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# $YM_2$: Continuum Expectations, Lattice Convergence, and Lassos

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**Abstract.** The two dimensional Yang-Mills theory ($YM_2$) is analyzed in both the continuum and the lattice. In the complete axial gauge the continuum theory may be defined in terms of a Lie algebra valued white noise, and parallel translation may be defined by stochastic differential equations. This machinery is used to compute the expectations of gauge invariant functions of the parallel translation operators along a collection of curves $\mathscr{C}$. The expectation values are expressed as finite dimensional integrals with densities that are products of the heat kernel on the structure group. The time parameters of the heat kernels are determined by the areas enclosed by the collection $\mathscr{C}$, and the arguments are determined by the crossing topologies of the curves in $\mathscr{C}$. The expectations for the Wilson lattice models have a similar structure, and from this it follows that in the limit of small lattice spacing the lattice expectations converge to the continuum expectations. It is also shown that the lasso variables advocated by L. Gross [36] exist and are sufficient to generate all the measurable functions on the $YM_2$-measure space.

## 1. Introduction

The informal expression for the Yang-Mills' measure is:

$$\mu(dA) = Z^{-1} \exp\left[ \frac{1}{2g_0^2} \int_{\mathbf{R}^d} \sum_{i<j} \operatorname{trace}_\rho (F_{ij}^A(x)^2) dx \right] \mathscr{D}A, \tag{1.1}$$

where A runs over a space of connection forms ($\mathscr{A}$) *on the trivial unitary vector bundle* $C^N \times R^d$, $F^A = dA + A \wedge A$ is the curvature of $A$, $\mathscr{D}A = \prod_{i=1}^{d} \prod_{x\in\mathbf{R}^d} d(A_i(x))$ is "infinite dimensional Lebesgue measure" on $\mathscr{A}$, $g_0^2$ is a positive "coupling" constant, and $Z$ is a normalization constant which makes $\mu$ a probability measure. The connection forms are restricted to take values in the Lie algebra $\mathscr{G}$ of the structure (or gauge) group $G$—a subgroup of $U(N)$. The trace is taken with respect to some representation $\rho$ of $G$.

It is well known that the expression (1.1) is ill defined, see Gross [35]. Despite the many technical problems, when $d = 2$ it is possible to define a "gauge fixed" version of $\mu$ as mean zero Gaussian measure on a space of generalized connection 1-forms, see Definitions 2.5 and 2.7. Heuristicly, this gauge fixed measure should agree with $\mu$ in (1.1) on gauge invariant functions, and be unrelated for non-gauge invariant functions. The purpose of this paper is to study this continuum $YM_2$-measure and its lattice approximations with an emphasis on using gauge invariant observables. By using gauge invariant functions, one is forced to consider non-linear functionals of the measure space. This makes the theory less trivial than one might first think.

There has been considerable effort in understanding (1.1) for $d = 3$ and $d = 4$, and in fact, much progress has been made by Balaban [7]–[17], and Federbush [24]–[31] in controlling the renormalization group flow of the lattice approximations (or in Federbush's case an infinite directed sequence of lattices). The group of R. Seneor et al. are trying to use a continuum regularization. Despite success in proving stability results for these flows, a $YM_3$ or $YM_4$ theory with gauge invariant observables is still missing.

One question is what should the gauge invariant observables be? The most common suggestion is the Wilson loop variables (traces of parallel translation operators), but these variables are quite singular in $d = 3$ and especially $d = 4$. As an alternative to these variables, L. Gross [36] has advocated using "lasso-variables." A lasso-variable is the curvature tensor transported back to the origin by the parallel translation operator along a curve, see Definition 9.1. It is shown in [36], with generalizations by Driver [21] to base spaces with non-trivial topologies, that the lassos generate the gauge invariant functions. So expectations of functions of the lassos would completely characterize the YM-measure on the gauge invariant sigma-field.

The lassos are smoothable random variables for the measure in (1.1) if the structure group $G = U(1)$. (In this abelian case it is well known how to interpret the measure $\mu$ as a generalized gaussian process, see for example [22, 37].) This is in contrast to the Wilson loops in $d = 4$, which do not seem to be smoothable. (There are proposals of how to work with the expectations of these singular Wilson loops, see Seiler [44].) Another success of the lassos is their use in proving that on the "current sector," the $U(1)$-lattice gauge theories converge to the continuum $U(1)$-theory, see Gross [37] when $d = 3$ and Driver [22] when $d = 4$.

When $d = 2$, parallel translation may be defined by stochastic differential equations, see Definition 3.3 and Gross, King, and Sen Gupta [34]. This means that in two space-time dimensions the Wilson loop variables are well defined. In this paper, I will give explicit formulas for the $YM_2$-expectations of functions depending on the parallel translation operators on a finite "admissible collection" ($\mathscr{C}$) of curves, see Theorem 4.12. The results are given as finite dimensional integrals involving densities that are certain products of the convolution heat kernel on $G$. If the function integrated is gauge invariant, the density simplifies in such a way that the arguments of the heat kernels are determined by the crossing topology of the curves in $\mathscr{C}$, and the time parameters are determined by the areas enclosed by the collection $\mathscr{C}$, see Theorem 6.4. This result immediately shows, Corollary 6.7, that gauge invariant expectations are Euclidean invariant—a fact that is not a priori obvious when using the gauge fixed measure.

Results showing that the expectations depend only on the enclosed areas and crossing topologies of the curves in $\mathscr{C}$ have been obtained by a number of different authors with varying degrees of rigor, see [19, 34, 39, 40, 41] and the references therein. The methods of this paper are a variation of the methods in Gross et al. [34] which were inspired by Bralić [19]. However, the form of the results in this paper were most influenced by Dosch and Müller [23], where it was shown that for $G = U(1)$ or $SU(2)$, the lattice theory converges, and the resulting answer depends on the collection ($\mathscr{C}$) only via the enclosed areas and crossing topologies.

Formulas similar to those given here for the expectations of gauge invariant functions have already been discovered. Most notably, Albeverio, Høegh-Krohn, and Holden [1–5] have such formulas and show that these formulas may be used as a definition of a random process indexed by planar "complexes." This point of view of defining the $YM_2$-theory has also appeared in Klimek and Kondracki [42].

It will also be shown that the (gauge fixed) Wilson [48] lattice approximations converge (on arbitrary functions) on gauge invariant functions to the continuum $YM_2$-measure, see Theorems 8.5 and 8.10 and their corollaries. This statement is valid for both the Villain and the Wilson action. These limits have been shown in less generality to exist if $G = U(1)$ and $G = SU(2)$ by Dosch and Müller [23] and Albeverio et al. [5]. The existence of the limit is also alluded to in Klimek and Kondracki [42]. However, a direct connection of the limiting values with the gauge fixed $YM_2$-measure seems to have been missing.

Finally, it will be shown that the lasso variables with "polygonal tails" exist for the continuum $YM_2$-theory, and do generate the full algebra of measurable functions on the gauge-fixed measure space, see Theorems 9.5 and 10.1. While this is not a conclusive test as to whether the lasso variables are smoother than the Wilson variables, it does show they are no worse.

## 2. The Continuum $YM_2$-Measure

The continuum $YM_2$-measure will be defined in this section, but first some basic notation will be given. For the purposes of this paper $G$ will denote a connected compact Lie group with Lie algebra $\mathscr{G}$. Without loss of generality, $G$ will be taken to be a closed subgroup of $U(N)$ for some $N$ and $\mathscr{G}$ a Lie subalgebra of the $N \times N$ skew adjoint matrices. If $\rho$ is any finite dimensional representation of $G$, let $\chi_\rho = \mathrm{trace}(\rho)$ be the character of $\rho$, $V_\rho$ be the representation space of $\rho$, $d_\rho = \dim V_\rho$, and $\rho_*(A) = d/dt|_0 \rho(\exp tA)$ for all $A \in \mathscr{G}$ be the differential of $\rho$. We define a real bi-linear form on $\mathscr{G}$ by

$$(A, B)_\rho \equiv -\mathrm{trace}(\rho_*(A)\rho_*(B)), \tag{2.1}$$

for all $A, B \in \mathscr{G}$. The bi-linear form in (2.1) is non-negative and in fact positive definite if $\rho_*$ is one to one. For the rest of this paper $\rho$ will be a fixed representation of $G$ for which $\rho_*$ is injective. Also let $\Gamma = \Gamma(G)$ be a complete collection of inequivalent irreducible representations of $G$.

**Definition 2.1.** *Let $\tau$ be any finite dimensional representation of $G$. The Casmir operator $C_\tau$ on $V_\tau$ is*

$$C_\tau = \sum_{a=1}^{\dim \mathscr{G}} \tau(T^a)^2,$$

*where* $\{T^a\}_{a=1}^{\dim(\mathscr{G})}$ *is an* $(\cdot,\cdot)_\rho$*–orthonormal basis for* $\mathscr{G}$, *and* $\tau(T^a) := d/dt|_0 \tau(e^{tT^a})$.

*Remarks 2.2.* 1. The Casmir operator $C_\tau$ is independent of the orthonormal basis $\{T^a\}_a$.
2. $[C_\tau, \tau(g)] = 0 \forall g \in G$, since $\{g^{-1}T^a g\}_a$ is also an orthonormal basis. Hence $C_\tau \tau(g) = \tau(g) \sum \tau(g^{-1}T^a g)^2 = \tau(g) C_\tau$.
3. If $\tau \in \Gamma(G)$ is irreducible, then by Shur's lemma, $C_\tau = c_\tau I$, where $c_\tau$ is a constant. Furthermore, with respect to a $G$-invariant inner product on $V_\tau$, $\tau(T^a)^* = -\tau(T^a)$, which shows that $c_\tau \leq 0$ with equality iff $\tau$ is the trivial representation.

The first problem to overcome in defining the measure in (1.1) is the "gauge" invariance of the exponent. Namely, if $g: \mathbf{R}^d \to G$ is a gauge transformation, under which the connection 1-form $A$ transforms to $A^g = g^{-1}Ag + g^{-1}dg$, then it is well known ([43]) that $F^{A^g} = g^{-1}F^A g$. Since traces are invariant under conjugation, it follows that the exponent of (1.1) is invariant under the action of $g \in GT$, where $GT$ is the set of gauge transformations. This has the serious consequence that even at an informal level it is impossible to choose the normalization constant $Z$ so that $\mu$ is a probability measure. As is well known, the first step in overcoming this problem is to interpret the Yang-Mills measure on the space $\mathscr{A}/GT$ of connections module the gauge transformations. This is a very appealing geometrical interpretation, however on a technical level it is easier to define the measure by choosing a "transversal" slice of the space of connections and restrict the measure $\mu$ to that slice. The price one pays for this procedure is to lose the manifest Euclidean invariance. Nevertheless, Euclidean invariance will be recovered, Corollary 6.7, also see Gross et al. [34] on this point.

We now restrict to $d = 2$, and let $GT = \{g \in C^\infty(\mathbf{R}^2, G) | g(0) = e \in G\}$ be the restricted gauge transformations, $\mathscr{A}$ be the set of $\mathscr{G}$-valued connection 1-forms on $\mathbf{R}^2$, and $\mathscr{A}_f = \{A \in \mathscr{A} | A = A_1 dx_1$ and $A_1 = 0$ on the $x$-axis$\}$. A element in $\mathscr{A}_f$ is said to be in the *complete axial gauge.*

The following theorems stated without proof are only to help to motivate the definition of the $\mathrm{YM}_2$-measure.

**Proposition 2.3.** *Define a left action of* $GT$ *on* $\mathscr{A}$ *by*

$$g \cdot A := A^{g^{-1}} := gAg^{-1} + gdg^{-1},$$

*then the map*

$$(g, A) \to g \cdot A : GT \times \mathscr{A}_f \to \mathscr{A} \tag{2.2}$$

*is a surjection. Furthermore the action of* $GT$ *on* $\mathscr{A}$ *is affine with the linear part leaving the subspace* $\mathscr{A}_f$ *invariant.*

This proposition indicates that it should be possible to "restrict" the measure $\mu$ to $\mathscr{A}_f$ provided one only integrates functions which are invariant under the left action of $GT$ on $\mathscr{A}$. Notice that

$$F = \text{curvature}(A) = -\partial_2 A_1 dx \wedge dy := F_{21} dy \wedge dx \tag{2.3}$$

for $A \in \mathscr{A}_f$.

**Meta-Theorem 2.4.** *The "measure"* $\mu$ *when "restricted" to* $\mathscr{A}_f$ *is given informally by*

$$\mu(dA) = Z^{-1} \exp\left[ \frac{-1}{2g_0^2} \int_{\mathbf{R}^2} (\partial_2 A_1(x), \partial_2 A_1(x))_\rho dx \right] \mathscr{D}A, \tag{2.4}$$

*where $A \in \mathscr{A}_f$. The content here is that the Jacobian factor coming from the change of variables from $\mathscr{A}$ coordinates to $GT \times \mathscr{A}_f$ coordinates is a constant.*

We now identify $F$ with $F_{21} = \partial_2 A_1$, and make the linear change of variables $F = \partial_2 A_1$ in Eq. (2.4) to find that

$$\mu(dF) = Z^{-1} \exp\left[ \frac{-1}{2g_0^2} \int_{\mathbf{R}^2} (F(x), F(x))_\rho dx \right] \mathscr{D}F. \tag{2.5}$$

But this last equation is the informal expression for $\dim(\mathscr{G})$-independent white noises on $\mathbf{R}^2$.

From now on the coupling constant $g_0^2$ will be set to one, this amounts to a trivial scaling of the $F$'s. Let $\{T^a\}$ be an orthonormal basis for $\mathscr{G}$ with respect to the inner product $(\cdot,\cdot)_\rho$, where $\rho_*$ is assumed to be injective.

**Definition 2.5.** *The $\mathscr{G}$-valued white noise $(\Omega, \mathscr{F}, E, F)$ is the process $F = \sum_{a=1}^{\dim \mathscr{G}} F_a T^a$ on a probability space $(\Omega, \mathscr{F}, E)$, where $\{F_a\}_{a=1}^{\dim \mathscr{G}}$ are independent real valued white noises on $\mathbf{R}^2$. That is, each $F_a$ satisfies*

1. *For each Borel subset $R \subset \mathbf{R}^2$ with $|R| :=$ Lebesgue measure $(R) < \infty$, $F_a(R)$ is a mean zero Gaussian random variable with variance $|R|$.*
2. *If $R$ and $S$ are disjoint subsets of $\mathbf{R}^2$ each with finite Lebesgue measure, then $F_a(R)$ and $F_a(S)$ are independent and $F_a(R \cup S) = F_a(R) + F_a(S)$.*

*Remark 2.6.* For a simple $L^2$-function $f = \sum_i c_i 1_{R_i}$, put $F(f) = \sum_i c_i F(R_i)$. By taking $L^2$-limits, it is possible to extend $F(f)$ to all functions $f \in L^2(\mathbf{R}^2)$.

Informally, the connection $A = A_1 dx$ may be recovered from the process $F$ by

$$A_1(x, y) = \int_0^y F(x, y')dy'. \tag{2.6}$$

Of course $F$ evaluated at a point is not well defined. To make this more precise, multiply Eq. (2.6) by a test function $g$, integrate over $\mathbf{R}^2$, and use Fubini's theorem (heuristicly) to get

$$A(g) := \int_{\mathbf{R}^2} A(x, y) g(x, y) dx dy = F(\tilde{g}),$$

where

$$\tilde{g}(x, y) = \int_{\mathbf{R}} g(x, y') [1_{0 \leq y \leq y'} - 1_{y' \leq y \leq 0}] dy'.$$

The definition $A(g) = F(\tilde{g})$ is well defined for $g \in \mathscr{S}(\mathbf{R}^2)$–the space of Schwartz test functions.

**Definition 2.7.** *The (gauge fixed) continuum $\mathrm{YM}_2$-measure $(\mu)$ is the distribution measure on $\mathrm{Re}\,\mathscr{S}'(\mathbf{R}^2) \otimes \mathscr{G}$ of the random variables $\{A(g)\}_{g \in \mathrm{Re}\,\mathscr{S}'(\mathbf{R}^2)}$.*

In the sequel, no distinction will be made between the measures $\mu$ and $E$, and both will be called the $YM_2$-measure.

## 3. Parallel Translation along Horizontal Paths

As already noted it is only the expectations of Gauge invariant functions that have the interpretation of being expectations with respect to the $YM_2$-measure informally defined in (1.1). The standard examples of such functions in the physics literature are the Wilson loop variables $W(\sigma)$, where $\sigma$ is a closed curve in the plane. The Wilson loop variable is defined to be the trace in some representation of the parallel translation operator around the curve $\sigma$. The purpose of this section is to define parallel translation with respect to the random connections $A$. Because of the singular nature of the random connections ($A$'s) it is necessary to define parallel translation by stochastic differential equations rather than ordinary differential equations.

**Definition 3.1.** A horizontal path in $\mathbf{R}^2$ is a path $\hat{\sigma}$ which may be written in the form $\hat{\sigma}(t) = (t, \sigma(t))$ for some continuous function $\sigma: [a, b] \to \mathbf{R}$.

To avoid a clutter of notation the function $\sigma$ and the path $\hat{\sigma}$ will both be denoted by $\sigma$. It should be clear from the context which meaning of $\sigma$ is being used. Parallel translation along an arbitrary curve will be defined in terms of parallel translation along these horizontal curves.

To motivate the definition of parallel translation along horizontal paths, I will follow the discussion in [34]. First recall that the parallel translation operator ($P_t = P_t^A(\sigma)$) with respect to a smooth connection $A$ along a curve $\sigma$ parameterized on $[a, b]$ is the solution to the differential equation

$$\frac{d}{dt} P_t + A\langle \sigma'(t) \rangle P_t = 0 \quad \text{with} \quad P_a = e \in G,$$

where $e$ is the identity in $G$. For $A \in \mathscr{A}_f$ and $\sigma$ a horizontal curve the differential equation becomes

$$\frac{d}{dt} P_t + A_1(t, \sigma(t)) P_t = 0.$$

Put $R^\sigma(t)$ to be the region in the plane bounded by the lines $x_1 = a$, $x_1 = t$, $x_2 = 0$ and the path $\sigma|_{[a,t]}$. Let $R_+^\sigma(t)(R_-^\sigma(t))$ be the portion of $R^\sigma(t)$ contained in the upper (lower) half plane. With this notation and the fundamental theorem of calculus we may rewrite $A_1(t, \sigma(t))$ as:

$$\begin{aligned} A_1(t, \sigma(t)) &= \int_0^{\sigma(t)} \partial_2 A_1(t, s) ds = \frac{d}{dt} \int_a^t \int_0^{\sigma(t)} F(u, s) du ds \\ &= \frac{d}{dt} \left\{ \int_{R_+^\sigma(t)} - \int_{R_-^\sigma(t)} F(u, s) du ds \right\} = \frac{d}{dt} \{ F(R_+^\sigma(t)) - F(R_-^\sigma(t)) \} \\ &=: \frac{d}{dt} M^\sigma(t), \end{aligned}$$

where $F = F_{21} = \partial_2 A_1$. So the equation for parallel translation may be written as

$$\frac{d}{dt}P_t + \frac{d}{dt}M^\sigma(t)\cdot P_t = 0.$$

In the case where $F$ is the $\mathscr{G}$-valued white noise, $M^\sigma(t)$ is a martingale, for which the paths are differentiable almost nowhere. Hence we are forced to consider the above equation as a stochastic differential equation. We now study the process $M^\sigma(t)$.

