Ricci Flow as a Test for AI

Bennett (Ben) Chow
UCSD and visiting member at IAS

Geometry of Machine Learning

Harvard CMSA

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UC San Diego

Abstract, I

We pose the question: *Is Ricci flow a good test for AI?*

Such a general question may be posed for any subfield of mathematics. The main reason I chose Ricci flow is because of my familiarity with aspects of this subfield.

(Riemannian) Ricci flow has climbed to great heights due to the works of Hamilton, Perelman, Bamler, Brendle, Schoen, Colding, Minicozzi, Kleiner, Lott, Böhm, Wilking, Munteanu, J. Wang, Shi, Q. Zhang, et al. There are also many expositions of Ricci flow.

There is potential for significant progress in the future. In particular, there is a topological conjecture (11/8) where Ricci flow (RF) seems to be a reasonable approach. **How can AI contribute to RF?**

For much of the talk, we will describe what Ricci flow is and why it is interesting. In the latter part of the talk, we will discuss some problems in Ricci flow that may be a good test for AI.

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Abstract, II

We invite the interested listener to study Ricci flow using AI and study AI using Ricci flow. Features of Ricci flow:

- Ricci flow is an exciting field with a lot of potential for further breakthroughs and important applications.
- Ricci flow connects to many different fields.
- The foundations of Ricci flow are elementary and concrete.
- Ricci flow is gradually moving toward a more abstract direction. E.g., Bamler's notion of metric flow.
- Ricci flow, like many subfields, is gradually getting more and more technically difficult.
- Good problems in Ricci flow are harder and harder to find. The reasons are probably mainly due to human limitations.

The arguments above apply to most subfields of mathematics.

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What is Ricci flow? Movies on the 2-sphere

The **Ricci flow** is a heat-type flow for Riemannian metrics. In 2D it provides a flow method for re-proving the differential geometric version of the uniformization theorem in complex analysis.

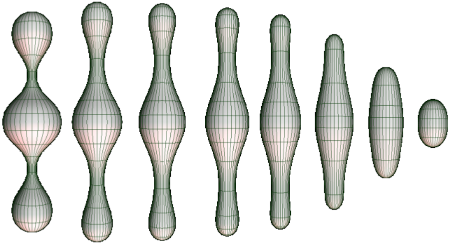

Figure: Under Ricci flow, any metric on a surface converges. From Wikipedia. Attribution: By CBM - Own work, Public Domain.

Ricci flow converging to a round sphere by Gabe Kahn. ▸ Play Video

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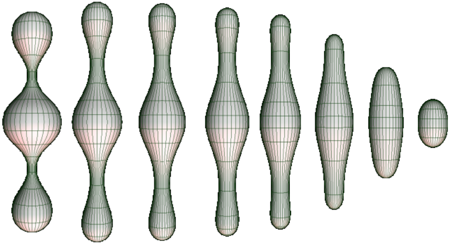

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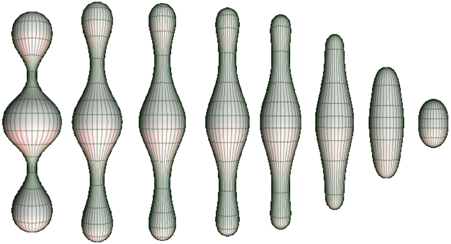

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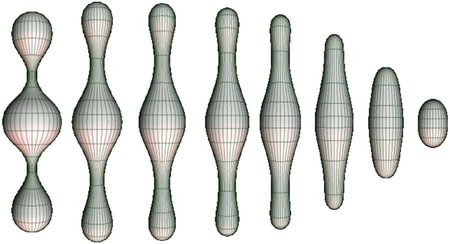

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Ricci flow Synthesizes Fields

How does Ricci flow synthesize geometry, analysis, and topology?

- Arguments are simultaneously analytic and geometric.
  - This was manifested in a plethora of ways in the works of Hamilton.
  - The degree of analytic-geometric synthesis was taken to another level by Perelman.
  - The recent works of Bamler has exhibited a continued deepening and broadening of this synthesis.
  - The developments in the last 40+ years exhibit an increasing degree of unreasonable effectiveness of Ricci flow in geometric analysis. Hopefully this trend will continue.
- Although topology is one of the main applications of Ricci flow, topology itself is also used to understand the behavior of Ricci flow.
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Background: Riemannian Manifolds

Smooth manifolds are spaces that are differential topologically Euclidean. Linearizing a manifold M^n at a point p yields the tangent space T_pM at that point. Euclidean geometry on $\mathbb{R}^n$ (distances and angles) is encoded by the inner product $v \cdot w = \sum_{i=1}^n v^i w^i$.

A Riemannian manifold is an infinitesimal assignment of a Euclidean geometry at each point. More precisely, a Riemannian metric g on M^n is a smoothly varying inner product g_p on each tangent space T_pM. This is the most straightforward way to extend Euclidean geometry to a nonlinear setting.

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Riemannian manifold, Cylinder

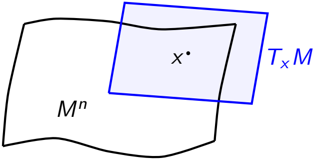

Figure: A Riemannian manifold and a tangent space.

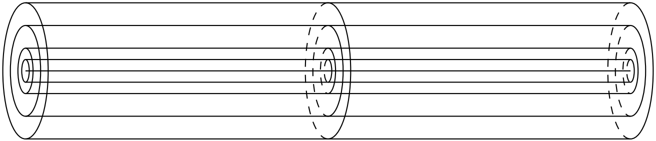

Figure: A cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$, $n \geq 3$, shrinking under Ricci flow. Notice that its size changes but its shape does not change.

Background: Connections and Curvatures

Given a smooth manifold M^n, we can differentiate functions: Let $f : M^n \to \mathbb{R}$ and let V be a vector field on M^n. Then the directional derivative $V(f)$ is a function on M^n. Endowing M^n with a metric g, there is a natural, metric-compatible way to differentiate vector fields $\nabla_V W$. This is called the Levi-Civita connection.

The Riemann curvature tensor Rm measures the non-commutativity of ∇ : $\mathrm{Rm}(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$. This defines a 4-tensor $\mathrm{Rm}(X,Y,Z,W) = \mathrm{Rm}(X,Y)Z \cdot W$.

