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Hypoelliptic heat kernel inequalities on the Heisenberg group

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Abstract

We study the existence of "L^p-type" gradient estimates for the heat kernel of the natural hypoelliptic "Laplacian" on the real three-dimensional Heisenberg Lie group. Using Malliavin calculus methods, we verify that these estimates hold in the case $p>1$. The gradient estimate for $p=2$ implies a corresponding Poincaré inequality for the heat kernel. The gradient estimate for $p=1$ is still open; if proved, this estimate would imply a logarithmic Sobolev inequality for the heat kernel.

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1. Introduction

1.1. Background

In the last 20 years or more, a fairly complete and very beautiful theory has been developed applying to elliptic operators on Riemannian manifolds. This theory relates

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properties of the solutions of elliptic and parabolic equations to properties of the Riemannian geometry. These geometric properties are determined by the principal symbol of the underlying elliptic operator. The following theorem (see for example [2]) is a typical example of the type of result we have in mind here.

Theorem 1.1. *Suppose* (M, g) *is a complete Riemannian manifold, and* ∇ *and* Δ *are the gradient and Laplace–Beltrami operators acting on* $C^\infty(M)$. *Let* $|v| := \sqrt{g(v,v)}$ *for all* $v \in TM$, Ric *denote the Ricci curvature tensor, and* k *denote a constant. Then the following are equivalent*:

(1) $\mathrm{Ric}(\nabla f, \nabla f) \geqslant -2k|\nabla f|^2$ *for all* $f \in C_c^\infty(M)$,
(2) $|\nabla e^{t\Delta/2} f| \leqslant e^{kt} e^{t\Delta/2} |\nabla f|$ *for all* $f \in C_c^\infty(M)$ *and* $t > 0$,
(3) $|\nabla e^{t\Delta/2} f|^2 \leqslant e^{2kt} e^{t\Delta/2} |\nabla f|^2$ *for all* $f \in C_c^\infty(M)$ *and* $t > 0$,
(4) *there is a function* $K(t) > 0$ *such that* $K(0) = 1$, $\dot{K}(0)$ *exists, and*

$$|\nabla e^{t\Delta/2} f|^2 \leqslant K(t) e^{t\Delta/2} |\nabla f|^2 \tag{1.1}$$

for all $f \in C_c^\infty(M)$ *and* $t > 0$.

Estimates like (1)–(4) are also equivalent to one parameter families of Poincaré and logarithmic Sobolev estimates for the heat kernel. The latter has implications for hypercontractivity of an associated semigroup; see Gross [8]. Also, in [1], Auscher, Coulhon, Duong, and Hofmann study inequalities of the form

$$|e^{t\Delta} f|^p \leqslant C e^{ct\Delta} |\nabla f|^2,$$

where C and c are positive constants, along with their relation to the Riesz transform on manifolds.

As a simple illustration of this theorem, consider the manifold $M = \mathbb{R}^3$ with vector fields

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y} \quad \text{and} \quad \partial_z = \frac{\partial}{\partial z}.$$

Let ∇ and Δ be the standard gradient and Laplacian on $\mathbb{R}^3$;

$$\nabla = (\partial_x, \partial_y, \partial_z) \quad \text{and} \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

In this case $e^{t\Delta/2}$ is convolution by the probability density

$$p_t(x) := \frac{1}{(2\pi t)^{3/2}} e^{-\frac{1}{2t}|x|^2_{\mathbb{R}^3}}$$

and

$$\nabla e^{t\Delta/2} f = e^{t\Delta/2} \nabla f \tag{1.2}$$

for all $f \in C_c^1(\mathbb{R}^3)$, as follows from basic properties of convolutions; more abstractly, this follows from the commutativity of the Euclidean gradient and Laplacian. Eq. (1.2) and an application of Hölder's inequality then imply that

$$\left|\nabla e^{t\Delta/2} f\right|^p \leqslant \left[e^{t\Delta/2} \ \nabla f|\right]^p \leqslant e^{t\Delta/2} |\nabla f|^p$$

for all $f \in C_c^1(\mathbb{R}^3)$, where $|\nabla f| := \sqrt{(\partial_x f)^2 + (\partial_y f)^2 + (\partial_z f)^2}$.

This paper is a first step toward extending Theorem 1.1 to hypoelliptic operators of the form

$$L = \sum_{i=1}^{n} X_i^2, \tag{1.3}$$

where $\{X_i\}_{i=1}^n$ is a collection of smooth vector fields on M satisfying the Hörmander bracket condition. Recall that the Hörmander bracket condition is the assumption

$$T_m M = \mathrm{span}(\{X(m) : X \in \mathcal{L}\}) \quad \forall m \in M,$$

where $\mathcal{L}$ is the Lie algebra of vector fields generated by the collection $\{X_i\}_{i=1}^n$.

By a celebrated theorem of Hörmander, L is hypoelliptic; however, the operator need not be elliptic. The principal symbol of L at $\xi \in T_m^* M$ is given by $\sigma_L(\xi) = \sum_{i=1}^n [\xi(X_i(m))]^2$. By definition, the operator L is degenerate at points $m \in M$ where there exists $0 \neq \xi \in T_m^* M$ such that $\sigma_L(\xi) = 0$. At points of degeneracy of L, the Ricci tensor is not well defined and should be interpreted to take the value $-\infty$ in some directions. Hence, it is not possible to directly generalize Theorem 1.1 in this setting. Nevertheless it is reasonable to ask if inequalities of the form (1.1) might still hold. More precisely, we let $\nabla = (X_1, \ldots, X_n)$ and address the following question: do functions $K_p(t) < \infty$ exist such that

$$|\nabla e^{tL/2} f|^p \leqslant K_p(t) e^{tL/2} |\nabla f|^p, \quad p \in [1, \infty)$$

for all $f \in C_c^\infty(M)$ and $t > 0$?

In this paper, we give an affirmative answer to this question for $p > 1$ in the model case of the Heisenberg Lie group; the case $p = 1$ remains open. Let $M = G$ be $\mathbb{R}^3$ equipped with the Heisenberg group operation given in Eq. (2.1). In this setting, we take $L = \tilde{X}^2 + \tilde{Y}^2$, where $\tilde{X}$ and $\tilde{Y}$ are the vector fields

$$\tilde{X} := \partial_x - \frac{1}{2} y \partial_z \quad \text{and} \quad \tilde{Y} := \partial_y + \frac{1}{2} x \partial_z. \tag{1.4}$$

We restrict to this simple case because the basic ideas can already be seen here without the added geometric complications appearing in more general formulations. However, much of the theory generalizes to certain classes of vector fields $\{X_i\}_{i=1}^n$ satisfying the Hörmander bracket condition on more general manifolds. These results will appear in forthcoming papers; see [17].

1.2. Statement of results

Notation 1.2. Let $C_p^\infty(G)$ denote those functions $f \in C^\infty(G)$ such that f and all its partial derivatives have at most polynomial growth.

Definition 1.3. The left-invariant *gradient* on $G = \mathbb{R}^3$ is the operator

$$\nabla = (\tilde{X}, \tilde{Y}).$$

The *subLaplacian* is

$$L = \tilde{X}^2 + \tilde{Y}^2$$

and we let $P_t = e^{tL/2}$ be the semigroup associated to L. Finally, $p_t(g) = P_t \delta_0(g) = e^{tL/2}\delta_0(g)$ denotes the fundamental solution associated to L, so that for $f \in C_p^\infty(G)$,

$$P_t f(g) = p_t * f(g) := \int_G f(gh) p_t(h)\, dh,$$

where dh denotes right Haar measure (i.e. Lebesgue measure) and gh is computed relative to the Heisenberg group multiplication in Eq. (2.1) below.