**Proposition 3.2 (Martingale).** *Let $F$ be the $\mathscr{G}$-valued white noise (Definition 2.5), and $\sigma$ be a horizontal path, then there exists a version $M^\sigma(t) = \sum_{a=1}^{\dim\mathscr{G}} M_a^\sigma(t) T^a$ of $F(R_+^\sigma(t)) - F(R_-^\sigma(t))$ such that the process $\{M^\sigma(t)\}_{t=a}^b$ is continuous $E$-a.s. Furthermore the process $M^\sigma(t)$ satisfies:*

1. *Each component $M_a^\sigma$ is a time changed Brownian motion.*
2. *$M^\sigma(t)$ has independent increments.*
3. *$M^\sigma(t)$ is a martingale with respect to the filtration $\mathscr{F}_t^a = \bigcap_{\varepsilon>0} \sigma(F(R)$: $R \subset \{(x_1, x_2): a \leqq x_2 < t + \varepsilon\}$.)*
4. *Suppose that $\tau$ is another horizontal curve on $[a, b]$, put*

$$dM^\tau(t)dM^\sigma(t) := \sum_{a,b} dM_a^\sigma(t)dM_b^\tau(t)T^aT^b,$$

*where $dM_a^\sigma(t)$ is the Itô differential of $M_a^\sigma$. Then*

$$dM^\sigma(t)dM^\tau(t) := d\langle M^\sigma, M^\tau\rangle(t) = C1_{\sigma(t)\tau(t)\geqq 0}\min(|\sigma(t)|, |\tau(t)|)dt, \tag{3.1}$$

*where $C := \sum_{a=1}^{\dim\mathscr{G}} (T^a)^2$.*

*Proof*. For the moment let $M^\sigma(t)$ be any version of $F(R_+^\sigma(t)) - F(R_-^\sigma(t))$. Set $\tau(t) = |R^\sigma(\cdot)|^{-1}(t)$, where $|S|$ denotes the Lebesgue measure of a subset $S \subset \mathbf{R}^2$. Then one checks that each component of $M^\sigma(\tau(t))$ is a mean zero Gaussian process with the same covariance as the Brownian motion, and so is a Brownian motion. It is standard that the Brownian motion can be chosen to be continuous, and this is done in such a way that $M^\sigma(t)$ is $\mathscr{F}_t^a$-measurable. From these facts and standard facts about Brownian motion, the first three items follow easily.

The martingales $M_a^\sigma$ and $M_b^\tau$ have independent increments, and hence their square differentials may be computed as $dM_a^\sigma dM_b^\tau = (d/dt)E(M_a^\sigma M_b^\tau)dt$. It is now an easy computation using the definition of the white noise to find

$$E(M_a^\sigma(t)M_b^\tau(t)) = \delta_{ab}\int_a^t 1_{\sigma(s)\tau(s)\geqq 0}\min(|\sigma(s)|, |\tau(s)|)ds. \quad \text{Q.E.D.}$$

**Definition 3.3.** *Stochastic parallel translation along the horizontal curve $\sigma$ on $[a, b]$ is the continuous process $P_t(\sigma)$ which solves the stochastic differential equation:*

$$dP_t(\sigma) + dM^\sigma(t)\circ P_t(\sigma) = 0, \quad \text{and} \quad P_a(\sigma) = e \in G,$$

*where $M^\sigma$ is the martingale defined in Proposition 3.2. The symbol "$\circ$" indicates that the differential are to be taken in the sence of Stratonovich. In terms of Ito differentials* *the equation becomes*:

$$dP_t(\sigma) + dM^\sigma(t)P_t(\sigma) - \tfrac{1}{2}da^\sigma(t)C_\rho P_t(\sigma) = 0,$$

*where* $a^\sigma(t) = |R^\sigma(t)|$. *We will abbreviate* $P_b(\sigma)$ *by* $P(\sigma)$.

Proposition 3.2 and the definition of the Stratonovich differential in terms of the Ito differential ($dX \circ Y = dX\,Y + \frac{1}{2}dX\,dY$) was used to go from the first equality to the second equality.

*Remark 3.4*. The existence and uniqueness to such stochastic differential equations is standard, see [38]. It is also well known with the choice of the Stratonovich differential, the operators $P_t(\sigma)$ will remain in the compact group $G$. For this reason there is no blow up in the solution to the parallel translation stochastic differential equation, and hence $P_t(\sigma)$ is defined on $[a, b]$. If the Ito differential were used rather than the Stratonovich differential, these remarks would no longer be true.

**Proposition 3.5.** *Suppose that* $\sigma \in C([a,b],\mathbf{R})$ *and* $\tau \in C([b,c],\mathbf{R})$ *are two functions satisfying* $\sigma(b) = \tau(b)$. *Let*

$$\tau\sigma(t) = \begin{cases} \tau(t), & \text{if } t \in [b,c]; \\ \sigma(t), & \text{if } t \in [a,b], \end{cases}$$

*and* $\hat{\tau}\hat{\sigma} := \widehat{\tau\sigma}$. *Then* $P(\tau\sigma) := P_c(\tau\sigma) = P_b(\tau)P_c(\sigma) =: P(\tau)P(\sigma)$.

*Proof*. Noting that $M^{\tau\sigma}(t) = M^\sigma(b) + M^\tau(t)$ for all $t \in [b, c]$, it follows that $P_t(\tau\sigma)$ and $g(t) := P_t(\tau)P_b(\sigma)$ satisfies the same stochastic parallel translation equation on $[b, c]$. Since $g(b) = P_b(\tau\sigma)$, the proposition follows from the uniqueness theorem for solution to stochastic differential equations. Q.E.D.

So far we have defined parallel translation for horizontal paths moving from left to right. The next proposition shows that, as one would expect, parallel translation along a horizontal path from right to left is the inverse of parallel translation along the same path from left to right.

**Proposition 3.6.** *Let* $\sigma \in C(I, \mathbf{R})$, *where* $I = [a, b]$, *then parallel translation along the path determined by the graph of* $\tau$ *moving from right to left is equal to* $P_b(\sigma)^{-1}$. *More precisely if* $g(t)$ *is the solution to the stochastic differential equation*

$$d_-g(t) + d_-M^\sigma(t) \circ g(t) = 0, \quad \text{and} \quad g(b) = e,$$

*then* $g(a) = P_b(\sigma)^{-1}$. *In this last equation,* $d_-$ *denotes the backwards pointing differential.*

*Remark 3.7*. Note that $M^\sigma(t) - M^\sigma(b)$ is a reverse martingale with respect to the filtration $\{\mathscr{F}_t^b\}_{t \in I}$. Hence the parallel translation equation for $g(t)$ is well defined.

*Proof*. Set $h(t) := P_t(\sigma)P_b(\sigma)^{-1}$, then because of Proposition 3.5, $h(t) = P_b(\sigma|_{[t,b]})$ and so is $\mathscr{F}_t^b$-measurable. It is also clear that $h(b) = g(b) = e$, so it suffices to show that $h$ satisfies the same stochastic differential equation as $g$. Because of the symmetry in the definition of the Stratonovich integral one has

$$P_t(\sigma) = e - \int_a^t dM^t(t) \circ P_t(\sigma) = e - \int_a^t d_-M^\sigma(t) \circ P_t(\sigma).$$

Multiply this last equation on the right by $P_b(\sigma)^{-1}$ to conclude that $h(t)$ satisfied $h(t) = e - \int_a^t d_-M^\sigma(t) \circ h(t)$. Hence one finds that

$$h(t) - e = h(t) - h(b) = \int_t^b d_-M^\sigma(t) \circ h(t) = -\int_t^b d_-M^\sigma(t) \circ h(t),$$

which is the same stochastic integral equation satisfied by $g$. So by uniqueness of solutions to stochastic differential equations, it follows that $h = g$ (a.s.). In particular, $g(a) = h(a) = P_b(\sigma)^{-1}$. Q.E.D.

**Definition 3.8.** *A continuous curve in the plane is called admissible if it can be broken into a finite number of pieces consisting of vertical line segments and $C^1$-horizontal curves.*

The parallel translation along an admissible curve is now defined to be products of parallel translations along the horizontal parts of the curve. These products are taken in the order determined by the path. Parallel translation along any vertical segments is defined to be the identity in $G$. This definition of parallel translation is consistent because of Propositions 3.5 and 3.6. Also because of the definitions and these last two propositions, parallel translation along any curve is determined by parallel translation along left to right moving horizontal curves. Because of the independence of $\mathscr{F}_a^b$ and $\mathscr{F}_c^d$ if $[a, b] \cap [c, d] =$ and the continuity of the parallel translation operators along the path, it follows that the random parallel translation operators restricted to curves lying in the vertical strip $\{(x_1, x_2): a \leqq x_1 \leqq b\}$ are independent of those restricted to curves lying in the strip $\{(x_1, x_2): b \leqq x_1 \leqq c\}$. Therefore, in order to understand expectations of parallel translation operators along any admissible curves it is enough to understand parallel translation along horizontal curves in a fixed vertical interval. The key to computing these expectations is Ito's Lemma. The computation of these expectations will be the subject of the next section.

## 4. Expectations of Functions of Parallel Translation

In order to facilitate the application of Itô's lemma, it is helpful to introduce some standard notation.

**Definition 4.1.** *Let $G$ be a Lie group with Lie algebra $\mathscr{G}$ which is taken to be the tangent space to $G$ at the identity. Also suppose that $f$ is a $C^\infty$-function on $G$. Then for each $A \in \mathscr{G}$ there is a unique right invariant vector field (again denoted by $A$) which agrees with $A \in \mathscr{G}$ at the identity. This vector field is given by*

$$Af(g) = \frac{d}{dt}\bigg|_0 f(\exp(tA)g),$$

*where* exp *is the exponential on $\mathscr{G}$.*

*Remark 4.2.* If $G$ is a group of matrices contained $\mathscr{M}$—the matrix algebra on some finite dimensional vector space, then the exponential function is the ordinary exponential. Furthermore, if $f$ is a $C^\infty$-function on $\mathscr{M}$ (a matrix algebra) and $A \in \mathscr{G}$

$\subset \mathcal{M}$, then

$$Af(g) = \frac{d}{dt}\bigg|_0 f(e^{tA}g) = \frac{d}{dt}\bigg|_0 f(g + tAg) = f'(g)\langle Ag\rangle, \tag{4.1}$$

where $f'(g)\langle\cdot\rangle$ is the differential as a function on the vector space $\mathcal{M}$.

It will also be necessary to know how to compute iterated applications of right invariant vector fields on functions defined on $\mathcal{M}$.

**Proposition 4.3.** *Let $f$ be a $C^\infty$-function on $\mathcal{M}$ and $A, B \in \mathcal{G}$, then*

$$ABf(g) = f''(g)\langle Ag, Bg\rangle + f'(g)\langle BAg\rangle.$$

*Proof.*

$$\begin{aligned} ABf(g) &:= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 f(e^{tB}e^{sA}g) = \frac{d}{ds'}\bigg|_0 f'(e^{sA}g)\langle Be^{sA}g\rangle \\ &= f''(g)\langle Ag, Bg\rangle + f'(g)\langle BAg\rangle. \quad \text{Q.E.D.} \end{aligned}$$

**Lemma 4.4.** *Suppose that G is a compact matrix group contained in $\mathcal{M}$ and $f$ is a $C^\infty$-function on G. Also suppose that $M(t)$ is a $\mathcal{G}$-valued continuous martingale (on some probability space $(\Omega, \mathcal{F}, E)$), and that $P_t$ is the solution to the stochastic parallel translation equation*

$$dP_t + dM(t)\circ P_t = 0, \quad \text{and} \quad P_0 = e \in G,$$

*then the differential of $f(P_t)$ is*

$$\begin{aligned} df(P_t) &= -dM(t)f(P_t) + \tfrac{1}{2}(dM(t))^2 f(P_t) \\ &:= -\sum_a T^a f(P_t)dM_a(t) + \tfrac{1}{2}\sum_{a,b} T^aT^b f(P_t)dM_a(t)dM_b(t) \\ &=: -dM(t)\circ f(P_t), \end{aligned}$$

*where $M(t) = \sum M_a(t)T^a$, and $\{T^a\}$ is a basis for $\mathcal{G}$.*

*Proof.* Since G is an embedded submanifold of $\mathcal{M}$, it is possible to choose a $C^\infty$-function $F$ on $\mathcal{M}$ which agrees with $f$ on G. Thus $f(P_t) = F(P_t)$, and so it is enough to compute the differential of $F(P_t)$. By Itô's lemma:

$$dF(P_t) = F'(P_t)\langle dP_t\rangle + \tfrac{1}{2}F''(P_t)\langle dP_t, dP_t\rangle,$$

which may be rewritten as

$$\begin{aligned} dF(P_t) &= -F'(P_t)\langle P_t dM(t)\rangle \\ &\quad + \tfrac{1}{2}(F'(P_t)\langle P_t(dM(t))^2\rangle + F''(P_t)\langle dM(t), dM(t)\rangle) \\ &= -dM(t)F(P_t) + \tfrac{1}{2}(dM(t))^2 F(P_t) \\ &= -dM(t)f(P_t) + \tfrac{1}{2}(dM(t))^2 f(P_t), \end{aligned}$$

using the stochastic differential equation for $P_t$, Remark 4.2 and Proposition 4.3. Q.E.D.

If one were to use a more intrinsic definition of solutions to stochastic differential equations taking values in a manifold, this last proposition would essentially become a definition. For this point of view the reader is referred to Ikeda and

Watanabe [38]. Now suppose that $\sigma_1, \ldots, \sigma_n$ are horizontal curves in $\mathbf{R}^2$ defined on the interval $I = [a, b]$. Set $M_i(t) = M^{\sigma_i}(t)$ and $P_t^i = P_t(\sigma_i)$ for $1 \leqq i \leqq n$. The next corollary is a direct consequence of the above proposition applied to the Lie group $G^n$ and the $\mathscr{G}^n$-martingale $M(t) = (M_1(t), \ldots, M_n(t))$. In order to state the result we need:

**Definition 4.5.** *For $A \in \mathscr{G}$, set $A_i$ to be the right invariant vector field on $G^n$ defined by $A_i f(g_1, \ldots, g_n) = d/dt|_0 f(g_1, \ldots, e^{tA} g_i, \ldots, g_n)$.*

Note that the vector fields $A_i$ and $B_j$ commute if $i \neq j$, where $A$ and $B$ are in $\mathscr{G}$.

**Corollary 4.6.** *Suppose that $f$ is a $C^\infty$-function on $G^n$, then*

$$\begin{aligned} df(P_t^1, \ldots, P_t^n) = {} & (dM_1(t) + \cdots + dM_n(t)) f(P_t^1, \ldots, P_t^n) \\ & + \tfrac{1}{2}(dM_1(t) + \cdots + dM_n(t))^2 f(P_t^1, \ldots, P_t^n), \end{aligned}$$

*where $dM_i(t) = \sum_a T_i^a (dM_i)_a$ is a right invariant differential vector field acting on the $i^{\text{th}}$ variable of $G^n$.*

The next step is to simplify the expression for the second order differential vector field $(dM_1(t) + \cdots + dM_n(t))^2$. This is easily done using Proposition 3.2. In order to state the result it is useful to introduce a number of different "Laplacians" on $G^n$.

**Definition 4.7.** *Let $\{T^a\}_{a=1}^{\dim \mathscr{G}}$ be a $(\cdot,\cdot)_\rho$-orthonormal basis for $\mathscr{G}$. The Laplacian $\Delta$ on G is defined to be the second order differential operator*

$$\Delta = \sum_{a=1}^{\dim \mathscr{G}} (T^a)^2,$$

*where $T^a$ is considered to be a right invariant vector field on G. More generally, the $i^{\text{th}}$ Laplacian $(\Delta_i)$ is the second order differential operator on $G^n$ defined by*

$$\Delta_i = \sum_{a=1}^{\dim \mathscr{G}} (T_1^a + \cdots + T_i^a)^2.$$

*Remark 4.8.* It is easy to check that the Laplacians defined above are independent of the orthonormal basis chosen for $\mathscr{G}$.

**Proposition 4.9.** *Suppose that $\sigma_i$'s are horizontal curves lying in the upper half plane parameterized by $t = x_1$. Also assume at time $t$ the $\sigma$'s are labeled such that $\sigma_1(t) \geqq \sigma_2(t) \geqq \cdots \geqq \sigma_n(t)$. Then, in the notation of Corollary 4.6:*

$$(dM_1(t) + \cdots + dM_n(t))^2 = \sum_{i=1}^{n} (\sigma_i(t) - \sigma_{i+1}(t)) \Delta_i dt,$$

*where $\sigma_{n+1}(t) := 0$ for all $t$.*

*Proof.* The proof will go by induction on $n$. $n := 1$,

$$\begin{aligned} (dM_1(t))^2/dt &= \left( \sum_a dM_a^{\sigma_1}(t) T_1^a \right)^2 \\ &= \sum_{a,b} dM_a^{\sigma_1}(t) dM_b^{\sigma_1}(t) T_1^a T_1^b \\ &= \sigma_1(t) \sum_a (T_1^a)^2 = \sigma_1(t) \Delta_1. \end{aligned}$$

Now assume that the proposition is true for $n$-curves, and consider the case of $(n+1)$-curves:

$$\begin{aligned}(dM_1(t)+\cdots+dM_{n+1}(t))^2 &= (dM_1(t)+\cdots+dM_n(t))^2+(dM_{n+1}(t))^2\\ &\quad+2\sum_{i=1}^{n} dM_i(t)dM_{n+1}(t)\\ &=\sum_{i=1}^{n-1}(\sigma_i(t)-\sigma_{i+1}(t))\Delta_i dt+\sigma_n(t)\Delta_n dt\\ &\quad+(dM_{n+1}(t))^2+2\sum_{i=1}^{n} dM_i(t)dM_{n+1}(t)\\ &=\sum_{i=1}^{n}(\sigma_i(t)-\sigma_{i+1}(t))\Delta_i dt+\sigma_{n+1}(t)\Delta_n dt\\ &\quad+\sigma_{n+1}(t)\left(\sum_a (T^a_{n+1})^2+2\sum_{i,a} T^a_i T^a_{n+1}\right)dt\\ &=\sum_{i=1}^{n}(\sigma_i(t)-\sigma_{i+1}(t))\Delta_i dt+\sigma_{n+1}(t)\Delta_{n+1}dt\\ &=\sum_{i=1}^{n+1}(\sigma_i(t)-\sigma_{i+1}(t))\Delta_i dt,\end{aligned}$$

where $\sigma_{n+2}(t)$ is now defined to be zero. To carry out these computations, we have made repeated use of Proposition 3.2 and the fact that $M_a^{\sigma_i}(t)$ and $M_b^{\sigma_j}(t)$ are independent and hence have zero bracket if $a \neq b$. Q.E.D.

