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Simplicity of Solutions of $x'(t) = bx(t-1)$

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It is well known that the solution of $x'(t) = bx(t-1)$ for $t \geq 0$ with $x(t) = 1$ on $[-1, 0]$ (or other initial function) can be approximated for "large t" by a linear combination of one or two exponentials $e^{\lambda t}$, where $\lambda = be^{-\lambda}$. (In case $b = -1/e$ the solution is approximated by cte^{-t}.) This paper gives estimates for the error in that approximation when $-5\pi/2 \leq b \leq 3\pi/2$. The smallness of the error even for modest values of $t > 0$ may be surprising. © 1991 Academic Press, Inc.

The linear delay differential equation

$$x'(t) = bx(t-1) \qquad \text{for} \quad t \geq 0, \tag{1}$$

where $b \neq 0$ is real, with initial function

$$x(t) = 1 \qquad \text{for} \quad -1 \leq t \leq 0 \tag{2}$$

is particularly easy to solve—at least in principle. On the interval $[0, 1]$ one integrates a constant function and obtains the first-degree polynomial $x(t) = 1 + bt$. Then on $[1, 2]$ one integrates a first-degree polynomial to get the quadratic $x(t) = 1 + bt + b^2(t-1)^2/2$. In general one gets the nth degree polynomial

$$x(t) = \sum_{k=0}^{n} \frac{b^k(t-k+1)^k}{k!} \qquad \text{on } (n-1, n] \text{ for } n = 1, 2, ... \tag{3}$$

The result of this step-by-step integration on $[0, 3]$ for the case $b = 1$ is displayed in Fig. 1.

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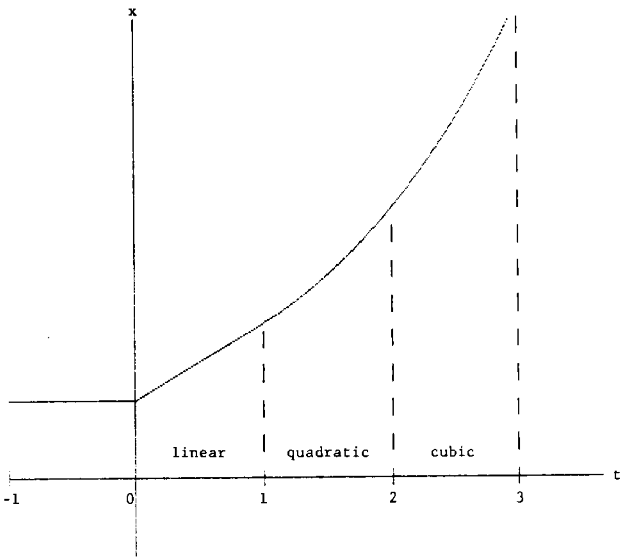

FIGURE 1

The exact solution is obtained using elementary calculus. But the result, Eq. (3), is awkward even when t is fairly small, and it does not appear to offer any insight into the nature of $x(t)$ for large t.

Other approaches to Eq. (1) involve the Laplace transform [1, 7, 9, 10, 13], semigroup theory [5, 12], and for the case $b > 0$, an ingenious advanced-calculus argument of deBruijn [2]. Using any of these, one gets a representation of the solution of (1) with a fairly general given initial function

$$x(t) = \varphi(t) \qquad \text{for} \quad t \in [-1, 0]$$

in terms of exponentials $e^{\lambda t}$, where λ satisfies the characteristic equation

$$D(\lambda) \equiv \lambda e^{\lambda} = b. \tag{4}$$

In fact, using either Laplace-transform or functional-analysis methods, one can show that asymptotically the solution is simple. It is well approximated "for sufficiently large t" by a linear combination of exponential solution(s)—often just one or two such exponentials—involving the root (or roots) of Eq. (4) with greatest real part(s).

But, what is "sufficiently large t?"

For example, if $b = 1$ the root of Eq. (4) with greatest real part is $\lambda = 0.567$. And a Laplace-transform argument, a functional-analysis

argument, deBruijn's argument [2], or the argument in [4] shows that the solution of Eqs. (1) and (2) is well approximated by the asymptotic solution

$$x_a(t) \equiv 1.13e^{0.567t}$$

"for sufficiently large t."

But if one wants to know what the solution really looks like, such a statement is virtually useless. How far must one use the cumbersome Eq. (3) before $x_a(t)$ becomes useable? And then with what error?

A simple exercise on a personal computer suggests an unexpected answer. For the case $b = 1$, Fig. 2 displays a graph of the solution of Eqs. (1) and (2) plus a graph of the function x_a defined above.

At $t = 0$ the two graphs differ by only 0.13; and for $t \geqslant 0.5$ they appear virtually indistinguishable.

Of course, the computer actually solved a discretized approximation to Eq. (1). So our purpose is to justify Fig. 2 analytically, and to determine whether such fast convergence of the solution to a simple asymptotic form should be expected for other values of b.

The proofs are based on the Laplace-transform solution of (1) and (2) as obtained in various standard references. But before one can even write down this solution one needs information about the roots of Eq. (4). In particular, one would like to know that the roots are countable.