The Ricci tensor is its trace: $\mathrm{Ric}(Y,Z) = \sum_i \mathrm{Rm}(e_i, Y)Z \cdot e_i$.

The scalar curvature is yet another trace: $R = \sum_i \mathrm{Ric}(e_i, e_i)$.

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Parallel Transport and Curvature

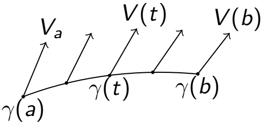

Figure: Parallel transport of a vector along a path.

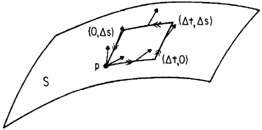

Fig. 3.3. The parallel transport of a vector v^a around a small closed loop. As derived in the text, to second order in Δt and Δs, the change in v^a is governed by the Riemann tensor at p.

Figure: Curvature as the infinitesimal change in a vector under parallel translation along small loops. From Wald's General Relativity page 37.

Decomposing Manifolds into Geometric Pieces

One can make an analogy between Riemannian metrics and smooth functions. Namely, in of itself, an arbitrary Riemannian metric is not of much significance. This leads to the question of the existence of canonical metrics/geometries on manifolds. E.g., we saw the space forms $\mathbb{R}^n/\Gamma$, $\mathbb{S}^n/\Gamma$, and $\mathbb{H}^n/\Gamma$. However, the condition of constant is topologically too restrictive for M^n. Following Thurston, a better question is the existence of decompositions of manifolds into pieces with canonical geometries.

Question: Does every closed manifold admit a decomposition into geometric pieces?

In 3D, this is Thurston's geometrization conjecture. We have the Hamilton–Perelman proof by Ricci flow of this conjecture.

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Hamilton's Ricci flow

Transition: From Motivation to Definition and Techniques

Given a Riemannian metric g_0 on M^n, we can deform it by an equation. Hamilton's Ricci flow is the equation

$$\partial_t g = -2\mathrm{Ric}$$

for a 1-parameter family of metrics $g(t)$. Initial condition: $g(0) = g_0$.

A solution always exists for a short time: For any g_0 there exists $\varepsilon > 0$ and a solution $g(t)_{t \in [0,\varepsilon)}$ to the Ricci flow with $g(0) = g_0$ (Hamilton 1982, and a simplified proof by DeTurck 1983).

Question: Under the Ricci flow, does something good happen for any initial metric on a closed 3-manifold?

The surprising answer is yes, as proved by Hamilton and Perelman. 3D Ricci flow (and Cheeger–Gromov collapsing theory) effects Thurston's conjectured geometric decomposition.

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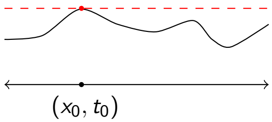
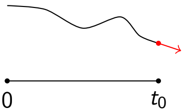
Techniques: First and Second Derivative Tests

Let $u(x,t) : M \times [0,T) \to \mathbb{R}$ be a solution to the heat equation $\partial_t u = \Delta u$. At time t_0 suppose the spatial maximum of u is attained at x_0. Then, by the first and second derivative tests, $\nabla u(x_0,t_0)$ and $\Delta u(x_0,t_0) \leq 0$. Thus $\partial_t u(x_0,t_0) \leq 0$.





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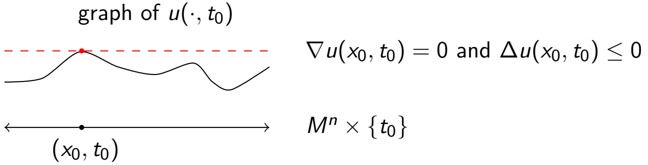

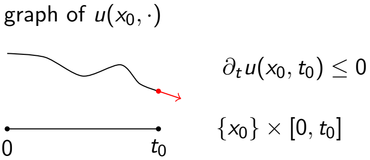

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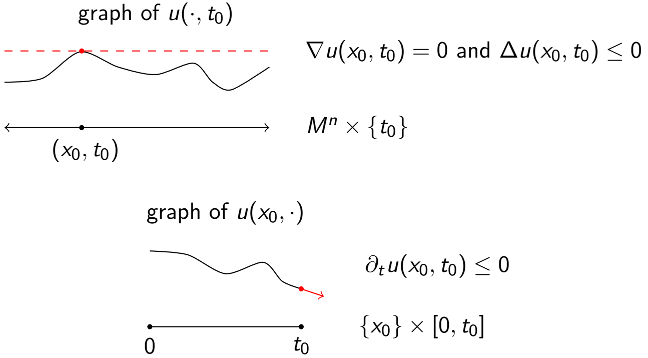

App. of Maximum Principle: Pointwise Monotonicity

Statement: Ricci flow is the heat equation for metrics (Hamilton 1995). This statement has been often repeated in the literature.

The simplicity of Riemannian geometry and the fundamental nature of the heat equation belie the importance of Ricci flow.

(Weak) maximum principle: Let $g(t)$ be a 1-parameter family of metrics on a manifold M^n, such as a Ricci flow. Let $u(x,t) : M^n \times [0,T) \to \mathbb{R}$ be a solution to a heat equation:

$$\partial_t u \leq \Delta_{g(t)} u + X(t) \cdot \nabla u.$$

The first and second derivative tests from multivariable calculus say that a spatial maximum of u, $\nabla u = 0$ and $\Delta u \leq 0$. Thus, at such a point, $\partial_t u \leq 0$. Spatial maximums decrease in time. So:

Corollary: If $u \leq C$ at $t = 0$, then $u \leq C$ for $t \in [0,T)$.

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Techniques: Evolution of Scalar Curvature

If $(M^n, g(t))$ is a solution to the Ricci flow, then the scalar curvature satisfies the heat equation:

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R. \tag{0.1}$$

Corollary. If $R \geq c$ at $t = 0$, then $R \geq c$ for $t \in [0, T)$.

This is useful, but controlling the scalar curvature in of itself doesn't necessarily get us very far.

Now, why does equation (0.1) hold?