Remark 1.4. Since $\{\tilde{X}, \tilde{Y}\}$ generates the tangent space at all points of G, Hörmander's theorem [9] implies that L is a hypoelliptic operator. Also Malliavin's techniques show p_t is a smooth positive function on $\mathbb{R}^3$; see Section 3. In this simple setting, an explicit formula for $p_t(g)$ is

$$p_t(g) = \frac{1}{8\pi^2} \int_{\mathbb{R}} \frac{w}{\sinh\left(\frac{wt}{2}\right)} \exp\left(-\frac{1}{4}\, |\vec{x}|^2 w \coth\left(\frac{wt}{2}\right)\right) e^{iwz}\, dw, \tag{1.5}$$

where $g = (x, y, z) \in G$ and $\vec{x} = (x, y)$; see for example [20].

Notation 1.5. For all $p \in [1, \infty)$ and $t > 0$, let $K_p(t)$ be the best function such that

$$|\nabla P_t f|^p \leqslant K_p(t) P_t |\nabla f|^p \tag{1.6}$$

for all $f \in C_p^\infty(G)$.

Theorem 1.6. *For all $p \in (1, \infty)$, $K_p(t)$ is independent of t, and $K_p(t) = K_p < \infty$. Moreover, $K_p > 1$ for all $p \in [1, \infty]$.*

Closely related results appear in Kusuoka and Stroock [15]. In particular, Theorem 2.18 of [15] states that for all $p \in (1, \infty)$ there exist finite constants C_p such that

$$|\nabla P_t f|^p \leqslant C_p t^{-p/2} P_t |f|^p,$$

for all smooth, bounded functions f with bounded derivatives of all orders and $t > 0$.

Section 2 justifies the choice of vector fields made here, a choice which corresponds to left-invariant vector fields on $\mathbb{R}^3$ under the Heisenberg group operation. We show that the left invariance of the vector fields leaves the inequality (1.6) translation invariant. Certain scaling arguments imply that the constants K_p are also independent of the t parameter. We also show that $K_p \geqslant \sqrt{2}$ when $1 \leqslant p \leqslant 2$ and, in general, that $K_p > 1$. Note that at $t = 0$ the inequality is an empty statement and certainly holds for constant 1. So unlike the elliptic case where $K_p(t)$ is continuous at $t = 0$, there is now a jump discontinuity in $K_p(t)$ at $t = 0$. Independence of the K_p with respect to t does not generalize to all Lie groups; however, the discontinuity of $K_p(t)$ at $t = 0$ should be a feature which persists in the general hypoelliptic setting.

Section 3 briefly reviews some infinite-dimensional calculus on Wiener space necessary for the proof of Theorem 1.6. The heat kernel $p_t(g)\,dg$ is the distribution in t of the process ξ satisfying Eq. (3.1). Using this representation of p_t, we may transform our finite-dimensional problem to a problem on Wiener space, where we then may apply Malliavin's probabilistic techniques on proving hypoellipticity. The advantage of the infinite-dimensional Wiener space representation of $p_t(g)\,dg$ over that in Eq. (1.5) is that it no longer involves an oscillatory integral.

Section 4 restates Theorem 1.6 and gives its proof and the proof that this result implies the following Poincaré inequality.

Theorem 1.7. *Let K_2 be the constant in Theorem 1.6 for $p = 2$. Then*

$$P_t f^2(0) - (P_t f)^2(0) \leqslant K_2 t P_t |\nabla f|^2(0)$$

for all $f \in C_p^\infty(G)$ and $t > 0$.

Finally, Section 4.2 shows that our method can not, without modification, be used to prove $K_1 < \infty$.

2. Real three-dimensional Heisenberg Lie group

2.1. Realization of the Heisenberg Lie group

Recall that the real Heisenberg Lie algebra is $\mathfrak{g} = \mathrm{span}\{X, Y, Z\}$, where $Z = [X, Y]$ and Z is in the center of $\mathfrak{g}$. Thus, $\mathfrak{g}_0 := \mathrm{span}\{X, Y\}$ is a hypoelliptic subspace of $\mathfrak{g}$; that is, the Lie algebra generated by $\mathfrak{g}_0$ is $\mathfrak{g}$. The Heisenberg group G is the simply connected real Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$. Letting $A = aX + bY + cZ$ and $A' = a'X + b'Y + c'Z$, we have by the Baker–Campbell–Hausdorff formula that

$$e^A e^{A'} = e^{A+A'+\frac{1}{2}[A,A']}.$$

Thus, we may realize G as $\mathbb{R}^3$ with the following group multiplication:

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + \tfrac{1}{2}\,(ab' - a'b)). \tag{2.1}$$

2.2. Differential operators on G

Notation 2.1. Given an element $A \in \mathfrak{g}$, let $\tilde{A}$ denote the left-invariant vector field on G such that $\tilde{A}(0) = A$. $\hat{A}$ will denote the right-invariant vector field associated to A.

Now let $X = (1, 0, 0)$, $Y = (0, 1, 0)$, and $Z = (0, 0, 1)$ at the identity $0 \in G$. We extend these to left-invariant vector fields on G in the standard way. For $g = (a, b, c) \in G$, let L_g denote left translation by g, and compute as follows:

$$\begin{aligned}\tilde{X}(a, b, c) = L_{(a,b,c)*}X &= \frac{d}{dt}\bigg|_0 (a, b, c)(t, 0, 0) \\ &= \frac{d}{dt}\bigg|_0 \left(a + t, b, c - \frac{1}{2}\,bt\right) = \left(1, 0, -\frac{1}{2}\,b\right).\end{aligned}$$

So if (x, y, z) are the standard coordinates on $G = \mathbb{R}^3$, for $f \in C^1(G)$,

$$(\tilde{X} f)(g) = \frac{d}{dt}\bigg|_0 f\,(g \cdot tX) = \frac{\partial f}{\partial x}\,(g) - \frac{1}{2}\,y\,\frac{\partial f}{\partial z}\,(g).$$

Performing similar computations for Y and Z, we then have

$$\tilde{X} = \partial_x - \tfrac{1}{2}\,y\partial_z,\quad \tilde{Y} = \partial_y + \tfrac{1}{2}\,x\partial_z \quad \text{and} \quad [\tilde{X}, \tilde{Y}] = \tilde{Z} = \partial_z;$$

compare with Eq. (1.4). Note then that $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$ forms a basis for the tangent space at every point of G. This combined with $[\tilde{X}, \tilde{Y}] = \tilde{Z}$ implies that $\{\tilde{X}, \tilde{Y}\}$ satisfies the

Hörmander bracket condition. One may also show that the right-invariant vector fields associated to X, Y, and Z are given by

$$\hat{X} = \partial_x + \frac{1}{2} y\partial_z, \quad \hat{Y} = \partial_y - \frac{1}{2} x\partial_z \quad \text{and} \quad [\hat{X}, \hat{Y}] = \hat{Z} = -\partial_z.$$

Remark 2.2. The right-invariant vector fields associated to X and Y may be expressed in terms of the left invariant vector fields, $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$, as:

$$\hat{X} = \tilde{X} + y\tilde{Z} \quad \text{and} \quad \hat{Y} = \tilde{Y} - x\tilde{Z}.$$

We will need the following straightforward results.

Lemma 2.3. *By the left invariance of* ∇ *and* P_t, *the inequality* (1.6) *holds for all* $g \in G$, $f \in C_p^\infty(G)$, *and* $t > 0$, *if and only if,*

$$|\nabla P_t f|^p(0) \leqslant K_p(t) P_t |\nabla f|^p(0) \tag{2.2}$$

for all $f \in C_p^\infty(G)$ *and* $t > 0$.

