The energy representation has no non-zero fixed vectors

Bruce K. Driver† and Brian C. Hall*

Abstract. We consider the "energy representation" W of the group $\mathcal{G}$ of smooth mappings of a Riemannian manifold M into a compact Lie group G. Our main result is that if $W(g) f = f$ for all $g \in \mathcal{G}$, then $f = 0$. In the language of quantum field theory this says that there are no "states." Our result follows from the irreducibility of the energy representation whenever the irreducibility theorems of Ismagilov, Gelfand–Graev–Veršik, Albeverio–Hoegh-Krohn–Testard, or Wallach apply. Our result, however, applies in general, even in cases where the energy representation is known to be reducible.

We work in the more general context of the "Gaussian regular representation" of the Euclidean group of a real separable Hilbert space. We show that if a function is invariant under the action of any subgroup of the Euclidean group that has unbounded orbits, then this function must be identically zero. Our result about the energy representation is a special case.

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1. Introduction and statement of result

Let M be a compact Riemannian manifold without boundary, with dimension at least one. Let G (the "structure group") be a compact connected Lie group, and let $\mathcal{G}$ (the "gauge group") be the set of smooth mappings of M into G, forming a group under pointwise multiplication. Let $\mathfrak{g}$ be the Lie algebra of G, and fix on $\mathfrak{g}$ an Ad-G-invariant inner product. In addition, fix a positive function ω on M.

We consider the real Hilbert space H of square-integrable, $\mathfrak{g}$-valued 1-forms on M. So an element A of H assigns to a point $m \in M$ an element of $T_m^* M \otimes \mathfrak{g} \cong \operatorname{Hom}(T_m M, \mathfrak{g})$. The inner product on H is given explicitly by

$$\langle A, B\rangle = \int_M \operatorname{tr}\left(A(m)^* B(m)\right) \omega(m)\, dm, \quad A, B \in H, \tag{1}$$

where $A(m)^*$ denotes the adjoint of $A(m) : T_m M \to \mathfrak{g}$ relative the given inner products on $T_m M$ and on $\mathfrak{g}$, and where dm is the Riemannian volume measure on M. If $\dim M \neq 2$, the function ω can be absorbed into the definition of the metric; if $\dim M = 2$, the inner product on H depends only on the conformal class of the metric on M.

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For $g \in \mathcal{G}$ and $A \in H$ we define the action of g on A by

$$(g \cdot A)(v_m) = \mathrm{Ad}_{g(m)} A(v_m) - dg(v_m)\, g(m)^{-1}, \tag{2}$$

where v is a vector field on M. The action of $\mathcal{G}$ on H is isometric, the first term being an invertible isometric linear transformation applied to A and the second term being a translation.

We now consider the Gaussian measure ρ associated to H, given formally by the expression

$$d\rho(A) = \frac{1}{Z} e^{-|A|^2/2} \mathcal{D}A,$$

where $|A|^2 := \langle A, A\rangle$, $\mathcal{D}A$ is the fictitious Lebesgue measure on H, and Z is supposed to be a normalizing constant. Rigorously, ρ is defined not on H itself, but on some extension B of H, in this case a suitable space of distributional ($\mathfrak{g}$-valued) 1-forms on M. The measure ρ may be characterized, for example, by the condition that

$$\int_B e^{i\phi(A)}\, d\rho(A) = \exp\left\{-\frac{1}{2}|\phi|^2_{H^*}\right\}$$

for all continuous linear functionals ϕ on B. Here $|\phi|_{H^*}$ is the norm of ϕ as a linear functional on $H \subset B$. The action of $\mathcal{G}$ on H extends continuously to an action of $\mathcal{G}$ on B, and this action leaves the measure ρ quasi-invariant.

Finally, we consider a unitary representation of $\mathcal{G}$ acting in $L^2(B, \rho)$ given by

$$W(g) f(A) = e^{-\left|dgg^{-1}\right|^2/4} e^{-\left\langle A, dgg^{-1}\right\rangle/2} f\left(g^{-1} \cdot A\right). \tag{3}$$

The "multiplier" in front of $f\left(g^{-1} \cdot A\right)$ is the square root of the Radon-Nikodym derivative $d\rho\left(g^{-1} \cdot A\right)/d\rho(A)$. More details on the definition of $W(g)$ may be found in the next section; in particular, $W(g) = U(Ad_g, -dg\, g^{-1})$ in the notation of Definition 8 below.

We may now state the main result of this paper.

Theorem 1. *If $f \in L^2(B, \rho)$ satisfies*

$$W(g) f = f$$

for all $g \in \mathcal{G}$, then $f = 0$.

Proof. This is an immediate consequence of Theorem 14 and Proposition 2. □

Thus in the language of quantum field theory, there are no "states." This result is expected in view of the non-compactness of the gauge orbits. However, we know of no proof of this in general in the literature.

The representation W is unitarily equivalent to the "energy representation" introduced by Ismagilov [**I**] in the case $G = SU(2)$ and by Gelfand, Graev, and Veršik [**GGV1**] in general. See Section 4 for details. If G is semisimple and $\dim M \geq 3$, or under additional assumptions on ω if G is semisimple and $\dim M = 2$, the energy representation is irreducible [**I, GGV1, GGV2, AKT, Wa**]. Thus, in these cases, W is also irreducible and Theorem 1 follows immediately. On the other hand, if $\dim M = 1$ or if G is commutative, then the energy representation is reducible; in any case it is desirable to have a simple direct proof of Theorem 1, which is much weaker than irreducibility.

