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On the Growth of Waves on Manifolds

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0. INTRODUCTION

If a steady current is initiated in a wire loop in $\mathbb{R}^3$ at $t = 0$, then after a short time, depending on the distance from the wire, there results an electromagnetic field that is subsequently independent of time.

The result can be very different if $\mathbb{R}^3$ is replaced by a 3-manifold V^3 having non-trivial homology in dimension 2, i.e., if there are closed surfaces in V^3 that do not bound 3-dimensional regions. An inward flux of current through a *bounding* cycle will, of course, lead to a build-up of charge inside and thus a growth of the electric field. As shown in [F, 1], a time-constant current flux through a *non-bounding* cycle must also lead to an electric field growth, even though there is no region "inside" in which charge can accumulate! One would not be surprised with unbounded fields if the source were oscillating at a "resonant frequency," but this source is essentially constant in time. The electromagnetic field has a vector potential A which is always assumed here to be in the "Lorentz gauge", i.e., "$d^*A = 0$". This 1-form satisfies a wave equation with the current serving as a source. In the present paper we show, using generalized Maxwell equations, that *this growth behavior is typical of a whole class of wave equations on manifolds of any dimension and driven by "closed" sources that need not be time independent* (Theorem 2). *Conversely, when the manifold V^n is compact, a closed source with a Heaviside (step function) time dependence and vanishing flux through each cycle will lead to fields that must remain bounded for all time* (see Theorem 1 for a more general result).

While the growth of the waves in the case of non-zero flux can be deduced directly from Stokes' theorem, the generalized Maxwell equations give added information.

Some applications to electromagnetism and linearized shallow water, sound, and elastic waves are considered, and we would like to make some comments about them at this time. We consider the main results of this paper, Theorems 1 and 2, to be an investigation in global analysis dealing

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with the long-time behavior of solutions of linear wave equations on manifolds, when the sources are not oscillatory. The applications presented give illustrations of these results, but their settings are sometimes exotic. Our hope is that their inclusion here might stimulate some readers to develop variants of our result and more realistic applications. For example, since the growth of electric flux through a wormhole depends only on Maxwell's equations, this makes sense in general relativity (see [F, 1]), but Theorem 1 does not apply to a spatial V^3 in relativity since the metric of V^n in Theorem 1 does not vary with time. We feel that extensions of our theorems to manifolds with boundary, e.g., bounded domains in $\mathbb{R}^3$, might lead to more compelling applications, especially with regard to electromagnetic and water waves. None of our remarks about gauge fields, after the proof of Theorem 1, apply to Yang-Mills fields since these fields do not have external sources and satisfy non-linear equations. Our discussion of shallow water waves is restricted in that we have assumed that rotational effects are negligible. We hope to return to some of these questions in the future.

1. Waves on a Compact Manifold V^n

Let V^n be an n-dimensional Riemannian manifold without boundary. In this section, if ω^p is a p-tensor, and $x \in V$, we let $\|\omega(x)\|^2 = \sum \omega_{i \dots k}(x)\, \omega^{i \cdots k}(x)$. This is essentially the sum of the squares of the components in an orthonormal coordinate system.

Theorem 1. *Let V^n be a compact Riemannian manifold without boundary. Let $\mathbf{J}^p(x)$ be a smooth p-form on V^n and let $A^p(x, t)$ be a solution to the driven wave equation (with $\Box = \partial^2/\partial t^2 - \nabla^2$)*

$$\Box \mathbf{A}^p(x, t) = \mathbf{J}^p(x)\, H(t)$$

$$\mathbf{A}^p(x, t) = 0 \qquad \text{for} \quad t < 0,$$

where H is the Heaviside step function. Then the norms $\|\mathbf{A}(x, t)\|$ and $\|\partial \mathbf{A}(x, t)/\partial t\|$ are bounded in time iff the harmonic part of $\mathbf{J}^p$ vanishes, i.e.,

$$(\mathbf{J}^p, h^p) = \int_V \mathbf{J}^p \wedge *h^p = 0$$

for all harmonic forms h^p. More generally, this same result holds when ∇^2 is replaced by any linear, non-positive, self-adjoint elliptic differential operator L (of any order) and where harmonic forms are replaced by the kernel of L.