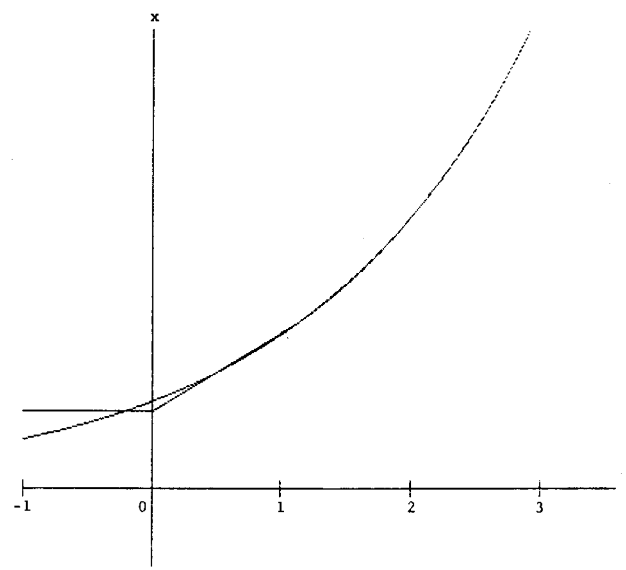

FIGURE 2

Equation (4) has been studied by many authors. For example see Lemeray [8], Pontrjagin [11], Hayes [6], Wright [13], or deBruijn [3]. Lemmas 1 and 2 below review some known properties of solutions of (4).

LEMMA 1 (The Real Roots of Eq. (4)). *All roots of Eq.* (4), *whether real or not, are simple roots except for a double root* $\lambda_0 = -1$ *when* $b = -1/e$.

(A) *If* $b > 0$ *there is a unique real root* λ_0, *and* $\lambda_0 > 0$. *Moreover,* $\lambda_0 < \ln b$ *when* $b > e$.

(B) *If* $-1/e < b < 0$, *there are exactly two real roots,* λ_0 *and* Λ_0, *and they satisfy* $\lambda_0 < \ln |b| < -1 \leqslant \Lambda_0 < 0$. *If* $b = -1/e$, *there is a unique real root,* $\lambda_0 = -1$, *and it is a double root.*

(C) *If* $b < -1/e$, *there are no real roots.*

Proof. These assertions follow from consideration of $D(0) = 0$, $D(\ln |b|) = |b| \ln |b|$, $D(-1) = -1/e$, $D(\infty) = \infty$, and $D(-\infty) = 0$ together with observations on the sign of $D'(\lambda) = (1 + \lambda) e^{\lambda}$. ∎

Since any non-real roots of Eq. (4) must occur in complex conjugate pairs, consider now roots of the form

$$\mu \pm i\omega \text{ where } \mu \text{ and } \omega \text{ are real with } \omega > 0.$$

LEMMA 2 (Imaginary Parts of the Complex Roots of Eq. (4)). *Equation* (4) *has countably many nonreal solutions. These occur in complex conjugate pairs* $\lambda_n = \mu_n + i\omega_n$ *and* $\bar{\lambda}_n = \mu_n - i\omega_n$ *for the values of n specified below.*

(A) *If* $b > 0$, *then* $n = 1, 2, \ldots$, *and* $(2n-1)\pi < \omega_n < 2n\pi$.

(B) *If* $-1/e \leqslant b < 0$, *then* $n = 1, 2, \ldots$, *and* $2n\pi < \omega_n < (2n+1)\pi$.

(C) *If* $b < -1/e$, *then* $n = 0, 1, 2, \ldots$, *and* $2n\pi < \omega_n < (2n+1)\pi$.

Proof. Write $\lambda = \mu + i\omega$, where μ and ω are real. Then taking real and imaginary parts of Eq. (4), written $\lambda = be^{-\lambda}$, yields

$$\mu e^{\mu} = b \cos \omega \tag{5}$$

and

$$\omega e^{\mu} = -b \sin \omega. \tag{6}$$

Considering $\omega > 0$, solve (6) for μ and substitute into (5) to get an equation for ω:

$$\Delta(\omega) \equiv -\frac{\omega \cos \omega}{\sin \omega} + \ln\left(\frac{\omega}{|\sin \omega|}\right) = \ln |b|. \tag{7}$$

Note that

$$\Delta'(\omega) = \frac{\omega^2 \sin^2 \omega + (\omega \cos \omega - \sin \omega)^2}{\omega \sin^2 \omega} > \omega > 0. \tag{8}$$

(A) Let $b > 0$. Then by (6), since $\omega > 0, \sin \omega < 0$. Hence $(2n-1)\pi < \omega < 2n\pi$ for some integer $n \geqslant 1$. And, since $\Delta'(\omega) > 0$, there is at most one such solution for each n. It will follow that Eq. (7) has exactly one solution ω_n in $(2n\pi - \pi, 2n\pi)$ for each $n = 1, 2, \ldots$ if

$$\lim_{\omega \to (2n-1)\pi+} \Delta(\omega) < \ln |b| \quad \text{and} \quad \lim_{\omega \to 2n\pi-} \Delta(\omega) > \ln |b|.$$

To show the first of these, let $(2n-1)\pi < \omega < (2n-1)\pi + \pi/3$. Then $\cos \omega < -\frac{1}{2}$ and $\sin \omega < 0$. So

$$\Delta(\omega) < \frac{\omega}{2 \sin \omega} + \ln\left(\frac{-\omega}{\sin \omega}\right)$$
$$< \frac{\omega}{2 \sin \omega} + 2\sqrt{\frac{-\omega}{\sin \omega}} - 2 \to -\infty \quad \text{as} \quad \omega \to (2n-1)\pi+.$$

For the limit at $2n\pi$ consider $2n\pi - \pi/3 < \omega < 2n\pi$. Then $\cos \omega > \frac{1}{2}$ and

$$\Delta(\omega) > \frac{-\omega}{2 \sin \omega} \to \infty \quad \text{as} \quad \omega \to 2n\pi-.$$

(B) and (C) Now let $b < 0$. Then Eq. (6) and the fact that $\omega > 0$ imply $\sin \omega > 0$, and hence $2n\pi < \omega < (2n+1)\pi$ for some integer $n \geqslant 0$. Since $\Delta'(\omega) > 0$, Eq. (7) has at most one solution in $(2n\pi, 2n\pi + \pi)$ for each $n = 0, 1, 2, \ldots$.