Theorem 1 can doubtless be proved in many ways, for example, using the same techniques as in the proofs of irreducibility. Our proof will make use of coherent states and the Segal–Bargmann transform. The only properties of $\mathcal{G}$ that we will use are that $\mathcal{G}$ acts isometrically on H and the following. (See Theorem 14 below.)

Proposition 2. *For any $A \in H$, the orbit of A under the action of $\mathcal{G}$ is unbounded in H.*

Proof. Since the inner product on $\mathfrak{g}$ is Ad-G-invariant, the norm of the first term in (2) is equal to the norm of A and hence independent of g. Meanwhile, the second term in (2) is independent of A. It follows that if any one $\mathcal{G}$-orbit is unbounded, then all the others are as well. Thus to show that the orbits of $\mathcal{G}$ are unbounded, we must merely show that $\sup_g \left|g^{-1}dg\right|_H = \infty$,which is evident, say, by considering g of the form $g(x) = e^{f(x)\xi}$ with $\xi \in \mathfrak{g}$ and $f \in C^\infty(M)$. □

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2. The regular representation and Segal–Bargmann transform

Having noted that the representation W arises from an isometric action of $\mathcal{G}$ on H, and having recorded Proposition 2, we no longer need to concern ourselves with the manifold M or with the precise nature of the gauge action in (2). We will prove a general result (Theorem 14) about subgroups of the isometry group of a Hilbert space H; Theorem 1 is an immediate consequence.

2.1. Finite dimensions. Let H be a real Hilbert space, which in this subsection we take to have finite dimension d. Let $\mathcal{E}(H)$ be the Euclidean group, that is, the set of transformations of the form

$$x \to Rx + h,$$

where R is an orthogonal linear transformation and h is an element of H. We denote this transformation as (R, h) and observe that the composition of Euclidean transformations takes the form $(R, h) \circ (S, k) = (RS, Rk + h)$. Let dx be the Lebesgue measure on H, normalized so that a unit cube has volume one. We now define a unitary representation $\tilde{U}$ of $\mathcal{E}(H)$ acting on $L^2(H, dx)$ by

$$\tilde{U}(R, h) f(x) = f((R, h)^{-1} x) = f\left(R^{-1}(x - h)\right).$$

In preparation for the passage to the infinite-dimensional limit, we make a unitary change from Lebesgue measure to Gaussian measure on H. So define the measure ρ on H to be

$$d\rho(x) = (2\pi)^{-d/2} e^{-|x|^2/2} dx,$$

where $d = \dim H$; and consider the unitary map $\Phi : L^2(H, dx) \to L^2(H, \rho)$ given by

$$\Phi f(x) = (2\pi)^{d/4} e^{|x|^2/4} f(x). \tag{4}$$

Then define a unitary representation of $\mathcal{E}(H)$ acting on $L^2(H, \rho)$ by

$$U(R, h) = \Phi \tilde{U}(R, h) \Phi^{-1}.$$

We will call $\tilde{U}$ the regular representation of $\mathcal{E}(H)$ and U the Gaussian regular representation. Computing U explicitly leads to the following definition.

Definition 3. *The **Gaussian regular representation** is the unitary representation U of $\mathcal{E}(H)$ acting on $L^2(H,\rho)$ given by*

$$U(R,h)f(x) = e^{-|h|^2/4}e^{\langle h,x\rangle/2}f\left(R^{-1}(x-h)\right).$$

We next turn to the Segal–Bargmann transform [**B, S1, S2, S3**], which converts the Gaussian regular representation to the "holomorphic regular representation." Let $H_{\mathbb{C}} = H + iH$ be the complexification of H and let μ be the measure on $H_{\mathbb{C}}$ given by

$$d\mu(z) = \pi^{-d}e^{-|z|^2}\,dz,$$

where $|x+iy|^2 = |x|^2 + |y|^2$ and dz denotes $2d$-dimensional Lebesgue measure on $H_{\mathbb{C}}$.

Definition 4. *Let $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ denote the space of entire holomorphic functions on $H_{\mathbb{C}}$ which are square-integrable with respect to μ.*

It is easily shown that $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ is a closed subspace of $L^2(H_{\mathbb{C}},\mu)$ and therefore a Hilbert space. We will now define a unitary representation of $\mathcal{E}(H)$ on $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$. Let (z,w) denote the complex *bilinear* (not Hermitian) extension of the inner product on H to $H_{\mathbb{C}}$. Given $R \in O(H)$ we consider the unique complex-linear extension of R to $H_{\mathbb{C}}$, also denoted R.

Definition 5. *The **holomorphic regular representation** of $\mathcal{E}(H)$ is the unitary representation V of $\mathcal{E}(H)$ on $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ by*

$$V(R,h)F(z) = e^{-|h|^2/8}e^{(h,z)/2}F\left(R^{-1}(z-h/2)\right).$$

It straightforward computation to check directly that V is unitary and a representation. This result also is a consequence of Proposition 7 below.

Theorem 6 (Segal–Bargmann transform). *For $f \in \mathcal{H}L^2(H_{\mathbb{C}},\mu)$, let Sf be the function on $H_{\mathbb{C}}$ given by*

$$Sf(z) = e^{-(z,z)/2}\int_H f(x)\,e^{(z,x)}\,d\rho(x).$$

Then the integral that defines Sf is absolutely convergent and depends holomorphically on z. Moreover, the map $f \to Sf$ is a unitary map of $L^2(H,\rho)$ onto $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$.