Before we can make use of this last result it will be necessary to know that the different $\Delta_i$'s commute with one another. This fact follows from the "infinitesimal braid relations" (see Fröhlich [32]) of the next proposition. These relations are also used in Gross, King, and Sen Gupta [34]. The author is grateful to L. Gross for showing me these relations.

**Proposition 4.10 (Infinitesimal Braid Relations).** *Let $T_i^a$ denote the action of $T^a$ acting on the $i^{th}$ variable of $G^n$ as in Definition 4.5. Then $\sum_a [T_i^a T_j^a, T_i^b + T_j^b] = 0$, for all $i$ and $j$.*

*Proof.* Let $\{f^{abc}\}$ be the structure constants for the $\mathscr{G}$ with respect to the $T^a$'s. Using the identities $[AB, C] = A[B, C] + [A, C]B$ valid for linear operators $A, B$ and $C$, and $[T_i^a, T_j^b] = 0$ if $i \neq j$, one shows that

$$\sum_a [T_i^a T_j^a, T_i^b + T_j^b] = \sum_{ac} (f^{abc} + f^{cba}) T_i^a T_j^b$$

for $i \neq j$ and twice the right-hand side if $i = j$. So to finish the proposition it suffices to show that $f^{abc} = -f^{cba}$. Now recall that the $T^a$'s were chosen such tat $-\operatorname{trace}(T^a T^b) = \delta_{ab}$, so that

$$f^{abc} = -\operatorname{trace}([T^a, T^b]T^c) = -\operatorname{trace}(T^a[T^b, T^c]) = f^{bca}.$$

Because of the skew symmetry of the bracket it follows that $f^{abc} = f^{bca} = -f^{cba}$. Q.E.D.

**Corollary 4.11.** *The operators ($\Delta_i$'s) commute with one another.*

*Proof.* Let $j > i$, then

$$\Delta_j = \Delta_i + \Delta_{>i} + 2\sum_a (T_1^a + \cdots + T_i^a)(T_{i+1}^a + \cdots + T_j^a),$$

where $\Delta_{>i} := \sum_a (T_{i+1}^a + \cdots + T_j^a)$. So $[\Delta_i, \Delta_j] = 2\sum_a [\Delta_i, T_1^a + \cdots + T_i^a](T_{i+1}^a + \cdots + T_j^a)$. Finally the expression $[\Delta_i, T_1^a + \cdots + T_i^a]$ is easily shown to be zero by expanding out $\Delta_i$ and then using the infinitesimal braid relations and the commutativity of $T_i^a$ and $T_j^b$ for $i \neq j$. Q.E.D.

**Theorem 4.12.** *Let $\{\sigma_i\}_i^n$ be a collection of horizontal curves in the upper half plane which, when considered as real value functions, satisfy $\sigma_1(t) \geqq \sigma_2(t) \geqq \cdots \geqq \sigma_n(t) \geqq 0$ for $t \in [a, b]$—the domain of the $\sigma_i$'s. Let $P_t^i = P_t(\sigma_i)$ be parallel translation along $\sigma_i$, and let $f$ be a bounded measurable function on $G^n$. Then*

$$Ef(P_t^1, \ldots, P_t^n) = \int_{G^n} f(g_1, \ldots, g_n) \prod_{i=1}^n Q_{A_{i,i+1}}(g_i g_{i+1}^{-1}) dg_1 \cdots dg_n, \tag{4.2}$$

*where $Q_t$ is the convolution kernel for the operator* $\exp t\Delta/2$ *and $A_{ij} := \int_a^b (\sigma_i(t) - \sigma_j(t))dt$—the signed area between the horizontal curves $\sigma_i(t)$ and $\sigma_j(t)$.*

*Remark 4.13.* The notation $e^{t\Delta/2}$ stands for the contraction semigroup associated with the heat equation $\partial_t u = \frac{1}{2}\Delta u$. The relationship between $Q_t$ and $e^{t\Delta/2}$ is $e^{t\Delta} f(g) = \int_G Q_t(h^{-1}g) f(h) dh$. The fact that this semigroup has a convolution kernel is a consequence of the Laplacian ($\Delta$) on $G$ commuting with left and right multiplication by elements of $G$. The explicit series expression for $Q_t$ is

$$Q_t(g) = \sum_{\tau \in \Gamma(G)} \exp(t c_\tau/2) \cdot d_\tau \chi_\tau(g),$$

where $c_\tau$ is defined in Remarks 2.2. Recall that $\Gamma$ is a set of inequivalent irreducible representations of $G$.

*Proof.* Define a new function $F(t, g_1, \ldots, g_n)$ on $I \times G^n$ as the solution to the backwards heat equation

$$\partial_t F(t, g_1, \ldots, g_n) = -\frac{1}{2}\sum_{i=1}^n \Delta_i F(t, g_1, \ldots, g_n),$$

$$F(b, g_1, \ldots, g_n) = f(g_1, \ldots, g_n).$$

The solution $F$ can be expressed more suggestively as

$$F(t, g_1, \ldots, g_n) = \prod_{i=1}^n (e^{(1/2)A_i(t)\Delta_i} f)(g_1, \ldots, g_n), \tag{4.3}$$

where $A_i(t) := \int_t^b \sigma_i(t) - \sigma_{i+1}(t)dt$—the area between the horizontal paths $\sigma_i(t)$ and $\sigma_{i+1}(t)$ on the $x_1$-interval $[t, b]$. The operator $e^{t1/2\Delta_i}$ denotes the semigroup on $G^n$ generated by $\Delta_i$. The representation in (4.3) for $F$ is valid because the different Laplacians all commute—Corollary 4.11.

At this point the careful reader may be concerned that $e^{t/2\Delta_i}$ may not be well defined because $\Delta_i$ is a degenerate elliptic operator. This problem is easily overcome by realizing that it is enough to prove the proposition for an algebra $\mathscr{A}$ of functions $f$ which generate the measurable function on $G^n$. A particularly nice algebra is the finite sums of functions of the form $f(g_1, \dots, g_n) = \prod_{i=1}^{n} f_i(g_i)$, where each $f_i$ is a matrix element of a finite dimensional representation of $G$. Since $A\tau_{ij} = \sum_k \tau_{ik}(A)\tau_{kj}$, if $\tau_{ij}$ are the matrix elements of a representation $\tau$ and $A \in \mathscr{G}$ is considered to be a right invariant vector field. Hence, it follows that this algebra is invariant under the action of $\Delta_i$ for each $i$. Since this algebra of functions is $\Delta_i$-invariant, the heat equation reduces to an ordinary linear differential equation. Finally, by the Peter-Weyl theorem (see [20]) the algebra $\mathscr{A}$ is uniformly dense in the continuous function on $G^n$, and hence the algebra generates the measurable functions on $G^n$.

Now apply Itô's lemma to compute the differential of the process $F(t, P_t^1, \dots, P_t^n)$. One gets the expression in Corollary 4.6 with $f \to F$ plus a term $\partial_t F(t, P_t^1, \dots, P_t^n)dt$. Because of Proposition 4.9 and the definition of $F$, only the martingale terms survive to leave:

$$dF(t, P_t^1, \dots, P_t^n) = (dM_1(t) + \cdots + dM_n(t))F(t, P_t^1, \dots, P_t^n).$$

This shows that the process $F(t, P_t^1, \dots, P_t^n)$ is a martingale, and in particular

$$\begin{aligned} Ef(P_b^1, \dots, P_b^n) &= EF(b, P_b^1, \dots, P_b^n) = EF(a, P_a^1, \dots, P_a^n) \\ &= EF(a, e, \dots, e) = F(a, e, \dots, e). \end{aligned}$$

The theorem now follows from the next lemma. Q.E.D.

**Lemma 4.14.**

$$\begin{aligned} F(a, e, \dots, e) &= \prod_{i=1}^{n} (e^{(A_i(a))/2\Delta_i} f)(e_1, \dots, e) \\ &= \int_{G^n} f(g_1, \dots, g_n) \prod_{i=1}^{n} Q_{A_{ij+1}}(g_i g_{i+1}^{-1}) dg_1 \cdots dg_n. \end{aligned}$$

*Proof.* Let $D_i \colon G^{n-i+1} \to G^n$ be defined by $D_i(g, g_{i+1}, \dots, g_n) := (g, \dots, g, g_{i+1}, \dots, g_n)$, and set $A_i := A_{ij+1} := A_i(a)$. The first step in the proof is to show that $(e^{t\Delta_i} f) \circ D_i = e^{t\Delta_1}(f \circ D_i)$. This follows from the fact that $(\Delta_i f) \circ D_i = \Delta_1(f \circ D_i)$ which is seen by the following computation:

$$\begin{aligned} (\Delta_i f)(g, \dots, g, g_{i+1}, \dots, g_n) &:= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \sum_a f(e^{(s+t)T^a} g, \dots, e^{(s+t)T^a} g, g_{i+1}, \dots, g_n) \\ &= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \sum_a f \circ D_i(e^{(s+t)T^a} g, g_{i+1}, \dots, g_n) \\ &= \Delta_1(f \circ D_i)(g, g_{i+1}, \dots, g_n). \end{aligned}$$

Now $F(a, e, \dots, e)$ may be expressed as

$$F(a,e,\ldots,e)=\left(e^{A_n/2\,\Delta_n}\prod_{i=1}^{n-1}e^{A_i/2\,\Delta_i}f\right)\circ D_n(e)$$

$$=e^{A_n/2\,\Delta_1}\left(\prod_{i=1}^{n-1}e^{A_i/2\,\Delta_i}f\circ D_n\right)(e)$$

$$=\int_G Q_{A_n}(g_n^{-1})\left(\prod_{i=1}^{n-1}e^{A_i\Delta_i}f\right)(g_n,\ldots,g_n)dg_n$$

$$=\int_G Q_{A_n}(g_n^{-1})\left(\prod_{i=1}^{n-1}e^{A_i/2\,\Delta_i}f\right)\circ D_{n-1}(g_n,g_n)dg_n$$

$$=\int_G Q_{A_n}(g_n^{-1})\left(e^{A_{n-1}\Delta_{n-1}}\prod_{i=1}^{n-2}e^{A_i/2\,\Delta_i}f\right)\circ D_{n-1}(g_n,g_n)dg_n$$

$$=\int_G Q_{A_n}(g_n^{-1})e^{A_{n-1}/2\,\Delta_1}\left(\prod_{i=1}^{n-2}(e^{A_i/2\,\Delta_i}f\circ D_{n-1}\right)(g_n,g_n)dg_n$$

$$=\int_G Q_{A_n}(g_n^{-1})Q_{A_{n-1}}(g_ng_{n-1}^{-1})\left(\prod_{i=1}^{n-2}e^{A_i/2\,\Delta_i}f\right)\circ D_{n-1}(g_{n-1},g_n)dg_n.$$

After iterating these steps $n$-times one arrives at the claimed result. Q.E.D.

Theorem 4.12 gives a method for computing the expectation $Ef(P(\sigma_1),\ldots,P(\sigma_n))$ for any function (f) and "admissible collection" of curves $\sigma_1,\ldots,\sigma_n$. The goal of the next two sections is to simplify these results for the special case that $f(P(\sigma_1),\ldots,P(\sigma_n))$ is a "gauge invariant" function. The idea will be to undo the gauge fixing which is inherent in Theorem 4.12. This will yield a simple expression for the final result. The main ideas will be borrowed from the lattice gauge theory techniques and will also be used for the lattice computations.

## 5. Tree Theorem

For the purposes of this section let $\Lambda$ be a finite set and let $B$ be an oriented graph on $\Lambda$. So $B$ is a finite set of "bonds" (b) satisfying:

1. There is a surjection (the orientation reversing map) $b\to\bar{b}:B\to B$ for which $\bar{b}\neq b$ and $\bar{\bar{b}}=b$.
2. There are mappings $b\to b^i$, and $b\to b^f$ from $B$ onto $\Lambda$ which satisfy $\bar{b}^i=b^f$. We say that $b^i(b^f)$ is the initial (final) point of the bond (b). The pair $\{b^i,b^f\}$ are called the end points of $b$.

A path $(\sigma)$ in $B$ is a finite sequence $(b_{n(\sigma)},\ldots,b_1)$ of bonds in $B$ which satisfy $b^i_{i+1}=b^f_i$ for $1\leq i<n(\sigma)$. Such a path will be denoted by $b_{n(\sigma)}\cdots b_1$ and will be called a path from $\sigma^i:=b^i_1$ to $\sigma^f:=b^f_n$. The orientation reversing map will be extended to paths by $\bar{\sigma}:=\bar{b}_1\cdots\bar{b}_{n(\sigma)}$.

If $T$ is a subset of $B$, let $T'$ denote the set of endpoints of all of the bonds in $T$. Finally if $b\in B$ let $[b]=\{b,\bar{b}\}$. We now are in a position to define a tree.

**Definition 5.1 (Tree).** *A subset $T$ of $B$ is called a tree if $\bar{T}=T$, and there are no loops in*

*T—that is there is no closed path $\sigma = b_n \cdots b_1$ with $b_i \in B$ and all $[b_i]$ distinct subsets of T. The tree is said to be connected if there is a path in T which joins any two points of $T'$.*

*Remark 5.2.* It is easily checked that for a connected tree $T$, there is a unique path $\sigma = b_n \cdots b_1$ in $T$ between any two points of $T'$ such that all of the $[b_i]$ are distinct.

Now suppose that $G$ is a compact Lie group as above. Set $\Omega = \{g: B \to G \mid g(\bar{b}) = g(b)^{-1} \forall b \in B\}$. If $T$ is a tree contained in $B$, set $D_{Tg} = \prod_{[b] \notin T} dg([b]) \prod_{[b] \in T} \delta_e(dg[b])$, where $dg([b]) = \text{Haar}(dg(b))$ and $\delta_e(dg) = \delta(g)\,\text{Haar}(dg)$ is the point mass at $e = id \in G$. (This is well defined, since the point mass at the identity and Haar measure are both invariant under $g \to g^{-1}$.) If $T = B$ we will write $Dg = D_B g$. The group element $g(b)$ is to be interpreted as parallel translation along the bond $b$.

In this setting the gauge transformations are elements of the set $G^\Lambda := \{\theta: \Lambda \to G\}$ which act on $\Omega$ via $g^\theta(b) = \theta(b^f)^{-1} g(b)\theta(b^i)$. Haar measure on $G^\Lambda$ will be denoted by $D\theta$ which is equal to $\prod_{x \in \Lambda} \text{Haar}(d\theta(x))$.

**Theorem 5.3 (Tree).** *Let $f$ be a measurable function on $\Omega$, $T = \bigcup T_i$ be a tree with connected components $T_i$ in $B$, and fix a "root" $x_i \in T_i'$ for each $i$. Then*

$$\int_\Omega f(g)Dg = \int_{\Omega \times G^\Lambda} f(g^\theta) D_T g D\theta = \int_{\Omega \times G^\Lambda} f(g^\theta) \prod_i \delta_e(\theta(x_i)) D_T g D\theta. \tag{5.1}$$

*Proof*. For simplicity I will give the proof for the case that the tree has only one connected component and the root of $T$ is labeled $x_r$. For the moment consider the third integral above with the $\theta$-integrals fixed. Make the change of variables $g(b) \to \theta(b^f) g(b) \theta(b^i)^{-1}$ for all $b \notin T$. By invariance of Haar measure, this operation leaves the integral unchanged. With this change of variables

$$g^\theta(b) \to \begin{cases} g(b), & \text{if } b \notin T; \\ \theta(b^f)\theta(b^i)^{-1}, & \text{otherwise,} \end{cases}$$

$D_T g$ almost everywhere. Now fix the $g$-integrals and choose an outer bond $\beta \in T$. That is $\beta$ is a bond in $T$ which satisfies the property; if $b$ is any bond in $T$ with $b^f = \beta^f$ then $b = \beta$. We further assume that $b^f \neq x_r$. It is always possible to find such a bond. Now make the change of variables $\theta(\beta^f) \to \theta(\beta^f)\theta(\beta^i)$, and rename $\theta(b^f) = g(\beta)$. Setting $T_1 = T \setminus [\beta]$, it is now easy to check that

$$\int_{\Omega \times G^\Lambda} f(g^\theta) \delta_e(\theta(x_r)) D_T g D\theta = \int_{\Omega \times G^\Lambda} f(g^\theta) \delta_e(\theta(x_r) D_{T_1} g D\theta.$$

The proof that the third and first integral in (5.1) are equal may now be completed by induction on the size of the tree $T$. From the argument given above it is easily checked, at the last step in the induction and upon doing all integrals except the $d\theta(x_r)$-integral, that the integrand of the $\theta(x_r)$-integral is of the form: constnat$\cdot \delta(\theta(x_r))$. Hence, one may remove the delta function without altering the value of the integral. Q.E.D.

As an immediate corollary we have:

**Corollary 5.4 (Tree).** *Let $f$ be a function on $\Omega$ for which $f(g^\theta) = f(g)$ for all $\theta \in G^\Lambda$ such* *that* $\theta(x_i) = e \in G$ *for all roots* $x_i$. *Then*

$$\int_\Omega f(g)Dg = \int_\Omega f(g)D_T g.$$

## 6. Gauge Invariant Expectations

In this section, the results of the last two sections will be used to compute the expectations of certain gauge invariant functions. The typical application is to gauge invariant functions of the form $f(P(\sigma_1), \ldots, P(\sigma_n))$, where $\{\sigma_1, \ldots, \sigma_n\}$ is an "admissible collection" of curves.

