Math 130A: Review for Final Exam

The final exam will cover possibly Sections 2.2–27, 3.1, 3.2, 3.4, 3.6, 4.1–4.3, 5.1, 5.2, 6.2, 6.3 (exclude the last part: Hyperbolic Fixed Points, Topological Equivalence, and Structural Stability), 6.5–6.7.

Chapter 2. Flows on the Line

1. Consider $\dot{x} = f(x)$. What is the vector field? What is a fixed point and the corresponding equilibrium solution? Determine the stability or instability of a fixed point. Linear stability analysis: if x^* is a fixed point and $f'(x^*) > 0$ (or < 0) then x^* is unstable (or stable)? Why?
2. A trick of finding fixed points: if $f(x) = f_1(x) - f_2(x)$, you can plot $f_1(x)$ and $f_2(x)$; and the intersection points of these two graphs are the fixed points of $\dot{x} = f(x)$. With the graphs of $f_1(x)$ and $f_2(x)$, how to determine the stability of these fixed points?
3. The definition of a potential V: $-V'(x) = f(x)$. If $x = x(t)$ is a solution to $\dot{x} = f(x)$ and V is a potential of f, then $(d/dt)V(x(t)) \leq 0$. Why? A fixed point x^* of f is the same as a critical point of V. Stable: a local minimum of V; unstable: a local maximum of V.
4. Prove this: If $x = x(t)$ is a solution of $\dot{x} = f(x)$ and $x(t_1) = x(t_2)$ for some t_1 and t_2 such that $t_1 < t_2$. Then $x(t) = x(t_1)$ for all t: $t_1 < t < t_2$. No oscillations!
5. Existence and uniqueness of solution to the initial-boundary-value problem $\dot{x} = f(x)$ and $x(0) = x_0$. Statement. Example of multiple solutions.
6. Finite-time blow up of a solution: Solve $\dot{x} = 1 + x^2$ and $x(0) = 0$. Solve $\dot{x} = x^2$ and $x(0) = 1$.

Chapter 3. Bifurcations

1. Consider $\dot{x} = f(x, r)$. When the parameter r varies, the number of fixed points and their stabilities often change; this is bifurcation. Study the bifurcation: (1) If the bifurcation occurs at (x^*, r_c), then $f(x^*, r_c) = 0$ and $\partial_x f(x^*, r_c) = 0$; (2) Fix $r < r_c$ and $r > r_c$, find fixed points and determine their stabilities; (3) Plot the bifurcation diagram: x vs. r.; and (4) Find the normal form, often using Taylor's expansion.
2. Study the following bifurcations: fixed points and their stabilities, the bifurcation diagram, etc.: (1) Saddle-node: $\dot{x} = r + x^2$ or $\dot{x} = r - x^2$; (2) Transcritical: $\dot{x} = rx - x^2$ or $\dot{x} = rx + x^2$; (3) Supercritical pitchfork: $\dot{x} = rx - x^3$; (4) Subcritical pitchfork: $\dot{x} = rx + x^3$. Regularization: $\dot{x} = rx + x^3 - x^5$.
3. Understand the imperfection bifurcation by studying $\dot{x} = h + rx - x^3$.

Chapter 4. Flows on the Circle

1. The difference between $\dot{\theta} = f(\theta)$ and $\dot{x} = g(x)$ is that θ and $\theta + 2k\pi$ (k: any integer) label the same point on the circle. What is the a vector field on the circle for $\dot{\theta} = f(\theta)$?
2. Uniform oscillator: $\dot{\theta} = \omega$. What is the period? Nonuniform oscillators. Study $\dot{\theta} = \omega - a \sin\theta$.

Chapter 5. Linear Systems

1. Linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is a 2×2 matrix. Classification of the fixed point $\mathbf{x} = \mathbf{0}$. Main ones: saddle points, nodes, and spirals. Borderline cases: centers, non-isolated fixed points, stars, and degenerate nodes. Typical trajectories. Understand the stability diagram Figure 5.2.8.

Chapter 6. Phase Plane

1. Consider $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. What is a fixed point? Relation of a fixed point to a solution. What are nullclines, phase plane, trajectories, closed orbits, homoclinic trajectories, and heteroclinic orbits?
2. Existence and uniqueness of solutions. Solution uniqueness implies that trajectories do not cross.
3. Linearization around a fixed point (x^*, y^*). The relation between the type of fixed point $(0, 0)$ for the linearized system and that of (x^*, y^*) for the nonlinear system: Same if it is not a borderline case.
4. Definition of a conservative system. A typical example: $m\ddot{x} = F(x)$; or equivalently $\dot{x} = y$ and $\dot{y} = F(x)/m$. Let $V'(x) = -F(x)$. Then $E(x) = (1/2)my^2 + V(x)$ is a conserved quantity.
5. A conservative system cannot have any attracting fixed points. Why? Theorem 6.5.1 on nonlinear centers for conservative systems.
6. Reversible systems: general definition. Some symmetric property of trajectories for such a system. Consider a reversible system. If $(x^*, y^*) = (0, 0)$ is a fixed point, and $(0, 0)$ is a linear center for the corresponding linearized system, then it is also a nonlinear center.
7. Pendulum: $\ddot{\theta} + \sin\theta = 0$ and $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$. Fixed points and their stabilities, phase portraits, and energies.