This is well known and will not be proved here. See [**S3, BSZ**], or [**B**], which considers $S\Phi$, where Φ is the change of measure (4). Note that for $z \in H$, $e^{-(z,z)/2}e^{(z,x)}$ is the Radon–Nikodym derivative $d\rho(z-x)/d\rho(x)$. So Sf is the analytic continuation to $H_{\mathbb{C}}$ of the function

$$Sf(z) = \int_H f(x)\,d\rho(z-x) = \int_H f(z-x)\,d\rho(x), \quad z \in H.$$

Proposition 7. *The Segal–Bargmann transform intertwines the Gaussian regular representation and the holomorphic regular representation. That is, for $(R,h) \in \mathcal{E}(H)$, $V(R,h) = SU(R,h)S^{-1}$.*

The proof of this result is an explicit computation which is carried out in the proof of Proposition 13 below in the infinite-dimensional case.

2.2. Infinite dimensions. Now let (H, B, ρ) be an abstract Wiener space [**G**]. This means that B is a real, separable Banach space; H is an infinite-dimensional, real, separable Hilbert space continuously embedded into B with dense image; and ρ is a probability measure on B such that

$$\int_B e^{i\phi(x)}\, d\rho\,(x) = \exp\left\{-\frac{1}{2}\,|\phi|^2_{H^*}\right\}$$

for all $\phi \in B^* \subset H^*$. In this the infinite-dimensional case, the Gaussian measure ρ does not exist on H, but must live on some larger space B; in particular, $\rho\,(H) = 0$.

As in the linear case, the Euclidean group $\mathcal{E}\,(H)$ is the set of transformations of H of the form $x \to Rx + h$, with $h \in H$ and R an invertible, isometric linear transformation on H. We want $\mathcal{E}\,(H)$ to act on $L^2\,(B, \rho)$ as in the finite-dimensional case. This requires some care, since now elements of $\mathcal{E}\,(H)$ are transformations of H, but our functions are defined on B.

First, there is a dense subspace of H, denoted B^*, such that for all $h \in B^*$, $\langle h, \cdot\rangle$ extends continuously from H to B. Given $h \in H$, choose $h_n \in X$ with $h_n \to h$. It is well known that $\langle h_n, \cdot\rangle$ converges in $L^2\,(B, \rho)$ to a limit $\langle h, \cdot\rangle \in L^2(H, \rho)$, independent of the choice of approximating sequence. It is easily shown that the map $h \to \langle h, \cdot\rangle$ is linear from H to $L^2\,(B, \rho)$, and that for all $h, k \in H$, $\langle h, x + k\rangle = \langle h, x\rangle + \langle h, k\rangle$ for ρ–a.e. x. The expression $\langle h, x + k\rangle$ is well defined for ρ – a.e. x because of the Cameron–Martin theorem, which asserts that

$$d\rho(x + h) = e^{-\langle h,x\rangle - |h|^2/2} d\rho(x) \qquad \forall\, h \in H. \tag{5}$$

Now we define an element f of $L^2\,(B, \rho)$ to be a **cylinder function** if there is a finite orthonormal set $\{h_1, \cdots, h_n\}$ in H and a measurable function ϕ on $\mathbb{R}^n$ such that

$$f\,(x) = \phi\,(\langle h_1, x\rangle\,, \cdots, \langle h_n, x\rangle)\,. \tag{6}$$

Suppose that R is an isometric invertible linear transformation of H and that f is a cylinder function given by (6). Then we *define* $f \circ R^{-1}$ to be the cylinder function given by

$$f \circ R^{-1}\,(x) = \phi\,(\langle Rh_1, x\rangle\,, \cdots, \langle Rh_n, x\rangle)\,. \tag{7}$$

Although a given cylinder function f can be represented in form (6) in many different ways, it is not hard to see that $f \circ R^{-1}$ is independent of the representation. It is straightforward to verify that the map $f \to f \circ R^{-1}$ is isometric on $L^2\,(B, \rho)$ and that $\left(f \circ R^{-1}\right) \circ R = f$. Hence this map extends to a unitary map of $L^2\,(B, \rho)$ to itself, which we still denote as $f \to f \circ R^{-1}$. If R^{-1} happens to have a continuous extension from H to B, then for all $f \in L^2\,(B, \rho)\,,\ f \circ R^{-1}$ will coincide with the composition of f with this extension.

Finally, by the Cameron–Martin theorem, the map $T_h : L^2\,(B, \rho) \to L^2\,(B, \rho)$ given by

$$T_h f\,(x) = e^{-|h|^2/4} e^{\langle h,x\rangle/2} f\,(x - h)$$

is unitary for all $h \in H$.

So we make the following definition.

Definition 8. *The **Gaussian regular representation** of $\mathcal{E}\,(H)$ on $L^2\,(B, \mu)$ is given by*

$$U\,(R, h)\, f = T_h\left(f \circ R^{-1}\right),$$

that is,

$$U(R,h)\,f(x) = e^{-|h|^2/4} e^{\langle h,x\rangle/2} f \circ R^{-1}(x-h).$$

We turn now to the infinite-dimensional version of the Segal–Bargmann transform. While this may be defined in terms of a complex abstract Wiener space as in [**DH, HS**], we follow [**S3**] and consider a Hilbert space of holomorphic functions on $H_{\mathbb{C}}$, where we use the standard notion of holomorphicity on a Banach space [**HP**].