It's a special case of the variation formula: If $g = g(s)$ satisfies $\partial_s g = v$, then

$$\partial_s R = -\Delta V + \mathrm{div}\,(\mathrm{div}\, v) - \langle v, \mathrm{Ric}\rangle\,, \tag{0.2}$$

Indeed, if $v = -2\mathrm{Ric}$, then $V = -2R$ and $\mathrm{div}(v) = -2\mathrm{div}(\mathrm{Ric}) = -\nabla R$, and (0.1) follows easily.

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Corollary. If $R \geq c$ at $t = 0$, then $R \geq c$ for $t \in [0, T)$.

This is useful, but controlling the scalar curvature in of itself doesn't necessarily get us very far.

Now, why does equation (0.1) hold?

It's a special case of the variation formula: If $g = g(s)$ satisfies $\partial_s g = v$, then

$$\partial_s R = -\Delta V + \mathrm{div}\,(\mathrm{div}\, v) - \langle v, \mathrm{Ric}\rangle\,, \tag{0.2}$$

Indeed, if $v = -2\mathrm{Ric}$, then $V = -2R$ and $\mathrm{div}(v) = -2\mathrm{div}(\mathrm{Ric}) = -\nabla R$, and (0.1) follows easily.

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Other Applications of the Maximum Principle

The maximum principle can be used to prove a plethora of monotonicity formulas for Ricci flow. Besides the preservation of lower bounds for the R we saw on the last slide, examples include:

- Hamilton's pinching estimate. In 3D: $|\mathrm{Ric}| \leq CR$; that is, Ricci curvature can be controlled by scalar curvature. There is an analogous estimate for the linearized Ricci flow.
- Hamilton–Ivey estimate. In 3D: At points where $R \gg 1$, we have $-\min \mathrm{sect}/\max \mathrm{sect} \ll 1$. I.e., almost non-negative sectional curvature. Consequence: 3D singularity models have nonnegative sectional curvature. The geometry and topology of 3D singularity models are well controlled, and in fact simple.
- Hamilton's trace Harnack. nD: If Rm is bounded and $\mathrm{Rm} \geq 0$, then $\partial_t(tR) \geq 0$. Consequence: For singularity models (which are ancient solutions), the scalar curvature is pointwise non-decreasing in time.

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Applications of the Maximum Principle, continued

- Perelman's reduced distance ℓ, which is a function on space-time, satisfies remarkable elliptic and parabolic equalities and inequalities.
- Perelman's differential Harnack for conjugate heat kernel. Integrated form says the conjugate heat kernel potential $f \leq \ell$.
- Bamler's estimates for the pointed Nash entropy $\mathcal{N}_{x,t}(r^2)$.

Lagrangian View: Ricci Flow as a Gradient Flow

Formula (0.2), with $\partial_s dv = \frac{1}{2} V dv$, implies $\mathcal{E} = \int R dv$. satisfies

$$\frac{d}{ds}\mathcal{E}(g(s)) = -\int v \cdot \left(\mathrm{Ric} - \frac{1}{2} Rg\right) dv. \tag{0.3}$$

Because of the term $-\frac{1}{2}Rg$, Ricci flow is not the gradient flow of $\mathcal{E}$. Perelman (2002) showed that if we consider Ricci flow as part of a slightly bigger system, then it is a gradient flow. He coupled the Ricci flow with the backward heat equation for a function f on M^n:

$$\partial_t f = -\Delta f - R + |\nabla f|^2. \tag{0.4}$$

Under the Ricci flow coupled to (0.4), he showed that his $\mathcal{F}$-energy

$$\mathcal{F}(g, f) = \int \left(R + |\nabla f|^2\right) \mathrm{e}^{-f} dv, \tag{0.5}$$

which is a variant of $\mathcal{E}$, satisfies

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int |\mathrm{Ric} + \nabla^2 f|^2 \mathrm{e}^{-f} dv \geq 0. \tag{0.6}$$

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Lagrangian View: $\mathcal{F}$-Energy Attributes

1. Perelman's $\mathcal{F}$-energy is a remarkable synthesis between the Einstein–Hilbert action $\mathcal{E} = \int R dv$ and the Dirichlet energy $\mathcal{D} = \int |\nabla(\mathrm{e}^{-f/2}|^2 dv$. Simultaneously geometric and analytic.
2. The right-hand side of (0.6) vanishes if and only if $\mathrm{Ric} + \nabla^2 f = 0$. A triple (M^n, g, f) satisfying this equation is called a steady Ricci soliton. A steady Ricci soliton is a fixed point of the Ricci flow in the space of all Riemannian metrics on M^n modulo the diffeomorphism group acting by pullback. This equation generalizes the Ricci flat equation $\mathrm{Ric} = 0$.
3. The monotonicity formula $\frac{d}{dt}\mathcal{F}(g(t), f(t)) \geq 0$ is completely general. In particular, no curvature condition is assumed.
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Applications of Monotonicity

- Evolution equation for R:
  1. Plus ancient solution imply R is nonnegative. Dichotomy:
  $$\mathrm{Ric} \equiv 0 \quad \text{or} \quad R > 0.$$
  2. Plus immortal solution imply Einstein limits.
- Perelman's $\mathcal{W}$-entropy monotonicity $\Rightarrow$ no local collapsing.
- Aforementioned applications of maximum principle: $|\mathrm{Ric}|/R$, Hamilton–Ivey, trace Harnack, ℓ, Harnack for conjugate heat kernel, Bamler's estimates for $\mathcal{N}_{x,t}(r^2)$.
  1. Assume $\mathrm{Rm} \geq 0$. Trace Harnack plus ancient $\Rightarrow \partial_t R \geq 0$.
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  1. Plus ancient solution imply R is nonnegative. Dichotomy:
  $$\mathrm{Ric} \equiv 0 \quad \text{or} \quad R > 0.$$
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- Aforementioned applications of maximum principle: $|\mathrm{Ric}|/R$, Hamilton–Ivey, trace Harnack, ℓ, Harnack for conjugate heat kernel, Bamler's estimates for $\mathcal{N}_{x,t}(r^2)$.
  1. Assume $\mathrm{Rm} \geq 0$. Trace Harnack plus ancient $\Rightarrow \partial_t R \geq 0$.
  2. Assume 3D. Hamilton–Ivey plus ancient $\Rightarrow \mathrm{Rm} \geq 0$.
  3. $(\partial_t + \Delta)\ell - |\nabla \ell|^2 + R + \frac{n}{2t} \leq 0$.

