Path integrals over Euclidean spaces

Bruce Driver

Visiting Miller Professor

(Permanent Address) Department of Mathematics, 0112

University of California at San Diego, USA

http://math.ucsd.edu/∼driver

Student Topology Seminar

University of California, Berkeley, August 29, 2007

Wiener Measure Heuristics and the Feynman-Kac formula

Theorem 1 (Trotter Product Formula)**.** *Let A and B be $d \times d$ matrices. Then* $e^{(A+B)} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n$.

Proof: By the chain rule,

$$\frac{d}{d\varepsilon}|_0 \log(e^{\varepsilon A} e^{\varepsilon B}) = A + B.$$

Hence by Taylor's theorem with remainder,

$$\log(e^{\varepsilon A} e^{\varepsilon B}) = \varepsilon\,(A+B) + O\left(\varepsilon^2\right)$$

which is equivalent to

$$e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon(A+B)+O\left(\varepsilon^2\right)}.$$

Taking $\varepsilon = 1/n$ and raising the result to the n^{th} – power gives

$$(e^{n^{-1}A} e^{n^{-1}B})^n = \left[e^{n^{-1}(A+B)+O\left(n^{-2}\right)}\right]^n$$
$$= e^{A+B+O\left(n^{-1}\right)} \to e^{(A+B)} \text{ as } n \to \infty.$$

Q.E.D.

Fact (Trotter product formula)**.** For "nice enough" V,

$$e^{T(\Delta/2-V)} = \text{strong}-\lim_{n\to\infty} [e^{\frac{T}{2n}\Delta} e^{-\frac{T}{n}V}]^n. \quad (1)$$

See [1] for a rigorous statement of this type.

Lemma 2. *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a continuous function which is bounded from below, then*

$$\left(\left(e^{\frac{T}{n}\Delta/2} e^{-\frac{T}{n}V}\right)^n f\right)(x_0)$$
$$= \int_{\mathbb{R}^{dn}} p_{\frac{T}{n}}(x_0, x_1) e^{-\frac{T}{n}V(x_1)} \dots p_{\frac{T}{n}}(x_{n-1}, x_n) e^{-\frac{T}{n}V(x_n)} f(x_n) dx_1 \dots dx_n$$
$$= \left(\frac{1}{\sqrt{2\pi\frac{T}{n}}}\right)^{dn} \int_{(\mathbb{R}^d)^n} e^{-\frac{n}{2T}\sum_{i=1}^n |x_i - x_{i-1}|^2 - \frac{T}{n}\sum_{i=1}^n V(x_i)} f(x_n) dx_1 \dots dx_n. \quad (2)$$

Notation 3. Given $T > 0$, and $n \in \mathbb{N}$, let $W_{n,T}$ denote the set of piecewise C^1 – paths, $\omega : [0,T] \to \mathbb{R}^d$ such that $\omega(0) = 0$ and $\omega''(\tau) = 0$ if $\tau \notin \left\{\frac{i}{n}T\right\}_{i=0}^n =: \mathcal{P}_n(T)$ – see Figure 1. Further let dm_n denote the unique translation invariant measure on $W_{n,T}$ which is well defined up to a multiplicative constant.

With this notation we may rewrite Lemma 2 as follows.

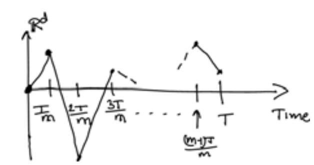

Figure 1: A typical path in $W_{m,T}$.

Theorem 4. *Let $T > 0$ and $n \in \mathbb{N}$ be given. For $\tau \in [0,T]$, let $\tau_+ = \frac{i}{n}T$ if $\tau \in (\frac{i-1}{n}T, \frac{i}{n}T]$. Then Eq. (2) may be written as,*

$$\left(\left(e^{\frac{T}{n}\Delta/2} e^{-\frac{T}{n}V}\right)^n f\right)(x_0)$$
$$= \frac{1}{Z_n(T)} \int_{W_{n,T}} e^{-\int_0^T \left[\frac{1}{2}|\omega'(\tau)|^2 + V(x_0+\omega(\tau_+))\right] d\tau} f\left(x_0 + \omega(T)\right) dm_n(\omega)$$

where

$$Z_n(T) := \int_{W_{n,T}} e^{-\frac{1}{2}\int_0^T |\omega'(\tau)|^2 d\tau} dm_n(\omega).$$