Definition 9. *A function $F : H_{\mathbb{C}} \to \mathbb{C}$ is* ***holomorphic*** *if F is locally bounded and the restriction of F to every finite-dimensional subspace of $H_{\mathbb{C}}$ is holomorphic.*

Definition 10. *The* ***Segal-Bargmann space*** *over $H_{\mathbb{C}}$, denoted $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$, is the set of holomorphic functions F on $H_{\mathbb{C}}$ such that*

$$\sup_X \int_X |F(z)|^2 \, d\mu_X(z) < \infty,$$

where the supremum is over finite-dimensional subspaces X of $H_{\mathbb{C}}$, and where μ_X is the Gaussian probability measure on X given by $d\mu_X(z) = \pi^{-\dim X} e^{-|z|^2}\, dz$. For $F \in \mathcal{H}L^2(H_{\mathbb{C}},\mu)$ we define $\|F\|$ to be the square root of the above supremum.

It is a consequence of the Krée Skeleton Theorem [**K**] that this defines a Hilbertian norm on $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ and that $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ is complete in this norm. Thus $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ is a complex Hilbert space. We remark that if $F : H_{\mathbb{C}} \to \mathbb{C}$ is holomorphic on each finite-dimensional subspace and the supremum in the above definition is finite, then F is automatically locally bounded. (See (11) below.)

Note that $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ is an abuse of notation, in that there does *not* exist a measure μ on $H_{\mathbb{C}}$ such that $\|F\|$ is the L^2 norm of F with respect to μ. We may regard μ either as a cylinder-set measure on $H_{\mathbb{C}}$ or as a honest measure on a suitable extension $B_{\mathbb{C}}$ of $H_{\mathbb{C}}$–see [**G**].

Since our holomorphic functions are truly functions on $H_{\mathbb{C}}$ (not on some Banach space containing $H_{\mathbb{C}}$), there is no trouble in extending the definition of the holomorphic regular representation to infinite dimensions, and in verifying that it is unitary.

Definition 11. *The* ***holomorphic regular representation*** *of $\mathcal{E}(H)$ is the unitary representation V of $\mathcal{E}(H)$ on $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ by*

$$V(R,h)\,F(z) = e^{-|h|^2/8} e^{(h,z)/2} F\left(R^{-1}(z-h/2)\right).$$

If we interpret (z,x) as $\langle \mathrm{Re} z, x\rangle + i\,\langle \mathrm{Im} z, x\rangle$ as an ρ-almost everywhere defined function on B, then the definition of the Segal–Bargmann transform carries over without change from the finite-dimensional case.

Theorem 12 (Segal–Bargmann transform). *For all $f \in L^2(B,\rho)$, define a function Sf on $H_{\mathbb{C}}$ by*

$$Sf(z) = e^{-(z,z)/2} \int_B e^{(z,x)} f(x)\, d\rho(x).$$

Then the integral that defines Sf is absolutely convergent and depends holomorphically on z. Moreover, S is a unitary map of $L^2(B,\rho)$ onto $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$.

For a proof see [**S3**].

PROPOSITION 13. *The Segal–Bargmann transform intertwines the Gaussian regular representation and the holomorphic regular representation. That is, for* $(R,h) \in \mathcal{E}(H)$, $V(R,h) = SU(R,h)S^{-1}$.

Proof. Note that any $(R,h) \in \mathcal{E}(H)$ may be factored as $(I,h)(R,0)$. So it suffices to consider rotations and translations separately. Note that by inspection, $U(R,h) = U(I,h)U(R,0)$ and similarly for V, as must be the case, since U and V are representations.

It is easily seen that a holomorphic function on $H_{\mathbb{C}}$ is determined by its values on H. So it suffices to show that $SU(R,h)f(z) = V(R,h)Sf(z)$ for all $f \in L^2(B,\rho)$ and for all $z \in H$. We begin by computing on a cylinder function f of the form (6). For any fixed $z \in H$ we have

$$\begin{aligned} Sf(z) &= \int_B \phi(\langle h_1, z-x\rangle, \cdots, \langle h_n, z-x\rangle)\, d\rho(x) \\ &= \int_B \phi(\langle h_1, z\rangle - \langle h_1, x\rangle, \cdots, \langle h_n, z\rangle - \langle h_n, x\rangle)\, d\rho(x) \\ &= \int_{\mathbb{R}^n} \phi(\langle h_1, z\rangle - x_1, \cdots, \langle h_n, z\rangle - x_n)\, e^{-\left(x_1^2+\cdots+x_n^2\right)/2} \frac{dx_1\cdots dx_n}{(2\pi)^{n/2}} \\ &= (S\phi)(\langle h_1, z\rangle, \cdots, \langle h_n, z\rangle). \end{aligned}$$

Here $S\phi$ means the Segal–Bargmann transform of ϕ defined on the finite-dimensional Hilbert space $\mathbb{C}^n$. The third equality is because the joint distribution of $\{\langle h_i, x\rangle\}$ is the given measure on $\mathbb{R}^n$.