Proof. If the inequality (2.2) holds, then

$$\begin{aligned} |\nabla P_t f|^p(g) &= |(\nabla P_t f) \circ L_g|^p(0) = |\nabla(P_t f \circ L_g)|^p(0) \\ &= |\nabla(P_t(f \circ L_g))|^p(0) \leqslant K_p(t) P_t |\nabla(f \circ L_g)|^p(0) \\ &= K_p(t) P_t |(\nabla f) \circ L_g|^p(0) = K_p(t) P_t |\nabla f|^p \circ L_g(0) \\ &= K_p(t) P_t |\nabla f|^p(g). \end{aligned}$$

The converse is trivial. $\square$

Lemma 2.4. *For* $A \in \mathfrak{g}$,

$$\tilde{A} P_t f(0) = P_t \hat{A} f(0)$$

for all $f \in C_p^\infty(G)$ *and* $t > 0$. *More generally,*

$$\hat{A} P_t f = P_t \hat{A} f,$$

from which the previous equation follows, since $\hat{A} = \tilde{A}$ *at* 0.

Proof. Heuristically, we know that $[\hat{A}, \tilde{B}] = 0$ for all $B \in \mathfrak{g}$, so that $[\hat{A}, L] = 0$, and thus $[\hat{A}, e^{tL/2}] = 0$. Rigorously

$$\begin{aligned}\tilde{A} P_t f(0) = \left.\frac{d}{d\varepsilon}\right|_0 P_t f(e^{\varepsilon A}) &= \left.\frac{d}{d\varepsilon}\right|_0 \int_G f(e^{\varepsilon A} g) p_t(g)\, dg \\ &= \int_G \left.\frac{d}{d\varepsilon}\right|_0 f(e^{\varepsilon A} g) p_t(g)\, dg \\ &= \int_G \hat{A} f(g) p_t(g)\, dg = P_t \hat{A} f(0).\end{aligned}$$

To differentiate under the integral, we have used the translation invariance of Haar measure (which is Lebesgue measure on $\mathbb{R}^3$) and the heat kernel bound

$$p_t(g) \leqslant C t^{-2} e^{-\rho^2(g)/Ct},$$

where $\rho(g) \geqslant C'(|x| + |y| + |z|^{1/2})$ is the Carnot–Carathéodory distance on G, and C and C' are some positive constants; see Theorem 5.4.3 in [19] and p. 27 of [4]. □

2.3. Dilations on G

Definition 2.5. A family of *dilations* on a Lie algebra $\mathfrak{g}$ is a family of algebra automorphisms $\{\phi_r\}_{r>0}$ on $\mathfrak{g}$ of the form $\phi_r = \exp(W \log r)$, where W is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues.

So let $r > 0$ and $g = (x, y, z)$, and define $\phi_r : G \to G$ by $\phi_r(x, y, z) = (rx, ry, r^2 z)$. Notice that

$$\begin{aligned}\phi_r((a, b, c)(x, y, z)) &= \phi_r\left((a + x, b + y, c + z + \frac{1}{2}(ay - xb)\right) \\ &= \phi_r\left((ra + rx, rb + ry, r^2 c + r^2 z + \frac{r^2}{2}(ay - xb)\right) \\ &= \phi_r(a, b, c)\phi_r(x, y, z).\end{aligned}$$

Thus ϕ_r is in fact an isomorphism of G. The generator W of ϕ_r is given by

$$\begin{aligned}W(x, y, z) &= \left.\frac{d}{dr}\right|_{r=1} \phi_r(x, y, z) = (x, y, 2z)_{(x,y,z)} \\ &= x\partial_x + y\partial_y + 2z\partial_z \\ &= x\left(\tilde{X} + \frac{1}{2} y\partial_z\right) + y\left(\tilde{Y} - \frac{1}{2} x\partial_z\right) + 2z\partial_z = x\tilde{X} + y\tilde{Y} + 2z\tilde{Z}.\end{aligned}$$

Using $e^{t\tilde{X}}(g) = g(t,0,0)$ and

$$\phi_{r*}\tilde{X} \circ \phi_r^{-1}(g) = \left.\frac{d}{dt}\right|_0 \phi_r(e^{t\tilde{X}}(\phi_r^{-1}(g))),$$

along with similar formulas involving $\tilde{Y}$, one shows

$$\phi_{r*}\tilde{X} \circ \phi_r^{-1} = r\tilde{X} \quad \text{and} \quad \phi_{r*}\tilde{Y} \circ \phi_r^{-1} = r\tilde{Y}. \tag{2.3}$$

The equations in (2.3) are equivalent to

$$\tilde{X}(f \circ \phi_r) = r(\tilde{X}f) \circ \phi_r \quad \text{and} \quad \tilde{Y}(f \circ \phi_r) = r(\tilde{Y}f) \circ \phi_r.$$

Therefore,

$$\begin{aligned} \nabla(f \circ \phi_r) &= r(\nabla f) \circ \phi_r, \\ L(f \circ \phi_r) &= r^2(Lf) \circ \phi_r \end{aligned} \tag{2.4}$$

and also, from Eq. (1.5), for $g = (x, y, z)$,

$$\begin{aligned} p_{r^2t}(g) &= \frac{1}{8\pi^2} \int_{\mathbb{R}} \frac{w}{\sinh\left(\frac{wr^2t}{2}\right)} \exp\left(-\frac{1}{4}|\vec{x}|^2 w \coth\left(\frac{wr^2t}{2}\right)\right) e^{iwz}\, dw \\ &= \frac{1}{8\pi^2} \int_{\mathbb{R}} \frac{w}{r^2 \sinh\left(\frac{wt}{2}\right)} \exp\left(-\frac{1}{4r^2}|\vec{x}|^2 w \coth\left(\frac{wt}{2}\right)\right) e^{iwz/r^2}\, r^{-2} dw \\ &= r^{-4}(p_t \circ \phi_{r^{-1}})(g) \end{aligned} \tag{2.5}$$

through the change of variables $w \mapsto r^{-2}w$. Thus,

$$\begin{aligned} P_t(f \circ \phi_r)(g) &= \int_G (f \circ \phi_r)(gh) p_t(h)\, dh = \int_G f(\phi_r(g)\phi_r(h)) p_t(h)\, dh \\ &= \int_G f(\phi_r(g)h) p_t(\phi_{r^{-1}}(h)) r^{-4}\, dh = \int_G f(\phi_r(g)h) p_{r^2t}(h)\, dh \\ &= (P_{r^2t} f \circ \phi_r)(g); \end{aligned}$$

that is,

$$P_t(f \circ \phi_r) = e^{tL/2}(f \circ \phi_r) = (e^{r^2tL/2} f) \circ \phi_r = (P_{r^2t} f) \circ \phi_r. \tag{2.6}$$

For a more general exposition on Lie groups which admit dilations, see [6]. The above remarks lead to the following proposition.

Proposition 2.6. *If K_p is the best constant such that*

$$|\nabla P_1 f|^p \leqslant K_p P_1 |\nabla f|^p$$

for all $f \in C_p^\infty(G)$, then $K_p(t) = K_p$ for all $t > 0$, where $K_p(t)$ is the function introduced in Notation 1.5.

Proof. By Eqs. (2.4) and (2.6),

$$\begin{aligned}|\nabla P_t(f \circ \phi_{r^{-1/2}})|^p &= |\nabla[(P_1 f) \circ \phi_{r^{-1/2}}]|^p = |r^{-1/2}(\nabla P_1 f) \circ \phi_{r^{-1/2}}|^p \\ &\leqslant K_p r^{-p/2}(P_1|\nabla f|^p)\circ\phi_{r^{-1/2}} = K_p r^{-p/2} P_t(|\nabla f|^p \circ \phi_{r^{-1/2}}) \\ &= K_p P_t(|\nabla f \circ \phi_{r^{-1/2}})|^p).\end{aligned}$$

Replacing f by $f \circ \phi_{r^{1/2}}$ in the above computation proves the assertion. Moreover, reversing the above argument shows that $|\nabla P_t f|^p \leqslant K_p P_t |\nabla f|^p$ implies that $|\nabla P_1 f|^p \leqslant K_p P_1 |\nabla f|^p$. $\square$

2.4. The constant $K_p > 1$

Proposition 2.7. *For $p \in [1, \infty)$, let K_p be the best constant such that*

$$|\nabla P_t f|^p \leqslant K_p P_t |\nabla f|^p \tag{2.7}$$

for all $f \in C_p^\infty(G)$ and $t > 0$. Then $K_p > 1$. In particular, $K_2 \geqslant 2$.