     I.e., $(\partial_t + \Delta - R)((4\pi|t|)^{-\frac{n}{2}} \mathrm{e}^{-\ell}) \geq 0$.

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Topological Surgery

Transition: From Analytic Tools $\rightarrow$ Topological Surgery

A k-**surgery** on a smooth oriented manifold M^n, where $0 \leq k \leq n-1$, removes a smoothly embedded $S^k \times B^{n-k}$ from the manifold, where the induced orientation on $S^k \times B^{n-k}$ is the standard orientation, and replaces it by $B^{k+1} \times S^{n-k-1}$. The replacement (or 'gluing') occurs along their shared smooth boundary $S^k \times S^{n-k-1}$.

We say that closed oriented smooth manifolds M_1^n and M_2^n are **oriented cobordant** if there exists a compact oriented smooth W^{n+1} such that $\partial W = \overline{M_1^n} \sqcup M_2^n$.

Two manifolds M_1^n and M_2^n are oriented cobordant if and only if there exists a finite sequence of surgeries that transforms M_1^n to M_2^n, where each 0-surgery is orientation preserving.

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Surgery Picture

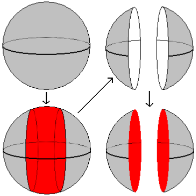

Figure: A 1-surgery on S^2. An embedded $S^1 \times B^1$ is removed and replaced by $B^2 \times S^0 = B^2 \sqcup B^2$. The picture is similar for an $(n-1)$-surgery on an n-manifold, where $S^{n-1} \times B^1$ is removed and replaced by $B^n \times S^0 = B^n \sqcup B^n$. Credit: Vivacissamamente. Wikipedia. Public domain.

Ricci Flow Neckpinch Singularity

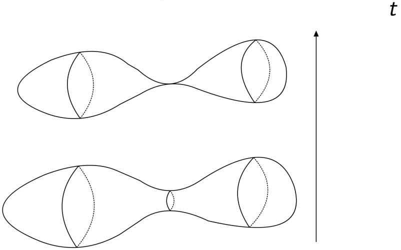

Figure: Formation of a **neckpinch** singularity on S^4. Time increases from bottom to top. The metric pinches along an embedded 3-sphere.

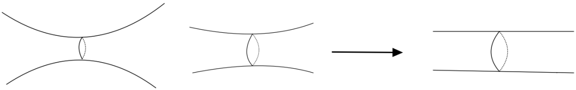

Figure: Rescalings of the forming neckpinch limit to the **cylinder singularity model** $S^3 \times \mathbb{R}$. Time increases from left to right.

Potential Ricci Flow Surgery in 4D: 3-Surgery

Ricci flow surgery associated to $S^3/\Gamma \times \mathbb{R}$ singularity model: Neckpinch. Let $\mathcal{O}^4$ be a smooth oriented 4D **orbifold** with isolated singularities.

- A **quotient** 3-**surgery** on $\mathcal{O}^4$ removes a smoothly embedded $(S^3/\Gamma) \times B^1$ from $\mathcal{O}^4$ and replaces it by $(B^4/\Gamma) \times S^0$, where $S^0 = \{\pm 1\}$. The gluing occurs along their shared smooth boundary $(S^3/\Gamma) \times S^0$. A **3-surgery** is when $\Gamma = \{1\}$.

Figure: Left: An embedded S^3/Γ in a quotient **neck region**. Right: **Neckpinch** at the singularity time.

Figure: Left: Removed $(S^3/\Gamma) \times B^1$. Right: Attached $B^4/\Gamma \times S^0$.

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Ricci Flow Degenerate Neckpinch Singularity

Ricci flow surgery associated to $S^3/\Gamma \times \mathbb{R}$ singularity model: Degnerate Neckpinch. Here, the $S^3/\Gamma \times \mathbb{R}$ singularity model occurs in conjunction with the Γ-quotient of the *Bryant soliton* singularity model. (*Appleton soliton* should also be possible.)

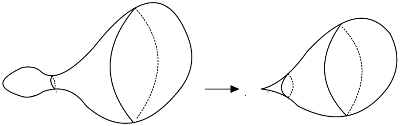

Figure: Formation of a degenerate neckpinch singularity ($\Gamma = \{1\}$). Time t increases from left to right.

Existence follows from a continuity argument (Gu and Zhu). This singularity formation interpolates between an S^4 shrinking to a round point and a neckpinch. Bryant soliton interpolates between S^4 and $S^3 \times \mathbb{R}$. Ricci flow surgery does not change the topology.

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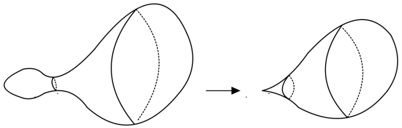

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Ricci Flow Pinching 2-Spheres

Bamler theory: In 4D, neckpinches are where 3-spheres pinch. We may also have 2-spheres pinching. Naively, if pinching 3-spheres correspond to 3-surgeries, then **pinching 2-spheres** should correspond to **2-surgeries**.

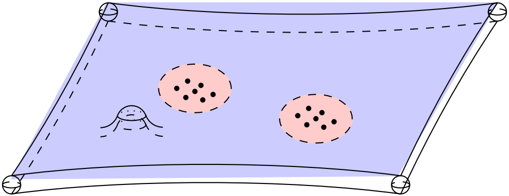

Figure: Locally, at the singularity time, the metric is like a warped product of S^2 over a surface $\mathcal{S}$, where the set of points in $\mathcal{S}$ (pictured as dots) where the 2-spheres pinch to points may be complicated (even a Cantor-type set?).