For the proof of this theorem it is necessary to introduce a number of Sobolev norms. If ω is a p-tensor, we define the L^2 norm of ω by

$$\|\omega\| = \left[\int_V \|\omega(x)\|^2 \, dx \right]^{1/2}$$

where dx is the volume measure on V. More generally, if s is a non-negative integer, we define the s-Sobolev norm of ω by

$$\|\omega\|_s = \|\nabla \cdots \nabla \omega\|,$$

i.e., the norm of the sth covariant derivative of ω. The Sobolev space consisting of the set of p-tensors with s-derivatives in L^2 will be denoted by H_s. Although we have defined H_s only for integer s, one may, in fact, define these spaces for any real s. For more on the facts needed about Sobolev spaces, the reader may consult, e.g., Gilkey [Gi].

Proof of Theorem 1. We shall give the proof in the case of the Laplacian, but the proof carries over to the general L.

The wave operator can be written as $\square = \partial^2/\partial t^2 - \nabla^2$, where $\nabla^2 := -(\mathbf{dd^*} + \mathbf{d^*d})$ is the usual Laplace operator for forms on V^n. It is known in this case of compact V^n (see [W], p. 256) that ∇^2 has an orthonormal family of eigenforms, $\{\alpha_k^p\}$, $k = 0, 1, \ldots$

$$\nabla^2 \alpha_k = -\lambda_k \alpha_k, \qquad 0 \leqslant (\lambda_0) < \lambda_1 < \cdots$$

(where the eigenvalue $\lambda_0 = 0$ occurs only if there are nontrivial harmonic p-forms on V) and, furthermore, the system of eigenforms is uniformly complete. The eigenvalues may be degenerate. Expand $\mathbf{A}^p = \sum a_k(t) \alpha_k$ and $\mathbf{J}^p = \sum j_k \alpha_k$, where $a_k(t)$ and j_k are r-tuples and α_k is an r-tuple of eigenforms if λ_k has multiplicity r. Substituting into $\square \mathbf{A}(r, \mathbf{x}) = \mathbf{J}(\mathbf{x}) \, h(t)$ we get the system of differential equations

$$\ddot{a}_k(t) + \lambda_k a_k(t) = j_k H(t) \tag{1}$$

with $a_k(t) = 0$ for $t < 0$. For $k > 0$ (i.e., $\lambda_k > 0$) we get (since $a(t) = 0$ for $t < 0$) either a trivial solution (if the Fourier coefficient $j_k = 0$) or an oscillatory solution

$$a_k(t) = [j_k H(t)/\lambda_k][1 - \cos(t \lambda_k^{1/2})]. \tag{2}$$

A problem can arise if $\lambda_0 = 0$, i.e., if there is a non-trivial *harmonic* form α_0. The constants j_0 arise as Hilbert space scalar products $(\alpha_0, \mathbf{J}^p)$ and if $\mathbf{J}^p$ has no harmonic part, $j_0 = 0$. We then get the trivial solution $a_0(t) = 0$. If, however, $\mathbf{J}^p$ has a non-trivial harmonic part, then some $j_0 \neq 0$. We then get a solution

$$a_0(t) = j_0 H(t)\, t^2/2 \tag{3}$$

that grows quadratically in time. Note also that the "electric field"

$$\mathbf{E} = (-1)^p\, \partial \mathbf{A}/\partial t$$

would grow linearly in time. We conclude that if $\mathbf{J}$ has no harmonic part, then

$$\mathbf{A}(x, t) = \sum_{k>0} a_k(t) \alpha_k, \tag{4}$$

where the a's are the oscillatory functions given in (2) and

$$\partial \mathbf{A}/\partial t = \sum_{k>0} (j_k/\lambda_k^{1/2}) \sin(t\lambda_k^{1/2}) \alpha_k. \tag{5}$$

On the other hand, if $\mathbf{J}$ does have a non-trivial harmonic part, $\mathbf{A}(x, t)$ is as above together with a new contribution

$$j_0 \alpha_0 H(t)\, t^2/2.$$

We shall be finished if we can show that (4) and (5) remain bounded for all time. We prove that (5) remains bounded; the proof for (4) is similar.