Proof. First consider the case $p = 2k$ for some positive integer k, and suppose the constant $K_{2k} = 1$. Then

$$|\nabla P_t f|^{2k} \leqslant P_t |\nabla f|^{2k}$$

for all $t \geqslant 0$, and $|\nabla f|^{2k} = |\nabla P_0 f|^{2k} = P_0 |\nabla f|^{2k} = |\nabla f|^{2k}$, together would imply that

$$k|\nabla f|^{2(k-1)} \nabla f \cdot \nabla L f = \left.\frac{d}{dt}\right|_0 |\nabla P_t f|^{2k} \leqslant \left.\frac{d}{dt}\right|_0 P_t |\nabla f|^{2k} = \frac{1}{2} L |\nabla f|^{2k}. \tag{2.8}$$

We now show that the function $f(x, y, z) = x + yz$ violates this inequality. Note that

$$Lf = \nabla \cdot \nabla f = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \cdot \nabla f = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \cdot \begin{pmatrix} 1 - \frac{1}{2} yy \\ z + \frac{1}{2} xy \end{pmatrix} = \frac{1}{2} x,$$

$$\nabla Lf = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}, \ \nabla f \cdot \nabla Lf = \frac{1}{2}\left(1 - \frac{1}{2}yy\right) \quad \text{and} \quad |\nabla f|^2(0) = 1.$$

Hence,

$$\left(k|\nabla f|^{2(k-1)}\nabla f \nabla Lf\right)(0) = \frac{k}{2}. \tag{2.9}$$

On the other hand,

$$L\phi(g) = \phi'(g)Lg + \phi''(g)|\nabla g|^2$$

and so setting $\phi(t) = t^k$ and $g = |\nabla f|^2$ gives

$$L|\nabla f|^{2k} = k|\nabla f|^{2(k-1)}L|\nabla f|^2 + k(k-1)|\nabla f|^{2(k-2)}\left|\nabla |\nabla f|^2\right|^2.$$

From the above,

$$\left|\nabla |\nabla f|^2\right|^2 = \left|\begin{pmatrix} yz + \frac{1}{2}xy^2 - \frac{1}{2}y(2z+xy) \\ -2y + y^3 + xz + \frac{1}{2}2x^2y + \frac{1}{2}x(2z+xy)\end{pmatrix}\right|^2$$

and hence

$$\left|\nabla |\nabla f|^2\right|^2(0) = 0,$$

while $(L\,|\nabla f|^2)(0) = -2$. Therefore

$$\tfrac{1}{2}\,(L|\nabla f|^{2k})(0) = -k. \tag{2.10}$$

Inserting the results of Eqs. (2.9) and (2.10) into Eq. (2.8) would imply that $\frac{k}{2} \leqslant -k$, which is absurd. Thus, $K_{2k} > 1$ for any positive integer k.

For any $p \in [1, \infty)$, there is some integer k such that $p \leqslant 2k$. Thus,

$$\begin{aligned} |\nabla P_t f|^{2k} &= (|\nabla P_t f|^p)^{2k/p} \\ &\leqslant K_p^{2k/p}(P_t|\nabla f|^p)^{2k/p} \leqslant K_p^{2k/p} P_t|\nabla f|^{2k}. \end{aligned} \tag{2.11}$$

Since K_{2k} is the optimal constant for which (2.11) holds and $K_{2k} > 1$,

$$1 < K_{2k} \leqslant K_p^{2k/p}$$

implies that $K_p > 1$.

We now quantify this estimate for $p = 2$. Since

$$K_2 = \sup_{F \in C_p^\infty(G)} \frac{|\nabla P_t F|^2}{P_t |\nabla F|^2}(0) \geqslant \frac{|\nabla P_t f|^2}{P_t |\nabla f|^2}(0) := C(t),$$

where $f(x, y, z) = x + yz$, it follows that $K_2 \geqslant \sup_{t>0} C(t)$. To finish the proof we compute $C(t)$ explicitly. Observe that P_t, when acting on polynomials, may be computed using

$$P_t = e^{tL/2} = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{tL}{2}\right)^n = I + \frac{t}{2} L + \frac{1}{2!}\frac{t^2}{4} L^2 + \cdots .$$

We then have

$$P_t f = f + \frac{t}{2} Lf = (x + yz) + \frac{t}{2} x, \quad \nabla P_t f = \begin{pmatrix} (1 + \frac{t}{2}) - \frac{1}{2} yy \\ z + \frac{1}{2} xy \end{pmatrix}$$

and

$$\begin{aligned} |\nabla P_t f|^2 &= \left(\left(1 + \frac{t}{2}\right) - \frac{1}{2} y^2\right)^2 + \left(z + \frac{1}{2} xy\right)^2 \\ &= \left(1 - y^2 + \frac{1}{4} y^4 + z^2 + xyz + \frac{1}{4} x^2 y^2\right) + \frac{t}{2}(2 - y^2) + \frac{t^2}{8} 2. \end{aligned}$$

Also, from before,

$$\nabla f = \begin{pmatrix} 1 - \frac{1}{2} yy \\ z + \frac{1}{2} xy \end{pmatrix}$$

and so

$$\begin{aligned} |\nabla f|^2 &= 1 - y^2 + \tfrac{1}{4} y^4 + z^2 + xyz + \tfrac{1}{4} x^2 y^2, \\ L|\nabla f|^2 &= -2 + 3y^2 + 2x^2 \end{aligned}$$

and

$$L^2 |\nabla f|^2 = 4 + 6 = 10.$$

Thus,

$$P_t |\nabla f|^2(0) = |\nabla f|^2(0) + \frac{t}{2} L|\nabla f|^2(0) + \frac{t^2}{8} L^2 |\nabla f|^2(0) = 1 - t + \frac{5}{4} t^2$$

and

$$|\nabla P_t f|^2(0) = 1 + t + \frac{1}{4}t^2.$$

We can find the maximum value of

$$C(t) = \frac{1 + t + \frac{1}{4}t^2}{1 - t + \frac{5}{4}t^2}$$

for $t \geqslant 0$ by taking derivatives in t to show that $C(t)$ takes on its maximum value of 2 at $t = \frac{2}{3}$. $\square$

3. Infinite-dimensional calculus

Let $(W(\mathbb{R}^2), \mathcal{F}, \mu)$ denote classical two-dimensional Wiener space. That is, $W = W(\mathbb{R}^2)$ is the space of continuous paths $\omega : [0, 1] \to \mathbb{R}^2$ such that $\omega(0) = 0$, equipped with the supremum norm

$$\|\omega\| = \max_{t \in [0,1]} |\omega(t)|,$$

μ is standard Wiener measure, and $\mathcal{F}$ is the completion of the Borel σ-field on W with respect to μ. $(W, \|\cdot\|)$ is a Banach space. By definition of μ, the process

$$b_t(\omega) = \left(b_t^1(\omega), b_t^2(\omega)\right) = \omega_t$$

is a two-dimensional Brownian motion. For those $\omega \in W$ which are absolutely continuous, let

$$E(\omega) := \int_0^1 |\dot{\omega}(s)|^2 \, ds$$

denote the energy of ω. The Cameron–Martin Hilbert space is the space of finite energy paths,

$$H^1 = H^1(\mathbb{R}^2) := \{\omega \in W(\mathbb{R}^2) : \omega \text{ is absolutely continuous and } E(\omega) < \infty\},$$

equipped with the inner product

$$(h, k)_{H^1} := \int_0^1 \dot{h}(s)\dot{k}(s) \, ds \quad \forall h, k \in H^1.$$

We may identify the Cameron–Martin space with $H = L^2\left([0,1], \mathbb{R}^2\right)$ in the obvious way

$$h \in H^1 \mapsto \dot{h} \in H.$$

In this way, the spaces are isomorphic, and in the sequel, we make this identification without further comment.