**Definition 6.1.** *A collection* $\{\sigma_i\}_{i=1}^n$ *of planar curves is an admissible collection if the following conditions hold*:

1. *Each of the curves* $\sigma_i$ *are piecewise* $C^1$.
2. *If* $S := S(\sigma_1, \ldots, \sigma_n)$ *is the union of the images of the curves* $\{\sigma_i\}$, *then the number of connected compounds* $\mathbf{R}^2 \backslash (S \cup \{x\text{-axis}\})$ *is required to be finite.*
3. *Each curve* $\sigma_i$ *is an admissible curve, Definition 3.8.*
4. *If* $\sigma_i$ *is parameterized by arc-length, then there is no time* $\tau$ *and* $\varepsilon > 0$ *such that* $\sigma_i(\tau - t) = \sigma_i(\tau + t)$ *for* $|t| < \varepsilon$.

The last requirement is a condition that the curves do not immediately retrace themselves at any time. This condition is not a real restriction, since by the definition of parallel translation, any such retraces may be removed without affecting the value of $P(\sigma)$.

**Definiton 6.2.** *A planar graph B is a finite directed graph on a discrete subset* $\Lambda$ *of* $\mathbf{R}^2$ *for which the bonds of B are directed curves joining the points of* $\Lambda$. *The bonds* (*curves*) *may only cross one another and themselves at the endpoints. The endpoint maps are required to correspond to the endpoints of the directed curves making up the bonds. Furthermore, it will be assumed that the bonds in B form an admissible collection of curves.*

Our goal is to compute $E[f(P|_B)]$ for any gauge invariant function $f$ on $\Omega = \Omega(B)$, where $B$ is a planar graph. In order to state the result, we need the following notation. If $B$ is a planar graph, set $\mathscr{R} = \mathscr{R}(B)$ to be the collection of bounded connected components of $\mathbf{R}^2 \backslash S(B)$, where $S(B)$ is the union of the images of all the bonds in $b$. For $R \in \mathscr{R}$, the boundary of $R$ may decompose into a number of connected pieces. Let $(\partial R)_i$ denote any path around the $i^{\text{th}}$ connected component of $\partial R$ consistent with an arbitrary but fixed orientation on $R$. Write formally $\partial R = \sum_i (\partial R)_i$, see Fig. 1 for an example. In this figure $R$ is the shaded region. There are three connected boundary components for the $\partial R$. An admissible choice for $\partial R$ is $\partial R = b_1 + b_4 + \bar{b}_2 b_3 b_2$. Notice the $b_2$ terms were included in this expression but according to Theorem 6.4 the $b_2$ bonds could have been omitted.

For any $g \in \Omega$, let $g(\partial R) := \prod_i g((\partial R)_i)$, where order of the product is taken in any fixed arbitrary manner. Admittedly, there is considerable ambiguity in the definition of $g(\partial R)$. It is part of the content of the next theorem that this ambiguity does not affect the final answer.

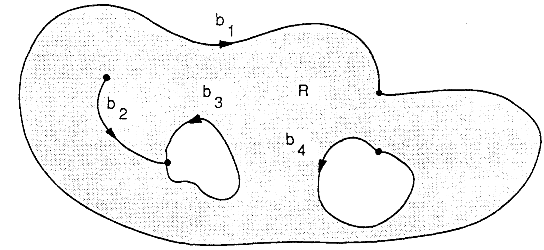


**Fig. 1.** Boundary of $R$

**Definition 6.3.** *Suppose that $B$ is a planar graph on $\Lambda$. A restricted gauge invariant function $f$ on $\Omega(B)=\{g: B\to G \mid g(\bar{b})=g(b)^{-1}\}$ is a function satisfying $f(g^\theta)=f(g)$ for all $\theta\in G^\Lambda$ with the restriction that $\theta(0)=e\in G$ if $0\in\Lambda$.*

**Theorem 6.4 (Gauge Invariant Expectation).** *Let $B$ be a planar graph, and $f$ be a restricted gauge invariant function on $\Omega(B)$. Then*

$$E[f(P|_B)] = \int_{\Omega(B)} f(g) \prod_{R\in\mathscr{R}} Q_{|R|}(g(\partial R))Dg,$$

*where $|R|$ is the area of the region $R$. The right-hand side of this equation is independent of any of the choices made in defining $g(\partial R)$. Furthermore, if $T$ is any tree in $B$, then the above integral is unchanged by "freezing" $g|_T$ to the identity, that is*

$$E[f(P|_B)] = \int_{\Omega(B)} f(g) \prod_{R\in\mathscr{R}} Q_{|R|}(g(\partial R))D_T g,$$

*where, as before, $D_T g = \prod_{[b]\in[T]} \delta_e(dg(b)) \prod_{[b]\notin[T]} dg(b)$.*

The idea of the proof of the theorem is to first compute the expectation using the Horizontal Expectation Theorem 4.12 and then to remove the gauge fixing using the Tree Corollary 5.4. The next step is to do a number of the integrals over the "spurious" bond variables. The remaining spurious variables are gauged away by an application of the Tree Theorem. In order to get the independence of the choices in $g(\partial R)$, it will actually be necessary to enlarge the graph $B$ before we start this procedure in such a way that the regions in $\mathscr{R}$ are all simply connected.

In order to better understand the theorem and the notation, we will pause for some examples. These examples will be concerned with computing $Ef(P(\sigma_1),\ldots,P(\sigma_n))$, where $\{\sigma_i\}_{i=1}^n$ is an admissible collection of planar curves. Given a collection of admissible curves $\{\sigma_i\}_{i=1}^n$ we associate a directed planar graph $B=B(\sigma_1,\ldots,\sigma_n)$ over a subset $\Lambda=\Lambda(\sigma_1,\ldots,\sigma_n)$ as follows. Let $S=S(\sigma_1,\ldots,\sigma_n)$ be as above, then a point $x\in S$ is in $\Lambda$ if either $x$ is an endpoint of some $\sigma_i$ or there is no open neighborhood $N$ of $x$ such that $N\cap S$ is homeomorphic to an open interval. The set $B$ is now composed of the directed curve segments of $S$ which join any two points of $\Lambda$ without passing through any other points of $\Lambda$. With these definitions and the obvious endpoint maps, the set $B$ becomes an oriented planar graph $\Lambda$. To each of the curves $\sigma_i$ there is a naturally defined path in $B$ which corresponds to breaking $\sigma_i$ into directed segments (i.e. bonds) in $B$. We will identify the curves $\sigma_i$ with their paths in $B$.

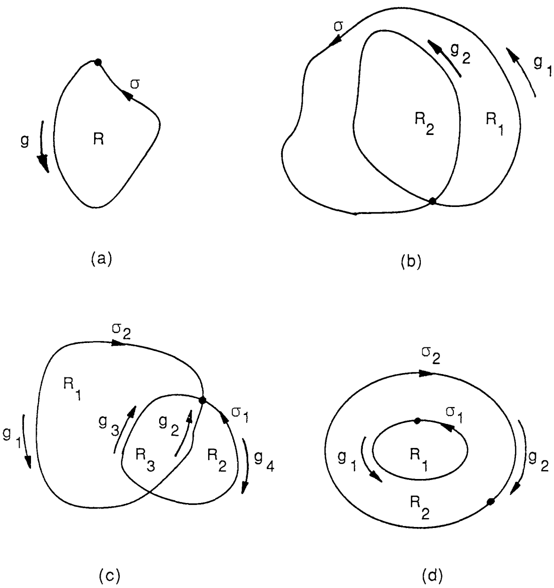


**Fig. 2.** Some simple Wilson loops

For the examples to follow, $\eta$ is taken to be a representation of $G$, and $\chi_1$ and $\chi_2$ are characters on $G$. The reader is referred to Fig. 2 when studying these examples.

(a) Let $\sigma$ be the curve in Fig. 2a then

$$E(\chi_\eta(P(\sigma)) = \int_G \chi_\eta(g) Q_{|R|}(g) dg.$$

If $\eta$ is an irreducible representation of $G$, then the result may be simplified to $\exp(c_\eta |R|/2)$, where $c_\eta$ is the eigenvalue of the Laplacian acting on $\eta$. If $\eta$ is not irreducible, then the result will have the form $\sum_\tau k_\tau \exp(c_\tau |R|/2)$, where the sum is over the irreducible representations of $G$. The coefficients $k_\tau$ are determined by the decomposition $\chi_\eta = \sum k_\tau \chi_\tau$ of the character $\chi_\eta$ into irreducible characters. Thus all but a finite number of the $k_\tau$'s are non-zero.

(b) Let $\sigma$ be the curve in Fig. 2b and $\eta$ be irreducible, then

$$E\chi_\eta(P(\sigma)) = \int_{G^2} \chi_\eta(g_1 g_2) Q_{|R_1|}(g_2^{-1} g_1) Q_{|R_2|}(g_2) dg_1 dg_2$$
$$= d_\eta^{-1} \int_G \chi_\eta(g^2) Q_{|R_2|}(g) dg \cdot \int_G \chi_\eta(g) Q_{|R_1|}(g) dg.$$

Since, $\chi_\eta(g^2) = \text{trace}(\eta(g) \otimes \eta(g)) - 2\text{trace}(\eta(g) \otimes \eta(g)|_{V_\eta \wedge V_\eta})$, it is possible to decompose $\chi_\eta$ into a finite sum of characters corresponding to the decomposition of the tensor representations. Hence, again the general form of the answer will be a finite sum of exponentials of the form $\exp(c_1 |R_1| + c_2 |R_2|)$.

(c) Let $\sigma_1$ and $\sigma_2$ be the curve in Fig. 2c, then

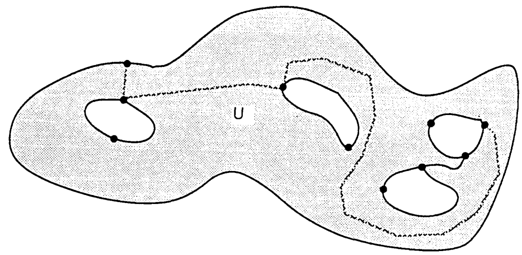


**Fig. 3.** Cutting the region $U$ to make it simple connected

$$E\chi_1(P(\sigma_1))\chi_2(P(\sigma_2))$$
$$= \int_{G^3} \chi_2(g_1^{-1}g_2^{-1})\chi_1(g_4^{-1}g_3^{-1})Q_{|R_1|}(g_3g_1)Q_{|R_3|}(g_3^{-1}g_2)Q_{|R_2|}(g_2g_4)dg_1\cdots dg_4$$
$$= \int_{G^3} Q_{|R_1|}(g_1)\cdots Q_{|R_3|}(g_3)\chi_1(g_3g_2)\chi_2(g_3^{-1}g_1)dg_1\cdots dg_3.$$

Again by representation theory this last result may be expressed as a finite sum of exponentials with arguments linear in the areas of $R_1,\ldots,R_3$.
(d) Let $\sigma_1$ and $\sigma_2$ be the curve in Fig. 2d, then

$$E\chi_1(P(\sigma_1))\chi_2(P(\sigma_2)) = \int_{G^2} \chi_1(g_1)\chi_2(g_2)Q_{|R_1|}(g_1)Q_{|R_2|}(g_2g_1)dg_1dg_2.$$

Notice that the density in this example is not a gauge invariant function.

We now return to the proof of the Gauge Invariant Expectation Theorem 6.4. The proof will proceed by a number of lemmas.

**Lemma 6.5.** *Let $B$ be a planar graph, and $U\in\mathscr{R}(B)$. Suppose that boundary of $U$ is disconnected and $\partial U = \sum_i (\partial U)_i$ with $(\partial U)_i$ a path around the $i^{th}$ component of the boundary of $U$. Assume that $\{\partial U\}$ is enumerated so that $(\partial U)_0$ is the path around the boundary component of the unbounded component of $\mathbf{R}^2\backslash U$. For each $i$ choose $x_i\in(\partial U)'_i$—recall that $(\partial U)'_i$ denotes the set of endpoints of all the bonds in $(\partial U)_i$. Then there exists non-crossing admissible paths $\{b_0,\ldots,b_N\}$ such that $b_i$ is a path from $x_i$ to $x_{i+1}$. The paths $b_i$ lie in $U$ except for the endpoints.*

Such a collection of paths will be called cuts, since they cut the region $U$ so as to make it simply connected, see Fig. 3. In this figure, $U$ is the shaded region, the dots indicate the endpoints of bonds, and the boundary of $U$ is shown by the solid lines. A possible collection of cuts which make $U$ simply connected are shown in dashed lines.

*Proof.* The set $\bar{U}$ is path connected. So the existence of a path $b_0$ from $x_0$ to $x_1$ is assured. This path can be chosen to be admissible with no self crossings. Now assume that $\{b_0,\ldots,b_j\}$ have been chosen to satisfy the lemma. One sees that the closure of $U\backslash$(Images of $b_0,\ldots,b_j$) is still path connected, for if not some pair of $\{b_0,\ldots,b_j\}$ must intersect at some point other than an endpoint. Hence, one may continue inductively and pick $b_{j+1}$. Q.E.D.

Let $B_c$ be the directed graph containing $B$ and a collection of cuts as described in the above lemma. Cuts are added for each $U \in \mathscr{R}$ for which the boundary is not connected—or in other words for which $U$ is not simply connected. This graph $B_c$ satisfies the following property: for each $R \in \mathscr{R}(B_c)$ the $\partial R$ is connected. A graph with this property will be called a BC-graph–BC for boundary connected. For regions $R$ with connected boundary it is easily seen that the only ambiguity in $g(\partial R)$ is a cyclic ordering of the group elements corresponding to a starting point for a path around the boundary. Since $Q_t$ is a class function, the function $Q_t(g(\partial R))$ is unambiguously defined once the orientation on $R$ is chosen. Since $Q_t(g) = Q_t(g^{-1})$, it follows that $Q_t(g(\partial R))$ is also independent of the orientation chosen for $R$. Then next theorem is a special case of Theorem 6.4, where $B$ is a BC-graph.

**Theorem 6.6** *Assume that $B$ is a* BC-*directed-planar-graph, and that $f$ is a gauge invariant function on $\Omega(B)$, then*

$$E[f(P|_B)] = \int_{\Omega(B)} f(g) \prod_{R \in \mathscr{R}} Q_{|R|}(g(\partial R)) Dg.$$

*Proof*. Set $S = S(B)$ to be union of the images of all the bonds in $B$. Let $X \subset \mathbf{R}$ be the set consisting of the union of the $x$-coordinates of the points in $\Lambda$ and the $x$-coordinates of the points where a bond in $B$ cross the $x$-axis or has a vertical tangent. Set $\sigma_0$ to be the left to right oriented line segment on the $x$-axis from $x = \min(X)$ to $x = \max(X)$. For each $x \in X$, set $a_x = \min\{y | (x, y) \in S\}$ and $b_x = \max\{y | (x, y) \in S\}$. Lable the points in $X$ by $x_1, \ldots, x_m$, and set $\sigma_k$ to be the upwards oriented vertical line segment at $x = x_k$ going from $y = a_{x_k}$ to $y = b_{x_k}$ for $1 \leqq k \leqq m$. Now set $V\Lambda = \Lambda(\{b\}_{b \in B}, \sigma_1, \ldots, \sigma_m)$, and $VB = B(\{b\}_{b \in B}, \sigma_1, \ldots, \sigma_m)$ as described in the discussion after Definition 6.1. Also put $V\mathscr{R} = \mathscr{R}(VB)$ and $V\Omega = \Omega(VB)$. Let $T$ be the tree in $VB$ consisting of all vertical bonds and any bonds on the $x$-axis.

A couple of remarks about $VB$ are in order. First, the directed graph $VB$ is still boundary connected. Second, there is a natural embedding of the graph $B$ into the paths on $VB$ arising from the fact that the bonds in $B$ are subdivided in the process of going to the graph $VB$. Thus if $g \in \Omega(VB)$ and $b \in B$, it makes sense to write $g(b)$. With this understanding, apply Theorem 4.12, use the independence of the continuum YM$_2$-measure over disjoint vertical strips, and the symmetry of the YM$_2$-measure with respect to reflections about the $x$-axis to conclude

$$Ef(P|_B) = \int_{V\Omega} f(g|_B) \prod_{R \in V\mathscr{R}} Q_{|R|}(g(\partial R)) D_T g. \tag{6.1}$$

(This last equation is valid for arbitrary functions $f$ on $\Omega(B)$.) Now apply the Tree Corollary 5.4 to this last integral (for $f$ gauge invariant) to conclude that

$$Ef(P|_B) = \int_{V\Omega} f(g|_B) \prod_{R \in V\mathscr{R}} Q_{|R|}(g(\partial R)) Dg. \tag{6.2}$$

Now do the integrals over bonds in $VB$ which are in the unbounded component of $\mathbf{R}^2 \backslash S$. Put $\bar{\mathscr{R}} := \bigcup_{R \in \mathscr{R}} R$, and set $V\mathscr{R}' = \{R \in V\mathscr{R} | R \subset \mathbf{R}^2 \backslash \bar{\mathscr{R}}\}$. If $V\mathscr{R}'$ is not empty, there is a region $R$ in $V\mathscr{R}'$ which contains a bond $b \in \partial R$ which is not contained in the boundaries of any other elements of $V\mathscr{R}$ or in the decomposition of the paths of any of the bonds in $B$. Thus the only place that $g(b)$ appears in the integrand of Eq. 6.2 is in the term $Q_{|R|}(g(\partial R))$. Owing to the fact that $\int_G Q_{|R|}(kh)dh = 1$ for all $k \in G$, the term $Q_{|R|}(g(\partial R))$ may be omitted from the integrand. Continuing inductively in this way, it is possible to omit all terms of the form $Q_{|R|}(g(\partial R))$ for which $R \in V\mathscr{R}'$.