But now recall the definition (7) of $f \circ R^{-1}$ on cylinder functions. Since R is isometric, $\{Rh_1, \cdots, Rh_n\}$ is again an orthonormal set, and so by the previous paragraph we have

$$\begin{aligned} S\left(f \circ R^{-1}\right)(z) &= (S\phi)(\langle Rh_1, z\rangle, \cdots, \langle Rh_n, z\rangle) \\ &= (S\phi)\left(\left\langle h_1, R^{-1}z\right\rangle, \cdots, \left\langle h_n, R^{-1}z\right\rangle\right) \\ &= Sf\left(R^{-1}z\right). \end{aligned}$$

Thus $S\left(f \circ R^{-1}\right) = Sf \circ R^{-1}$ on cylinder functions, and therefore for all functions.

Meanwhile, for $z \in H_{\mathbb{C}}$ we have, using the Cameron–Martin theorem,

$$\begin{aligned} SU(I,h)f(z) &= e^{-(z,z)/2} \int_B e^{(z,x)} e^{-|h|^2/4} e^{\langle h,x\rangle/2} f(x-h)\, d\rho(x) \\ &= e^{-|h|^2/4} e^{-(z,z)/2} \int_B e^{(z,x+h)} e^{\langle h,x+h\rangle/2} f(x)\, e^{-\langle h,x\rangle - |h|^2/2} d\rho(x) \\ &= e^{-|h|^2/4} e^{-(z,z)/2} e^{(z,h)} \int_B e^{(z-h/2,x)} f(x)\, d\rho(x) \\ &= e^{-|h|^2/4} e^{-(z,z)/2} e^{(z,h)} e^{(z-h/2,z-h/2)/2} Sf(z-h/2) \\ &= e^{-|h|^2/8} e^{(h,z)/2} Sf(z-h/2) = V(I,h)Sf(z). \end{aligned}$$

Combining the results of this paragraph and the previous one gives the theorem. □

3. The gauge-fixed subspace is trivial

We now present an abstract theorem about subgroups $\mathcal{G}$ of $\mathcal{E}(H)$. Now, $\mathcal{G}$ is a set of transformations of the form (R_g, h_g), $g \in \mathcal{G}$. An orbit of $\mathcal{G}$ is thus of the form

$\{R_g x + h_g\}$. Since $|R_g x| = |x|$, we see that if any one orbit is unbounded then all others are as well.

THEOREM 14. *Suppose $\mathcal{G}$ is a subgroup of $\mathcal{E}(H)$ with the property that one (and hence every) orbit of $\mathcal{G}$ is unbounded in H. Suppose $f \in L^2(B,\rho)$ is such that*

$$U(R,h)f = f$$

for all $(R,h) \in \mathcal{G}$. Then $f = 0$.

Proof. In light of Proposition 13 it is equivalent to assume that $F := Sf$ satisfies $V(R,h)F = F$ for all $(R,h) \in \mathcal{G}$. The hypothesis on $\mathcal{G}$ implies that there is a sequence $\{(R_n,h_n)\}$ in $\mathcal{G}$ with $|h_n| \to \infty$. Then the assumption on F says that for all $z \in H_{\mathbb{C}}$,

$$F(z) = e^{-|h_n|^2/8} e^{(h_n,z)/2} F\left((R_n,h_n/2)^{-1} z\right).$$

Fixing some $w \in H_{\mathbb{C}}$ and choosing $z = z_n := (R_n,h_n/2)\,w = R_n w + h_n/2$ we get

$$F(z_n) = e^{-|h_n|^2/8} e^{(h_n,z_n)/2} F(w).$$

Doing a little algebra, this implies that

$$\frac{|F(z_n)|^2}{e^{|z_n|^2}} = \frac{|F(w)|^2}{e^{|w|^2}}. \tag{8}$$

Now, it is well known (e.g. [**B, BSZ**]) that the pointwise evaluation map

$$F \to F(z)$$

is a continuous linear functional on $\mathcal{H}L^2(H_{\mathbb{C}},\mu)$ for all $z \in H_{\mathbb{C}}$. So there is a unique $\chi_z \in \mathcal{H}L^2(H_{\mathbb{C}},\mu)$ such that

$$F(z) = \langle \chi_z, F\rangle \tag{9}$$

for all $F \in \mathcal{H}L^2(H_{\mathbb{C}},\mu)$. (The inner product is complex linear on the right.) The χ_z's are called **coherent states** and are given explicitly by

$$\chi_z(w) = e^{(\bar{z},w)}. \tag{10}$$

As a consequence of (9) and (10) we have

$$\langle \chi_w, \chi_z\rangle = e^{(\bar{z},w)}.$$

In particular $\|\chi_z\|^2 = \exp|z|^2$, and so by (9)

$$|F(z)|^2 \le \|F\|^2 e^{|z|^2}. \tag{11}$$

We now introduce the normalized coherent states

$$\tilde{\chi}_z := \frac{\chi_z}{\|\chi_z\|}$$

and observe that

$$\begin{aligned} |\langle \tilde{\chi}_w, \tilde{\chi}_z\rangle| &= e^{\mathrm{Re}(\bar{z},w)} e^{-|w|^2/2} e^{-|z|^2/2} \\ &= e^{-|z-w|^2/2}. \end{aligned} \tag{12}$$