To define a notion of differentiation for functions on W, let $B = \{B(h), h \in H\}$ be the process given by

$$B(h) = \int_0^1 h(t)db_t.$$

B is an isonormal Gaussian process associated to the Hilbert space H. Denote by $\mathcal{S}$ the class of smooth Wiener functionals; that is, random variables $F : W \to \mathbb{R}$ such that

$$F = f(B(h_1), \ldots, B(h_n))$$

for some $n \geqslant 1$, $h_1, \ldots, h_n \in H$, and function $f \in C_p^\infty(\mathbb{R}^n)$.

Definition 3.1. The derivative of a smooth functional $F \in \mathcal{S}$ is the random process defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n))h_i(t).$$

Iterations of the derivative for smooth functionals F are given by

$$D^k_{t_1,\ldots,t_k} F = D_{t_1} \cdots D_{t_k} F,$$

and are measurable functions defined almost everywhere on $[0,1]^k \times W$. We will denote the domain of D^k in $L^p\left([0,1]^k \times W\right)$ by $\mathcal{D}^{k,p}$, which is the completion of the family of smooth Wiener functionals $\mathcal{S}$ with respect to the seminorm $\|\cdot\|_{k,p}$ on $\mathcal{S}$ defined by

$$\|F\|_{k,p} = \left[\mathbb{E}(|F|^p) + \sum_{j=1}^k \mathbb{E}\left(\|D^j F\|^p_{L^2([0,1]^j)}\right)\right]^{1/p}.$$

Let

$$\mathcal{D}^\infty = \bigcap_{p \geqslant 1} \bigcap_{k \geqslant 1} \mathcal{D}^{k,p}.$$

One may generalize these Sobolev spaces to Hilbert-valued functions, again, given an appropriate notion of differentiation. So let $\mathcal{S}_H$ be the set of H-valued Wiener functions of the form

$$F = \sum_{j=1}^{n} F_j h_j, \quad h_j \in H, \quad F_j \in \mathcal{S}.$$

Define $D^k F = \sum_{j=1}^{n} D^k F_j \otimes h_j$ for $k \geqslant 1$. Then, as in the Euclidean case, we may define the seminorm

$$\|F\|_{k,p,H} = \left[\mathbb{E}\left(\|F\|_H^p\right) + \sum_{j=1}^{k} \mathbb{E}\left(\|D^j F\|^p_{L^2([0,1]^j, H)}\right) \right]^{1/p}$$

on $\mathcal{S}_H$ for any $p \geqslant 1$, and let $\mathcal{D}^{k,p}(H)$ be the completion of $\mathcal{S}_H$ in the norm $\|\cdot\|_{k,p,H}$, and

$$\mathcal{D}^\infty(H) = \bigcap_{p \geqslant 1} \bigcap_{k \geqslant 1} \mathcal{D}^{k,p}(H).$$

Definition 3.2. Let D^* denote the adjoint of the derivative operator D, which has domain in $L^2(W \times [0,1])$ consisting of functions G such that

$$|\mathbb{E}[(DF, G)_H]| \leqslant C \|F\|_{L^2(\mu)}$$

for all $F \in \mathcal{D}^{1,2}$, where C is a constant depending on G. For those functions G in the domain of D^*, $D^* G$ is the element of $L^2(\mu)$ such that

$$\mathbb{E}[F D^* G] = \mathbb{E}[(DF, G)_H].$$

It is known that D is a continuous operator from $\mathcal{D}^\infty$ to $\mathcal{D}^\infty(H)$, and similarly, D^* is continuous from $\mathcal{D}^\infty(H)$ to $\mathcal{D}^\infty$; see for example Proposition 1.5.4 from Nualart [18]. For a more complete exposition of the above definitions, we refer the reader to [5,10–14,16,18,20] and references contained therein.

3.1. *The stochastic differential equation*

Let $\xi : [0,1] \times W \to G$ denote the solution to the Stratonovich stochastic differential equation

$$\begin{aligned}d\xi_t &= L_{\xi_t *} X \circ db_t^1 + L_{\xi_t *} Y \circ db_t^2 \\ &= \tilde{X}(\xi_t) \circ db_t^1 + \tilde{Y}(\xi_t) \circ db_t^2, \\ \xi_0 &= 0. \end{aligned} \tag{3.1}$$

Remark 3.3. Since $\tilde{X}$ and $\tilde{Y}$ have smooth coefficients with bounded partial derivatives, Theorem 2.2.2 in Nualart [18] implies that $\xi_t^i \in \mathcal{D}^\infty$, for $i = 1, 2, 3$ and all $t \in [0, 1]$.

Because G is a nilpotent Lie group, we may determine an explicit solution of the given SDE.

$$\begin{aligned}d\xi_t &= \tilde{X}\left(\xi_t^1, \xi_t^2, \xi_t^3\right) \circ db_t^1 + \tilde{Y}\left(\xi_t^1, \xi_t^2, \xi_t^3\right) \circ db_t^2 \\ &= \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}\xi_t^2 \end{pmatrix} \circ db_t^1 + \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}\xi_t^1 \end{pmatrix} \circ db_t^2.\end{aligned}$$

Thus,

$$d\xi_t^1 = db_t^1, \quad d\xi_t^2 = db_t^2 \quad \text{and} \quad d\xi_t^3 = -\frac{1}{2}\xi_t^2 \circ db_t^1 + \frac{1}{2}\xi_t^1 \circ db_t^2$$

and one may verify directly that

$$\xi_t = \left(b_t^1, b_t^2, \frac{1}{2}\int_0^t \left[b_s^1\, db_s^2 - b_s^2\, db_s^1\right]\right) \tag{3.2}$$

satisfies the required SDE. Note that the third component of ξ may be recognized as Lévy's stochastic area integral.

From Section 3.9 in Gīhman and Skorohod [7] and Theorem 1.22 in Bell [3], the solution $\xi = \left(\xi^1, \xi^2, \xi^3\right)$ is a time homogenous Markov process, and $P_t = e^{tL/2}$ with $L = \tilde{X}^2 + \tilde{Y}^2$ is the associated Markov diffusion semigroup to ξ; that is, $v_t := (\xi_t)_* \mu = p_t(g)\, dg$ is the density of the transition probability of the diffusion process ξ_t, and

$$(P_t f)(0) = \mathbb{E}\left[f(\xi_t)\right] \tag{3.3}$$

for any $f \in C_p^\infty(G)$, where the right hand side is expectation conditioned on $\xi_0 = 0$.

Proposition 3.4. *The Malliavin covariance matrix of ξ_t*

$$\sigma_t = \left(\left(D\xi_t^i, D\xi_t^j\right)_H\right)_{1 \leqslant i,j \leqslant 3}$$

is invertible a.s. for $t > 0$, *and*

$$(\det \sigma)^{-1} \in \bigcap_{p \geqslant 1} L^p(\mu) =: L^{\infty-}(\mu).$$

This statement follows from the proof of Theorem 2.3.3 in Nualart [18] which relies on the satisfaction of the Hörmander bracket condition, $\mathrm{Lie}\{X, Y\} = \mathfrak{g}$.

Remark 3.5. By the general theory, Proposition 3.4 implies $\nu_t = \mathrm{Law}(\xi_t)$ is a smooth measure; see for example Theorem 2.12 and Remark 2.13 in Bell [3].