For $n \geqslant 1$, arguments analogous to those of case (A) show that

$$\lim_{\omega \to 2n\pi+} \Delta(\omega) = -\infty \quad \text{and} \quad \lim_{\omega \to (2n+1)\pi-} \Delta(\omega) = \infty.$$

For $n = 0$,

$$\lim_{\omega \to 0+} \Delta(\omega) = -1 \quad \text{and} \quad \lim_{\omega \to \pi-} \Delta(\omega) = \infty.$$

From these limits we draw the following conclusions.

(B) Let $-1/e \leqslant b < 0$. Then $\Delta(\omega) > \ln |b|$ on $(0, \pi)$, while Eq. (7) has exactly one solution in $(2n\pi, 2n\pi + \pi)$ for each $n = 1, 2, \ldots$.

(C) Let $b < -1/e$. Then Eq. (7) has exactly one solution in $(2n\pi, 2n\pi + \pi)$ for each $n = 0, 1, 2, \ldots$. ∎

The notation introduced in Lemmas 1 and 2 for the real and complex roots of Eq. (4) is used henceforth without further comment. In addition, when real λ_0 exists write $\mu_0 \equiv \lambda_0$ and $\omega_0 \equiv 0$.

LEMMA 3 (Ordering the Real Parts of the Roots of Eq. (4)). *Let* $b \neq 0$. *Then for* $n = 0, 1, 2, \ldots$, $\mu_n < \ln|b|$, $\mu_n \leqslant \ln|b| - \ln\omega_n$, *and* $\mu_{n+1} < \mu_n$. *More specifically,*

$$\mu_{n+1} \leqslant \mu_n - \frac{\omega_{n+1}^2 - \omega_n^2}{2(\omega_{n+1}^2 + \mu_{n+1}^2 + \mu_{n+1})}. \tag{9}$$

Proof. If $\mu_0 = \lambda_0$ and $\omega_0 = 0$, then $\mu_0 < \ln|b|$ by Lemma 1, and $\mu_0 \leqslant \ln|b| - \ln\omega_0$ is trivial. If $\omega_n > 0$, then Eq. (6) gives

$$\mu_n = \ln\left(\frac{-b\sin\omega_n}{\omega_n}\right),$$

which yields $\mu_n < \ln|b|$ and $\mu_n \leqslant \ln|b| - \ln\omega_n$.

From Eq. (4) itself, $\mu = \mu_n$ and $\omega = \omega_n$ must satisfy

$$\omega^2 = f(\mu) \qquad \text{where} \quad f(\mu) \equiv b^2 e^{-2\mu} - \mu^2. \tag{10}$$

(Note that this also holds for $\mu = \lambda_0$ and $\omega = 0$ when λ_0 is real.)

If we can show that $f'(\mu) < 0$ then, since $\omega_{n+1}^2 > \omega_n^2$, it will follow that $\mu_{n+1} < \mu_n$ for $n = 0, 1, 2, \ldots$.

Now

$$f'(\mu) = -2b^2 e^{-2\mu} - 2\mu \qquad \text{and} \qquad f''(\mu) = 4b^2 e^{-2\mu} - 2.$$

Since we need only consider $\mu \leqslant \max_{n \geqslant 0} \mu_n \leqslant \ln|b|$, $f''(\mu) \geqslant 2$. Hence $f'(\mu) \leqslant f'(\ln|b|) = -2 - 2\ln|b|$. If $|b| \geqslant 1/e$ then $f'(\mu) \leqslant 0$. If $0 < |b| < 1/e$, then $\mu < \ln|b|$ implies $\mu < -1$, and so $f'(\mu) = -2f(\mu) - 2(\mu^2 + \mu) < -2f(\mu)$. Thus $(d/d\mu)[f(\mu)\, e^{2\mu}] < 0$. So, for $\mu \leqslant \max_{n \geqslant 0} \mu_n$

$$f(\mu)\, e^{2\mu} \geqslant f(\max \mu_n)\, e^{2\max\mu_n} \geqslant 0,$$

and hence again $f'(\mu) < 0$.

To establish inequality (9) consider $\mu_{n+1} \leqslant \mu \leqslant \mu_n$. Then

$$-2(\omega_{n+1}^2 + \mu_{n+1}^2 + \mu_{n+1}) = f'(\mu_{n+1}) \leqslant f'(\mu) \leqslant f'(\mu_n) < 0.$$

By the mean value theorem, for some $\xi \in (\mu_{n+1}, \mu_n)$,

$$\omega_{n+1}^2 - \omega_n^2 = f'(\xi)(\mu_{n+1} - \mu_n) \leqslant 2(\omega_{n+1}^2 + \mu_{n+1}^2 + \mu_{n+1})(\mu_n - \mu_{n+1}),$$

which yields inequality (9). ∎

Assume $b \neq -1/e$ and let $z_1, z_2, \ldots$ be an enumeration of all the roots of Eq. (4) ordered, say, so that $|\mathrm{Im}\, z_k|$ is a nondecreasing function of k. Then the solution of Eq. (1) with an arbitrary continuous initial function $x(t) = \varphi(t)$ on $[-1, 0]$ is found with the aid of the Laplace transform to be

$$x(t) = \sum_{k=1}^{\infty} \frac{p(z_k)}{1 + z_k} e^{z_k t} \qquad \text{for} \quad t \geqslant 0, \tag{11}$$

where

$$p(z) = \varphi(0) + b e^{-z} \int_{-1}^{0} e^{-zs} \varphi(s)\, ds.$$

See, for example, [7, 9, 13] or Theorem 4.2 of [1].