Let $t \geqslant 0$ be fixed and set

$$E := \sum_{k>0} (\lambda_k)^{-1/2} \sin(t\lambda_k^{1/2})(\mathbf{J}, \alpha_k) \alpha_k, \tag{6}$$

where again the α's are orthonormal eigenforms of the Laplacian

$$\Delta \alpha_k = (dd^* + d^*d)\, \alpha_k = \lambda_k \alpha_k$$

on a compact Riemannian V^n. Let $\sigma(\Delta)$ be the set of eigenvalues and let $\lambda_1 > 0$ be the smallest non-zero eigenvalue. Set

$$\begin{aligned} S(\lambda; t) &= (\lambda)^{-1/2} \sin(t\lambda^{1/2}) \qquad \text{if} \quad \lambda \geqslant \lambda_1 \\ &= 0 \qquad \text{if} \quad \lambda < \lambda_1. \end{aligned} \tag{7}$$

Equation (6) can then be written as an operator equation

$$E = S(\Delta; t)\mathbf{J}. \tag{8}$$

It is well known, and easy to check, that if $f: \sigma(\Delta) \to \mathbb{R}$ is a bounded function, then the operator defined by

$$f(\Delta)\, u := \sum_{k \geq 0} (u, \alpha_k)\, f(\lambda_k) \alpha_k$$

maps L^2 into itself with the operator norm of $f(\Delta)$ being given by $\sup_{\lambda \in \sigma(\Delta)} |f(\lambda)|$. We then have

$$\begin{aligned} \|\Delta^{s/2} E\| &= \|\Delta^{s/2} S(\Delta; t) \mathbf{J}\| = \|S(\Delta; t)\, \Delta^{s/2} \mathbf{J}\| \\ &= \|S(\Delta; t)\, \Delta^{1/2} \Delta^{(s-1)/2} \mathbf{J}\| \leq \sup_{\lambda} |S(\lambda; t)\, \lambda^{1/2}|\ \|\Delta^{(s-1)/2} \mathbf{J}\| \\ &\leq \|\Delta^{(s-1)/2} \mathbf{J}\|. \end{aligned}$$

Also,

$$\|E\| = \|S(\Delta; t) \mathbf{J}\| \leq \sup_{\lambda \in [\sigma(\Delta) - 0]} |\lambda^{-1/2} \sin \lambda^{1/2} t|\ \|\mathbf{J}\| \leq (\lambda_1)^{-1/2}\ \|\mathbf{J}\|$$

and thus

$$\|\Delta^{s/2} E\| + \|E\| \leq (\lambda_1)^{-1/2}\ \|\mathbf{J}\| + \|\Delta^{(s-1)/2} \mathbf{J}\|. \tag{9}$$

Recall now the elliptic estimates

$$\|f\|_s \leq c_1 \{ \|f\|_0 + \|\Delta^{s/2} f\|_0 \} \tag{e_1}$$

$$\|f\|_0 + \|\Delta^{s/2} f\|_0 \leq c_2\ \|f\|_s. \tag{e_2}$$

We also have the Sobolev embedding theorem:

If $f \in H_s$ and if $s > (n/2) + k$, where n is the dimension of the compact manifold V, then $f \in C^k$ and

$$\|f\|_{\infty, k} \leq C_3\ \|f\|_s,$$

where $\|f\|_{\infty, k}$ is the sup norm of derivatives of f of order $\leq k$.

By the elliptic estimates and (9) we have

$$\begin{aligned} \|E\|_s &\leq c\{ \|E\| + \|\Delta^{s/2} E\| \} \leq c\{ (\lambda_1)^{-1/2}\ \|\mathbf{J}\| + \|\Delta^{(s-1)/2} \mathbf{J}\| \} \\ &\leq c(\lambda_1) \{ \|\mathbf{J}\| + \|\Delta^{(s-1)/2} \mathbf{J}\| \}, \end{aligned}$$

where $c(\lambda_1)$ now depends on the lowest eigenvalue of Δ. From (e_2) we now have

$$\|E\|_s \leq c(s, \Delta, V)\ \|\mathbf{J}\|_{s-1}. \tag{10}$$

From the Sobolev embedding we then get that if $\mathbf{J} \in H_{s-1}$ with $s > (n/2) + k$, then $E \in C^k$ and

$$\|E\|_{\infty, k} \leqslant c \, \|\mathbf{J}\|_{s-1}. \tag{11}$$

In particular, if $\mathbf{J} \in C^{s-1}$ with $s > (n/2) + k$, then $E \in C^k$ and

$$\|E\|_{\infty, k} \leqslant c \, \|\mathbf{J}\|_{\infty, s-1}. \quad \blacksquare$$

We wish to make two remarks about Theorem 1.