3.2. Lifted vector fields and their L^2-adjoints

Given $A \in \mathfrak{g}$, let $\tilde{A}^i$ be the ith component of the left-invariant vector field $\tilde{A}$, hence $\tilde{A} = \left(\tilde{A}^1, \tilde{A}^2, \tilde{A}^3\right)$. In particular, we are interested in the vector fields $\tilde{X}(x, y, z) = \left(1, 0, -\frac{1}{2}y\right)$ and $\tilde{Y}(x, y, z) = \left(0, 1, \frac{1}{2}x\right)$. We define the "lifted vector field" $\mathbf{A}$ of $\tilde{A}$ as

$$\mathbf{A} = \mathbf{A}^t := \sum_{i,j=1}^{3} \sigma_{ij}^{-1} \tilde{A}^j(\xi_t) D\xi_t^i \in H, \tag{3.4}$$

acting on functions $F \in \mathcal{D}^{1,2}$ by

$$\mathbf{A}F = (DF, \mathbf{A})_H.$$

Remark 3.6. Recall that D is a continuous operator from $\mathcal{D}^\infty$ to $\mathcal{D}^\infty(H)$. Thus, Remark 3.3 implies that $\tilde{A}^j(\xi_t) \in \mathcal{D}^\infty$ and $D\xi_t^i \in \mathcal{D}^\infty(H)$, for all $t \in [0, 1]$. So $\sigma_{ij} \in \mathcal{D}^\infty$ for $i, j = 1, 2, 3$, and this along with Proposition 3.4 implies that $\sigma_{ij}^{-1} \in \mathcal{D}^\infty$. Hence, $\mathbf{A} \in \mathcal{D}^\infty(H)$.

Proposition 3.7. *For all* $f \in C_p^\infty(G)$, $\mathbf{A}[f(\xi_t)] = (\tilde{A}f)(\xi_t)$.

Proof. For any function $f \in C_p^\infty(G)$, $f(\xi_t) \in \mathcal{D}^\infty$ and

$$D[f(\xi_t)] = \sum_{k=1}^{3} \frac{\partial f}{\partial x_k}(\xi_t) D\xi_t^k;$$

see Proposition 1.2.3 from Nualart [18]. Then using Eq. (3.4) and the definition of the Malliavin matrix σ, we have

$$\begin{aligned}\mathbf{A}[f(\xi_t)] &= (Df(\xi_t), \mathbf{A})_H \\ &= \sum_{i,j,k=1}^{3} \left(\frac{\partial f}{\partial x_k}(\xi_t) D\xi_t^k, \sigma_{ij}^{-1} \tilde{A}^j(\xi_t) D\xi_t^i\right)_H\end{aligned}$$

$$= \sum_{i,j,k=1}^{3} \tilde{A}^j(\xi_t) \frac{\partial f}{\partial x_k}(\xi_t) \left(D\xi_t^k, D\xi_t^i \right)_H \sigma_{ij}^{-1}$$

$$= \sum_{j,k=1}^{3} \tilde{A}^j(\xi_t) \frac{\partial f}{\partial x_k}(\xi_t)\delta_{kj} = \sum_{j=1}^{3} \tilde{A}^j(\xi_t) \frac{\partial f}{\partial x_j}(\xi_t) = (\tilde{A} f)(\xi_t)$$

as desired. □

Definition 3.8. For a vector field $\mathbf{A}$ acting on functions of W, we will denote the adjoint of $\mathbf{A}$ in the $L^2(\mu)$ inner product by $\mathbf{A}^*$, which has domain in $L^2(\mu)$ consisting of functions G such that

$$|\mathbb{E}[(\mathbf{A}F)G]| \leqslant C \|F\|_{L^2(\mu)},$$

for all $F \in \mathcal{D}^{1,2}$, for some constant C. For functions G in the domain of $\mathbf{A}^*$,

$$\mathbb{E}[F(\mathbf{A}^* G)] = \mathbb{E}[(\mathbf{A}F)G],$$

for all $F \in \mathcal{D}^{1,2}$.

Note that for any $F \in \mathcal{D}^{1,2}$,

$$\mathbb{E}[\mathbf{A}F] = \mathbb{E}[(DF, \mathbf{A})_H] = \mathbb{E}[F D^* \mathbf{A}].$$

Thus, we must have that $\mathbf{A}^* = \mathbf{A}^* 1 = D^* \mathbf{A}$ a.s. Recall that D^* is a continuous operator from $\mathcal{D}^\infty(H)$ into $\mathcal{D}^\infty$. Thus, for $\mathbf{A}$ a vector field on W as defined in Eq. (3.4), Remark 3.6 implies that

$$D^* \mathbf{A} = \sum_{i,j=1}^{3} D^* \left(\sigma_{ij}^{-1} \tilde{A}^j(\xi_t) D\xi_t^i \right) \in \mathcal{D}^\infty.$$

Thus, we have the following proposition.

Proposition 3.9. *Let $\tilde{A}$ be a left-invariant vector field on G with lifted vector field $\mathbf{A}$ on W as defined by Eq. (3.4). Then $\mathbf{A}^*$, the $L^2(\mu)$-adjoint of $\mathbf{A}$, is an element of $\mathcal{D}^\infty$.*

4. Heat kernel inequalities

4.1. An L^p-type gradient estimate ($p > 1$) and a Poincaré inequality

Theorem 4.1. *For all* $p > 1$,

$$|\nabla P_t f|^p \leqslant K_p P_t |\nabla f|^p \tag{4.1}$$

for all $f \in C_p^\infty(G)$ *and* $t > 0$, *where*

$$K_p := 2^{p/q} + 2^{p\left(\frac{1}{q}+\frac{1}{2}\right)} \left[\|\mathbf{X}^* \xi_1^1\|_{L^q(\mu)}^2 + \|\mathbf{X}^* \xi_1^2\|_{L^q(\mu)}^2 \right]^{p/2} < \infty,$$

with $\mathbf{X}^*$ *the adjoint of the lifted vector field* $\mathbf{X}$ *as in Eq.* (3.4) *with* $t = 1$, *and* $q = \frac{p}{p-1}$.

Proof. By Proposition 2.6, we know the constants K_p are independent of t. Also, Lemma 2.3 states that the inequality is translation invariant. Thus, the proof is reduced to verifying the inequality at the identity for $t = 1$; that is, we must find finite constants K_p such that

$$|\nabla P_1 f|^p(0) \leqslant K_p P_1 |\nabla f|^p(0), \tag{4.2}$$

for all $f \in C_p^\infty(G)$. So applying Remark 2.2 and Lemma 2.4, consider

$$\begin{aligned}\tilde{X} P_1 f(0) &= P_1 \hat{X} f(0) \\ &= P_1(\tilde{X} + y\tilde{Z}) f(0) = P_1(\tilde{X} f)(0) + P_1(y\tilde{Z} f)(0).\end{aligned}$$

Similarly,

$$\tilde{Y} P_1 f(0) = P_1(\tilde{Y} f)(0) - P_1(x\tilde{Z} f)(0).$$

Thus,

$$\begin{aligned}|\nabla P_1 f|^p(0) &= \left| P_1 \nabla f + P_1 \left(\begin{pmatrix} y \\ -x \end{pmatrix} \tilde{Z} f \right) \right|^p (0) \\ &\leqslant \left(|P_1 \nabla f| + \left| P_1 \left(\begin{pmatrix} y \\ -x \end{pmatrix} \tilde{Z} f \right) \right| \right)^p (0) \\ &\leqslant 2^{p/q} \left(|P_1 \nabla f|^p(0) + \left| P_1 \left(\begin{pmatrix} y \\ -x \end{pmatrix} \tilde{Z} f \right) \right|^p (0) \right),\end{aligned} \tag{4.3}$$

where

$$\left|P_1\left(\begin{pmatrix} y \\ -x \end{pmatrix}\tilde{Z}f\right)\right|^p(0) = [|P_1(y\tilde{Z}f)|^2(0) + |P_1(x\tilde{Z}f)|^2(0)]^{p/2}$$

and $q = \frac{p}{p-1}$ is the conjugate exponent to p. Let $F = (F_1, F_2, F_3) := \xi_1$ and recall that $\tilde{Z} = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$. By Eq. (3.3),