In case $b = -1/e$, Eq. (4) has a double root $\lambda_0 = -1$ and the term in (11) corresponding to $z_k = -1$ becomes

$$[2tp(-1) + \tfrac{2}{3} p(-1) + 2p'(-1)]\, e^{-t}.$$

For $\varphi(t) \equiv 1$, $p(z_k) = b/z_k$ for each k, and $p'(-1) = -b = 1/e$ when $b = -1/e$. Thus, if $b \neq -1/e$, Eq. (11) becomes

$$x(t) = \sum_{k=1}^{\infty} \frac{b}{z_k(1 + z_k)} e^{z_k t} \qquad \text{for} \quad t \geqslant 0. \tag{12}$$

If $b = -1/e$ the term in (12) involving $z_k = -1$ becomes

$$\left(\frac{2t}{e} + \frac{8}{3e}\right) e^{-t}.$$

Incidentally, the convergence of the series in (12) for all $t \geqslant 0$ will be a corollary of the estimates to follow.

Returning to the notation introduced in Lemmas 1 and 2, define $x_a(t)$ by

$$x(t) = x_a(t) + g(t) \qquad \text{where} \quad g(t) = 2 \sum_{n=1}^{\infty} \mathrm{Re}\left[\frac{b}{\lambda_n(1 + \lambda_n)} e^{\lambda_n t}\right]. \tag{13}$$

Since $\mu_{n+1} < \mu_n$ for $n = 1, 2, 3, \ldots$,

$$|g(t)| \leqslant 2\,|b| \sum_{n=1}^{\infty} \frac{1}{\omega_n^2} e^{\mu_1 t}. \tag{14}$$

Sometimes one might want the more explicit estimate for this "error"

$$|g(t)| \leqslant \frac{2\,|b|}{|\lambda_1|\, |1 + \lambda_1|} e^{\mu_1 t} + 2\,|b| \sum_{n=2}^{\infty} \frac{1}{\omega_n^2} e^{\mu_2 t}. \tag{15}$$

As shown in Lemma 3, $\mu_1 < \mu_0$ (or $\mu_1 < \lambda_0$ if λ_0 is real). So one can say that the error $g(t)$ always decays faster, or grows more slowly, than $x_a(t)$.

We shall find explicit bounds for $|g(t)|$, especially in the case $-5\pi/2 < b < 3\pi/2$.

To estimate the sums in (14) and (15), we use the identities

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8}. \tag{16}$$

LEMMA 4 (Further Estimates for μ_n). *Let* $n = 1, 2, \ldots$.

(A) *If* $b > 0$, *then*

$$-1.3 \ln \frac{(4n-1)\pi}{2b} \leqslant \mu_n \leqslant -0.95 \ln \frac{(4n-1)\pi}{2b} \quad \textit{when} \quad 0 < b \leqslant \frac{(4n-1)\pi}{2}$$

$$0.82 \ln \frac{2b}{(4n-1)\pi} \leqslant \mu_n \leqslant \ln \frac{2b}{(4n-1)\pi} \quad \textit{when} \quad b \geqslant \frac{(4n-1)\pi}{2}.$$

(B), (C) *If* $b < 0$, *then*

$$-1.15 \ln \frac{(4n+1)\pi}{2|b|} \leqslant \mu_n \leqslant -0.98 \ln \frac{(4n+1)\pi}{2|b|} \quad \textit{when} \quad -\frac{(4n+1)\pi}{2} \leqslant b < 0$$

$$0.88 \ln \frac{2|b|}{(4n+1)\pi} \leqslant \mu_n \leqslant \ln \frac{2|b|}{(4n+1)\pi} \quad \textit{when} \quad b \leqslant -\frac{(4n+1)\pi}{2}.$$

Proof. For a fixed integer $n \geqslant 1$, consider $\mu = \mu_n$ and $\omega = \omega_n$ as functions of b.

From Eqs. (5) and (6), $\mu = -\omega \cot \omega$, which yields

$$\frac{d\mu}{d\omega} = \frac{\omega - \sin \omega \cos \omega}{\sin^2 \omega}.$$

Also (6) and (7) show that $b\, d\omega/db = 1/\Delta'(\omega) > 0$. Thus

$$b \frac{d\mu}{db} = b \frac{d\mu}{d\omega} \frac{d\omega}{db} = \frac{\omega^2 - \omega \sin \omega \cos \omega}{\omega^2 - 2\omega \sin \omega \cos \omega + \sin^2 \omega}. \tag{17}$$

It would be nice to find sharp upper and lower bounds for the right-hand side of (17) considered as a function of ω on appropriate intervals. This appears difficult. However, useful estimates can be obtained by a cruder process.

Consider the dependence of the right-hand side of (17) on $\cos \omega$ alone. One finds it to be an increasing function of $\cos \omega$ when $\sin \omega > 0$ and a decreasing function of $\cos \omega$ when $\sin \omega < 0$.

(A) Let $0 < b \leqslant (2n - 1/2)\pi$. Then by Lemma 2(A) and the fact that $\Delta(2n\pi - \pi/2) \geqslant \ln b$ it follows that $(2n-1)\pi < \omega \leqslant (2n - 1/2)\pi$. Thus $\sin \omega < 0$ and $-1 < \cos \omega < 0$, and hence

$$\frac{1}{1 + (\sin^2 \omega)/\omega^2} \leqslant b \frac{d\mu}{db} < \frac{1}{1 + (\sin \omega)/\omega}.$$

Now $(\sin \omega)/\omega$ is negative, and it attains a minimum in $(2n\pi - \pi, 2n\pi - \pi/2)$ when $\tan \omega = \omega$. This occurs for some $\bar{\omega} \in (2n\pi - \pi + 1.35, 2n\pi - \pi/2)$. Thus $(\sin \omega)/\omega \geqslant -1/(2n\pi - \pi + 1.35) > -1/(\pi + 1.35)$ and

$$0.95 < b \frac{d\mu}{db} < 1.3. \tag{18}$$

Using (18) and observing from (5) and (7) that $\mu = 0$ if $b = 2n\pi - \pi/2$, it is easy to compute

$$\mu = 0 - \int_b^{2n\pi - \pi/2} (d\mu/d\beta)\, d\beta < -\int_b^{2n\pi - \pi/2} (0.95/\beta)\, d\beta = -0.95 \ln \frac{(4n-1)\pi}{2b}.$$