First, it is not surprising that $\mathbf{A}$ grows when $\mathbf{J}$ has a harmonic part. If, e.g., $\mathbf{J}$ itself is harmonic, then $\mathbf{J}$ is a solution to the homogeneous wave equation and we expect a "resonance."

Second, our conclusion on the time derivative $\partial \mathbf{A}/\partial t$ is of special importance for the following reason. It may be that $\mathbf{A}$ is merely the spatial part of a gauge field $A^p = \phi^{p-1} \wedge dt + \mathbf{A}^p$ on a classical space-time $M^{n+1} = V^n \times \mathbb{R}$. In this case $\mathbf{A}$ itself is of no physical significance, but $\partial \mathbf{A}/\partial t$ is an ingredient (as we shall see) of the "field strength" $F = dA$ that *is* of significance. For example, in the electromagnetic case in which the charge density vanishes, one may put $\phi = 0$ and then $-\partial \mathbf{A}/\partial t$ is the electric field.

2. The Wave Equation for Gauge Fields on a General V^n

We now turn our attention to the case when the space V^n is not compact. The use of harmonic forms and projections is perhaps then problematical. In the compact case, the statement that $\mathbf{J}^p$ has a trivial harmonic part is equivalent (at least when $\mathbf{J}$ is co-closed) to the statement that the "flux of $\ast\mathbf{J}$" through each $(n-p)$-cycle W vanishes. This restatement allows us to give a version of Theorem 1 in the non-compact case. We state our results for a gauge-type field, but or course we recover the usual non-gauge situation by putting $\phi = 0$.

In a space-time, the exterior differentials d and $\mathbf{d}$, in $M = V \times \mathbb{R}$ and V, respectively, are related by $d = dt \wedge \partial/\partial t + \mathbf{d}$. We also let $*$ and $\ast$ be the Hodge duality operators in M and V. Then $\square = dd^* + d^*d$.

Observation. *Let* $M^{n+1} = V^n \times \mathbb{R}$, *with product metric* $ds^2 = ds_V^2 - dt^2$. *Let*

$$A(x, t) = \phi^{p-1}(x, t) \wedge dt + \mathbf{A}^p(x, t)$$

vanish for $t < 0$ *and let*

$$S^{n+1-p} = \boldsymbol{\sigma}^{n+1-p}(x, t) - \mathbf{j}^{n-p}(x, t) \wedge dt$$

be closed, $dS = 0$. Assume that A satisfies the wave equation

$$\Box A = *S. \tag{12}$$

Then $d^*A = 0$ and the field strength $F := dA$ satisfies the "generalized Maxwell equations"

$$dF = 0, \qquad \text{and} \qquad d^*F = *S. \tag{13}$$

Proof. From (12) we have $\Box d^*A = d^*\Box A = d^**S = \pm *dS = 0$. Thus d^*A satisfies the homogeneous wave equation. Since $A = 0$ for $t < 0$, we conclude that $d^*A = 0$ for all t. Then $d^*F = d^*dA = \Box A = *S$.

We remark that Günther [G], has used the wave equation to study solutions of the generalized Maxwell equations. We are more concerned with studying the wave equation via Maxwell's equations.