This shows that $\tilde{\chi}_w$ and $\tilde{\chi}_z$ are nearly orthogonal whenever the distance between w and z is large. Equation (8), re-stated in terms of coherent states, says

$$|\langle \tilde{\chi}_{z_n}, F\rangle|^2 = |\langle \tilde{\chi}_w, F\rangle|^2 = |F(w)|^2 e^{-|w|^2}. \tag{13}$$

We are assuming that $|h_n| \to \infty$ and so that $|z_n| \to \infty$. The idea is now to pass to a subsequence on which $|z_n - z_m|$ is large for all $n \neq m$. Then the $\tilde{\chi}_{z_n}$'s will be essentially orthonormal, and so (13) will tell us that

$$\|F\|^2 \gtrsim \sum_{n=1}^{\infty} |\langle \tilde{\chi}_{z_n}, F\rangle|^2 = \infty$$

unless $\langle \tilde{\chi}_w, F\rangle = F(w)\exp(-|w|^2) = 0$. Since this argument holds for all w, we see that $F \equiv 0$.

To be more precise, suppose that we are given a positive integer N. Since $\lim z_n = \infty$ we may choose $\{w_k\}_{k=1}^N \subset \{z_n\}_{n=1}^\infty$ such that $|w_n - w_m|$ is large enough that (by (12))

$$|\langle \tilde{\chi}_{w_n}, \tilde{\chi}_{w_m}\rangle - \delta_{m,n}| \leq \frac{1}{4N} \ \forall\ m, n \leq N.$$

Let $C_{n,m} := \langle \tilde{\chi}_{w_n}, \tilde{\chi}_{w_m}\rangle$ be the Gram matrix of $\{\tilde{\chi}_{w_n}\}_{n=1}^N$, $A = I - C$ and $D = C^{-1} - I$. Then A is an $N \times N$ matrix whose Hilbert–Schmidt norm, and therefore it's operator norm $\|A\|$, is bounded by $1/4$. Using the geometric series expansion for C^{-1}, we conclude that

$$\|D\| = \left\|C^{-1} - I\right\| \leq \frac{1}{4}(1 - 1/4)^{-1} = 1/3.$$

By the previous considerations and the fact that the norm-squared of the projection of F onto the span of $\{\tilde{\chi}_{w_n}\}_{n=1}^N$ is given by $\sum_{m,n=1}^N C_{n,m}^{-1}\overline{\langle \tilde{\chi}_{w_n}, F\rangle}\langle \tilde{\chi}_{w_m}, F\rangle$, we find that

$$\begin{aligned}
\|F\|^2 &\geq \sum_{m,n=1}^N C_{n,m}^{-1}\overline{\langle \tilde{\chi}_{w_n}, F\rangle}\langle \tilde{\chi}_{w_m}, F\rangle \\
&= \sum_{m=1}^N |\langle \tilde{\chi}_{w_m}, F\rangle|^2 + \sum_{m,n=1}^N D_{n,m}\overline{\langle \tilde{\chi}_{w_n}, F\rangle}\langle \tilde{\chi}_{w_m}, F\rangle \\
&\geq \sum_{m=1}^N |\langle \tilde{\chi}_{w_m}, F\rangle|^2 - \|D\| \sum_{m=1}^N |\langle \tilde{\chi}_{w_m}, F\rangle|^2 \\
&= \frac{2}{3}N|F(w)|^2 e^{-|w|^2}.
\end{aligned}$$

Since $\|F\|^2 < \infty$ and N is arbitrary in the above equation, it follows that $|F(w)| = 0$. This finishes the proof since w was arbitrary as well. ☐

Corollary 15. *If* $F \in \mathcal{H}L^2(H_{\mathbb{C}}, \mu)$ *then*

$$\lim_{|z| \to \infty} |F(z)|^2 e^{-|z|^2} = 0.$$

This result is reasonable, since in order for F to be in $\mathcal{H}L^2(H_{\mathbb{C}}, \mu)$, the integral of $|F(z)|^2 e^{-|z|^2}$ over every finite-dimensional subspace of $H_{\mathbb{C}}$ must be finite. Nevertheless, there is no obvious way to prove this directly from the definitions.

Proof. If the corollary were not true, then there would exist $\varepsilon > 0$ and a sequence $\{z_n\}$ with $|z_n| \to \infty$ such that

$$\frac{|F(z_n)|^2}{e^{|z_n|^2}} = |\langle \tilde{\chi}_{z_n}, F\rangle|^2 \geq \varepsilon,$$

and so the above argument would show that $\|F\| = \infty$. □

Remark. It would be possible to prove Theorem 14 directly in $L^2(B,\rho)$, by considering the coherent states $\psi_z \in L^2(B,\rho)$ defined as $\psi_z = S^{-1}(\chi_z)$, which may be computed explicitly as

$$\psi_z(x) = e^{(\bar{z},x)-(\bar{z},\bar{z})/2}.$$

However, it seems that the most natural "home" for the coherent states is the Segal–Bargmann space, especially in light of the natural interpretation of the χ_z's as representing the pointwise evaluation maps.

4. Comparison with the Energy Representation

4.1. Energy Representation. Recall the notation of Section 1. We now introduce a 1-parameter family ρ_t of Gaussian measures on B, characterized by the condition that

$$\int_B e^{i\phi(A)}\, d\rho_t(A) = \exp\left\{-\frac{t}{2}\,|\phi|^2_{H^*}\right\}$$

for all $\phi \in B^* \subset H^*$. Here t is an arbitrary positive number. Thus the measure ρ of the previous sections is the same as ρ_1.