The next step is to now do integral with respect to the variables $g(vb)$ for all bonds $vb \in VB$ for which neither $vb$ or $\overline{vb}$ occur in the decomposition of any of the bonds in $B$—these $vb$ will all come from the added vertical bonds. These bond variables appear in the integrand of (6.2) in the form

$$\cdots Q_{|R|}(g(\partial R))Q_{|U|}(g(\partial U))\cdots,$$

where $vb \in \partial R \cap \partial U$. Now using the basic semigroup property of $Q_t$, namely

$$\int_G Q_t(gh)Q_s(h^{-1}k)dh = Q_{t+s}(gk) \forall g, k \in G,$$

one finds that

$$\int_{V\Omega} f(g|_B) \prod_{R \in V\mathscr{R}} Q_{|R|}(g(\partial R))Dg = \int_{\Omega(B^\#)} f(g|_B) \prod_{R \in \mathscr{R}(B')} Q_{|R|}(g(\partial R))Dg,$$

where $B^\#$ is the subgraph of $VB$ gotten by removing the vertical bonds which do not occur in the decomposition of any bond in $B$. (Notice that $B$ is still naturally embedded in the paths in $B^\#$.)

In order to finish the proof it is necessary to show that $\Omega^\# = \Omega(B^\#)$ may be replaced by $\Omega$. Notice that the bond variables in $\Omega^\#$ in the decomposition of $g(b)$ for a bond $b \in B$ always occur as definite ordered product, and these variables do not occur in the ordered products for any other bond variable desides $\bar{b}$. So by the invariance of Haar measure, all but any one of the integrals of the bond variables associated to a bond $b \in B$ is redundant. Hence, the value of the second integral in the above equation is unchanged by replacing $\Omega^\#$ by $\Omega$. Q.E.D.

*Completion of the proof of the Gauge invariant Expectation Theorem 6.4.* Let $B$ be as in Theorem 6.4 and $B_c = B \cup (\text{cuts})$ as described after Lemma 6.5. Then by Theorem 6.6,

$$E[f(P|_B)] = \int_{\Omega(B_c)} f(g|_B) \prod_{R \in \mathscr{R}(B_c)} Q_{|R|}(g(\partial R))Dg. \tag{6.3}$$

Now suppose that $T$ is a tree in $B$, let $\hat{T} = T \cup (B_c \backslash B)$. The set $\hat{T}$ is still a tree, for if $\hat{T}$ were not a tree there would be a simple loop $\sigma$ in $\hat{T}$. Since both $\hat{T}$ and $B_c \backslash B$ are trees, there must be bonds from both $\hat{T}$ and $B_c \backslash B$ in the decomposition of $\sigma$. But there are no simple loops in $B_c$ which contains a cut (an element of $B_c \backslash B$), because each point in a simple loop is in the boundary of at least two distinct regions of $B_c$. But by construction, the region on each side of a cut is the same, see Fig. 3. Here we have used the fact that the region inside of $\sigma$ is made up of the closure of elements in $\mathscr{R}(B_c)$.

Because $B_c$ is a BC-graph, each of the functions $Q_{|R|}(g(\partial R))$ are gauge invariant functions on $\Omega(B_c)$. Thus the integrand in the right-hand side of (6.3) is gauge invariant. Hence the Tree Theorem 5.3 applies and we conclude from (6.3) that

$$\begin{aligned} E[f(P|_B)] &= \int_{\Omega(B_c)} f(g|_B) \prod_{R \in \mathscr{R}(B_c)} Q_{|R|}(g(\partial R))D_{\hat{T}}g \\ &= \int_{\Omega(B)} f(g) \prod_{R \in \mathscr{R}} Q_{|R|}(g(\partial R))Dg, \end{aligned}$$

where the ordering of $g(\partial R)$ is determined by the particular cuts used to create $B_c$. One may now check that the ambiguity choosing the cuts corresponds precisely with the ambiguity in the expression $g(\partial R)$. But this last equation is valid for any system of cuts, and hence the final result is independent of the ambiguities in $g(\partial R)$.

Q.E.D.

It follows immediately from the Gauge Invariant Expectation Theorem 6.4 that:

**Corollary 6.7.** *The expectation value $E[f(P|_B)]$ for gauge invariant functions $f$ is invariant under area preserving diffeomorphisms of $\mathbf{R}^2$ provided the image graph is still admissible. Notice that a diffeomorphism $(D)$ acts naturally on a bond $b$ as $D \circ b$, so that $D$ transforms $B$ to a new graph $DB$.*

## 7. Lattice YM$_2$ Expectations

In this section, the lattice YM$_2$-measures will be introduced and it will be shown that expectations with respect to these measures have the identical structure as expectations with respect to the continuum measure. Once this fact is established it will be an easy matter to show that the appropriate lattice measures converge to the corresponding continuum measures as the lattice spacing tends to zero.

Throughout this section let $G$ be a compact Lie group, $\mathbf{Z}^2$ be the unit square lattice in $\mathbf{R}^2$, $B_\infty$ be the infinite directed graph on $\mathbf{Z}^2$ consisting of nearest neighbor directed bonds, and $\Omega_\infty = \Omega(B_\infty) = \{g : B_\infty \to G \mid g(\bar{b}) = g(b)^{-1}\}$. Also for each positive integer $n$, let $\Lambda_n$ be the closed square of side $2n$ centered at zero in $\mathbf{R}^2$ and put $B_n = \{b \in B_\infty \mid b^i \text{ or } b^f \in \Lambda_{n-1}\}$ and $\overline{B_n} = \{b \in B_\infty \mid b^i, b^f \in \Lambda_n\}$.

**Definition 7.1.** *An action function $A$ is a continuous positive class function on $G$ for which $A(g^{-1}) = A(g) \forall g \in G$. We further assume that $A$ has been normalized so that $\int_G A(g) dg = 1$.*

*We also set $A_n$ to be the $n$-fold convolution of $A$.*

Associated to an action $A$ we will introduce two measures on $\Omega_\infty$, one of the measures will be a gauge fixed version of the other. These measures will be the unique infinite volume limits of the finite volume measures to be defined presently.

For each "boundary condition" $h \in \Omega_\infty$, define $\mu_n(\cdot\,; h)$ to be the unique probability measure on $\Omega_\infty$ such that

$$\mu_n(f; h) = \frac{1}{Z_n(h)} \int_{\Omega_\infty} f(g) \prod_{P \in \mathscr{R}(\bar{B}_n)} A(g(\partial P)) \delta(g = h \text{ on } B_n^c) Dg \tag{7.1}$$

for all continuous functions $f$ on $\Omega_\infty$, where $Dg = D^g_{B_\infty} = \prod_{[b] \in [B]} dH\text{aar}(g(b))$ (a probability measure on $\Omega_\infty$), and $Z_n(h)$ is the normalization constant to make $\mu_n(\cdot\,; h)$ a probability measure. Because of the delta-function, the integral in (7.1) is finite dimensional.

In order to define the gauge fixed version of this measure, let $T$ be the tree in $B_\infty$ consisting of all vertical bonds and all bonds which are on the $x$-axis. Then the finite volume axial-gauge fixed measure is defined analogously for each $h \in \Omega_\infty$ satisfying $h|_T = \text{id}$ to be the unique measure $\mu^a_n(\cdot\,; h)$ on $\Omega_\infty$ such that

$$\mu_n^a(f;h) = \frac{1}{Z_n^a(h)} \int_{\Omega_\infty} f(g) \prod_{P \in \mathscr{R}(\bar{B}_n)} A(g(\partial P))\delta(g = h \text{ on } B_n^c) D_T g \tag{7.2}$$

for all continuous functions $f$ on $\Omega_\infty$.

**Theorem 7.2.** *The weak infinite volume limits of the measures defined in (7.1) and (7.2) exist and are independent of the boundary conditions h. (These limits will be denoted by $\mu(f)$ and $\mu^a(f)$ respectively.) Furthermore, the resulting measures when restricted to functions depending on finitely many bond variables are the corresponding finite volume measures with "free" boundary conditions. More precisely, if $\Lambda_n$, $B_n$ are as above the measures with free boundary conditions are*

$$\mu_n(f) := \frac{1}{Z_n} \int_{\Omega_\infty} f(g) \prod_{P \in \mathscr{R}(\bar{B}_n)} A(g(\partial P)) Dg, \tag{7.3}$$

*and*

$$\mu_n^a(f) := \frac{1}{Z_n^a} \int_{\Omega_\infty} f(g) \prod_{P \in \mathscr{R}(\bar{B}_n)} A(g(\partial P)) D_T g. \tag{7.4}$$

*With this notation the theorem states that*

$$\mu(f) = \lim_{N \to \infty} \mu_N(f;h) = \mu_n(f), \tag{7.5}$$

*and*

$$\mu^a(f) = \lim_{N \to \infty} \mu_N^a(f;h) = \mu_n^a(f) \tag{7.6}$$

*for all continuous functions f on $\Omega_\infty$ which depend only on the bond variables over $B_n$.*

This theorem is basically contained in Dosch and Müller [23]. The proof is instructive so it will be given. We do however borrow the following elementary lemma from [23].

**Lemma 7.3.** *Let A be an action, then $A_k \to 1$ uniformly as $k \to \infty$.*

*Proof of Theorem 7.2.* Since the continuous functions depending on only finitely many variables is dense in all continuous functions on $\Omega_\infty$ with the sup-norm topology (Stone Weierstrass Theorem), it suffices, by an $\varepsilon/3$-argument, to show that Eqs. (7.5) and (7.6) hold.

Suppose that $f$ is a continuous function on $\Omega_\infty$ which depends only on the bond variables over $B_n$ – that is $f(g) = F(g|_{B_n})$ for some function $F$ on $\Omega_n := \Omega(B_n)$.

I will first concentrate on the non-gauge fixed measures. Suppose that $N > n$, and let $c$ be the path in $B_N$ corresponding to the directed line segment on the $y$-axis of $\mathbf{R}^2$ going from $y = n$ to $y = N$. The region $R = \Lambda_N \backslash (\Lambda_n \cup S(c))$ is simply connected. By integrating over all of the bond variables in $B_N \backslash (B_n \cup (\text{the bonds in } c))$ and using the invariance of Haar measure one finds that

$$\mu_N(f;h) = \frac{1}{Z_n(h)} \int_{\{k \in G\}} \int_{\{g \in \Omega_n\}} F(g) \cdot \prod_{P \in \mathscr{R}(\bar{B}_n)} A(g(\partial P)) A_{\#(R)}(k^{-1} g(\sigma_n) k h(\bar{\sigma}_N)) Dg\, dk, \tag{7.7}$$

where $\#(R) = N^2 - n^2$ is the number of plaquettes in $R$, and $\sigma_k$ is the counterclockwise path around the boundary of $\Lambda_k$ starting at $(0,k)\in\mathbf{Z}^2$. So by the Dominated Convergence Theorem and Lemma 7.3,

$$\int_{\{k\in G\}}\int_{\{g\in\Omega_n\}} F(g)\prod_{P\in\mathscr{R}(\bar{B}_n)} A(g(\partial P))A_{\#(R)}(k^{-1}g(\sigma_n)kh(\bar{\sigma}_N)Dg\,dk$$
$$\to \int_{\{g\in\Omega_n\}} F(g)\prod_{P\in\mathscr{R}(\bar{B}_n)} A(g(\partial P))Dg \quad \text{as} \quad n\to\infty.$$

Taking $n=0$ and $F:=1$ in this last equation shows that $\lim_{N\to\infty} Z_N(h)=1$. The combination of these last two limits along with (7.7) gives (7.5).

Now to the proof for the gauge fixed measure. Again let $N>n$ and $f(g)=F(g|_{B_n})$ for some continuous function $F$ on $\Omega_n$. Because of the gauge fixing, the bond variables in different vertical strips are jointly independent with respect to the measure $\mu_N^a(\cdot;h)$. Using this fact and performing the bond variable integrations for bonds outside of $\bar{B}_n$, one finds that

$$\mu_N^a(f;h)=\int_{\Omega_n} F(g)\prod_{P\in\mathscr{R}(\bar{B}_n)} A(g(\partial P))\prod_{i=-n}^{n-1} A_{(N-n)}(g(b(i,n))h(b(i,N))^{-1}), \tag{7.8}$$

$$A_{(N-n)}(g(b(i,-n))h(b(i,-N))^{-1})D_T g, \tag{7.9}$$

where $b(i,j)$ is the left to right directed horizontal bond in $B_\infty$ starting at $(i,j)\in\mathbf{Z}^2$. The proof now follows by the same reasoning as the non-gauged fixed measure case.

Q.E.D.

The next two theorems show that the structure of these lattice measures are in close correspondence with continuum $\mathrm{YM}_2$-measure.

**Theorem 7.4.** *Let $B$ be a directed planar graph of paths in $B_\infty$ and $f$ be a function on $\Omega_\infty$ of the form $f(g)=F(g|_B)$ for some function $F$ on $\Omega(B)$. Then*

$$\mu^a(f)=\int_{\Omega(VB)} F(g|_B)\prod_{R\in\mathscr{R}} A_{|R|}(g(\partial R))D_T g,$$

*where $VB$ is directed graph derived from $B$ by subdividing the original bonds and adding certain vertical bonds as in the proof of Theorem 6.6 and $T$ is the tree in $VB$ consisting of the vertical bonds and bonds on the x-axis.*

*Proof.* Choose $n$ to be a sufficiently large integer such that $B$ may be embedded in $B_n$. Then by Theorem 7.2, $\mu^a(f)=\mu_n^a(F|_B)$. Let $VB^\#$ be the collection of bonds $(b)$ in $B_n$ such that $b$ occurs in the decomposition of some bond in $VB$. (Notice that, like $B$, $VB$ is a directed planar graph consisting of paths in $B_\infty$.) Using the definition of $A_n$ and the assumption that the integral of $A$ is normalized to one, it is easy to integrate over all of the variables corresponding to the horizontal bonds in $B_n\backslash VB^\#$ and conclude that

$$\mu^a(f)=\int_{\Omega(VB^\#)} F(g|_B)\prod_{R\in\mathscr{R}(VB^\#)} A_{|R|}(g(\partial R))D_{T^\#}g.$$

In this last expression, $T^\#$ is the tree in $VB^\#$ consisting of any bonds on the $x$-axis and any vertical bonds in $VB^\#$. In going from the graph $VB$ to $VB^\#$, the bonds of $VB$ were subdivided into unit bonds in $B_\infty$. This process is easily reversed with the aid of the following formula,

$$\int_{G^{2k}} H(u_1 v_1, \ldots, u_k v_k) \prod_{i=1}^{k} A_{M_i}(u_i u_{i+1}^{-1}) A_{N_i}(v_i v_{i+1}^{-1}) du_1 \cdots du_k dv_1 \cdots dv_k$$
$$= \int_{G^k} H(v_1, \ldots, v_k) \prod_{i=1}^{k} A_{M_i+N_i}(v_i v_{i+1}^{-1}) dv_1 \cdots dv_k,$$

which is valid for an arbitrary function $H$, and non-negative integers $M_i$ and $N_i$. This formula is easily verified by using the definition of $A_n$, making the change of variables $v_i \to u_i^{-1} v_i$, and then do the $u$-integrals in the order that they are labeled. Repeated use of this formula "splices" the split bonds together to yield

$$\mu^a(f) = \int_{\Omega(VB)} F(g|_B) \prod_{R \in \mathscr{R}(VB)} A_{|R|}(g(\partial R)) D_T g. \quad \text{Q.E.D.}$$

**Theorem 7.5.** *Let $B$ be a directed planar graph of paths in $B_\infty$ and $f$ be a function on $\Omega_\infty$ of the form $f(g) = F(g|_B)$ for some gauge invariant function $F$ on $\Omega(B)$. Then*

$$\mu(f) = \int_{\Omega(B)} F(g|_B) \prod_{R \in \mathscr{R}} A_{|R|}(g(\partial R)) Dg$$
$$= \int_{\Omega(B)} F(g|_B) \prod_{R \in \mathscr{R}} A_{|R|}(g(\partial R)) D_T g,$$

*where $T$ is any tree in $B$.*

*Proof.* By Theorem 7.2, if $n$ is chosen sufficiently large, then $\mu(f) = \mu_n(F|_B)$. Now for the moment forget that $B$ is embedded in $\mathbf{R}^2$, and introduce the graph $B_c = B \cup (\text{cuts})$, where the cuts are paths as described after Lemma 6.5. Overlay these cuts onto $B_n$, to create a new directed graph $B_n^\#$ by subdividing the bonds of $B_n$ for each crossing of a cut. It may be checked that

$$\mu_n(f) = \int_{\Omega(B_n^\#)} F(g|_B) \prod_{R \in \mathscr{R}(B_n^\#)} A_{|R|}(g(\partial R)) Dg. \tag{7.10}$$

This is done by integrating out the bond variables which make up the cuts and then using the invariance of Haar measure to get rid of the subdivisions of the bonds in $B_n^\#$ as was done at the end of the proof of Theorem 6.6.

Consider the right-hand side of Eq. (7.10). By integrating out the bond variables in $B_n^\#$ which do not occur in the decomposition of the cuts or the bonds in $B$ (see proof of Theorem 6.6) one finds that

$$\mu(f) = \mu_n(F|_B) = \int_{\Omega(B_c)} F(g|_B) \prod_{R \in \mathscr{R}(B_c)} A_{|R|}(g(\partial R)) Dg,$$

where $B_c$ is the planar graph consisting of the bonds in $B$ along with the cuts. Finally as in the conclusion of the proof of Theorem 6.4, the Tree Corollary 5.4 may be applied to this last equation to prove the theorem. Q.E.D.

## 8. Continuum Limit

The goal of this section is to show that the continuum $\mathrm{YM}_2$-measure may be recovered by choosing appropriate actions $A$ and letting the lattice spacing tend to zero. Dosch and Müller [23] have shown that the continuum limit exists for the $\mathrm{YM}_2$-lattice measures if the structure group is $U(1)$ of $SU(2)$. But their expression for the limit is rather complicated and is hard to compare with the continuum $YM_2$-measure. Basically, their result is the same as ours except that the heat kernels are always expanded in terms of the characters.

Throughout this section $\varepsilon\mathbf{Z}^2$ will be the $\varepsilon$-square lattice in $\mathbf{R}^2$, $B_\infty(\varepsilon)$ will be the infinite directed graph on $\varepsilon\mathbf{Z}^2$ consisting of nearest neighbor directed bonds, and $\Omega_\infty(\varepsilon)=\Omega(B_\infty(\varepsilon))$.