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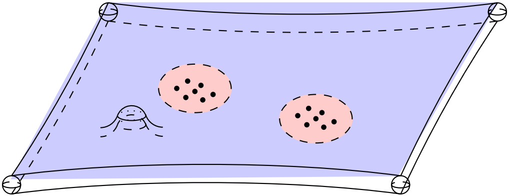

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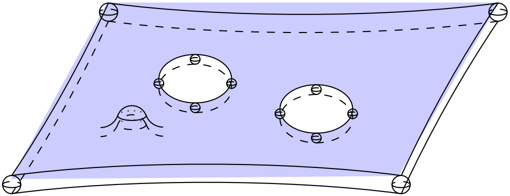
Potential Ricci Flow Surgery in 4D: 2-Surgery

Ricci flow surgery associated to $S^2 \times \mathbb{R}^2$ singularity model:

- Let Σ^2 be a smooth compact surface with nonempty boundary (which is a disjoint union of circles). A **prophylactic** 2-**surgery** on $\mathcal{O}^4$ removes a smoothly embedded $S^2 \times (\Sigma^2)^\circ$ from $\mathcal{O}^4$ and replaces it by $B^3 \times \partial\Sigma$. When Σ is a 2-ball, we have a 2-surgery.



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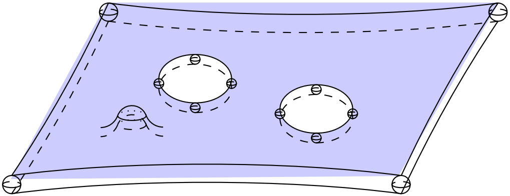

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Potential Ricci Flow Surgery in 4D: $S^2 \times \mathbb{R}^2$

Prophylactic 2-surgery, continued:

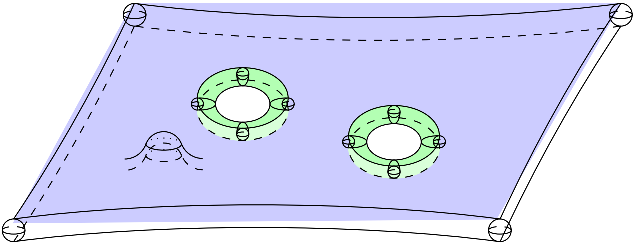

Figure: Attached $B^3 \times \partial\Sigma$.

Geometrically, the two sheets are very close together. Locally, at a small scale, the attached $B^3 \times \partial\Sigma$ looks like the 3-dimensional **Perelman's standard solution** producted with a line.

3D Perelman's Standard Solution

- 3D Perelman's standard solution is a rotationally symmetric solution to Ricci flow with underlying manifold $\mathbb{R}^3$.
- It is asymptotic to a round cylinder $S^2 \times \mathbb{R}$.
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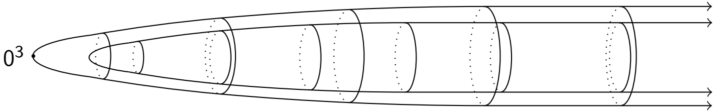

Figure: Perelman's 3D Standard Solution on $\mathbb{R}^3$.

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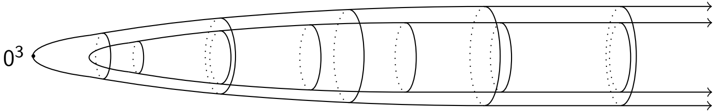

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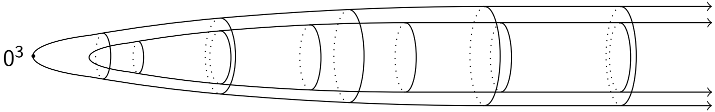

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Close-up Look at Ricci Flow Prophylactic 2-Surgery

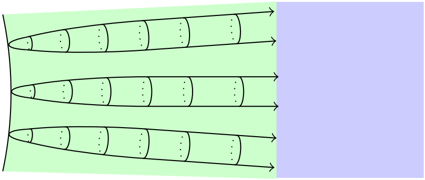

Figure: Close-up view of attached $B^3 \times \partial\Sigma$. The attached piece is locally like the product of Perelman's standard solution and $\mathbb{R}$.

Potential Ricci Flow Surgery in 4D: $\mathbb{C}P^2$ Model

- A **smooth flowdown** on $\mathcal{O}^4$ removes a smoothly embedded $\overline{\mathbb{C}P^2} \setminus \overline{B}^4$ and replaces it by B^4. The topological effect of a smooth flowdown is to remove a $\overline{\mathbb{C}P^2}$ topological summand. Geometrically, the Feldman–Ilmanen–Knopf shrinking Kähler Ricci soliton forms as a singularity model (complex line bundle over $\mathbb{C}P^1$ with first Chern class -1).

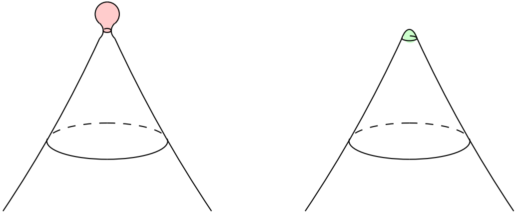

Figure: A smooth flowdown. Left: Pre-surgery. Right: Post-surgery.

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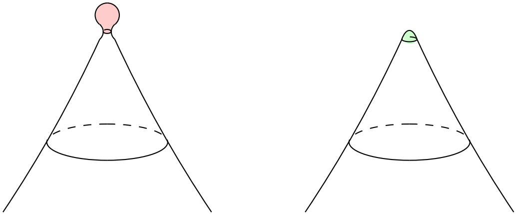

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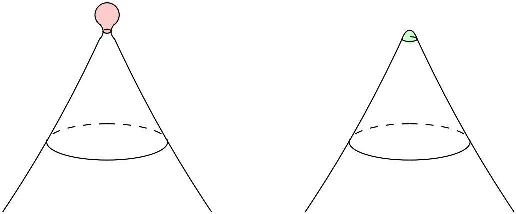

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Asymptotically Conical Shrinking Soliton Singularity Models

A shrinking Ricci soliton is **asymptotically conical** if its associated Ricci flow converges to the Riemannian cone over a compact smooth Riemannian manifold.