We consider the "energy representation" E_t, a unitary representation of $\mathcal{G}$ acting on $L^2(B,\rho_t)$, given by the formula

$$E_t(g)\, f(A) = e^{i\langle A, dg\, g^{-1}\rangle} f\left(\mathrm{Ad}_{g^{-1}}(A)\right) \tag{14}$$

as introduced in [**I**] in the case $G = SU(2)$ and in [**GGV1**] in general. (Although the right hand side of (14) appears to be independent of t, the Hilbert space in which f lives depends on t.) Note that for g smooth, the adjoint action of $\mathcal{G}$ on H extends continuously to B. Equation (14) is the form of the energy representation used in [**GGV1**, Sect. 5] and in [**Wa**]. A unitarily equivalent form is used in some of the other references. Note also that parameter t could be absorbed into the density ω in (1), or into the choice of inner product on $\mathfrak{g}$.

The main theorem of this section is the following.

Theorem 16. *The representations W and $E_{1/4}$ are unitarily equivalent.*

The proof of this theorem is an easy consequence of Proposition 18 below with $r = 1$. The intertwining operator between W and $E_{1/4}$ is a variant of the Fourier transform called the Fourier–Wiener transform, which we now introduce. The details of the proof are given at the end of the paper.

4.2. The Fourier–Wiener Transform. The Fourier–Wiener transform was introduced by Cameron and Martin [**C, CM1, CM2**]–see especially Theorem 1 of [**CM2**]. See also [**BSZ**]. The Fourier–Wiener transform is given by essentially the same formula as the Segal–Bargmann transform–see [**GM**, Sect. 4]. In fact, the unitarity of the Fourier–Wiener transform can be viewed as a limiting case of the unitarity of the Segal–Bargmann transform [**H**, Sect. 2].

For the sake of motivation, first assume that $\dim(H) < \infty$. For any $r > 0$ and $f \in L^2(H,\lambda)$ let

$$\mathcal{F}_r f(x) = \left(\frac{r}{2\pi}\right)^{d/2} \int_H e^{-ir\langle x,y\rangle} f(y)\, d\lambda(y) \tag{15}$$

where the integral is (as usual in the theory of the Fourier transform) to be interpreted as an improper Lebesgue integral. Since $r^{d/2}d\lambda$ is Lebesgue measure on H

relative to the inner product $r\langle\cdot,\cdot\rangle$, it follows $\mathcal{F}_r : L^2(H, r^{d/2}\lambda) \to L^2(H, r^{d/2}\lambda)$ is unitary for all $r > 0$. This easily implies that $\mathcal{F}_r : L^2(H,\lambda) \to L^2(H,\lambda)$ is unitary for all $r > 0$. We now define the Fourier–Wiener transform $\mathcal{W}_r$ so that the following diagram commutes

$$\begin{array}{ccccc} & L^2(H,\lambda) & \xrightarrow{\mathcal{F}_r} & L^2(H,\lambda) & \\ \Phi_1 & \downarrow & & \downarrow & \Phi_{1/4r^2} \\ & L^2(H,\rho) & \xrightarrow[\mathcal{W}_r]{} & L^2(H,\rho_{1/4r^2}) & \end{array}$$

where

$$\Phi_t f(x) = \left(\frac{d\rho_t(x)}{dx}\right)^{-1/2} f(x) = (2\pi t)^{d/4}\, e^{|x|^2/4t} f(x).$$

A simple computation shows that

$$\mathcal{W}_r f(x) = e^{r^2|x|^2} \int_H e^{-ir\sqrt{2}\langle x,y\rangle} f(\sqrt{2}y)d\rho(y). \tag{16}$$

The choice of the target space $L^2(H,\rho_{1/4r^2})$ is necessary for the passage to infinite-dimensional H. Specifically, the target space is chosen so that the following property will hold. Suppose $H = H_1 \oplus H_2$ is an orthogonal direct sum decomposition, and let P_1 be the projection onto the first factor. Then our definition of $\mathcal{W}_r$ guarantees that

$$\mathcal{W}_r^H \left[f \circ P_1\right] = \left[\mathcal{W}_r^{H_1} f\right] \circ P_1, \tag{17}$$

where $\mathcal{W}_r^H$ is computed on H and $\mathcal{W}_r^{H_1}$ is computed on H_1. This consistency condition is necessary in the definition of $\mathcal{W}_r$ when $\dim H = \infty$, and is easily verified from (16) once one checks that $\mathcal{W}_r$ of the constant function $\mathbf{1}$ is $\mathbf{1}$.

With this as background we are now in a position to define the Fourier–Wiener transform in general.

Definition 17 (Fourier–Wiener Transform). *The Fourier–Wiener transform is the unique unitary map $\mathcal{W}_r : L^2(B,\rho) \to L^2(B,\rho_{1/4r^2})$ defined on cylinder functions as follows. If $f \in L^2(B,\rho)$ is of the form*

$$f(x) = \phi\left(\langle h_1, x\rangle, \cdots, \langle h_n, x\rangle\right) \tag{18}$$

for some finite orthonormal set $\{h_1, \cdots, h_n\}$ in H, then we set

$$\mathcal{W}_r f(x) = \left(\mathcal{W}_r^{\mathbb{R}^n}\phi\right)\left(\langle h_1, x\rangle, \cdots, \langle h_n, x\rangle\right), \tag{19}$$

where $\mathcal{W}_r^{\mathbb{R}^n}$ is computed on $\mathbb{R}^n$.