Similarly

$$\mu > -1.3 \ln \frac{(4n-1)\pi}{2b}.$$

Let $b \geqslant (2n - 1/2)\pi$. Then $\Delta(2n\pi - \pi/2) < \ln b$. This implies $2n\pi - \pi/2 < \omega < 2n\pi$ and hence $\sin \omega < 0$ and $0 < \cos \omega < 1$. It follows that

$$0.82 \leqslant \frac{1}{1 + 1/(2n\pi - \pi/2)} \leqslant \frac{1}{1 - (\sin \omega)/\omega}$$
$$< b \frac{d\mu}{db} < \frac{1}{1 + (\sin^2 \omega)/\omega^2} < 1. \tag{19}$$

The corresponding bounds for $\mu = \mu_n$ are again obtained by integrating $d\mu/d\beta$.

(B), (C). The cases for $b < 0$ are similar: Let $-(2n + 1/2)\pi \leqslant b < 0$. By Lemma 2(B) and (C) and the fact that $\Delta(2n\pi + \pi/2) \geqslant \ln |b|$, $2n\pi < \omega \leqslant 2n\pi + \pi/2$. Thus $\sin \omega > 0$ and $0 < \cos \omega < 1$. So

$$\frac{1}{1 + (\sin^2 \omega)/\omega^2} < b \frac{d\mu}{db} < \frac{1}{1 - (\sin \omega)/\omega}.$$

This time $(\sin\omega)/\omega$ is positive and attains a maximum in $(2n\pi, 2n\pi+\pi/2)$ when $\tan\omega=\omega$. This occurs for some $\bar\omega\in(2n\pi+1.44, 2n\pi+\pi/2)$. Thus $(\sin\omega)/\omega\leqslant 1/(2n\pi+1.44)<0.13$ and

$$0.98<b\frac{d\mu}{db}<1.15. \tag{20}$$

Since $\mu=0$ if $b=-2n\pi-\pi/2$,

$$\mu=-\int_b^{-2n\pi-\pi/2}(d\mu/d\beta)\,d\beta\in\left(-1.15\ln\frac{(4n+1)\pi}{2|b|},\ -0.98\ln\frac{(4n+1)\pi}{2|b|}\right).$$

Finally, for $b<-(2n+1/2)\pi$ one finds $(2n+1/2)\pi<\omega<(2n+1)\pi$, and hence

$$0.88<\frac{1}{1+1/(2n\pi+\pi/2)}\leqslant\frac{1}{1+(\sin\omega)/\omega}<b\frac{d\mu}{db}$$
$$<\frac{1}{1+(\sin^2\omega)/\omega^2}\leqslant 1. \tag{21}$$

The final assertion of the lemma now follows as before. ∎

Remarks. The constants in (18)–(21), which then appear as the coefficients of ln in the assertions of the lemma, are fairly good for the case $n=1$. However they can be sharpened for larger n. For example, if $n\geqslant 2$, $0<b\leqslant 2n\pi-\pi/2$ and $\mu=\mu_n$, then $(\sin\omega)/\omega>-0.1$ for $\omega\in(2n\pi-\pi, 2n\pi-\pi/2)$. Thus (18) can be replaced by

$$0.99<b\frac{d\mu}{db}<1.1.$$

Similarly, if $n\geqslant 2$ (19)–(21) can be replaced respectively by:

$$0.91<b\frac{d\mu}{db}<1,\qquad 0.99<b\frac{d\mu}{db}<1.08,\qquad\text{and}\qquad 0.93<b\frac{d\mu}{db}<1.$$

In a sense, none of the estimates in Lemma 4 or Lemma 5 below are really needed. Any root of Eq. (4) can be found to any desired accuracy via Newton's method. However the bounds presented here will yield error estimates valid over fairly large ranges of values of b in the forthcoming theorems.

LEMMA 5 (Some Bounds for λ_0 and μ_0).

(A) *Let $b>0$. Then $\lambda_0=b\max_{t\geqslant 0}(1+t)/(b+e^t)$. Thus, for example,*

$$\lambda_0\geqslant\max\left\{\frac{b}{b+1},\frac{2b}{b+e},\frac{\ln b}{1+1/e},\frac{1+\ln k}{1+k/b}\right\}\qquad\textit{for each}\quad k\geqslant 1,$$

and

$$\lambda_0\leqslant\min\left\{\ln(b+1),\ \ln\left(\frac{b+e}{2}\right)\right\}.$$

(C) *Let $b<-1/e$. Then*

$$-\ln\left(\frac{\pi}{2|b|}\right)\leqslant\mu_0\leqslant-0.5\ln\left(\frac{\pi}{2|b|}\right)\qquad\textit{if}\quad -\pi/2\leqslant b<-1/e$$

and

$$0.6\ln(2|b|/\pi)\leqslant\mu_0\leqslant\ln(2|b|/\pi)\qquad\textit{if}\quad b\leqslant-\pi/2.$$

Proof. (A) Let $b>0$. Then $h(t)\equiv b(1+t)/(b+e^t)$ is defined and differentiable for all t. Moreover $h(t)>0$ for $t\geqslant 0$, $h(\infty)=0$, and $h'(t)=b(b-te^t)/(b+e^t)^2$ is zero if and only if $t=\lambda_0$. So $h(t)$ attains its maximum when $t=\lambda_0$. Now note that $h(\lambda_0)=\lambda_0$.