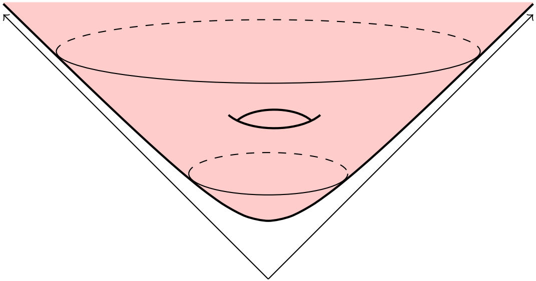

Figure: An asymptotically conical shrinking soliton.

Potential Ricci Flow Surgery in 4D: Asymptotically Conical Shrinking Soliton Singularity Models

- Munteanu and J. Wang: If a 4D shrinking Ricci soliton has $R \to 0$, then $|\mathrm{Rm}| \leq Cr^{-2}$. Kotschwar and L. Wang: Then the Ricci soliton is **asymptotically conical**. In fact, the Ricci flow of an asymptotically conical shrinking soliton converges to a Riemannian cone over a closed smooth 3-manifold.
- So, more generally than the previous slide, Ricci flow with surgery should effect **geometric-topological surgeries resolving conical singularities**: These surgeries correspond to the formation of asymptotically conical singularity models, combined with the *Bamler–Chen resolution* via expanding solitons.

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Expanding Soliton Resolving an Asymptotically Conical Singularity Model

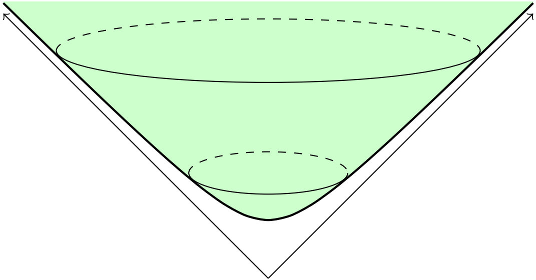

Figure: An asymptotically conical *expanding* soliton resolving the asymptotically conical *shrinking* soliton Ricci flow singularity .

Potential Ricci Flow Surgery in 4D: The BCCD Shrinking Soliton

The Bamler–Cifarelli–Conlon–Deruelle (**BCCD**) **shrinking Ricci soliton**. It is weakly asymptotic to $S^2 \times \mathbb{R}^2$ and diffeomorphic to $\mathbb{C}^2$ connected sum with $\mathbb{C}P^2$. It is a singularity model of the Ricci flow.

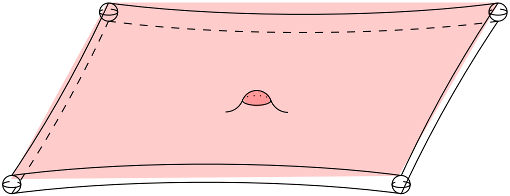

Figure: The BCCD shrinking soliton.

Ricci flow surgery removes the $\mathbb{C}P^2$ summand. However, this does not occur in the spin (even intersection form) case.

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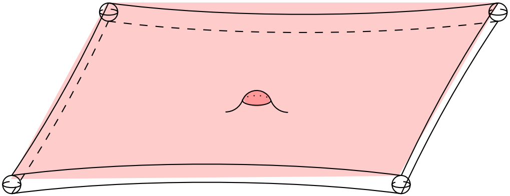

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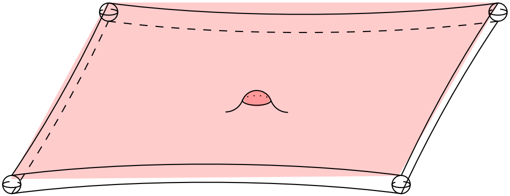

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Potential Ricci Flow Surgery in 4D: The BCCD Shrinking Soliton

The Bamler–Cifarelli–Conlon–Deruelle (**BCCD**) **shrinking Ricci soliton**. It is weakly asymptotic to $S^2 \times \mathbb{R}^2$ and diffeomorphic to $\mathbb{C}^2$ connected sum with $\mathbb{C}P^2$. It is a singularity model of the Ricci flow.

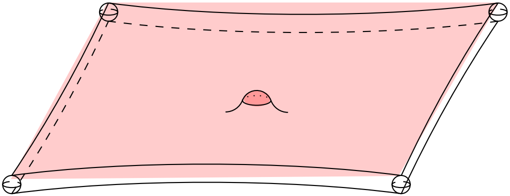

Figure: The BCCD shrinking soliton.

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Resolving the BCCD Shrinking Soliton Singularity

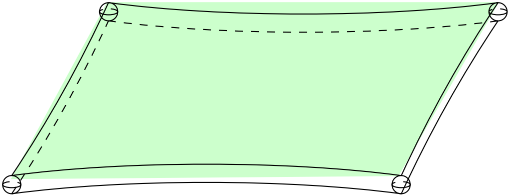
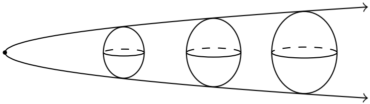


Figure: An approximate $S^2 \times \mathbb{R}^2$ resolving the BCCD shrinking soliton singularity.

Potential Ricci Flow Surgery in 4D: Steady Solitons with Tangent Flow $S^3/\Gamma \times \mathbb{R}$

- $\Gamma = \{1\}$. 4D Bryant steady soliton is rotationally symmetric and opens up like a paraboloid. Asymptotically cylindrical.



- **Appleton** constructed noncollapsed, cohomogeneity-one **steady Ricci solitons** on the total space M_k^4 of the complex line bundle $\mathcal{O}(-k)$ over $\mathbb{C}P^1$ with degree $-k$, $k \geq 3$. M_k^4 is diffeomorphic to $(\mathbb{S}^3/\mathbb{Z}_k) \times (0, \infty)$ with a 2-sphere attached at the origin. The tangent flow is $S^3/\Gamma \times \mathbb{R}$. One can take $\mathbb{Z}_2$ quotients to obtain real plane bundle bundles over $\mathbb{R}P^2$ (Law).
- Surgery, by gluing in the $\mathbb{Z}_k$-quotient of Perelman's standard solution, results in a singular point with $\mathbb{Z}_k$ isotropy group.

Potential Ricci Flow Surgery in 4D: Steady Solitons with Tangent Flow $S^2 \times \mathbb{R}^2$

Flying wing-type 4D steady solitons.