Note that in the formula for $\mathcal{W}_r f$, $\langle h_i, x\rangle$ is to be computed by a limiting procedure in $L^2(B,\rho_{1/4r^2})$ and so technically does not mean the same thing as in the formula for f, where it is computed in $L^2(B,\rho)$. Although a given cylinder function f can be represented as in (18) in many different ways, it follows from (17) that $\mathcal{W}_r f$ is independent of the representation. Also note that $\mathcal{W}_r$ is an isometric transform of cylinder functions in $L^2(B,\rho)$ onto cylinder functions in $L^2(B,\rho_{1/4r^2})$ and that $\mathcal{W}_r$ is formally given by (16).

Proposition 18. *For $f \in L^2(H,\rho)$ and $(R,h) \in \mathcal{E}(H)$,*

$$\mathcal{W}_r U(R,h) f(x) = \exp(-ir\langle h, x\rangle)\left(\mathcal{W}_r f\right) \circ R^{-1}(x). \tag{20}$$

Proof. If f is a cylinder function, then using (19) we reason precisely as in the proof of Proposition 13 to show that $\mathcal{W}_r\left(f\circ R^{-1}\right)=\mathcal{W}_r f\circ R^{-1}$. By continuity, this holds for all $f\in L^2\left(B,\rho\right)$.

To consider the effect of a translation $U\left(I,h\right)=T_h$, we consider a cylinder function f as in (18). By enlarging the set $\{h_1,\cdots,h_n\}$ if necessary we may write $h=\sum_{i=1}^n a_i h_i$ with $a_i=\langle h,h_i\rangle\in\mathbb{R}$. Let $a=(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n$, then

$$T_h f\left(x\right)=e^{-|h|^2/4}e^{\langle h,x\rangle/2}f\left(x-h\right)=\psi(\left\langle h_1,x\right\rangle,\cdots,\left\langle h_n,x\right\rangle),$$

where

$$\psi(u):=e^{-|h|^2/4}e^{a\cdot u/2}\phi\left(u-a\right)$$

for $u\in\mathbb{R}^n$. Using the definition of $\mathcal{W}_r^{\mathbb{R}^n}$, the change of variables $u\to u+a/\sqrt{2}$ and the fact that $a\cdot a=|h|^2$, we find that

$$\begin{aligned}
\left(\mathcal{W}_r^{\mathbb{R}^n}\psi\right)(v) &= e^{r^2|v|^2}\int_H e^{-ir\sqrt{2}v\cdot u}e^{-|h|^2/4}e^{a\cdot\sqrt{2}u/2}\phi\left(\sqrt{2}u-a\right)d\rho(u)\\
&= e^{r^2|v|^2}\int_H e^{-ir\sqrt{2}v\cdot(u+a/\sqrt{2})}e^{-|h|^2/4}e^{a\cdot\sqrt{2}(u+a/\sqrt{2})/2}\times\\
&\qquad \phi(\sqrt{2}u)e^{-u\cdot a/\sqrt{2}-|a|^2/4}d\rho(u)\\
&= e^{r^2|v|^2}\int_H e^{-ir\sqrt{2}v\cdot(u+a/\sqrt{2})}\phi(\sqrt{2}u)d\rho(u)\\
&= e^{-irv\cdot a}\left(\mathcal{W}_r^{\mathbb{R}^n}\phi\right)(v),
\end{aligned}$$

where ρ is now the standard normal distribution on $\mathbb{R}^n$. Since $\sum_{i=1}^n\langle h_i,x\rangle a_i=\langle h,x\rangle$, evaluating this last expression at $v=(\left\langle h_1,x\right\rangle,\ldots,\left\langle h_n,x\right\rangle)$ implies that

$$\mathcal{W}_r T_h f\left(x\right)=\left(\mathcal{W}_r^{\mathbb{R}^n}\psi\right)(\left\langle h_1,x\right\rangle,\ldots,\left\langle h_n,x\right\rangle)=e^{-ir\langle h,x\rangle}\mathcal{W}_r f\left(x\right).$$

□

Proof of Theorem 16. As noted prior to Theorem 1, $W(g)=U(Ad_g,-dg\,g^{-1})$. Hence by Proposition 18 and the definition of $E_{1/4}$ in (14),

$$\begin{aligned}
\mathcal{W}_1 W(g)f(A) &= \mathcal{W}_1 U(Ad_g,-dg\,g^{-1})f(A)\\
&= \exp(i\langle -dg\,g^{-1},A\rangle)\left(\mathcal{W}_1 f\right)\circ Ad_g^{-1}\left(A\right)\\
&= E_{1/4}(g)\mathcal{W}_1 f(A).
\end{aligned}$$

Note that in order to make the formulas come out right we need to use $\mathcal{W}_r$ with $r=1$, and that $\mathcal{W}_1$ maps the Hilbert space on which $W(g)$ is defined to Hilbert space on which $E_{1/4}$ is defined – see Definition 17.

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Department of Mathematics-0112, University of California at San Diego, La Jolla, CA 92093-0112 USA
E-mail address: `driver@math.ucsd.edu`

University of Notre Dame, Department of Mathematics, Notre Dame, IN 46556 USA
E-mail address: `bhall@nd.edu`