$$\begin{aligned} P_1(y\tilde{Z}f)(0) &= P_1(y\tilde{X}\tilde{Y}f)(0) - P_1(y\tilde{Y}\tilde{X}f)(0) \\ &= \mathbb{E}[F_2(\tilde{X}\tilde{Y}f)(F)] - \mathbb{E}[F_2(\tilde{Y}\tilde{X}f)(F)] \\ &= \mathbb{E}[F_2\mathbf{X}((\tilde{Y}f)(F))] - \mathbb{E}[F_2\mathbf{Y}((\tilde{X}f)(F))] \\ &= \mathbb{E}[\mathbf{X}^*F_2 \cdot (\tilde{Y}f)(F)] - \mathbb{E}[\mathbf{Y}^*F_2 \cdot (\tilde{X}f)(F)], \end{aligned} \tag{4.4}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are the lifted vector fields of $\tilde{X}$ and $\tilde{Y}$, as in Eq. (3.4), with $t = 1$. Hence,

$$\begin{aligned} |P_1(y\tilde{Z}f)|^2(0) &\leqslant (|\mathbb{E}[\mathbf{X}^*F_2 \cdot (\tilde{Y}f)(F)]| + |\mathbb{E}[\mathbf{Y}^*F_2 \cdot (\tilde{X}f)(F)]|)^2 \\ &\leqslant 2(|\mathbb{E}[\mathbf{X}^*F_2 \cdot (\tilde{Y}f)(F)]|^2 + |\mathbb{E}[\mathbf{Y}^*F_2 \cdot (\tilde{X}f)(F)]|^2) \\ &\leqslant 2[(\mathbb{E}|\mathbf{X}^*F_2|^q)^{2/q}(P_1|\tilde{Y}f|^p)^{2/p}(0) + (\mathbb{E}|\mathbf{Y}^*F_2|^q)^{2/q}(P_1|\tilde{X}f|^p)^{2/p}(0)] \end{aligned}$$

by Hölder's inequality. Similarly,

$$\begin{aligned} |P_1(x\tilde{Z}f)|^2(0) \leqslant 2[&(\mathbb{E}|\mathbf{X}^*F_1|^q)^{2/q}(P_1|\tilde{Y}f|^p)^{2/p}(0) \\ &+ (\mathbb{E}|\mathbf{Y}^*F_1|^q)^{2/q}(P_1|\tilde{X}f|^p)^{2/p}(0)]. \end{aligned}$$

Combining this with Eq. (4.3), we have

$$\begin{aligned} |\nabla P_1 f|^p(0) \leqslant\; & 2^{p/q}\Big(|P_1\nabla f|^p(0) + \Big[2(\mathbb{E}|\mathbf{X}^*F_2|^q)^{2/q}(P_1|\tilde{Y}f|^p)^{2/p}(0) \\ & + 2(\mathbb{E}|\mathbf{Y}^*F_2|^q)^{2/q}(P_1|\tilde{X}f|^p)^{2/p}(0) \\ & + 2(\mathbb{E}|\mathbf{X}^*F_1|^q)^{2/q}(P_1|\tilde{Y}f|^p)^{2/p}(0) \\ & + 2(\mathbb{E}|\mathbf{Y}^*F_1|^q)^{2/q}(P_1|\tilde{X}f|^p)^{2/p}(0)\Big]^{p/2}\Big) \\ \leqslant\; & 2^{p/q}\big(P_1|\nabla f|^p(0) \\ & + 2^{p/2}[(P_1|\tilde{X}f|^p)^{2/p}(0)][(\mathbb{E}|\mathbf{Y}^*F_1|^q)^{2/q} + (\mathbb{E}|\mathbf{Y}^*F_2|^q)^{2/q}] \\ & + (P_1|\tilde{Y}f|^p)^{2/p}(0)[(\mathbb{E}|\mathbf{X}^*F_1|^q)^{2/q} + (\mathbb{E}|\mathbf{X}^*F_2|^q)^{2/q}]^{p/2}\big), \end{aligned}$$

where we use Hölder's inequality and that $p_1(g)\,dg$ is a probability measure to get

$$|P_1\nabla f|^p(0) \leqslant P_1|\nabla f|^p(0).$$

So let

$$C_p := (\mathbb{E}|\mathbf{X}^* F_1|^q)^{2/q} + (\mathbb{E}|\mathbf{X}^* F_2|^q)^{2/q}$$

or equivalently by symmetry,

$$C_p = (\mathbb{E}|\mathbf{Y}^* F_1|^q)^{2/q} + (\mathbb{E}|\mathbf{Y}^* F_2|^q)^{2/q}.$$

Note that C_p is a finite constant for all $p > 1$ by Hölder's inequality, Remark 3.3, and Proposition 3.9, since

$$\mathbf{A}^* F = D^*(F\mathbf{A})$$

for any vector field $\mathbf{A}$ on W and $F \in \mathcal{D}^\infty$. Thus,

$$\begin{aligned} |\nabla P_1 f|^p(0) &\leqslant 2^{p/q} P_1 |\nabla f|^p(0) + (2C_p)^{p/2} [(P_1|\tilde{X} f|^p)^{2/p}(0) + (P_1|\tilde{Y} f|^p)^{2/p}(0)]^{p/2} \\ &\leqslant (2^{p/q} + 2^{p(\frac{1}{q}+\frac{1}{2})} C_p^{p/2}) P_1 |\nabla f|^p(0), \end{aligned}$$

which proves Eq. (4.2), and hence, the theorem. □

Theorem 4.2 (*Poincaré inequality*). *Let K_2 be the constant in Eq.* (4.1) *for $p = 2$ and $p_t(g)\,dg$ be the Heisenberg group heat kernel. Then*

$$\int_{\mathbb{R}^3} f^2(g) p_t(g)\,dg - \left(\int_{\mathbb{R}^3} f(g) p_t(g)\,dg\right)^2 \leqslant K_2 t \int_{\mathbb{R}^3} |\nabla f|^2(g) p_t(g)\,dg$$

for all $f \in C_p^\infty(G)$ and $t > 0$.

Proof. Let $F_t(g) = (P_t f)(g)$. Then

$$\frac{d}{ds} P_{t-s} F_s^2 = P_{t-s}\left(-\frac{1}{2} L F_s^2 + F_s L F_s\right) = -P_{t-s}|\nabla F_s|^2.$$

Integrating this equation on t implies that

$$\begin{aligned} P_t f^2 - (P_t f)^2 = \int_0^t P_{t-s} |\nabla F_s|^2\,ds &= \int_0^t P_{t-s} |\nabla P_s f|^2\,ds \\ &\leqslant K_2 \int_0^t P_{t-s} P_s |\nabla f|^2\,ds \\ &= K_2 \int_0^t P_t |\nabla f|^2\,ds = K_2 t P_t |\nabla f|^2, \end{aligned}$$

wherein we have made use of Theorem 4.1. Evaluating the above at 0 gives the desired result. □

4.2. Method fails for the $p = 1$ case

In this section, we show that the argument in the proof of Theorem 4.1 can not be used to prove the inequality (4.1) for $p = 1$.