**Definition 8.1.** *Suppose that $B$ is a directed planar graph. A lattice approximating sequence to $B$ is a collection $\{B(\varepsilon)\}_{\varepsilon_0>\varepsilon>0}$ of directed planar graphs $B(\varepsilon)$ of paths in $B_\infty(\varepsilon)$ with surjections $(b\to b(\varepsilon)):B\to B(\varepsilon)$ and $(R\to R(\varepsilon)):\mathscr{R}(B)\to\mathscr{R}(B(\varepsilon))$ satisfying the following conditions.*

1. *The area $|R\backslash R(\varepsilon)|+|R(\varepsilon)\backslash R|$ is of order $\varepsilon$.*
2. *If the surjection $(b\to b(\varepsilon))$ is denoted by $i_\varepsilon$, and $\partial R$ denotes an admissible sum of paths for the boundary $R\in\mathscr{R}$ then $i_\varepsilon(\partial R)$ should be an admissible sum of paths for the boundary of $R(\varepsilon)$.*

The following lemma is easy to prove and will be stated without proof.

**Lemma 8.2.** *Given a directed planar graph $B$, there exist lattice approximating sequences.*

Now suppose that $\{A^\varepsilon\}_{\varepsilon>0}$ is a collection of actions. By Theorem 7.2 we may define unique measures $\mu_\varepsilon^a$ and $\mu_\varepsilon$ on $\Omega_\infty(\varepsilon)$ associated to the given action $A^\varepsilon$. ($\mathbf{Z}^2$ has been trivially replaced by $\varepsilon\mathbf{Z}^2$.) The next theorem asserts that the measure $\mu_\varepsilon^a$ converges to the continuum $YM_2$-measure, and the measure $\mu_\varepsilon$ converges to the continuum measure on gauge invariant functions. The two actions that will interest us are the Villain action and the Wilson action.

**Definition 8.3 (Villain).** *A Villain action is an action of the form $A^\varepsilon(g)=Q_{\varepsilon^2}(g)$, where $Q_t$ is the convolution heat-kernel for $e^{t\Delta/2}$, where $\Delta$ is defined in Definition 4.7.*

**Definition 8.4 (Wilson).** *A Wilson action is an action of the form $A_\chi^\varepsilon(g)=Z_\varepsilon^{-1}\exp\operatorname{Re}\chi(g)$, where $\chi$ is the character of a unitary representation of $G$ and $Z_\varepsilon$ is chosen to normalize $A_\chi^\varepsilon$ to have integral one.*

**Theorem 8.5.** *Suppose that $\rho$ is a representation of $G$ for which $\rho_*$ is injective. Let $Q_t$ denote the usual convolution semigroup associated to the representation $\rho$, and $A^\varepsilon$ be the Villain action based on $Q$. Suppose that $B$ is a directed planar graph, $\{B(\varepsilon)\}$ is a lattice approximating sequence to $B$, and $f$ is a continuous function on $\Omega(B)$. Then*

$$\lim_{\varepsilon\to 0}\mu_\varepsilon^a(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)})=EF(P|_B),$$

*where $\mu_\varepsilon^a$ is the axial gauge fixed measure on $\varepsilon\mathbf{Z}^2$ with action $A^\varepsilon$.*

*Proof.* Suppose that $VB$ and $VB(\varepsilon)$ are as described in the proof of Theorem 6.6, then it may be shown that $\{VB(\varepsilon)\}$ is an approximating sequence to the directed graph $VB$. If $T(T(\varepsilon))$ is the tree in $VB(VB(\varepsilon))$ consisting of the vertical bonds and the bonds on the $x$-axis, then by Theorem 7.4, the definition $A^\varepsilon$, and the definition of an approximating sequence, we find,

$$\mu_\varepsilon^a(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = \int_{\Omega(VB(\varepsilon))} f(g|_B) \prod_{R\in\mathscr{R}(VB(\varepsilon))} A^\varepsilon_{|R|/\varepsilon^2}(g(\partial R))D_{T(\varepsilon)}g$$
$$= \int_{\Omega(VB)} f(g|_B) \prod_{R\in\mathscr{R}(VB)} A^\varepsilon_{|R(\varepsilon)|/\varepsilon^2}(g(\partial R(\varepsilon)))D_{T(\varepsilon)}g$$
$$= \int_{\Omega(VB)} f(g|_B) \prod_{R\in\mathscr{R}(VB)} Q_{|R|+O(\varepsilon)}(g(\partial R))D_T g.$$

The limit as $\varepsilon$ tends to zero is now easily taken to yield,

$$\lim_{\varepsilon\to 0}\mu_\varepsilon^a(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = \int_{\Omega(VB)} f(g|_B) \prod_{R\in\mathscr{R}(VB)} Q_{|R|}D_T g,$$

which is precisely the answer one gets by computing $EF(P|_B)$ by Theorem 4.12.

Q.E.D.

**Corollary 8.6.** *Assuming the same hypothesis of the above theorem, with the additional hypothesis that $f$ is now gauge invariant, then*

$$\lim_{\varepsilon\to 0}\mu_\varepsilon(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = EF(P|_B),$$

*Proof.* By the Tree Theorem 5.3 and Theorem 7.2 it follows that

$$\mu_\varepsilon^a(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = \mu_\varepsilon(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)}).$$

This along with the Theorem 8.5, and Theorem 7.5 proves the Corollary.

Q.E.D.

*Remark 8.7.* This corollary could have been proved directly by using Theorem 7.5 and Theorem 6.4.

The analogues of Theorem 8.5 and Corollary 8.6 hold in case the Villain action is replaced by the Wilson action. The key added ingredient in the proof is a "central limit theorem" for group valued random variables. The version that we will need is essentially Theorem A.2. of Borgs and Seiler [18].

**Theorem 8.8 (Borgs and Seiler).** *Let $\rho$ be a faithful representation of a compact Lie Group $G$, and let $A^\varepsilon$ be the Wilson action associated with $\rho$. Suppose that $t(\varepsilon)$ is a positive function such that $t(\varepsilon)/\varepsilon^2$ is a positive integer for all $\varepsilon$. Also suppose that $t(\varepsilon) = t + O(\varepsilon)$ as $\varepsilon\to 0$, where $t > 0$, then $A^\varepsilon_{t(\varepsilon)/\varepsilon^2}\to Q_t$ uniformly as $\varepsilon\to 0$.*

*Proof.* Decompose $A^\varepsilon$ and $Q_t$ into irreducible characters as

$$A^\varepsilon = \sum_\tau d_\tau a_\tau(\varepsilon)\chi_\tau, \quad \text{and} \quad Q_t = \sum_\tau d_\tau \exp\frac{tc_\tau}{2}\chi_\tau,$$

where the sums are over $\tau\in\Gamma$—a complete set of irreducible representations of $G$. The eigenvalues of $A^\varepsilon$ and $Q_t$ as convolution operators are $a_\tau$ and $\exp(tc_\tau/2)$ respectively. The Laplacian $\Delta$ is negative so that $c_\tau \leqq 0$, see also Remarks 2.2. Also it is easy to show that $\operatorname{Re}\chi_\tau$ is a positive semi-definite function in the sense that for $\{g_i\}_{i=1}^m \subset G$ the matrix $\{\operatorname{Re}\chi_\tau(g_i g_j^{-1})\}_{ij}$ is positive semi-definite. Therefore, the Wilson action $A^\varepsilon$ is positive semi-definite, and hence as a convolution operator $A^\varepsilon$ is non-negative. This shows that $a_\tau(\varepsilon)\geqq 0$, see Borgs and Seiler [18] Lemma II.1 for another proof of this fact. Furthermore, by the orthogonality of the $\chi_\tau$'s, $a_\tau(\varepsilon) = (1/d_\tau)\int_G A^\varepsilon(g)\overline{\chi_\tau(g)}dg$, from which it follows that $a_\tau(\varepsilon)\leqq 1$.

In Appendix A of [18] it is shown that

$$\operatorname{Trace}[A^{\varepsilon}_{t/\varepsilon^2}* - Q_t*] \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{8.1}$$

(where $F*$ denotes the operator on $L^2(G)$ given by convolution by the function $F$), and that

$$\left| a_\tau(\varepsilon) - \exp\frac{-c_\tau \varepsilon^2}{2} \right| \leqq \operatorname{const} c_\tau^2 \varepsilon^4. \tag{8.2}$$

Using the fact that the degeneracy of the (simultaneous) eigenspaces for $A^\varepsilon *$ and $Q_t *$ are $d_\tau^2$, Eq. (8.1) may be rewritten as

$$\sum_\tau d_\tau^2 \left[ a_\tau(\varepsilon)^{t/\varepsilon^2} - \exp\frac{tc_\tau}{2} \right] \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{8.3}$$

It is easy to conclude from Eqs. (8.2) and (8.3), and the fact that $0 \leqq a_\tau(\varepsilon)$, $\exp(tc_\tau/2) \leqq 1$, that

$$\sum_\tau d_\tau^2 \left| a_\tau(\varepsilon)^{t(\varepsilon)/\varepsilon^2} - \exp\frac{tc_\tau}{2} \right| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

(*This is a special case of Grümm's convergence theorem, see Simon [46] Theorem* 2.19.) But from the series representations for $A^\varepsilon$ and $Q_t$ and the estimate $\|\chi_\tau\|_\infty \leqq d_\tau$ one finds

$$\| Q_t - A^{\varepsilon}_{t(\varepsilon)/\varepsilon^2} \|_\infty \leqq \sum_\tau d_\tau^2 \left| a_\tau(\varepsilon)^{t(\varepsilon)/\varepsilon^2} - \exp\frac{tc_\tau}{2} \right|. \quad \text{Q.E.D.}$$

*Remark 8.9.* The full strength of Theorem 8.8 is not needed here. The equation (8.2) and Lemma 9.6 of the next section would be adequate for the proof of the next theorem.

**Theorem 8.10.** *Suppose that $\rho$ is a faithful representation of $G$. Let $Q_t$ denote the usual convolution semigroup associated to the representation $\rho$, and $A^\varepsilon = A^\varepsilon_{\chi_\rho}$ be the Wilson action. Suppose that $B$ is a directed planar graph, $\{B(\varepsilon)\}$ is a lattice approximating sequence to $B$, and $f$ is a bounded measurable function on $\Omega(B)$. Then*

$$\lim_{\varepsilon \to 0} \mu^a_\varepsilon(f \circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = EF(P|_B),$$

*where $E$ denotes the continuum* YM$_2$*-expectation.*

*Proof.* Proceeding with the notation and the beginning of the proof of Theorem 8.5 one computes that

$$\mu^a_\varepsilon(f \circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = \int_{\Omega(VB)} f(g|_B) \prod_{R \in \mathscr{R}(VB)} A^\varepsilon_{|R| + O(\varepsilon)}(g(\partial R)) D_T g.$$

With the aid of Theorem 8.8, the conclusion of the proof follows identically to the proof of Theorem 8.5. Q.E.D.

Again we have the immediate corollary.

**Corollary 8.11.** *Assuming the same hypothesis of the above theorem, with the* *additional hypothesis that f is now gauge invariant, then*

$$\lim_{\varepsilon\to 0} \mu_\varepsilon(f\circ i_\varepsilon^{-1}|_{B(\varepsilon)}) = EF(P|_B).$$

## 9. Existence of Lassos

One of the motivations for this paper is the extreme singularity of the Wilson loop variables in the four dimensional theory. Using the $U_4(1)$-model as a test case, it would seem that there is no reasonable way to smooth the Wilson loop variables to get genuine random variables. This meta-fact initiated the work in Gross [36], where it was shown that "Lassos and integrated Lassos" could be used to parameterize smooth gauge theories. These results were generalized in Driver [21] to show that the lassos and integrated lassos may be used to classify bundle connection pairs over a fixed simply connected manifold.

We now recall the definition of a lasso in the special case of a trivial vector bundle $E = \mathbf{C}^N \times \mathbf{R}^d$ with $\mathscr{G}$-valued connection 1-form $A = \sum_{i=1}^{d} A_i dx^i$.

**Definition 9.1.** *The Lasso $L = L^A$ associated to the pair $(E, A)$ is the path two form $L = \sum_{1\leqq i\leqq j\leqq d} L_{ij}dx^i dx^j$, where $L_{ij}$ is the $\mathscr{G}$-valued function on the paths $(\sigma)$ in $\mathbf{R}^d$ starting at zero given by*

$$L_{ij}(\sigma) = P(\sigma)^{-1}F_{ij}(\sigma^f)P(\sigma),$$

*where $F = F^A = dA + A\wedge A$ is the curvature 2-form.*

If the structure group $G$ of the bundle is $U(1)$, then the definition of $L$ essentially reduces to the curvature $F$. For the $U(1)$-Yang-Mills-measure, the curvature is a smoothable random variable. Furthermore, it was shown in Gross [37] for $d = 3$ and Driver [22] for $d = 4$ that, on the "current sector" $(d^*F)$, the Wilson lattice gauge theories converge to the continuum theory. So at least for the abelian models Lasso's behave better than the Wilson loops.

In this section, it will be shown that lassos are smoothable random variables with respect to the $\mathrm{YM}_2$-measure. The key to this result is the computation of the lasso two point function. The result may be summarized informally as:

$$E(L(\sigma)L(\tau)) = \sum_{a=1}^{\dim\mathscr{G}} E[\mathrm{Ad}_{P(\sigma)^{-1}P(\tau)}(T^a)\cdot T^a]\cdot\delta(\sigma^f - \tau^f) + J(\sigma,\tau), \tag{9.1}$$

where $J(\cdot,\cdot)$ is a bounded function when restricted to polygonal paths of a fixed "order." Strictly speaking, $L(\sigma)$ is not defined, since this requires evaluating the white noise $F$ at the point $\sigma^f\in\mathbf{R}^2$. Therefore to get started we introduce the regularized lasso variables. In the following discussion $d$ will always be two.

**Definition 9.2.** *The $\varepsilon$-regularized lasso $(L_\varepsilon)$ is the random variable given by*

$$L_\varepsilon(\sigma) = \frac{P(\sigma)^{-1}[P(l_\varepsilon(\sigma)) - P(l_\varepsilon(\sigma)^{-1})]P(\sigma)}{2\varepsilon^2},$$

*where $l_\varepsilon(\sigma)$ is the clockwise oriented square loop of side $\varepsilon$ with the lower left-hand corner at $\sigma^f\in\mathbf{R}^2$.*

*Remark 9.3.* If $A$ is a smooth connection 1-form on $\mathbf{R}^2$ then it is well known that

$$\lim_{\varepsilon\to 0}\frac{[P(l_\varepsilon(\sigma)-P(l_\varepsilon(\sigma)^{-1}]}{2\varepsilon^2}=F_{12}(\sigma),$$

so that $\lim_{\varepsilon\to 0}L_\varepsilon = L_{12}$, with $L_{12}$ defined as above for smooth connections.

The main goal of this section is to show that $\lim\limits_{\varepsilon\to 0}\int_{\mathscr{P}} L_\varepsilon(\sigma)dm(\sigma)$ exists for a suitable class of measures on $\mathscr{P}$—the continuous paths on $\mathbf{R}^2$ starting at $0\in\mathbf{R}^2$. I will now describe two classes of measures for which the desired limit exists. Fix a positive integer $N$, and write the elements of $\mathbf{R}^{2N}$ as $x=(x_1,\ldots,x_N)$ with each $x_i$ in $\mathbf{R}^2$. Let $\sigma_N:\mathbf{R}^{2N}\to\mathscr{P}$ be the map which takes the $N$-points $(x_1,\ldots,x_N)$ and assigns the piecewise linear path in $\mathscr{P}$ found by connecting $0\in\mathbf{R}^2$ to $x_1, x_1$ to $x_2$, etc. Each segment of the path is directed from $x_i$ to $x_{i+1}$, where $x_0=0$. The measures to be considered will come from pushing measures on $\mathbf{R}^{2N}$ to $\mathscr{P}$ by the map $\sigma_N$. So given a measure (m) on $\mathbf{R}^{2N}$ it will be identified with the measure $m\circ\sigma_N^{-1}$ on $\mathscr{P}$. It should be clear from context whether $m$ refers to the measure on $\mathbf{R}^{2N}$ or the push forward to $\mathscr{P}$.

**Definition 9.4.** *A measure $m$ on $\mathbf{R}^{2N}$ is said to be of type I, if $m$ has an $L^1$ continuous density with respect to Lebesgue measure. The measure $m$ is of type II if it has the form $m(dx_1\cdots dx_N)=\hat{m}_{x_N}(dx_1\cdots dx_{N-1})dx_N$, where $\hat{m}_x$ for each $x\in\mathbf{R}^2$ is a measure on $\mathbf{R}^{N-1}$ satisfying:*

1. *The measure $m$ should be a finite signed measure.*
2. *If $K$ is a compact set of $\mathbf{R}^2$ then $\hat{m}_x$ is concentrated on a fixed compact subset of $\mathbf{R}^{N-1}$ for $x\in K$.*
3. *The mapping*

$$(x,f)\to\hat{m}_x(f):\mathbf{R}^2\times C_b(\mathbf{R}^{N-1})\to\mathbf{R}$$

*should be continuous, where $C_b(\mathbf{R}^{N-1})$ are the bounded continuous functions on $\mathbf{R}^{N-1}$ with the sup-norm topology.*

**Theorem 9.5 (Existence of Lassos).** *Let $m$ be a measure of type I or type II, then $L^2-\lim\limits_{\varepsilon\to 0}\int_{\mathscr{P}} L_\varepsilon(\sigma)dm(\sigma)=:L(m)$ exists. Furthermore if $n$ is another measure of type I or II, then*

$$E(L(m)L(n))=\int_{\mathscr{P}\times\mathscr{P}}\left[\sum_{a=1}^{\dim\mathscr{G}}E(\mathrm{Ad}_{P(\sigma)^{-1}P(\tau)})(T^a)\cdot T^a\cdot\delta(\sigma^f-\tau^f)+J(\sigma,\tau)\right]dm(\sigma)dn(\tau), \tag{9.2}$$

*where the first term is interpreted in the obvious way. For example, if $m$ and $n$ are both of type II, then*

$$\int_{\mathscr{P}\times\mathscr{P}}F(\sigma,\tau)\delta(\sigma^f-\tau^f)dm(\sigma)dn(\tau)=\int_{\mathbf{R}^{2N-1}\times\mathbf{R}^{2N}}F(x,y)\hat{m}_{x_N}(dx_<)\hat{n}_{x_N}(dy_<)dx_N,$$

*where $F(x,y):=F(\sigma_N(x),\tau_N(y))$, $x_<=(x_1,\ldots,x_{N-1})$, and $y=(y_<,x_N)$.*

The proof will be given after a number of preparatory results. A key ingredient in the proof is that $\Delta$ preserves the space of polynomial functions on $G$ with degree less than some fixed number. Therefore, the $\Delta$ restricted to such a space of polynomials is a bounded operator. From this observation, one may deduce smoothness and good bounds on the function $e^{t\Delta/2}p(g)=\int_G Q_t(gh^{-1})p(h)dh$ and its derivatives when $p$ is a polynomial. The next lemma clears the way for using these facts. In the sequel, $K(\cdots)$ will denote a positive function which is non-decreasing in its arguments.