The lower bounds for λ_0 offered as examples are obtained by evaluating $h(t)$ at various values of $t\geqslant 0$. Upper bounds then follow from the identity $\lambda_0=\ln(b/\lambda_0)$.

(C) Let $b<-1/e$ and consider $\omega=\omega_0$ and $\mu=\mu_0$ as functions of b. Then, as in the proof of Lemma 4, Eq. (17) holds. By Lemma 2, $0<\omega<\pi$ and so $\sin\omega-\omega\cos\omega>0$. Hence

$$b\frac{d\mu}{db}=\frac{\omega^2-\omega\sin\omega\cos\omega}{\omega^2-\omega\sin\omega\cos\omega+\sin\omega(\sin\omega-\omega\cos\omega)}<1.$$

If $-\pi/2\leqslant b<-1/e$, then $0<\omega\leqslant\pi/2$ and, as in the proof of Lemma 4,

$$b\frac{d\mu}{db}\geqslant\frac{1}{1+(\sin^2\omega)/\omega^2}\geqslant\frac{1}{2}.$$

Using the fact that $\mu=0$ when $b=-\pi/2$, the above yield

$$-\ln\left(\frac{\pi}{2|b|}\right)\leqslant\mu_0\leqslant-0.5\ln\left(\frac{\pi}{2|b|}\right).$$

If $b\leqslant-\pi/2$, then $\pi/2\leqslant\omega<\pi$. So, as in the proof of Lemma 4,

$$b\frac{d\mu}{db}\geqslant\frac{1}{1+(\sin\omega)/\omega}\geqslant\frac{1}{1+2/\pi}>0.6.$$

Hence

$$0.6\ln(2|b|/\pi)\leqslant\mu_0\leqslant\ln(2|b|/\pi).\ ∎$$

In Example 2 shall we take $b=1$. For that case, Lemmas 4 and 5 yield

$$\mu_1 \leqslant -0.95 \ln \frac{3\pi}{2b} < -1.47 \quad \text{and} \quad \lambda_0 \geqslant \frac{1+\ln 2}{1+2/b} \geqslant 0.56.$$

THEOREM A. *Let $b>0$ (and hence $\lambda_0>0$). Then for all $t \geqslant 0$*

$$x(t)=\frac{b}{\lambda_0(1+\lambda_0)} e^{\lambda_0 t}+g(t), \quad \textit{where} \quad |g(t)| \leqslant \frac{b}{4} e^{\mu_1 t}.$$

If $0<b \leqslant 3\pi/2$, then $\mu_1 \leqslant -0.95 \ln(3\pi/2b)$.
If $b \geqslant 3\pi/2$, then $0<\mu_1 \leqslant \ln(2b/3\pi)$ and

$$\mu_1 \leqslant \lambda_0-\frac{9\pi^2}{4\pi^2+\left(\ln \frac{2b}{3\pi}\right)^2+\ln \frac{2b}{3\pi}}.$$

Proof. From Lemma 2, each $\omega_n>(2n-1)\pi$. So, (14) and (16) yield the first statement of the theorem.

The estimate for μ_1 when $b \leqslant 3\pi/2$ comes from Lemma 4.

If $b \geqslant 3\pi/2$ then, from Lemma 4 and the proof of Lemma 4, $3\pi/2 \leqslant \omega_1<2\pi$ and $0<\mu_1 \leqslant \ln(2b/3\pi)$. These observations applied in Lemma 3 produce the final assertion about μ_1. ∎

EXAMPLE 1. Let $b=e$. Then $\lambda_0=1$, and for $t \geqslant 0$

$$x_a(t)=\frac{e}{2} e^t \quad \text{and} \quad |g(t)| \leqslant 0.68 e^{-0.52 t}.$$

EXAMPLE 2. Let $b=1$. Then $\lambda_0=0.567$, and for $t \geqslant 0$

$$x_a(t)=1.13 e^{0.567 t} \quad \text{and} \quad |g(t)| \leqslant 0.25 e^{-1.47 t}.$$

See Fig. 2 for the graphs of x and x_a.

Remarks. Estimates can always be improved with more effort. When $b=1$, $\omega_1>4.37$, since $\Delta(4.37)<\ln b$. (See the proof of Lemma 2.) This together with $\mu_1<-1.47$ gives $|\lambda_1|>4.61$. From a remark following Lemma 4, $\mu_2 \leqslant -0.99 \ln(7\pi/2)<-2.37$. Thus (15) gives

$$|g(t)|<0.1 e^{-1.47 t}+0.05 e^{-2.37 t} \quad \text{for} \quad t \geqslant 0.$$

This improved estimate for Example 2 comes very close to predicting the fast convergence suggested by Fig. 2.

THEOREM B. *Let $-1/e<b<0$. Then for $t \geqslant 0$*

$$x(t)=\frac{b}{\Lambda_0(1+\Lambda_0)} e^{\Lambda_0 t}+\frac{b}{\lambda_0(1+\lambda_0)} e^{\lambda_0 t}+g(t),$$

where $|g(t)| \leqslant (|b|/12) e^{\lambda_1 t}<(|b|/12) e^{-[0.98 \ln(5\pi/2|b|)] t} \leqslant (|b|/12) e^{-3t}$.

Let $b=-1/e$. Then for $t \geqslant 0$

$$x(t)=\left(\frac{2}{e} t+\frac{8}{3e}\right) e^{-t}+g(t) \quad \textit{where} \quad |g(t)|<\frac{1}{12e} e^{-3t}.$$

Proof. From (11), if $-1/e<b<0$, then for all $t \geqslant 0$

$$x(t)=\frac{b}{\Lambda_0(1+\Lambda_0)} e^{\Lambda_0 t}+\frac{b}{\lambda_0(1+\lambda_0)} e^{\lambda_0 t}+2 \sum_{n=1}^{\infty} \operatorname{Re}\left[\frac{b}{\lambda_n(1+\lambda_n)} e^{\lambda_n t}\right].$$

Since $\omega_n>2n\pi$, and invoking Eq. (16),

$$|g(t)| \leqslant 2|b| \sum_{n=1}^{\infty} \frac{1}{\omega_n^2} e^{\mu_n t}<\frac{|b|}{12} e^{\mu_1 t}.$$

Lemma 4 gives the estimate for μ_1.