Proposition 4.3. *Let* $F = (F_1, F_2, F_3) := \xi_1$. *Then*

$$\|\mathbf{X}^* F_1\|_{L^\infty(\mu)} + \|\mathbf{X}^* F_2\|_{L^\infty(\mu)} = \infty. \tag{4.5}$$

Proof. Let $\sigma(F)$ denote the σ–algebra generated by $F: W \to G$ and $p_t(g)\,dg$ denote the Heisenberg group heat kernel. Then for $f \in C_c^1(\mathbb{R}^3)$

$$\begin{aligned}
\mathbb{E}[\mathbf{X}^* F_1 f(F)] &= \mathbb{E}[F_1(\tilde{X} f)(F)] = P_1(x\tilde{X} f)(0) \\
&= \int_G x\tilde{X} f(g) p_1(g)\,dg \\
&= -\int_G f(g)\tilde{X}(x p_1(g))\,dg \\
&= -\int_G f(g)(1 + x\tilde{X} \ln p_1(g)) p_1(g)\,dg \\
&= -\mathbb{E}[f(F)(1 + x\tilde{X} \ln p_1)(F)],
\end{aligned}$$

where in the third line we have applied standard integration by parts. Consequently, we have shown

$$\mathbb{E}[\mathbf{X}^* F_1 | \sigma(F)] = -(1 + x\tilde{X} \ln p_1)(F).$$

By a similar computation one also shows

$$\mathbb{E}[\mathbf{X}^* F_2 | \sigma(F)] = -(y\tilde{X} \ln p_1)(F).$$

Since conditional expectation is L^p-contractive and the law of F is absolutely continuous relative to Lebesgue measure, it now follows that

$$\begin{aligned}
&\|\mathbf{X}^* F_1\|_{L^\infty(\mu)} + \|\mathbf{X}^* F_2\|_{L^\infty(\mu)} \\
&\quad \geqslant \|\mathbb{E}[\mathbf{X}^* F_1 | \sigma(F)]\|_{L^\infty(\mu)} + \|\mathbb{E}[\mathbf{X}^* F_2 | \sigma(F)]\|_{L^\infty(\mu)} \\
&\quad = \|1 + x\tilde{X} \ln p_1\|_{L^\infty(\mathbb{R}^3, m)} + \|y\tilde{X} \ln p_1\|_{L^\infty(\mathbb{R}^3, m)},
\end{aligned}$$

where m is Lebesgue measure. Hence, it suffices to show that either $x\tilde{X} \ln p_1$ or $y\tilde{X} \ln p_1$ is unbounded. We will show $x\tilde{X} \ln p_1$ is unbounded by making use of the formula for $p_t(g)$ in Eq. (1.5). Letting $t = 1$ in Eq. (1.5) and making the change of variables $w \mapsto 2w$, we have

$$p_1(g) = \frac{1}{2\pi^2} \int_{\mathbb{R}} \frac{w}{\sinh w} \exp\left(-\frac{1}{2} |\vec{x}|^2 w \coth w\right) e^{2iwz}\, dw.$$

Then applying $\tilde{X} = \partial_x - \frac{1}{2} y \partial_z$ yields

$$\tilde{X} p_1(g) = -\frac{1}{2\pi^2} \int_{\mathbb{R}} (xw \coth w + iyw) \frac{w}{\sinh w} \times \exp\left(-\frac{1}{2} |\vec{x}|^2 w \coth w\right) e^{2iwz}\, dw.$$

Setting $y = z = 0$, it follows that

$$\tilde{X} \ln p_1(x, 0, 0) = -x \int_{\mathbb{R}} w \coth w \, d\nu_x(w),$$

where

$$d\nu_x(w) := \frac{1}{z_x} \frac{w}{\sinh w} \exp\left(-\frac{1}{2} x^2 w \coth w\right) dw \tag{4.6}$$

and z_x is the normalizing constant

$$z_x := \int_{\mathbb{R}} \frac{w}{\sinh w} \exp\left(-\frac{1}{2} x^2 w \coth w\right) dw.$$

By Lemma 4.4 below,

$$\lim_{x \to \infty} \int_{\mathbb{R}} w \coth w \, d\nu_x(w) = 1$$

and so

$$\lim_{x \to \infty} \tilde{X} \ln p_1(x, 0, 0) = \lim_{x \to \infty} \left(-x \int_{\mathbb{R}} w \coth w \, d\nu_x(w)\right) = -\infty. \qquad \square$$

Lemma 4.4. *Let $\psi(w) = w \coth w - 1$ and ν_x be as in Eq. (4.6). Then*

$$\lim_{x \to \infty} \int \psi \, d\nu_x = \psi(0) = 0. \tag{4.7}$$

Proof. Since $\psi(0) = 0$ and ψ is continuous, to prove Eq. (4.7) it suffices to show by the usual approximation of δ – function arguments that

$$\lim_{x\to\infty} \int_{|w|\geqslant\varepsilon} \psi(w)\, d\nu_x(w) = 0$$

holds for every $\varepsilon > 0$. We begin by rewriting Eq. (4.6) as

$$d\nu_x(w) = \frac{1}{Z_x} \frac{w}{\sinh\, w} \exp\left(-\frac{1}{2} x^2 \psi(w)\right) dw$$

where

$$Z_x := \int_{\mathbb{R}} \frac{w}{\sinh\, w} \exp\left(-\frac{1}{2} x^2 \psi(w)\right) dw.$$

A glance at the graph of ψ will convince the reader that there are constants $\alpha, \beta > 0$ (depending on $\varepsilon > 0$) such that $\alpha|w| \leqslant \psi(w) \leqslant \beta|w|$ for all $|w| \geqslant \varepsilon$. (In fact, one could take $\beta = 1$ independent of ε). Thus

$$\begin{aligned}\int_{|w|\geqslant\varepsilon} \psi(w) \frac{w}{\sinh\, w} \exp\left(-\frac{1}{2} x^2 \psi(w)\right) dw &\leqslant 2 \int_{w\geqslant\varepsilon} \beta w e^{-\alpha x^2 w/2}\, dw \\ &= \frac{4\beta}{x^2\alpha}\left(\varepsilon + \frac{2}{x^2\alpha}\right) e^{-\alpha x^2 \varepsilon/2},\end{aligned}$$

where in the inequality we have also used that $\frac{w}{\sinh\, w} \leqslant 1$.

Now consider the constant Z_x. We know that for w small, there exists a constant $\gamma > 0$ such that $\psi(w) \leqslant \gamma w^2$. So letting $\varphi(w) = \frac{w}{\sinh\, w}$,

$$\begin{aligned}Z_x &\geqslant \int_{|w|\leqslant\varepsilon} \varphi(w) \exp\left(-\frac{1}{2} x^2 \psi(w)\right) dw \\ &\geqslant \int_{-\varepsilon}^{\varepsilon} \varphi(w) e^{-\gamma x^2 w^2/2}\, dw = \frac{1}{x} \int_{-\varepsilon x}^{\varepsilon x} \varphi\left(\frac{w}{x}\right) e^{-\gamma w^2/2}\, dw,\end{aligned}$$

where we have made the change of variables $w \mapsto \frac{w}{x}$. So, by the dominated convergence theorem,

$$\begin{aligned}\liminf_{x\to\infty} (x Z_x) &\geqslant \liminf_{x\to\infty} \int_{-\varepsilon x}^{\varepsilon x} \varphi\left(\frac{w}{x}\right) e^{-\gamma w^2/2}\, dw \\ &= \varphi(0) \int_{-\infty}^{\infty} e^{-\gamma w^2/2}\, dw = \sqrt{\frac{2\pi}{\gamma}}.\end{aligned}$$

Thus, $Z_x \geqslant \frac{1}{2}\sqrt{\frac{2\pi}{\gamma}}\,\frac{1}{x}$ for x sufficiently large, and so

$$\lim_{x\to\infty}\int_{|w|\geqslant\varepsilon}\psi(w)\,d\nu_x(w) = \lim_{x\to\infty}\frac{1}{Z_x}\int_{|w|\geqslant\varepsilon}\psi(w)\,\frac{w}{\sinh\,w}\,\exp\left(-\frac{1}{2}x^2\psi(w)\right)dw$$

$$\leqslant 2\lim_{x\to\infty}\frac{\frac{4\beta}{x^2\alpha}\left(\varepsilon+\frac{2}{x^2\alpha}\right)e^{-\alpha x^2\varepsilon/2}}{\sqrt{\frac{2\pi}{\gamma}}\,\frac{1}{x}} = 0$$

as desired. □

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