delta_{x_0} \in \mathcal{P}_2(X),
\]

\[
\overline{\mu}_n = (1 - \Sigma_n) \mu + \Sigma_n \delta_{x_n} \in \mathcal{P}_2(X),
\]

\[
\underline{\gamma}_n = (1 - \Sigma_n) (\text{Id} \times \text{Id}) \# \mu + \Sigma_n (\text{Id} \times x_0) \# \mu,
\]

\[
\overline{\gamma}_n = (1 - \Sigma_n) (\text{Id} \times \text{Id}) \# \mu + \Sigma_n \delta_{x_n} \times \delta_{x_0}.
\]

[Recall: \( \text{Id} \times T : X \to X \times X, \, x \mapsto (x, T(x)) \).

If \( T \# \nu_1 = \nu_2 \) then \( \chi = (\text{Id} \times T) \# \nu \in \mathcal{A}(\nu_1, \nu_2) \).]

It can be verified that \( \underline{\gamma}_n \in \mathcal{A}(\underline{\mu}_n, \overline{\mu}_n) \) and \( \overline{\gamma}_n \in \mathcal{A}(\underline{\mu}_n, \overline{\mu}_n) \). Thus,

\[
W^2_2(\underline{\mu}_n, \overline{\mu}_n) \leq 2 W^2_2(\underline{\mu}_n, \overline{\gamma}_n) + 2 W^2_2(\underline{\gamma}_n, \overline{\gamma}_n)
\]

\[
= 2 \int_{X \times X} d^2(x, y) \, d\overline{\gamma}_n(x, y) + 2 \int_{X \times X} d^2(x, y) \, d\underline{\gamma}_n(x, y)
\]

\[
= 2 \Sigma_n d^2(x_n, x_0) + 2 \Sigma_n \int_X d^2(x, x_0) \, d\mu(x)
\]

\[
= R^2 \quad (n = 1, 2, \ldots)
\]

Thus, \( \underline{\mu}_n \in B(\overline{\mu}, R) \) w.r.t. \( W_2 \) \((n = 1, 2, \ldots)\). Moreover,
$$\liminf_{n \to \infty} \int X d^2(x, x_0) \, d\mu_n(x)$$

$$= \liminf_{n \to \infty} \left[ (1 - \varepsilon_n) \int X d^2(x, x_0) \, d\mu + \varepsilon_n \int X d^2(x_n, x_0) \right]$$

$$= \int X d^2(x, x_0) \, d\mu + \frac{\varepsilon_n}{2}.$$  \hfill (**) 

Now, if there exists a subseq. of \(\mu_n, \mu_{n_j} \overset{W_2}{\to} \mu \subset \mathcal{P}_2(X)\). Then, \(\mu_{n_j} \overset{W_2}{\to} \mu\) narrowly. But clearly \(\mu_n \overset{W_2}{\to} \mu\) narrowly. Thus \(\mu = \mu\). Then, \(\mu_{n_j} \overset{W_2}{\to} \mu\) which should imply that

$$\int X d^2(x, x_0) \, d\mu_{n_j}(x) \to \int X d^2(x, x_0) \, d\mu(x).$$

This contradicts (**). \textbf{QED} 

The example below demonstrates two points.

1. The condition (*) for some \(x_0\) in the main theorem of convergence is necessary. If (*) fails for some \(x_0 \in X\), then \(\mu_n \overset{W_2}{\to} \mu \subset \mathcal{P}_2(X)\) but it is still possible that \(\mu_n \overset{W_2}{\to} \mu\) narrowly.

2. If \(X\) is a Polish space, and is also proper, then \((\mathcal{P}_2(X), W_2)\) may not be proper. Recall that a metric space is proper, if bounded and closed sets are compact. In a proper space, any bounded sequence has a convergent subsequence.

Example \(X = IR, \mu_n = (1 - \frac{1}{n}) \delta_0 + \frac{1}{n} \delta_n \subset \mathcal{P}(IR) (n=1, 2, \ldots)\), and \(\mu = \delta_0\). Then, \(\mu \subset \mathcal{P}_2(X)\) as
\[
\int d^2(x,0) m_n(x) = \int 1 \times 1^2 d\delta(x) = 0. 
\]
Also, each \( m_n \in \mathcal{D}(\mathbb{R}) \), since
\[
\int d^2(x,0) m_n(x) = \int 1 \times 1^2 d\left( (x,0)^2 + \frac{1}{n^2} \delta(x) \right) = \frac{1}{n^2} n^2 = 1. 
\]
By the condition (*) in the convergence theorem, no subseq. of \( \{m_n\} \) is \( L^2 \)-convergent. In particular, \( m_n \rightarrow m \) in \( L^2 \). But, \( m_n \rightarrow m \) narrowly, as
\[
\int f d\mu_n = (1 - \frac{1}{n}) f(0) + \frac{1}{n^2} f(0) \rightarrow f(0) = \int f d\mu \quad \forall f \in C_0(\mathbb{R}). 
\]
If \( x \in \mathcal{D}(m_n, n) \) then \( x_n(\mathbb{R} \times \{0\}) = m(\{0\}) = 1 \), and hence,
\[
\int_{\mathbb{R} \times \mathbb{R}} (x - y)^2 \delta(x, y) = \int_{\mathbb{R} \times \mathbb{R}} x^2 \delta(x, y) = \int_{\mathbb{R} \times \mathbb{R}} x^2 \delta_m(x) = \frac{n^2}{n^2} = 1. 
\]
Hence, \( W_2(m_n, m) = 1 \ (n = 1, 2, \ldots) \), and \( \{m_n\} \) is \( L^2 \)-bounded.