If $b=-1/e$, then $x_a(t)=(2t+\frac{8}{3}) e^{-t-1}$, as given after Eq. (10), and the estimate for $|g(t)|$ is as above but with $b=-1/e$. ∎

EXAMPLE 3. Let $b=-0.36$. Then $\Lambda_0=-0.806$ and $\lambda_0=-1.223$. Thus for $t \geqslant 0$

$$x_a(t)=2.30 e^{-0.806 t}-1.32 e^{-1.223 t} \quad \text{and} \quad |g(t)|<0.03 e^{-3t}.$$

The solution x and the function x_a are displayed in Fig. 3.

Remark. When $b<0$ and $|b|$ is small, the $\exp(\Lambda_0 t)$ term may give an adequate representation of the solution. For example, if $b=-0.012$, then $\Lambda_0=-0.012$, $\lambda_0=-6.3$, and for $t \geqslant 0$,

$$x_a(t)=e^{-0.0122 t}-0.0004 e^{-6.3 t} \quad \text{and} \quad |g(t)|<0.001 e^{-6.35 t}.$$

For Case C, $b<-1/e$, we begin with an example:

EXAMPLE 4. Let $b=-\pi/2$. Then Eq. (7) yields $\omega_0=\pi/2$. From this and Eq. (6) $\mu_0=0$. From Lemma 4, $\mu_1 \leqslant -0.98 \ln 5$. Thus for $t \geqslant 0$

$$x_a(t)=\frac{4\pi}{4+\pi^2} \cos \frac{\pi t}{2}-\frac{8}{4+\pi^2} \sin \frac{\pi t}{2},$$

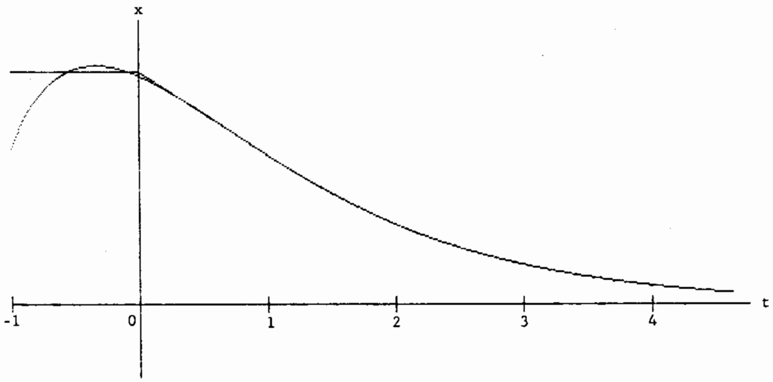

FIGURE 3

and

$$|g(t)| \leqslant \sum_{n=1}^{\infty} \frac{\pi}{\omega_n^2} e^{\mu_1 t} \leqslant \sum_{n=1}^{\infty} \frac{1/\pi}{4n^2} e^{\mu_1 t}$$

$$= \frac{\pi}{24} e^{\mu_1 t} < 0.131 e^{-1.57t}.$$

A sharper result is obtained using the fact that $\omega_1 > 7.64$ and $\mu_2 < -0.99 \ln 9$. Then

$$|g(t)| \leqslant \frac{\pi}{\omega_1^2} e^{\mu_1 t} + \sum_{n=2}^{\infty} \frac{\pi}{\omega_n^2} e^{\mu_2 t} \leqslant 0.054 e^{-1.57t} + 0.051 e^{-2.17t}.$$

The solution is displayed in Fig. 4 along with the graph of x_a.

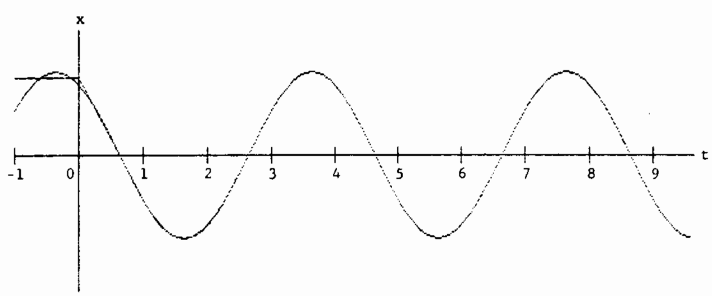

FIGURE 4

THEOREM C. *Let $b < -1/e$. Then for $t \geqslant 0$*

$$x(t) = \frac{2b}{|\lambda_0(1+\lambda_0)|^2} [(\mu_0 + \mu_0^2 - \omega_0^2) \cos \omega_0 t$$
$$+ (\omega_0 + 2\omega_0 \mu_0) \sin \omega_0 t] \, e^{\mu_0 t} + g(t),$$

where $|g(t)| \leqslant (|b|/12) e^{\mu_1 t}$.
If $-5\pi/2 \leqslant b < -1/e$, then $\mu_1 \leqslant -0.98 \ln(5\pi/2\,|b|)$.
If $b \leqslant -5\pi/2$, then $0 < \mu_1 \leqslant \ln(2\,|b|/5\pi)$ and

$$\mu_1 \leqslant \mu_0 - \frac{21\pi^2/8}{9\pi^2 + \left(\ln \frac{2|b|}{5\pi}\right)^2 + \ln \frac{2|b|}{5\pi}}.$$