Moreover, by Trotter's product formula,

$$e^{T(\Delta/2-V)} f(x_0)$$
$$= \lim_{n\to\infty} \frac{1}{Z_n(T)} \int_{W_{n,T}} e^{-\int_0^T \left[\frac{1}{2}|\omega'(\tau)|^2 + V(x_0+\omega(\tau_+))\right] d\tau} f\left(x_0 + \omega(T)\right) dm_n(\omega). \quad (3)$$

Following Feynman, at an informal level (see Figure 2), $W_{n,T} \to W_T$ as $n \to \infty$, where

$$W_T := \left\{\omega \in C\left([0,T] \to \mathbb{R}^d\right) : \omega(0) = 0\right\}.$$

Moreover, formally passing to the limit in Eq. (3) leads us to the following heuristic

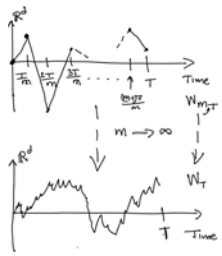

Figure 2: A typical path in W_T may be approximated better and better by paths in $W_{m,T}$ as $m \to \infty$.

expression for $\left(e^{T(\Delta/2-V)}f\right)(x_0)$;

$$\left(e^{T(\Delta/2-V)}f\right)(x_0) = \text{“}\frac{1}{Z(T)}\int_{W_T} e^{-\int_0^T\left[\frac{1}{2}|\omega'(\tau)|^2+V(x_0+\omega(\tau))\right]d\tau} f(x_0+\omega(T))\,\mathcal{D}\omega\text{”} \quad (4)$$

where $\mathcal{D}\omega$ is the **non-existent** Lebesgue measure on W_T, and $Z(T)$ is the

“normalization” constant (or partition function) given by

$$Z(T) = \text{“}\int_{W_T} e^{-\frac{1}{2}\int_0^T|\omega'(\tau)|^2 d\tau}\mathcal{D}\omega.\text{”}$$

This expression may also be written in the **Feynman – Kac** form as

$$e^{T(\Delta/2-V)}f(x_0) = \int_{W_T} e^{-\int_0^T V(x_0+\omega(\tau))d\tau} f(x_0+\omega(T))\,d\mu(\omega),$$

where

$$d\mu(\omega) = \text{“}\frac{1}{Z(T)}e^{-\frac{1}{2}\int_0^T|\omega'(\tau)|^2 d\tau}\mathcal{D}\omega\text{”} \quad (5)$$

is the informal expression for **Wiener measure** on W_T. Thus our immediate goal is to make sense out of Eq. (5).

Let

$$H_T := \left\{h \in W_T : \int_0^T |h'(\tau)|^2 d\tau < \infty\right\}$$

with the convention that $\int_0^T |h'(\tau)|^2 d\tau := \infty$ if h is not absolutely continuous. Further let

$$\langle h, k\rangle_T := \int_0^T h'(\tau)\cdot k'(\tau)\,d\tau \text{ for all } h, k \in H_T$$

and $X_h(\omega) := \langle h, \omega\rangle_T$ for $h \in H_T$. Since

$$d\mu(\omega) = \text{“}\frac{1}{Z(T)}e^{-\frac{1}{2}\|\omega\|_{H_T}^2}\mathcal{D}\omega,\text{”} \quad (6)$$

$d\mu(\omega)$ should be a Gaussian measure on H_T and hence we expect,

$$\int_{H_T} e^{i\langle h,\omega\rangle_T} d\mu(\omega) = \exp\left(-\frac{1}{2}\|h\|_{H_T}^2\right). \quad (7)$$

Gaussian Measures “on” Hilbert spaces

Goal Given a Hilbert space H, we would ideally like to define a probability measure μ on $\mathcal{B}(H)$ such that

$$\hat{\mu}(h) := \int_H e^{i(\lambda,x)}d\mu(x) = e^{-\frac{1}{2}\|\lambda\|^2} \text{ for all } \lambda \in H \quad (8)$$

so that, informally,

$$d\mu(x) = \frac{1}{Z}e^{-\frac{1}{2}|x|_H^2}\mathcal{D}x. \quad (9)$$

The next proposition shows that this is impossible when $\dim(H) = \infty$.