THEOREM 2. *Let $\Box A^p = *S^{n+1-p}$, where $dS = 0$. Assume that $d^*A = 0$ (e.g., it is enough, from Observation 1, to assume that $A = 0$ for $t < 0$). Then if W^{n-p} is a transversally oriented cycle on V^n (i.e., a cycle of "even kind" in the sense of de Rham)*

$$d/dt \int_W *F = \pm \int_W i_{\partial/\partial t} S. \tag{14}$$

In particular, if a "current flux is maintained through W", i.e., if

$$\int_0^t dt \int_W i_{\partial/\partial t} S \to \infty \qquad as \quad t \to \infty,$$

then

$$\int_W *F \to \pm\infty \qquad as \quad t \to \infty.$$

Proof. As we shall see, the proof (as in [F, 1]) is an immediate application of the generalized law of Ampere-Maxwell.

As in electromagnetism, we may write out the genealized Maxwell equations in their usual spatial form (see e.g., [F, 2]). We start out by expressing $F^{p+1} = \mathbf{E}^p \wedge dt + \mathbf{B}^{p+1}$ in terms of the "electric" and "magnetic" form fields. From $A^p = \phi^{p-1} \wedge dt + \mathbf{A}^p$ and $F = (dt \wedge \partial/\partial t + \mathbf{d})A$ one gets

$$\mathbf{B}^{p+1} = \mathbf{dA}^p \qquad \text{and} \qquad \mathbf{E}^p = \mathbf{d}\phi^{p-1} + (-1)^p \,\partial \mathbf{A}^p/\partial t.$$

There will be a number of sign differences from the usual expressions in electromagnetism, where $n = 3$ and $p = 1$; e.g., $(-1)^p$ replaces the usual (-1)

in the above expression for **E**. These sign differences play no role in our problem. From $dF=0$ we get

$$\mathbf{dB}=0 \text{ (absence of magnetic monopoles)}$$

and

$$\mathbf{dE}=(-1)^p\, \partial\mathbf{B}/\partial t \text{ (Faraday's law)}.$$

If we define the spatial duals

$$\mathbf{D}^{n-p}=\star\mathbf{E}^p \quad \text{and} \quad \mathbf{H}^{n-p-1}=\star\mathbf{B}^{p+1},$$

then one gets

$$*F^{p+1}=(-1)^n\,\mathbf{H}^{n-p-1}\wedge dt+(-1)^{p+1}\,\mathbf{D}^{n-p}.$$

For calculations such as these it is helpful to note the relation between the spatial and space-time duals

$$*\mathbf{B}^q=(-1)^n\,(\star\mathbf{B}^q)\wedge dt$$
$$*(\mathbf{E}^p\wedge dt)=(-1)^{p+1}\,\star\mathbf{E}.$$

The second set of Maxwell equations $d{*}F^{p+1}={*}S^{n+1-p}={*}(\boldsymbol{\sigma}-\mathbf{j}^{n-p}\wedge dt)$ yields

$$\mathbf{dD}=(-1)^{np-1}\,\boldsymbol{\sigma} \qquad \text{(Gauss' law)}$$

and

$$\mathbf{dH}=(-1)^{(n+1)(p+1)}\,\mathbf{j}+\partial\mathbf{D}/\partial t \qquad \text{(Ampere-Maxwell)}.$$

From Ampere-Maxwell we get, since $\partial W=0$,

$$0=\int_{\partial W}\mathbf{H}=\int_W \mathbf{dH}=\pm\int_W \mathbf{j}+\int_W \partial\mathbf{D}/\partial t$$

or

$$d/dt\int_W \mathbf{D}=\pm\int_W \mathbf{j}. \tag{15}$$

The proof is then completed on noting that $\int_W *F=\pm\int_W \mathbf{D}$ (since W is space-like, i.e., $t=0$ on W) and $\mathbf{j}=(-1)^{n-p+1}\,i_{(\partial/\partial t)}S$.

Theorem 2 gives a geometric form of "resonance" condition that is applicable even in non-compact manifolds V^n. In Theorem 1, when this resonance condition was *not* satisfied, we were assured that the fields would remain bounded when the source $\mathbf{J}^p$ (which corresponds to $*\mathbf{j}^{n-p}$) has step function time dependence. We suspect that this behavior holds also in the non-compact case:

Conjecture. If the current has a Heaviside time dependence, compact support in V^n, and zero flux through each cycle W^{n-p}, then $\mathbf{A}$ and $\partial \mathbf{A}/\partial t$ remain bounded for all t.