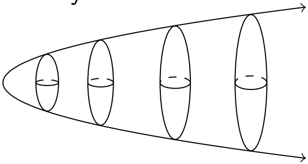


Figure: Lai's 4-dimensional steady soliton. The slices are ovaloids that elongate and should asymptotically approach Perelman's ancient solution.

Building on works by Angenent, Brendle, Daskalopoulos, Sesum, there is an asymptotic analysis by Ma, Mahmoudian, and Sesum.

Ricci flow surgery should be analogous to surgery on the Bryant soliton singularity formation, where instead a 3D Perelman ancient solution cartesian product with $\mathbb{R}$ is capped off.

Potential Ricci Flow Surgery in 4D: General Theory

An aim of Ricci flow in higher dimensions is to prove the **existence of Ricci flow past singularity times** (e.g. Ricci flow with surgery). To this end there is recent striking progress due to Richard Bamler.

Theorem (Bamler)

*Let $(M^4, g(t))$, $t \in [0, T)$, be a Ricci flow on a closed 4-manifold that develops a singularity at time $T < \infty$. If M^4 is not diffeomorphic to a shrinking Ricci soliton, then there **exists an associated singularity model** which is one of the following:*

1. 2-***cylinder***: $S^2 \times \mathbb{R}^2$,
2. 3-***cylinder***: $(S^3/\Gamma) \times \mathbb{R}$,
3. ***Riemannian Cone***: *A Riemannian cone with $R \geq 0$, that is either a flat orbifold $\mathbb{R}^4/\Gamma$ or the asymptotic cone of an asymptotically conical shrinking Ricci soliton.*

Flowbordism

Given a 4D Ricci flow with surgery defined on $[t_1, t_2]$, let $\mathcal{O}_t^4$ denote the 4D smooth orbifold at time t. We say that $\mathcal{O}_{t_2}^4$ is **flowbordant** to $\mathcal{O}_{t_1}^4$, and we call the space-time of the Ricci flow a **flowbordism**. The relation of being flowbordant is reflexive and transitive, but not symmetric. An objective is to understand the topological effects of flowbordisms on the underlying orbifolds.

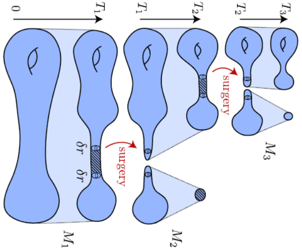
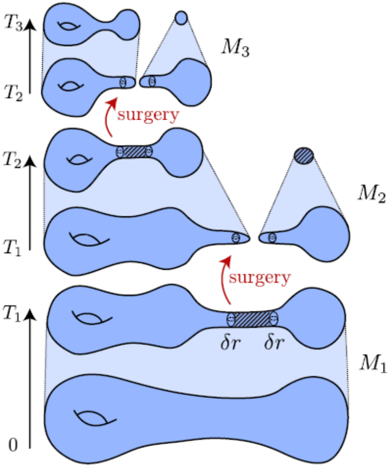


Figure: Figure 3 in Bamler's 2022 ICM talk.

Flowbordism Picture



11/8 Conjecture: Bamler's Program, I

Transition: Potential Applications to 4-Manifold Topology

The 11/8-**conjecture** says that the **signature-to-second-Betti-number ratio** satisfies

$$\mathcal{R} := \frac{|\sigma|}{b_2} \leq \frac{8}{11},$$

where b_2 is the second Betti number and σ is the signature.

Ideally, under Ricci flow, if $\mathcal{R} \leq 8/11$ holds *post-surgery*, one might hope that this would imply that $\mathcal{R} \leq 8/11$ holds *pre-surgery* as well. A sufficient condition for this implication is the monotonicity inequality:

$$\text{If } \mathcal{O}_2^4 \text{ is flowbordant to } \mathcal{O}_1^4, \text{ then } \mathcal{R}(\mathcal{O}_2^4) \geq \mathcal{R}(\mathcal{O}_1^4).$$

However, the situation is not quite this simple.

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- Monotonicity of the signature to second-Betti-number ratio under quotient 3-surgeries and prophylactic 2-surgeries.
- Positive scalar curvature and Lichnerowicz vanishing theorem: Implications for compact and noncompact singularity models.
- If a singularity model of a connected component of a Ricci flow has its underlying manifold being compact, then the singularity model must be a shrinking gradient Ricci soliton. Such a singularity model must have $R > 0$ and hence, if it is spin, $\sigma = 0$ by the Lichnerowicz vanishing theorem.
- In the spin case, if a Ricci flow has finite-time extinction, then each connected component of the Ricci flow that has no orbifold points (i.e., is a manifold) will have vanishing signature. What can one say about the components that contain orbifold points?

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- Tracking signature-to-second-Betti-number ratio under Ricci-flat ALE bubbling. If Ricci flow with surgery produces only the bubbling of Ricci-flat ALEs, and the resulting orbifold satisfies the Hitchin–Thorpe inequality (e.g., by admitting an Einstein metric), then M^4 must satisfy the Hitchin–Thorpe inequality as well.
- A moral is the Hitchin–Thorpe inequality bounds the signature of Einstein manifolds, and in particular it bounds the signature of Ricci-flat ALE bubbles, which yields a favorable inequality direction regarding Ricci-flat ALE bubbling under Ricci flow.
- There is the well-known conjecture that any 4D Ricci-flat ALE space is a Kronheimer space, i.e., hyperkähler. This should not be needed for 11/8, but would be important for any 4D geometrization conjecture.

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  2. Collapsed Riemannian orbifolds with boundary.
- Understanding the topology of complete finite-volume Einstein 4-orbifolds: Does a Hitchin–Thorpe type-inequality hold for such manifolds? What can one say about the geometry of ends of such orbifolds?
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General Questions: What can Ricci flow do?