Proposition 5. *Suppose that H is an infinite dimensional Hilbert space. Then there is no probability measure μ on $\mathcal{B} = \mathcal{B}(H)$ such that Eq. (8) holds.*

Proof: Suppose such a Gaussian measure were to exist. Let $\{\lambda_i\}_{i=1}^\infty$ be an orthonormal set in H and for $M > 0$ and $n \in \mathbb{N}$ let

$$W_n^M = \{x \in H : |(\lambda_i, x)| \le M \text{ for } i = 1, 2, \ldots, n\}.$$

Let μ_n be the standard Gaussian measure on $\mathbb{R}^n$,

$$d\mu_n(y) = (2\pi)^{-n/2}e^{-y\cdot y/2}dy.$$

Then for all bounded measurable functions $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\int_H f((\lambda_1, x), \ldots, (\lambda_n, x))\, d\mu(x) = \int_{\mathbb{R}^n} f(y) d\mu_n(y)$$

and therefore,

$$\begin{aligned}\mu(W_n^M) &= \mu_n(\{y \in \mathbb{R}^n : |y_i| \le M \text{ for } i = 1, \ldots, n\}) \\ &= \left((2\pi)^{-1/2} \int_{-M}^{M} e^{-\frac{1}{2}y^2} dy\right)^n \to 0 \text{ as } n \to \infty.\end{aligned}$$

Because

$$W_n^M \downarrow H_M := \{x \in H : |(\lambda_i, x)| \le M \; \forall\, i = 1, 2, \ldots\},$$

$\mu(H_M) = \lim_{n\to\infty} \mu(W_n^M) = 0$ for all $M > 0$. Since $H_M \uparrow H$ as $M \uparrow \infty$ we learn that $\mu(H) = \lim_{M\to\infty} \mu(H_M) = 0$, i.e. $\mu \equiv 0$. Q.E.D.

Moral: The measure μ must be defined on a larger space. This is somewhat analogous to trying to define Lebesgue measure on the rational numbers. In each case the measure can only be defined on a certain completion of the naive initial space.

Guassian Measure for ℓ^2

Remark 6. Suppose that $H = \ell^2$ the space of square summable sequences $\{x_n\}_{n=1}^\infty$. In this case the Gaussian measure that we are trying to construct is formally given by the expression

$$\begin{aligned}d\mu(x) &= \frac{1}{(\sqrt{2\pi})^\infty} \exp\left(-\frac{1}{2}\Sigma_{i=1}^\infty x_i^2\right) \prod_{i=1}^\infty dx_i \\ &= \prod_{i=1}^\infty \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}\, dx_i\right) =: \prod_{i=1}^\infty p_1(dx_i),\end{aligned}$$

where $p_1(dx)$ is the heat kernel defined by

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}x^2\right).$$

This suggests that we define $\mu = p_1^{\otimes\mathbb{N}}$, the infinite product measure on $\mathbb{R}^\mathbb{N}$.

Theorem 7. *Let $\mu = p_1^{\otimes\mathbb{N}}$ be the infinite product measure on $(\mathbb{R}^\mathbb{N}, \mathcal{F})$ where μ and $\mathcal{F}$. Also let $a = (a_1, a_2, \ldots) \in (0, \infty)^\mathbb{N}$ be a sequence and define*

$$X_a = \ell^2(a) = \{x \in \mathbb{R}^\mathbb{N} : \sqrt{\textstyle\sum_{i=1}^\infty a_i x_i^2} := \|x\|_a < \infty\}.$$