**Lemma 9.6.** *Let $B$ be a directed planar graph, and $p:\Omega(B)\to\mathbf{C}$ be polynomial function of the matrix elements of $g(b)$ for $b\in B$. Then there is a choice of bonds $\{b_R\}_{R\in\mathscr{R}}$ such that*

$$I_B(p):=\int_{\Omega(B)} p(g)\prod_{R\in\mathscr{R}} Q_{|R|}(g(\partial R))Dg=\int_{\Omega(B)} \tilde{p}(g)\prod_{R\in\mathscr{R}(B)} Q_{|R|}(g(b_R))Dg,$$

*where $\tilde{p}$ is another polynomial with $\deg(\tilde{p})\leqq K(\deg(p),\#(B))$ for some function $K$. Furthermore if $p$ is a product of $g(b)$'s with possible repetition, then $\tilde{p}$ is also of this form.*

*Proof.* Choose $R_1\in\mathscr{R}=\mathscr{R}(B)$ such that $\partial R_1$ meets the boundary of the unbounded component of $\mathbf{R}^2\backslash S(B)$, and let $b_1$ be a bond in this intersection. Notice that the only term in the product $\prod_{R\in\mathscr{R}} Q_{|R|}(g(\partial R))$ that the bond variable $g_1:=g(b_1)$ enters is $Q_{|R_1|}(g(\partial R_1))$, and this term has the form $Q_{|R_1|}(g_1 h)$, where $h$ is a function on $\Omega(B)$ not depending on $g_1=g(b_1)$. Thus by making the change of variables $(g_1\to g_1 h^{-1})$ and using the invariance of Haar measure, the integral may be written as

$$I_B(p)=\int_{\Omega(B)}\prod_{R\in\mathscr{R}(B_1)} Q_{|R|}(g(\partial R))\cdot Q_{|R_1|}(g(b_1))p_1(g)Dg,$$

where $p_1$ is another polynomial, and $B_1=B\backslash[b_1]$. If $p$ is a product of $g(b)'s$, then so is $p_1$. Also one sees that $\deg(p_1)\leqq\deg(p)+\#(B)^{\deg(p)}$. Continuing this process inductively one finds bonds $\{b_i\}_{i=1}^{\#(\mathscr{R})}$ and polynomials $\{p_i\}_{i=1}^{\#(\mathscr{R})}$ such that

$$I_B(p)=\int_{\Omega(B)}\prod_{R\in\mathscr{R}(B_j)} Q_{|R|}(g(\partial R))\cdot\prod_{i=1}^{j} Q_{|R_i|}(g(b_i))p_j(g)Dg\quad\text{for}\quad 1\leqq j\leqq\#(\mathscr{R}),$$

where $B_i=B\backslash(\bigcup_{k=1}^{i}[b_k])$. The polynomial $\tilde{p}=p_{\#(\mathscr{R})}$ is the desired polynomial.
Q.E.D.

**Corollary 9.7.** *Assuming the same notation as the lemma, there exists a $C^\infty$-function $F$ on $\mathbf{R}^{\mathscr{R}}$ such that $I_B(p)=F(\{|R|\}_{R\in\mathscr{R}})$. Furthermore, if $p$ is a product of $g(b)'s$, then $\|F^{(k)}\|_\infty\leqq K(k,\deg(p),\#(B))$, for some function $K$, where $F^{(k)}$ is the $k^{\text{th}}$ fold differential of $F$, and the sup-norm is over all $S\in\mathbf{R}^{\mathscr{R}}$ such that $S(R)\geqq 0$ for all $R\in\mathscr{R}$.*

*Proof.* The fact that such a $C^\infty$-function (in fact analytic) function $F$ exists follows from Lemma 9.6 and the comments prior to it. The norm estimate holds because taking a derivative with respect to an $S$-variable brings down a $\Delta$ acting on $p$, and hence the sup-norm of the resulting polynomial may be estimated as $C\|p\|_\infty$ for some constant $C=C(\deg(p))$. Since, there is only a finite number of polynomials of a fixed degree which are products of $g(b)'s$, one gets the estimate that $\|\Delta^k p\|_\infty\leqq K(k,\deg(p),\#(B))$, where $\Delta^k$ denotes the product of $k-\Delta$'s each acting on possibly different variables. This estimate and the fact that $\int_G Q_t(h)dh=1$ for $t\geqq 0$, gives the norm estimate. Q.E.D.

These last results along with the next elementary facts about $C^\infty$-functions will prove to be a powerful tool in approximating $E(L_\varepsilon(\sigma)L_\delta(\tau))$ for $\varepsilon$ and $\delta$ small.

**Lemma 9.8.** *All functions in this lemma are assumed to be smooth.*

1. *Suppose that $F$ is a function on $\mathbf{R}^N$, then*

$$F(x) = F(0) + \int_0^1 F'(tx)\langle x\rangle dt = F(0) + \int_0^1 \partial_x F(tx)dt.$$

*(Recall that $\partial_v F(x) := (d/dt)|_0 F(x+tv)$.)*

2. *Next suppose that $F$ is a function on $\mathbf{R}^N \times \mathbf{R}^M$, and that $F(x,0) = F(0,y) = 0$ for all $x$ and $y$, then $F(x,y) = \int_{I\times I} \partial_x\partial_y F(sx,ty)ds\,dt$, where $I=[0,1]$.*

*Proof.* The first item follows from the Fundamental Theorem of Calculus applied to the function $t \to F(tx)$. The second item follows from two applications of the first:

$$F(x,y) = \int_I \partial_y F(x,ty)dt = \int_{I\times I} \partial_x\partial_y F(sx,ty)ds\,dt. \quad \text{Q.E.D.}$$

**Theorem 9.9.** *Let $\sigma$ and $\tau$ be two piecewise linear curves in $\mathscr{P}$, each with at most $N$ linear segments. Set $A_\varepsilon(\sigma)$ $(A_\delta(\tau))$ to be the $\varepsilon$ $(\tau)$ square in $\mathbf{R}^2$ with left lower corner at $\sigma^f$ $(\tau^f)$. Then on the set where $A_\varepsilon(\sigma)\cap A_\delta(\tau)$ is empty we have $\| E(L_\varepsilon(\sigma)L_\delta(\tau))\| \leqq K(N)$ and if $\sigma^f \neq \tau^f$, then $\lim_{\varepsilon,\delta\to 0} E(L_\varepsilon(\sigma)L_\delta(\tau)) =: J(\sigma,\tau)$ exists.*

*Proof.* Assume that $A_\varepsilon(\sigma)\cap A_\delta(\tau)$ is empty. Put $B = B(\sigma,\tau,l_\varepsilon(\sigma),l_\delta(\tau))$, $\mathscr{R} = \mathscr{R}(B)$, $\mathscr{R}_1 = \{R\in\mathscr{R} \mid R \subset A_\varepsilon(\sigma)\}$, and $\mathscr{R}_2 = \{R\in\mathscr{R} \mid R\subset A_\delta(\tau)\}$. Let $I = E(L_\varepsilon(\sigma)L_\delta(\tau))$, and $F$ be the $C^\infty$-function on $\mathbf{R}^{\mathscr{R}}$ (via Lemma 9.6) such that $4\varepsilon^2\delta^2 I = F(\{|R|\}_{R\in\mathscr{R}})$, so $F$ is

$$\begin{aligned} F(S) = \int_{\Omega(B)} &\mathrm{Ad}_{g(\sigma)^{-1}}(g(l_\varepsilon(\sigma)) - g(l_\varepsilon(\sigma))^{-1})\cdot \mathrm{Ad}_{g(\tau)^{-1}}(g(l_\delta(\tau)) - g(l_\delta(\tau))^{-1}) \\ &\cdot \prod_{R\in\mathscr{R}} Q_{S(R)}(g(\partial R))Dg. \end{aligned} \tag{9.3}$$

Notice, that if $S$ is such that $S|_{\mathscr{R}_1} = 0$, then the terms $Q_{S(R)} = Q_0 = \delta$ for $R\in\mathscr{R}_1$. This is easily seen to force $g(L_\varepsilon) = \mathrm{id}$ in the integrand above, and hence integral is zero. The same argument applies to $S$ such that $S|_{\mathscr{R}_2} = 0$, which shows that $F(S) = 0$ if either $S|_{\mathscr{R}_1} = 0$ or $S|_{\mathscr{R}_2} = 0$.

For $S\in\mathbf{R}^{\mathscr{R}}$ and $u,v\in I$ put

$$S_{u,v}(R) = \begin{cases} S(R), & \text{if} \quad R\notin \mathscr{R}_1\cup\mathscr{R}_2; \\ uS(R), & \text{if} \quad R\in\mathscr{R}_1; \\ vS(R), & \text{if} \quad R\in\mathscr{R}_2. \end{cases}$$

Then by Lemma 9.8 it follows that

$$I = \sum_{R_1\in\mathscr{R}_1, R_2\in\mathscr{R}_2} \frac{|R_1||R_2|}{4\varepsilon^2\delta^2} \int_{I\times I} \partial_{S(R_1)}\partial_{S(R_2)} F(S_{u,v})du\,dv, \tag{9.4}$$

where $S(R) = |R|$ for all $R\in\mathscr{R}$. But clearly $|R_1||R_2|/(4\varepsilon^2\delta^2) \leqq 1/4$, so that $|I| \leqq \#(\mathscr{R})\|F\|_\infty \leqq K(N)$. The last inequality was obtained with the help of Corollary 9.7.

So it only remains to show that the limit of $I = I(\varepsilon,\delta)$ exists as $\varepsilon,\delta\to 0$. The key to this is to note that for $\varepsilon$ and $\delta$ small enough the topological type of the graph $B$

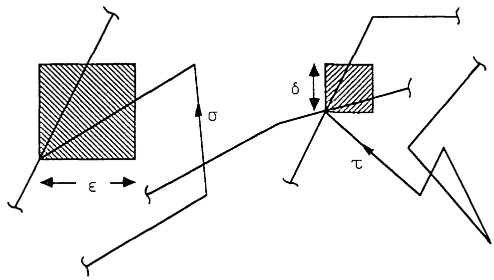


**Fig. 4.** A possible configuration near the heads of seperated lassos for small epsilon and delta

stabilizes and must have the form depicted in Fig. 4. From Fig. 4 one sees that there are constants $C_{R_1}$ and $C_{R_2}$ such that $|R_1| = C_{R_1}\varepsilon^2$ and $|R_2| = C_{R_2}\delta^2$ for $R_1 \in \mathscr{R}_1$ and $R_2 \in \mathscr{R}_2$. Hence, taking the limit of Eq. (9.4) gives

$$\lim_{\varepsilon,\delta\to 0} I = \sum_{R_1,R_2} C_{R_1} C_{R_2} \partial_{S(R_1)} \partial_{S(R_2)} F(S_{0,0}) =: J(\sigma,\tau),$$

where $S(R) = |R|$ for $R \in \mathscr{R}(B(\sigma,\tau))$. We have been tacitly using the fact that for $\varepsilon$ and $\delta$ small, the graphs $B(\sigma, \tau, l_\varepsilon(\sigma), l_\delta(\tau))$ are all isomorphic. Q.E.D.

**Theorem 9.10.** *Let $\sigma$ and $\tau$ be the piecewise linear paths in $\mathscr{P}$. Denote by $\bar{\sigma}\tau$ the piecewise linear path consisting $\tau$ followed by the straight line segment from $\tau^f \to \sigma^f$ and then following $\sigma$ backwards back to the origin in the plane. There exist a constant $K(N)$, where $N$ is the number of line segments in $\sigma$ and $\tau$, such that*

$$\left\| E[L_\varepsilon(\sigma)L_\delta(\tau)] - E\left[\sum_{a=1}^{\dim\mathscr{G}} \mathrm{Ad}_{P(\bar{\sigma}\tau)} T_a \cdot T_a\right] \cdot \frac{|A_\varepsilon(\sigma) \cap A_\delta(\tau)|}{\varepsilon^2\delta^2} \right\| \leqq K(N)$$

*on the set where $A_\varepsilon(\sigma) \cap A_\delta(\tau) \neq \emptyset$.*

*Proof.* Let $B = B(\sigma, \tau, l_\varepsilon(\sigma), l_\delta(\tau))$, and $F(S)$ be given by (9.3). Also define the following subsets of $\mathscr{R}$:

$$\mathscr{R}_0 = \{R \in \mathscr{R} \mid R \subset A_\varepsilon(\sigma) \cap A_\delta(\tau)\},$$
$$\mathscr{R}_1 = \{R \in \mathscr{R} \mid R \subset A_\varepsilon(\sigma) \backslash A_\varepsilon(\sigma) \cap A_\delta(\tau)\},$$
$$\mathscr{R}_2 = \{R \in \mathscr{R} \mid R \subset A_\delta(\tau) \backslash A_\varepsilon(\sigma) \cap A_\delta(\tau)\},$$
$$\mathscr{R}_3 = \mathscr{R} \backslash (\mathscr{R}_0 \cup \mathscr{R}_1 \cup \mathscr{R}_2),$$

so that $\mathscr{R}$ is the disjoint union of the $\mathscr{R}_i$'s. Corresponding to this splitting of $\mathscr{R}$, write $S = (S_0, S_1, S_2, S_3)$ and by abuse of notation $F(S) = F(S_0, S_1, S_2, S_3)$, where $S_i = S|_{\mathscr{R}_i}$. Suppose that $S_0 = 0$ and $S_1 = 0$, then the terms in (9.3) $Q_{S(R)}(g(\partial R))$ for $R \in \mathscr{R}_0 \cup \mathscr{R}_1$ are now $\delta$-functions, which forces $g(l_\varepsilon(\sigma)) = \mathrm{id}$. This causes $g(l_\varepsilon(\sigma)) - g(l_\varepsilon(\sigma))^{-1} = 0$, and hence makes $F = 0$. A similar argument shows that $F = 0$ if $S_0 = 0$, and $S_2 = 0$. Suppressing $S_3$ from the notation for the moment, this implies by Lemma 9.8 that

$$F(S_0, S_1, S_2) = F(0, S_1, S_2) + \int_0^1 \partial_{S_0} F(uS_0, S_1, S_2)\,du \tag{9.5}$$

$$= \int_0^1\int_0^1 \partial_{S_1}\partial_{S_2} F(0, uS_1, vS_2)\,du\,dv + \int_0^1 \partial_{S_0} F(uS_0, S_1, S_2)\,du. \tag{9.6}$$

To simplify notation write $I := E[L_\varepsilon(\sigma)L_\delta(\tau)]$, and $I \approx H(\sigma,\tau,\varepsilon,\delta)$ if $\|I - H(\sigma,\tau,\varepsilon,\delta)\| \leqq K(N)$ for some constant $K(N)$ only depending on $N$. Then by (9.6) and Lemma 9.6, it follows that

$$I \approx \frac{1}{4\varepsilon^2\delta^2}\int_0^1 \partial_{S_0}F(uS_0, S_1, S_2)du,$$

where $S(R) = |R|$. This is because $|R_1||R_2| \leqq \varepsilon^2\delta^2$ for $R_1 \in \mathscr{R}_1$ and $R_2 \in \mathscr{R}_2$. By similar arguments we may also expand $\partial_{S_0}F$ with respect to the $S_0, S_1$ and $S_2$ arguments to conclude $I \approx (1/4\varepsilon^2\delta^2)\partial_{S_0}F(0,0,0,S_3)$.

Let $l$ be the directed straight line segment from $\sigma^f \to \tau^f$. There are two possibilities; either $l$ is contained in $A_\varepsilon(\sigma) \cup A_\delta(\tau)$ or it is not. In the latter case, we split $\mathscr{R}_3$ into the disjoint union $\mathscr{R}_3' \cup \mathscr{R}_3''$ where $\mathscr{R}_3'$ consists of the regions in $\mathscr{R}_3$ which are contained in the triangular region (TR) bounded by $l$ and $\partial(A_\varepsilon(\sigma) \cup A_\delta(\tau))$. In order to treat both cases at once, when in the first case write $S_3'' = S_3$ and interpret any statement involving $S_3'$ to be vacuous. Using the fact that the area of this triangular region is no more than $\varepsilon^2\delta^2/2$, it follows, by expanding with respect to $S_3'$, that

$$I \approx \frac{1}{4\varepsilon^2\delta^2}\partial_{S_0}F(0,0,0,S_3' = 0, S_3''). \tag{9.7}$$

Now we are in a position to compute the derivative $\partial_{S_0}F$. Enumerate $\mathscr{R}_0 = \bigcup_{i=1}^{m} R^i$ and write $S_0 = s = (s^1, \ldots, s^m)$ and write $f(s) = F(s,0,0,S_3' = 0, S_3'')$. The immediate goal is to show

$$\partial_{s^i}f(0) = 4\int_{\Omega(B)}\left[\sum_{a=1}^{\dim\mathscr{G}} \mathrm{Ad}_{g(\bar\sigma\tau)}T_a \cdot T_a\right]\prod_{R\in\mathscr{R}} Q_{S(R)}(g(\partial R))Dg, \tag{9.8}$$

where $S = (0,0,0,S_3' = 0, S_3'')$. To take this derivative, we may assume that $s_j = 0$ if $j \neq i$. For such $s$, the $\delta$-functions in the integrand of (9.3) enable one to conclude that variables $g(c)$ may be set to the identity, where $c$ is a closed path in $B$ which lies in the region $(A_\varepsilon(\sigma) \cup A_\delta(\tau) \cup \mathrm{TR})\backslash R^i$. This coupled with the fact that the regions $R^j$ are simply connected, enables one to deform the paths $l_\varepsilon(\sigma)$ and $l_\delta(\tau)$ to paths bordering the boundary of $R^i$ in such a way that $f(s)$ may be written as

$$\begin{aligned} f(0,\ldots,s_i,\ldots,0) = \int_{\Omega(B)} &\mathrm{Ad}_{g(l\sigma)^{-1}}\mathrm{Ad}_{g(\eta)}(g(\partial R^i) - g(\partial R^i)^{-1}) \\ &\cdot \mathrm{Ad}_{g(\tau)^{-1}}\mathrm{Ad}_{g(\eta)}(g(\partial R^i) - g(\partial R^i)^{-1})\prod_{R\in\mathscr{R}} Q_{S(R)}(g(\partial R))Dg. \end{aligned} \tag{9.9}$$

Where, in this last equation, $\eta$ is a path in $B$, $\partial R^i$ is a path around the boundary of $R^i$, and $l\sigma$ is the path $\sigma$ followed by the path $l$. In view of this last expression, the derivative to be computed may be written in the form:

$$J := \frac{d}{dt}\bigg|_0 \int_G Q_t(h)\mathrm{Ad}_{g(h)}(h - h^{-1})\mathrm{Ad}_{k(h)}(h - h^{-1})D(h)dh,$$

where $g, k\colon G \to G$ and $D\colon G \to \mathbf{R}$ is a smooth density. A straightforward calculation shows:

$$J = \int_G \tfrac{1}{2}\Delta\delta(h)\mathrm{Ad}_{g(h)}(h-h^{-1})\mathrm{Ad}_{k(h)}(h-h^{-1})D(h)dh$$
$$= \tfrac{1}{2}\Delta[\mathrm{Ad}_{g(h)}(h-h^{-1})\mathrm{Ad}_{k(h)}(h-h^{-1})D(h)]|_{h=\mathrm{id}}$$
$$= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_0 \sum_{a=1}^{\dim\mathscr{G}} [\mathrm{Ad}_{g(e^{sT_a})}(e^{sT_a}-e^{-sT_a})\mathrm{Ad}_{k(e^{sT_a})}(e^{sT_a}-e^{-sT_a})D(e^{sT_a})]$$
$$= \sum_{a=1}^{\dim\mathscr{G}} [\mathrm{Ad}_{g(e)}(T_a)\mathrm{Ad}_{k(e)}T_a \cdot D(e)]. \tag{9.10}$$

From this last computation and Eq. (9.9) we conclude that

$$\partial_{s^i} f(0) = 4 \int_{\Omega(B)} \mathrm{Ad}_{g(l\sigma)^{-1}g(\eta)} T_a \cdot \mathrm{Ad}_{g(\tau)^{-1}g(\eta)} T_a \cdot \prod_{R\in\mathscr{R}} Q_{S(R)}(g(\partial R))Dg.$$

This last equation is identical to Eq. 9.8, because for any $k$ and $g$ in $G$,

$$\sum_a \mathrm{Ad}_g(T_a)\mathrm{Ad}_k(T_a) = \sum_a \mathrm{Ad}_{gk^{-1}}(T_a)T_a.$$

This is proven by noticing that $\sum_a \mathrm{Ad}_g(T_a)\mathrm{Ad}_k(T_a)$ is independent of the orthonormal basis $\{T_a\}$ of $\mathscr{G}$ and so one may choose the orthonormal basis $\{\mathrm{Ad}_k^{-1} T_a\}$ instead of $\{T_a\}$.