Transition: From Technical Frontier → Open Questions

- (Bamler) Can Ricci flow be used to prove the $11/8$ conjecture?
- More generally, can Ricci flow be used to geometrize 4-manifolds, independent of possible further topological consequences (which may be difficult to infer)? Is it even possible to generalize manifolds of any dimension (and this presumably would need the help of AI)?
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Specific Questions: Lean Formalization

What to formalize for Ricci flow:

- For Ricci flow, it will be important to formalize in the category of smooth orbifolds with isolated singularities (where the metric is smooth over these singularities, i.e., the local lifts of g to $\mathbb{R}^n$ are smooth. This seems to be the category that 4D Ricci flow stays in, and also necessarily goes into even when starting with a smooth Riemannian 4-manifold.
- Also, it will be important to formalize things in the metric measure space category. Much of Bamler's higher dimensional, more abstract, theory is formulated in this category, including his essentially completely general compactness theorem. Then he uses some of the Cheeger-Colding–Naber (CCN) theory to prove regularity. It will be important to geometric analysis to formalize the CCN theory also.

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- For Ricci flow, it will be important to formalize in the category of smooth orbifolds with isolated singularities (where the metric is smooth over these singularities, i.e., the local lifts of g to $\mathbb{R}^n$ are smooth. This seems to be the category that 4D Ricci flow stays in, and also necessarily goes into even when starting with a smooth Riemannian 4-manifold.
- Also, it will be important to formalize things in the metric measure space category. Much of Bamler's higher dimensional, more abstract, theory is formulated in this category, including his essentially completely general compactness theorem. Then he uses some of the Cheeger-Colding–Naber (CCN) theory to prove regularity. It will be important to geometric analysis to formalize the CCN theory also.

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Specific Questions: Singularity Models

- Classification of singularity models. Can AI help?
  1. Is every singularity model of Ricci flow necessarily either a shrinking or steady Ricci soliton? This is true in 3D by the works of Hamilton, Perelman, and Brendle. Is it true in 4D?
  2. Can one classify shrinking and steady Ricci soliton singularity models in dimension 4? There is a well-developed qualitative theory of O. Munteanu and J. Wang. There are recent examples of Bamler, Cifarelli, Conlon, and Deruelle, and Y. Lai.
  3. Can a Ricci soliton singularity model be an exotic $\mathbb{R}^4$? Is the small scale differential topology of Ricci flows simple?
  4. In what sense are the topologies of (4D) singularity models small? E.g., nD singularity models necessary have finite fundamental group (Yokota, Bamler). Bounds for b_2 in 4D?
  5. Compact singularity models are necessarily shrinking Ricci solitons (Z.-L. Zhang). So, if spin, then the signature $\sigma = 0$. Bounds for signatures of noncompact singularity models?

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Specific Questions: A shrinking flying wing?

A hypothetical flying wing shrinking soliton is pictured below. Bamler asked if there exists a shrinking soliton analogous to Lai's steady soliton. In the central region, as points tend to infinity, the shrinker would look like $S^2 \times \mathbb{R}^2$ (of an asymptotically fixed scale). Along the edges of the soliton, the rescalings tend to the product of a 3D Bryant steady soliton and $\mathbb{R}$. The scales of these Bryant solitons decrease to zero as they go to infinity. Thus, this shrinking soliton on $\mathbb{R}^4$ would have unbounded curvature.

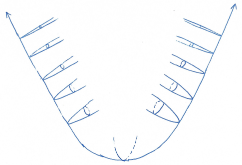

Figure: A hypothetical flying wing shrinking Ricci soliton.

Specific Questions

- Formulation of 4D Ricci flow with surgery or weak Ricci flow.
  1. This is dependent on having a sufficiently good singularity model classification.
  2. 'Canonical neighborhood theorem' for high curvature regions of Ricci flows. Does each portion of a high curvature region look like a singularity model?
- Are the singularity times of Ricci flow discrete in nD, $n \geq 4$? (In 3D, they are (Perelman), and in fact they are finite (Bamler).)
- Long time behavior of Ricci flow. Understanding limits:
  1. Noncompact Einstein manifolds with negative scalar curvature and finite volume.
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General Technically Oriented Questions

- What yet undiscovered fundamental monotonicity formulas are still out there? Perhaps these will more come from new points of view rather than technical improvements. E.g., Perelman's revolution was a laser sharp **geometric**-analytic space-time synthesis of Ricci flow. Bamler's breakthroughs were a return to more **analytic**-geometric arguments, with the introduction of probablistic ideas. Much of Ricci flow tensor calculation has an algebraic flavor.
- Can we insert more topological methods in Ricci flow? E.g., Bamler and Chen: uses a fixed point theorem.
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The Art of the *A Priori* Estimate

- Hamilton's principle is that the quantity you want to estimate should vanish or be constant in the model case.
- Essentially every (fine vs. coarse) estimate in Ricci flow follows this principle. E.g., Harnack estimates are fine. Derivative of curvature estimates are coarse.
  1. Fine estimates are algebraic in nature. Equalities throughout. Even when intermediate steps are complicated, the final formula is a simple equality.
  2. Coarse estimates are analytic in nature. Inequalities throughout, such as Cauchy–Schwarz, Peter–Paul, etc.
- In fact, the model case has been Ricci solitons. Recall that these are the fixed points of the Ricci flow in the space of metrics modulo diffeomorphisms and scalings. I.e., two metrics are equivalent if and only if one is isometric to a constant times the other.

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- Is the following meta-belief correct? Singularity models are as simple as possible geometrically and topologically given the complexity of the landscape of smooth manifolds of a given dimension and Ricci flow's role as a smoothing operator on the space of metrics.

- For topological applications, is the Riemann curvature tensor really necessary in the study of Ricci flow? If the answer is no in some sense, can we interpret this as reflecting that the full Riemann curvature tensor, as a quantity, is not physical?

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Ricci Flow and AI

- Can we teach AI to do tensor calculations? Many Ricci flow calculations are such calculations. Typical calculations include commuting Laplace-type operators with covariant derivatives. Usually, a heat-type equation is sought for various geometric quantities.
  1. Scalar quantities are often norms (squared) of tensors. E.g., evolution of scalar curvature, curvature pinching quantities, etc. For Ricci solitons (M^n, g, f), various quantities involving the curvature invariants of g and the potential function f and its covariant derivatives.
  2. Hamilton's matrix Harnack estimate, albeit a bit complicated, follows only a couple of basic principles to carry out the calculation. One such principle is that the quantity should vanish in the model (Ricci soliton) case. In this case, some of the covariant derivative commutator formulas can be considered as a space-time second Bianchi identity.

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THANK YOU!