So $X = L^2(\mathbb{N}, a)$ where a now denotes the measure on $\mathbb{N}$ determined by $a(\{i\}) = a_i$ for all $i \in \mathbb{N}$. Then $X_a \in \mathcal{F}$, $\mathcal{F}_{X_a} := \{A \cap X_a : A \in \mathcal{F}\} = \mathcal{B}(X_a)$ ($\mathcal{B}(X_a)$ is the Borel σ – field on X_a) and

$$\mu(X_a) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^\infty a_i < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Assuming that $\sum_{i=1}^\infty a_i < \infty$, $\mu_a := \mu|_{\mathcal{B}(X_a)}$ is a the unique probability measure on $(X_a, \mathcal{B}(X_a))$ which satisfies

$$\int_{X_a} f(x_1, \ldots, x_n) d\mu_a(x) = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) p_1(dx_i) \ldots p_1(dx_n) \tag{11}$$

for all $f \in (\mathcal{B}(\mathbb{R}^n))_b$ and $n = 1, 2, 3, \ldots$

Proof: For $N \in \mathbb{N}$, let $q_N : \mathbb{R}^\mathbb{N} \to \mathbb{R}$ be defined by $q_N(x) = \sum_{i=1}^N a_i x_i^2$. Then it is easily seen that q_N is $\mathcal{F}$ – measurable. Therefore, $q := \sup_{N\in\mathbb{N}} q_N$ (also notice that $q_N \uparrow q$ as $N \to \infty$) is $\mathcal{F}$ – measurable as well and hence

$$X_a = \{x \in \mathbb{R}^\mathbb{N} : q(x) < \infty\} \in \mathcal{F}.$$

Similarly, if $x_0 \in X_a$, then $q(\cdot - x_0) = \sup_{N\in\mathbb{N}} q_N(\cdot - x_0)$ is $\mathcal{F}$ – measurable and therefore for $r > 0$,

$$B(x_0, r) = \{x \in X_a : \|x - x_0\|_a < r\} = \{x \in \mathbb{R}^\mathbb{N} : q(\cdot - x_0) < r^2\} \in \mathcal{F}$$

which shows that $\mathcal{B}(X_a) \subset \mathcal{F}$ and hence $\mathcal{B}(X_a) \subset \mathcal{F}_{X_a}$. To prove the reverse inclusion, let $i : X_a \to \mathbb{R}^\mathbb{N}$ be the inclusion map and recall that

$$\begin{aligned}\mathcal{F}_{X_a} &= i^{-1}(\mathcal{F}) = i^{-1}\left(\sigma\left(\pi_j^{-1}(B) : j \in \mathbb{N} \text{ and } B \in \mathcal{B}_\mathbb{R}\right)\right) \\ &= \sigma\left(i^{-1}\pi_j^{-1}(B) : j \in \mathbb{N} \text{ and } B \in \mathcal{B}_\mathbb{R}\right) \\ &= \sigma\left((\pi_j \circ i)^{-1}(B) : j \in \mathbb{N} \text{ and } B \in \mathcal{B}_\mathbb{R}\right) = \sigma(\pi_j \circ i : j \in \mathbb{N}).\end{aligned}$$

Since $\pi_j \circ i \in X_a^*$ for all j, we see from this expression that

$$\mathcal{F}_{X_a} \subset \sigma(X_a^*) = \mathcal{B}(X_a).$$

Let us now prove Eq. (10). Letting q and q_N be as defined above, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^\mathbb{N}} e^{-\varepsilon q/2} d\mu &= \int_{\mathbb{R}^\mathbb{N}} \lim_{N\to\infty} e^{-\varepsilon q_N/2} d\mu \overset{\text{M.C.T.}}{=} \lim_{N\to\infty} \int_{\mathbb{R}^\mathbb{N}} e^{-\varepsilon q_N/2} d\mu \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^N} e^{-\frac{\varepsilon}{2}\sum_1^N a_i x_i^2} \prod_{i=1}^N p_1(dx_i) \\ &= \lim_{N\to\infty} \prod_{i=1}^N \int_\mathbb{R} e^{-\frac{\varepsilon}{2}a_i x^2} p_1(dx).\end{aligned} \tag{12}$$