Proof. For all $t \geqslant 0$,

$$x(t) = \operatorname{Re}\left(\frac{2b}{\lambda_0(1+\lambda_0)} e^{i\omega_0 t}\right) e^{\mu_0 t} + g(t),$$

where $g(t)$ is as defined in Eq. (13). When the required real part is evaluated, this is equivalent to the form stated in the theorem; and, by (14) and (16),

$$|g(t)| = 2\left|\sum_{n=1}^{\infty} \operatorname{Re}\left[\frac{b}{\lambda_n(1+\lambda_n)} e^{\lambda_n t}\right]\right|$$
$$\leqslant 2 \sum_{n=1}^{\infty} \frac{|b|}{(2n\pi)^2} e^{\mu_n t} \leqslant \frac{|b|}{12} e^{\mu_1 t}.$$

The assertions about μ_1 follow from Lemma 4. ∎

EXAMPLE 5. Let $b = -2$. Then Theorem C yields for $t \geqslant 0$

$$x_a(t) = [0.88 \cos 1.67t - 0.76 \sin 1.67t] \, e^{0.173t}, \qquad |g(t)| \leqslant \frac{1}{6} e^{-1.34t}.$$

The graphs of x and x_a are shown in Fig. 5.

Theorems A, B, and C show that if $-5\pi/2 \leqslant b \leqslant 3\pi/2$ then the solution of Eq. (1) with constant initial function is well approximated by the "zeroth order terms" of the series representation.

For other values of b the error $g(t)$ may not decay exponentially. But at least it will not grow as fast as $x_a(t)$. To get an approximation in which the error actually decays, one must add more terms in x_a. Include in x_a all terms through the $(n-1)$'st order where n is choosen such that $\mu_n < 0$.

An obvious question remains: What about a nonconstant initial function?

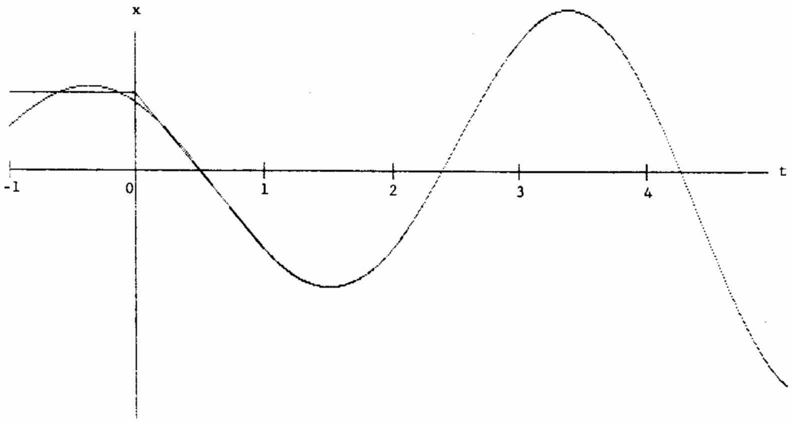

FIGURE 5

The genesis of this study was actually a computer experiment on Eq. (1) with "random" initial functions, i.e., finite sequences of random numbers produced by the computer. The results were unexpected. Figure 6 shows a typical solution for the case $b = -\pi/2$ with a "random" initial function taking values in $[-1, 1]$.

A sufficiently random initial function with average value over small intervals approximately zero leads to something close to a constant function on $[0, 1]$. (The "constant" is the value of $\varphi(0)$.) From then on we have essentially Eqs. (1) and (2) again. In fact, the more random the initial function the closer its integral will be to zero on small intervals, and the closer the solution on $[0, 1]$ will be to a constant.

Considering Eq. (1) with an arbitrary continuous (or merely integrable) initial function

$$x(t) = \varphi(t) \qquad \text{for} \quad -1 \leqslant t \leqslant 0, \tag{22}$$

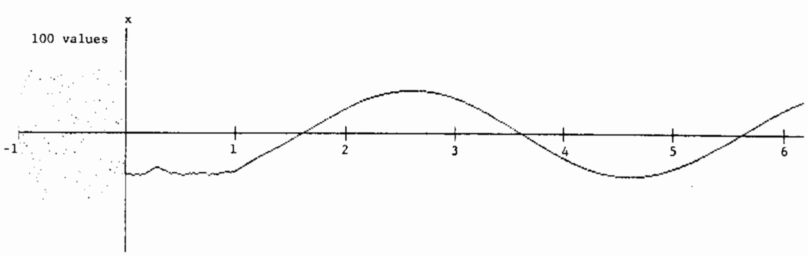

FIGURE 6

one could try to pursue the analysis of this paper using the series representation (11) for the general solution instead of the special one in (12). The estimates for μ_1 would still be valid. But one would not get such convenient general estimates for $|g(0)|$.

Of course it is possible to choose a special nontrivial initial function φ for which the leading terms $p(\lambda_0)$, $p(\Lambda_0)$, or $p(\mu_0 \pm i\omega_0)$ in (11), or in case $b = -1/e$ the terms $p(-1)$ and $p'(-1)$, vanish. Then the rest of the series representation would be anything but negligible. However, this will "practically never" happen by accident since such a φ would have to lie in subspace of codimension one or codimension two of the space of initial functions.

As further justification for studying the case of a constant initial function, we mention the variation-of-parameters approach to Eq. (1) with arbitrary initial function. Let u be the solution of Eq. (1) with initial function being the (discontinuous) unit step function

$$u(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t = 0. \end{cases}$$

Clearly $u(t) = 1$ for $0 \leqslant t \leqslant 1$. So, except for a time translation, u is the solution of Eqs. (1) and (2) again. The solution of Eq. (1) with general initial function (22) can now be represented as

$$x(t) = u(t)\, \varphi(0) + \int_{-1}^{0} bu(t-s-1)\, \varphi(s)\, ds;$$

cf. Hale [5, p. 22].

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