3. Energy and the Electromagnetic Analogy

We remark that the growth of the waves, as described in Theorem 2, can be deduced from Stokes' theorem without introducing the electromagnetic analogy. We hope, however, that this analogy, together with over a century and a half of electromagnetic investigations by physicists, engineers, and mathematicians, might lead to increased insight into the behavior of waves of all sorts. For a first example, note that from the wave equation (12),

$$\begin{aligned}\Box(\phi^{p-1} \wedge dt + \mathbf{A}^p) &= *S^{n+1-p}\\ &= (-1)^n\,[*\sigma^{n+1-p}(x,t)] \wedge dt + (-1)^{n-p}\,[*\mathbf{j}^{n-p}(x,t)].\end{aligned}$$

We conclude

$$\begin{aligned}\Box\phi^{p-1} &= (-1)^n *\sigma^{n+1-p}\\ \Box\mathbf{A}^p &= (-1)^{n-p}\,[*\mathbf{j}^{n-p}(x,t)]\end{aligned} \tag{16}$$

and

$$\Box\mathbf{B}^{p+1} = \mathbf{d}\Box\mathbf{A}^p = \mathbf{d}(-1)^{n-p}\,[*\mathbf{j}^{n-p}(x,t)].$$

Thus the "magnetic" field $\mathbf{B}^{p+1}$ itself satisfies a wave equation, just as in electromagnatism and *the source on the right hand side is harmonically trivial.* Thus

THEOREM 3. *If M^n is compact and if $\mathbf{j}$ has Heaviside time dependence, then $\mathbf{B}^{p+1}$ and $\partial\mathbf{B}^{p+1}/\partial t$ are bounded fields for $t>0$, independent of the growth of $\mathbf{E}^p$.*

As with Poynting in electromagnetism, we may compute, for any compact region U on the spatial manifold V^n,

$$d/dt\left[\tfrac{1}{2}\int_U \mathbf{E}\wedge *\mathbf{E} + \mathbf{B}\wedge *\mathbf{B}\right] = (-1)^p \int_{\partial U} \mathbf{E}\wedge\mathbf{H} + (-1)^n \int_U \mathbf{j}\wedge\mathbf{E}. \tag{17}$$

We may then interpret the non-negative integrand on the left-hand side, $1/2[\mathbf{E} \wedge *\mathbf{E} + \mathbf{B} \wedge *\mathbf{B}]$ as the *energy* form. If $(-1)^{p-1} \mathbf{E} \wedge \mathbf{H}$ is the Poynting *energy flux* or momentum form, then

$$d\mathscr{E}/dt = (-1)^n \int_U \mathbf{j} \wedge \mathbf{E}$$

represents the rate at which energy is supplied to the field by the current. Consider the general wave equation $\Box A^p = *S^{n+1-p}$ envisioned in Theorem 2, but on a *compact* V. Each of the fields ϕ and $\mathbf{A}$ satisfies wave equations (16). We may apply Theorem 1 to both fields, each with its Heaviside time dependence. We have seen in Theorem 1 that a Heaviside current $\mathbf{j}^{n-p}$ *with no harmonic part*, in a compact V^n, will lead to a potential $\mathbf{A}^p$ which, together with its time derivative, is bounded for all t. From Gauss' law, $\boldsymbol{\sigma}$ is essentially $\mathbf{dD}$, and so $*\boldsymbol{\sigma}$ has no harmonic part; consequently ϕ and $\partial\phi/\partial t$ are also bounded. Thus, since $\mathbf{E}^p = \mathbf{d}\phi \pm \partial\mathbf{A}^p/\partial t$, we have, for $t > 0$, $d\mathscr{E}/dt = \pm\int_V \mathbf{j} \wedge (\mathbf{d}\phi \pm \partial\mathbf{A}^p/\partial t)$ which is of the form $\pm\int_V \mathbf{dj} \wedge \phi \pm d/dt \int_V \mathbf{j} \wedge \mathbf{A}$, since $\mathbf{j}$ is independent of $t > 0$. Furthermore, "conservation of charge," i.e., $dS = 0$, has the familiar form $\partial\boldsymbol{\sigma}/\partial t \pm \mathbf{dj} = 0$, which, since $\boldsymbol{\sigma}$ is independent of time, gives $\mathbf{dj} = 0$. We conclude that $d\mathscr{E}/dt = \pm d/dt \int_V \mathbf{j} \wedge \mathbf{A}$ for all $t > 0$. Since $\mathbf{j} \wedge \mathbf{A}$ is bounded for $t > 0$, we may conclude the following.

