Math 130A: Ordinary Differential Equations, Winter 2015

Midterm Review

The midterm exam will cover possibly Sections 2.1–27, 3.1, 3.2, 3.4, 3.6, 3.7, 4.1, and 4.2.

Chapter 2. Flows on the Line

1. Consider $\dot{x} = f(x)$. What is a fixed point and the corresponding equilibrium solution? What is the vector field for this system? What is the meaning that a fixed point is stable (or unstable) and how to determine that? Notation: a solid dot (bullet) or a circle.
2. Linear stability analysis: if x^* is a fixed point and $f'(x^*) > 0$ (or < 0) then x^* is unstable (or stable)? Why?
3. The definition of a potential V: $-V'(x) = f(x)$. Note the minus sign. If $x = x(t)$ is a solution to $\dot{x} = f(x)$ and V is a potential of f, then $(d/dt)V(x(t)) \leq 0$. Why?
4. Let V be a potential of f, i.e., $-V'(x) = f(x)$ for all x. Then a fixed x^* of f is the same as a critical point of V. Stable: a local minimum of V; unstable: a local maximum of V.
5. Prove this: If $x = x(t)$ is a solution of $\dot{x} = f(x)$ and $x(t_1) = x(t_2)$ for some t_1 and t_2 such that $t_1 < t_2$. Then $x(t) = x(t_1)$ for all t: $t_1 < t < t_2$. No oscillations!
6. Existence and uniqueness of solution to the initial-boundary-value problem $\dot{x} = f(x)$ and $x(0) = x_0$. Statement. Example of multiple solutions.
7. Finite-time blow up of a solution: Solve $\dot{x} = 1 + x^2$ and $x(0) = 0$. Solve $\dot{x} = x^2$ and $x(0) = 1$.

Chapter 3. Bifurcations

1. Consider $\dot{x} = f(x, r)$, where r is a parameter. When r varies, the number of fixed points and their stabilities often change; this is bifurcation. Generally, there are a few steps in studying the bifurcation: (1) Fix r in certain range, find fixed points and determine their stabilities; (2) plot the bifurcation diagram: x vs. r.; (3) find the normal form.
2. A technique of finding fixed points: if $f(x) = f_1(x) - f_(x)$, you can plot $f_1(x)$ and $f_2(x)$; and the intersection points of these two graphs are the fixed points of $\dot{x} = f(x)$. With the graphs of $f_1(x)$ and $f_2(x)$, how to determine the stability of these fixed points?
3. A technique of finding the normal form: use Taylor's expansion; cf. Section 3.1.
4. Study the following bifurcations: fixed points and their stabilities, the bifurcation diagram, etc.
   (a) Saddle-node: $\dot{x} = r + x^2$ or $\dot{x} = r - x^2$.
   (b) Transcritical: $\dot{x} = rx - x^2$ or $\dot{x} = rx + x^2$.
   (c) Supercritical pitchfork: $\dot{x} = rx - x^3$.
   (d) Subcritical pitchfork: $\dot{x} = rx + x^3$. Regularization: $\dot{x} = rx + x^3 - x^5$.
   (e) Examples of bifurcation of system with two parameters: $\dot{x} = h + rx - x^3$; $\dot{x} = r(1 - x/k) - x^2/(1 + x^2)$.

Chapter 4. Flows on the Circle

1. The difference between $\dot{\theta} = f(\theta)$ and $\dot{x} = g(x)$ is that θ and $\theta + 2k\pi$ (k: any integer) label the same point on the circle.
2. What is the a vector field on the circle for $\dot{\theta} = f(\theta)$?
3. Uniform oscillator: $\dot{\theta} = \omega$. What is the period?