Using

$$\int e^{-\frac{\lambda}{2}x^2} p_1(x) dx = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{\lambda+1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{\lambda+1}} = \frac{1}{\sqrt{\lambda+1}}$$

in Eq. (12) we learn that

$$\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu = \lim_{N\to\infty} \prod_1^N \frac{1}{\sqrt{1+\varepsilon a_i}} = \sqrt{\lim_{N\to\infty} \prod_1^N (1+\varepsilon a_i)^{-1}}$$

or equivalently that

$$-\log\left(\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu\right) = \frac{1}{2}\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i). \tag{13}$$

Notice that there is $\delta > 0$ such that

$$\ln(1+x) \le x \; \forall x \ge 0 \text{ and } \ln(1+x) \ge x/2 \text{ for } x \in [0,\delta]. \tag{14}$$

If $\limsup_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i) = \infty$ for all $\varepsilon > 0$. If $\lim_{i\to\infty} a_i = 0$ but $\sum_{i=1}^{\infty} a_i = \infty$, then using Eq. (14), $\ln(1+\varepsilon a_i) \ge \varepsilon a_i/2$ for all i large and hence again $\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i) = \infty$. If $\sum_{i=1}^{\infty} a_i < \infty$ then by Eq. (14),

$$\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i) \le \varepsilon \sum_{i=1}^{\infty} a_i \to 0 \text{ as } \varepsilon \downarrow 0.$$

In summary,

$$-\lim_{\varepsilon\downarrow 0} \log\left(\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu\right) = \begin{cases} \infty & \text{if } \sum_{i=1}^{\infty} a_i = \infty \\ 0 & \text{if } \sum_{i=1}^{\infty} a_i < \infty \end{cases}$$

or equivalently,

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} a_i = \infty \\ 1 & \text{if } \sum_{i=1}^{\infty} a_i < \infty. \end{cases}$$

Since $e^{-\varepsilon q/2} \le 1$ and $\lim_{\varepsilon\downarrow 0} e^{-\varepsilon q/2} = 1_{X_a}$, the previous equation along with the dominated convergence theorem shows that

$$\mu(X_a) = \int_{\mathbb{R}^{\mathbb{N}}} 1_{X_a} d\mu = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} a_i = \infty \\ 1 & \text{if } \sum_{i=1}^{\infty} a_i < \infty. \end{cases}$$

proving Eq. (10inally Eq. (11) follows from the definition of μ and the fact that

$$\int_{X_a} f(x_1,\ldots,x_n) d\mu_a(x) = \int_{\mathbb{R}^{\mathbb{N}}} f(x_1,\ldots,x_n) d\mu(x).$$

Q.E.D.

Classical Wiener Measure

Let $W = \{\omega \in C([0,1] \to \mathbb{R}) : \omega(0) = 0\}$ and let H denote the set of functions $h \in W$ which are absolutely continuous and satisfy $(h,h) = \int_0^1 |h'(s)|^2 ds < \infty$. The space H is called the Cameron-Martin space and is a Hilbert space when equipped with the inner product

$$(h,k) = \int_0^1 h'(s)k'(s)ds \text{ for all } h,k \in H.$$

The space W is a Banach space when equipped with the sup-norm,

$$\|f\| = \max_{x\in[0,1]} |f(x)|.$$

Theorem 8 (Wiener, 1923)**.** *There exists a unique Gaussian measure μ on W such that*

$$\int_W e^{i\varphi(x)} d\mu(x) = e^{-q(\varphi)/2}, \tag{15}$$

where $\varphi = (\cdot, h_\varphi)$, and $q(\varphi,\psi) := (h_\varphi, h_\psi)_H$ is the dual inner product to H.

Theorem 9 (Feynman-Kac Formula)**.** *Suppose that $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth function such that $k := \inf_{x\in\mathbb{R}^d} V(x) > -\infty$. Then for $f \in L^2(m)$,*

$$\left(e^{-t\left(-\frac{1}{2}\Delta + V\right)} f\right)(x) = \int_W e^{-\int_0^t V(x+\omega_\tau)d\tau} f(x+\omega_t)\, d\mu(\omega).$$

References

[1] Barry Simon, *Functional integration and quantum physics*, second ed., AMS Chelsea Publishing, Providence, RI, 2005. MR MR2105995 (2005f:81003)