THEOREM 4. *Let A^p satisfy the wave equation $\Box A = *S$ on the compact V^n. If $S^{n+1-p} = \boldsymbol{\sigma}^{n+1-p}(x, t) - \mathbf{j}^{n-p}(x, t) \wedge dt$ is closed with Heaviside time dependence and if* $\mathbf{j}$ *has zero flux through each cycle W^{n-p}, then only a finite amount of work $\int_V \mathbf{j} \wedge \mathbf{A}$ is needed to maintain the constant current for all time $t > 0$!*

Consider a wire loop in a V^3 carrying a current. If the loop bounds as a real 1-cycle, e.g., if V^3 is simply connected, there will be no current flux through any 2-cycle. Thus

COROLLARY. *Let a perfectly conducting wire loop C bound as a real 1-cycle in a closed V^3. Then only a finite amount of energy expenditure is required to maintain a constant current in C for all $t > 0$.*

We get a result similar to Theorem 4 for the solutions of the generalized wave equation considered in Theorem 1, $\partial^2\mathbf{A}/\partial t^2 = L\mathbf{A} + \mathbf{j}$, where L is elliptic, self-adjoint, and non-positive. One defines the *energy* of a spatial p-form solution $\mathbf{A}$ by

$$\mathscr{E} := 1/2[(\partial\mathbf{A}/\partial t, \partial\mathbf{A}/\partial t) - (L\mathbf{A}, \mathbf{A})] \tag{18}$$

since $\mathscr{E} \geqslant 0$ and

$$d\mathscr{E}/dt = (\mathbf{j}, \partial \mathbf{A}/\partial t)$$

vanishes when there is no source.

The electromagnetic analogy may be used also in the following way. Theorem 2 tells us that a non-trivial current flux (with Heaviside time dependence) through a cycle will lead to an electric flux through this cycle that grows linearly with time. Suppose we consider now a current $\mathbf{j}^{n-p}(x, t) = \mathbf{j}^{n-p}(x)\,\delta(t)$ that has a Dirac $\delta(t)$ time dependence instead of a Heaviside dependence. From Ampere-Maxwell (15) we see

THEOREM 5. *If* $\mathbf{j}^{n-p}(x, t) = \mathbf{j}^{n-p}(x)\,\delta(t)$, *then for any transversally oriented* $(n-p)$ *cycle* W,

$$\int_W \mathbf{D}^{n-p}(t) = \pm \int_W \mathbf{j}^{n-p}(x)$$

is constant for $t > 0$.

Thus a $\delta(t)$ current through W yields an "electric" field that is *trapped by the topology*, i.e., the cycle W. For example, in electromagnetism, if W^2 is the throat of a wormhole, i.e., a closed surface that does not bound, then by passing a charged particle through W we change the flux of $\mathbf{D}$ through W and this flux remains constant until some other charge passes through. We have "charged" the wormhole!

4. Linearized Shallow Water, Sound, and Elastic Waves

Consider a "planet" whose entire surface (which may be a surface of arbitrary genus) is covered with a thin ocean of water. Measure the height ζ of the ocean surface, the (variable) depth h of the ocean floor, and the height z of a typical water particle all from the *surface* M^2 *of the undisturbed ocean*, assumed undisturbed for $t < 0$. *We shall assume that the effects of rotation are negligible.* The *linearized* version of the shallow water equations yields a "wave equation" for ζ

$$\partial^2\zeta/\partial t^2 - \operatorname{div}[(hg)\nabla\zeta] = \operatorname{div}[h\nabla(p_0) - hf]. \tag{19}$$

Here div and ∇ are the *surface* divergence and gradient operators associated with the surface M^2, g is the acceleration due to gravity at points of M^2 (assumed to be a slowly varying function of the local coordinates (x, y) on M^2), p_0 is the atmospheric pressure at the ocean surface (a function of x and y), and f is an external horizontal force (assumed independent of z). This equation can be derived in a manner similar to Lamb's derivation in [L, section 189 and 198] for a spherical Earth. This equation is used in discussing the tides; we shall not discuss this because time dependence of the tidal forces f plays a key role. We envision, rather, waves caused by a sudden localized change in atmospheric pressure p_0 and by other external forces f. It is believed that such atmospheric disturbances have in the past been the cause of catastrophic waves in lakes (see [S, p. 423]). As mentioned above, we are concerned, as in Noah's era, with a sea that completely covers the planet's surface. In such a situation of total immersion, we have $gh > 0$, the operator $\zeta \mapsto L(\zeta) := \operatorname{div}[gh\nabla\zeta]$ is self-adjoint, elliptic and non-positive, with kernel consisting of constant functions, and thus the generalized version of Theorem 1 applies. Since $\operatorname{div}[h\nabla(p_0) - hf]$ is Hilbert space orthogonal to the kernel of L, we conclude

THEOREM 6. *If f and the atmospheric pressure p_0 have smooth spatial distributions with Heaviside time dependence, then the resulting water waves will be of bounded height for all time.*

Note that the linearized equation for the propagation of *sound waves* in a 3-manifold is of the form

$$\partial^2 s/\partial t^2 - c^2\nabla^2 s = -\operatorname{div} \mathbf{f}$$

(see [L], p. 502). This again has a harmonically trivial source. We conclude that the "condensation" s in a compact V^3 with Heaviside source is bounded for $t > 0$.

The linearized equations of *elasticity* for a 3-dimensional body of constant density ρ are (see [M, H] p. 238)

$$\rho\ddot{\mathbf{u}} = \operatorname{Div}(c \cdot \nabla\mathbf{u}) + \mathbf{b} \tag{20}$$

i.e.,

$$\rho\ddot{u}^r = (c^{rijk}u_{j|k})_{|i} + b^r.$$

Here $\mathbf{u}$ is the displacement vector, c is the elasticity tensor, $\mathbf{b}$ is the external force field, and $_|$ denotes covariant differentiation. These equations make sense in any Riemannian manifold. As they stand, these equations are not in the form of a standard vector wave equation, and our previous results do not apply directly.

Assume that V^3 is a compact Riemannian manifold of constant sectional curvature $K = \pm 1$ or 0; thus *V is locally isotropic.* It makes sense then to consider V as made of an isotropic elastic substance. This means that the elastic constants c are of the form

$$c^{rijk} = \mu(g^{rj}g^{ik} + g^{rk}g^{ij}) + \lambda g^{ri}g^{jk},$$

where $\mu > 0$ and λ are the "Lamé constants." If then one assumes, as usual, that the "modulus of compression" $(3\lambda + 2\mu)$ is positive, then the operator $\mathbf{u} \to K\mathbf{u} := \operatorname{Div}(c \cdot \nabla \mathbf{u})$ is a self-adjoint, non-positive, elliptic operator whose kernel consists of Killing vector fields, i.e., infinitesimal isometries ([M, H, pp. 240, 321). (This is to be compared with the Laplace operator on 1-forms, whose kernel consists of the harmonic forms.)

The generalized version of Theorem 1 will apply to this situation, with globally defined Killing fields taking the place of harmonic ones. We conclude

THEOREM 7. *If V^3 is compact with constant curvature, and if the external force 1-form* **b** *has Heaviside time dependence and is Hilbert space orthogonal to the space of Killing fields, then the displacement* **u** *remains bounded for all time. In particular, this is always so if V^3 has negative curvature, since there are then no global Killing fields (Bochner's theorem). On the other hand, if* **b** *has a non-trivial component along the Killing fields, then* **u** *will be of the form of a bounded function plus a function growing quadratically with time.*

In local coordinates Eq. (20) becomes

$$\rho \ddot{u}_r = \mu(\nabla^2 u)_r + (\lambda + \mu)\, u^k_{\,|kr} + 2\mu u^s R_{sr} + b_r, \tag{21}$$

where $R_{sr} = 2Kg_{sr}$ is the Ricci tensor of the locally isotropic space V^3.

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