From (9.8) we conclude that

$$\partial_{S_0} F(0,0,0,S_3' = 0, S_3'') = \left(\sum_{i=1}^m s^i\right) \int_{\Omega(B)} \sum_{a=1}^{\dim\mathscr{G}} \mathrm{Ad}_{g(\bar\sigma\tau)} T_a T_a \cdot \prod_{R\in\mathscr{R}} Q_{S(R)}(g(\partial R))Dg,$$

where $S = (0,0,0,S_3' = 0, S_3'')$, and $S_0 = s$. So setting $s^i = |R^i|$, it follows that

$$I \approx \frac{|A_\varepsilon(\sigma)\cap A_\delta(\tau)|}{\varepsilon^2\delta^2} \int_{\Omega(B)} \sum_{a=1}^{\dim\mathscr{G}} \mathrm{Ad}_{g(\bar\sigma\tau)} T_a \cdot T_a \prod_{R\in\mathscr{R}} Q_{S(R)}(g(\partial R))Dg,$$

where $S$ is still $S = (0,0,0,S_3' = 0, S_3'')$. Finally, by reversing the arguments that led to (9.7), this last equation is still valid for $S$ such that $S(R) = |R|$ for all $R\in\mathscr{R}$. But for $S(R) = |R|$, the right-hand side of this last approximation is exactly

$$E\left[\sum_{a=1}^{\dim\mathscr{G}} \mathrm{Ad}_{P(\bar\sigma\tau)} T_a \cdot T_a\right]. \quad \text{Q.E.D.}$$

We now return to the proof of Theorem 9.5.

*Proof of Theorem 9.5.* Let $m$ and $n$ be two measures of type I or II, and write $L_\varepsilon(m) = \int_{\mathscr{P}} L_\varepsilon(\sigma)dm(\sigma)$. Also put

$$J_{\varepsilon,\delta}(\sigma,\tau) = \begin{cases} E(L_\varepsilon(\sigma)L_\delta(\tau)), & \text{if } A_\varepsilon(\sigma)\cap A_\delta(\tau) \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

Then according to Theorems 9.9 and 9.10,

$$E(L_\varepsilon(m)L_\delta(n)) = \int_{\mathscr{P}\times\mathscr{P}} \left\{ \frac{A_\varepsilon(\sigma)\cap A_\delta(\tau)}{\varepsilon^2\delta^2} E\left[\sum_{a=1}^{\dim\mathscr{G}} \mathrm{Ad}_{P(\bar{\sigma}\tau)} T_a \cdot T_a\right] + J_{\varepsilon,\delta}(\sigma,\tau) \right.$$
$$\left. + 1_{A_\varepsilon(\sigma)\cap A_\delta(\tau)\neq\varnothing}\cdot O(1) \right\} dm(\sigma)dn(\tau). \tag{9.11}$$

The last term in this equation goes to zero as $\varepsilon$ and $\delta$ go to zero, since

$$\operatorname{limsup}_{\varepsilon,\delta\to 0} | 1_{A_\varepsilon(\sigma)\cap A_\delta(\tau)\neq\varnothing}\cdot O(1)| \leqq K(N)\cdot 1_{\sigma^f=\tau^f}$$

which is zero $m\times n$–almost everywhere. The second term in the equation converges to $\int_{\mathscr{P}\times\mathscr{P}} J(\sigma,\tau)dm(\sigma)dn(\tau)$ by the dominated convergence theorem and Theorem 9.9. So it is only the first term that needs attention.

A short computation shows that $\int_{\mathbf{R}^2}((|A_\varepsilon(x)\cap A_\delta(y)|)/\varepsilon^2\delta^2)dx = 1$ for $\varepsilon,\delta>0$ and $y\in\mathbf{R}^2$. This shows that $((|A_\varepsilon(x)\cap A_\delta(y)|)/\varepsilon^2\delta^2)$ is a sequence of approximate $\delta$-functions. Now it is easy to check that $E\,\mathrm{Ad}_{P(\bar{\sigma}_N(x))P(\sigma_N(y))}$ is a continuous and uniformly bounded function of $(x,y)\in\mathbf{R}^N\times\mathbf{R}^N$. This fact along with the properties assumed on $m$ and $n$ implies, by standard techniques of approximate $\delta$-functions, that the first term in (9.11) converges to the first term in (9.2). Finally, to see that $L_\varepsilon(m)$ converges in $L^2$ take $m=n$ in the above computations, to show the $\lim_{\varepsilon,\delta\to 0} E(-\operatorname{trace}(L_\varepsilon(m)L_\delta(m))$ exists. This proves the $L^2$-limit exists, since $-\operatorname{trace}(AB)=\operatorname{trace}(A^*B)$ is the Hilbert Schmidt norm on $\mathscr{G}$. Q.E.D.

## 10. Lassos Generate the Measurable Functions

In this last section, we will show that the $\mathscr{G}$-valued white noise can be recovered from the lassos. Or in other words, the lassos generated all measurable functions on $(\Omega,\mathscr{F},E)$. Let $g$ be a real valued $C^\infty$-function with compact support on $\mathbf{R}^2$. Let $m$ be the measure on $\mathbf{R}^2\times\mathbf{R}^2$ given by $m(du,dv)=\delta(u-(v_1,0))g(v)du\,dv$. Then the corresponding measure on $\mathscr{P}$ is concentrated on "L-shaped" paths.

**Theorem 10.1.** *Assuming the above notation, then* $L(m) = -F(g)$ *E-almost surely.*

This theorem is what we should expect from Remark 9.3. Recall that $F$ was identified with $F_{21}$, which explains the sign discrepancy. The key fact needed for the proof is:

**Lemma 10.2.** *Let* $\sigma$ *be an L-shaped path, and* $x=\sigma^f$ *be the final point of the path. Then*

$$E(L_\varepsilon(\sigma)F(g)) = (-1/\varepsilon^2)(g, 1_{A_\varepsilon(\sigma)})C + 1_{\mathrm{supp}(g)}(x)O_g(\varepsilon),$$

*where* $C=T\cdot T := \sum_a T^a T^a$, $(\cdot,\cdot)$ *denotes the* $L^2$*-inner product on* $\mathbf{R}^2$, *and* $O_g(\varepsilon)$ *is a function depending on* $g$ *which when divided by* $\varepsilon$ *remains bounded uniformly in* $\sigma$ *which has been suppressed from the notation. The same estimate also holds for* $E(F(g)L_\varepsilon(\sigma))$.

*Proof of Theorem 10.1.* We will now prove the theorem assuming the lemma. In order to do this we will show that $E(L(m)+F(g))^2=0$. By the definition of the white noise it follows that $EF(g)^2 = (g,g)C$. By Eq. (9.2) of Theorem 9.5, one finds that

$$EL(m)^2 = (g,g)C + \int_{\mathscr{P}\times\mathscr{P}} J(\sigma,\tau)m(d\sigma)m(d\tau),$$

where we have noticed that if $\sigma$ and $\tau$ are $L$-shaped curves which agree at their final points, then $\sigma = \tau$. Since, $m$ concentrates on $L$-shaped curves, it follows that

$$E\,\mathrm{Ad}_{P(\sigma)^{-1}P(\tau)}\cdot\delta(\sigma^f - \tau^f) = E\,\mathrm{Ad}_{P(\sigma)^{-1}P(\sigma)}\cdot\delta(\sigma^f - \tau^f) = \mathrm{Id}\cdot\delta(\sigma^f - \tau^f)\, m\times m\text{-a.s.}$$

Now we show that $J(\sigma,\tau)=0$ for two non-equal $L$-shaped curves. Let $J_{\varepsilon,\varepsilon}(\sigma,\tau)$ be as in the proof of Theorem 9.5, then for $\varepsilon$ sufficiently small it follows by Theorem 6.4 (with $T$ equal to the bonds in the decomposition of $\sigma$ and $\tau$) that

$$J_{\varepsilon,\varepsilon}(\sigma,\tau) = \int_{G^2}(k-k^{-1})(h-h^{-1})Q_{\varepsilon^2}(k)Q_{\varepsilon^2}(h)dk\,dh = 0.$$

This shows that $J(\sigma,\tau) = \lim_{\varepsilon\to 0} J_{\varepsilon,\varepsilon}(\sigma,\tau) = 0$, and so $EL(m)^2 = (g,g)C$.

Therefore $E(L(m)+F(g))^2 = 2(g,g)C + E(L(m)F(g)) + E(F(g)L(m))$, and so it only remains to show $E(L(m)F(g)) = E(F(g)L(m)) = -(g,g)C$. Now compute

$$\begin{aligned}
E(L(m)F(g)) &= \lim_{\varepsilon\to 0}\int_{\mathscr{P}} E[L_\varepsilon(\sigma)F(g)]dm(\sigma)\\
&= -\lim_{\varepsilon\to 0}\int_{\mathscr{P}}\left[\frac{1}{\varepsilon^2}(1_{A_\varepsilon(\sigma^f)}, g)C + O_g(\varepsilon)1_{\mathrm{supp}(g)}(\sigma^f)\right]dm(\sigma)\\
&= -\lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}\left[\frac{1}{\varepsilon^2}(1_{A_\varepsilon(x)}, g)\right]Cg(x)dx\\
&= -(g,g)C,
\end{aligned}$$

where in the second inequality we used Lemma 10.2, and in the last inequality we used the fact the $(1/\varepsilon^2)1_{A_\varepsilon(x)}$ is a sequence approaching the delta-function at $x\in\mathbf{R}^2$. The computation for $E(F(g)L(m))$ is identical. Q.E.D.

*Proof of Lemma 10.2.* Only the statement concerning $E(L_\varepsilon(\sigma)F(g))$ will be proved, since the estimate for $E(F(g)L_\varepsilon(\sigma))$ has a similar proof. Because of symmetry considerations, there is no loss in generality assuming that $x = \sigma^f$ is in the first quadrant. It is now necessary to introduce a considerable amount of notation. For what follows it will be understood that the variable $t$ is in $[x_1, x_1+\varepsilon]$.

First we define three regions in the plane: $A(t) = [x_1, t]\times[x_2, x_2+\varepsilon]$, $B(t) = [x_1, t]\times[0, x_2]$, and $D(t) = A(t)\cup B(t)$. Now we need the six martingales: $M(t) = F(D(t))$, $N(t) = F(B(t))$, $\delta(t) = F(A(t)) = M(t) - N(t)$, $f(t) = F(g1_{D(t)})$, $k(t) = F(g1_{B(t)})$, and $\alpha(t) = F(g1_{A(t)}) = f(t) - k(t)$. (All of these martingales are taken to be continuous in $t$, the existence of such versions can be deduced from Theorem 4.3 of [38].) Next, let $P_t$ and $Q_t$ be parallel translation with respect to the martingales $M(t)$ and $N(t)$ respectively, see Definition 3.3. Finally, let $a(t) = (g, 1_{A(t)})$, $b(t) = (g, 1_{B(t)})$, and $d(t) = (g^2, 1_{D(t)})$.

Because we are using the complete axial gauge, parallel translation along $L$-shaped curves is equal to the identity operator. Therefore

$$L_\varepsilon(\sigma) = \frac{1}{2\varepsilon^2}([Q_{x_1+\varepsilon}^{-1}P_{x_1+\varepsilon}] - [\cdots]^{-1}),$$

which is measurable with respect to the $\sigma$-algebra generated by the functions $\{F(A)\}_{A\subset D(x_1+\varepsilon)}$. By the independence properties of the white noise, this implies that

$$\frac{1}{2\varepsilon^2}\cdot I := E(L_\varepsilon(\sigma)F(g)) = \frac{1}{2\varepsilon^2}E(([Q_\beta^{-1}P_\beta]-[\cdots]^{-1})F(g1_{D(t)})) \tag{10.1}$$

$$= \frac{1}{2\varepsilon^2}\cdot EH(P_\beta, Q_\beta, f(\beta)),$$

where $\beta = x_1+\varepsilon$, and $H$ is the function on $G^2\times\mathscr{G}$ given by $H(p,q,f):= [q^{-1}p - p^{-1}q]f$. So following the method used in proving Theorem 4.12, we need to compute $dH_t := dH(P_t, Q_t, f(t))$. This can be done in the same manner as the proof of Proposition 4.9. To state the final result of Itô's lemma, we define for each $T\in\mathscr{G}$ the vector fields $T_1, T_2$ and $T_3$ on $G^2\times\mathscr{G}$ by $T_1H(p,q,f) = (d/dt)|_0 H(e^{tT}p,q,f)$, $T_2H(p,q,f) = (d/dt)|_0 H(p,e^{tT}q,f)$, and $T_3H(p,q,f) = (d/dt)|_0 H(p,q,f+tT)$. Also let $\mathscr{M} = M_1 + N_2 - f_3$—a vector field valued martingale. With this notation

$$dH_t = -d\mathscr{M}(t)\circ H_t := -d\mathscr{M}(t)H(P_t,Q_t,f(t)) + \tfrac{1}{2}d\mathscr{M}(t)^2 H(P_t,Q_t,f(t)).$$

Arguing as in the proof of Theorem 4.12, we conclude that

$$I = e^{\frac{1}{2}\int_{x_1}^{\beta} d\mathscr{M}(t)^2} H(e,e,0).$$

So compute

$$\begin{aligned} d\mathscr{M}^2 &:= d(M_1 + N_2 - df_3)^2 \\ &= [d(N_1+N_2-k_3) + d(\delta_1-\alpha_3)]^2 \\ &= d(N_1+N_2-k_3)^2 + d(\delta_1-\alpha_3)^2 \\ &= d(N_1+N_2)^2 + dk_3^2 + d\delta_1^2 + d\alpha_3^2 - 2d(N_1+N_2)dk_3 - 2d\delta_1 d\alpha_3 \\ &= d(N_1+N_2)^2 + df_3^2 + d\delta_1^2 - 2d(N_1+N_2)dk_3 - 2d\delta_1 d\alpha_3, \end{aligned}$$

where the independence of $(N_1+N_2-k_3)$ and $(\delta_1-\alpha_3)$ were used in the third equality, and the independence of $k_3$ and $\alpha_3$ were used in the last equality. Now define the following second order differential operators on $G^2\times\mathscr{G}$: $C_i = T_i\cdot T_i = \sum_a (T_i^a)^2$, $C_{12} = T_{12}\cdot T_{12} := \sum_a (T_1^a + T_2^a)^2$, $T_1\cdot T_3 = \sum_a T_1^a T_3^a$, and $T_{12}\cdot T_3 = \sum_a (T_1^a + T_2^a)T_3^a$. Using the fact that all of the martingales considered have independent increments, the above equation for $d\mathscr{M}^2$ is easily shown to be equal to:

$$\begin{aligned} d\mathscr{M}(t)^2 &= \{y\,dt\,C_{12} - 2da(t)T_1\cdot T_3\} + [d(d(t))C_3 + \varepsilon\,dt\,C_1 - 2db(t)T_{12}\cdot T_3] \\ &=: dV_1(t) + dV_2(t). \end{aligned}$$

Now all the terms in this last equation commute with one another except for the terms in $\{\cdots\}$, which do commute with the terms in $[\cdots]$ but not necessarily with one another. The desired commutators are shown to be zero with the aid of the infinitesimal braid relations (Proposition 4.10), and the fact that the $T_3^a$'s commute with everything. Since $dV_1$ commutes with $dV_2$, it follows that

$$I = \left(e^{\frac{1}{2}\int_{x_1}^{\beta} dV_1(t)}\, e^{\frac{1}{2}\int_{x_1}^{\beta} dV_2(t)}\, H\right)(e,e,0),$$

where $y := x_2$ is the second component of $\sigma^f$.

Now $dV_2(t)H = \varepsilon dt C_1 H = \varepsilon dt CH$, since $C_3 H = 0$, $(T_1 + T_2)H = 0$, and $C_1 H = CH$. Since $C$ commutes with the matrices in $G$ and $\mathscr{G}$ (Remarks 2.2), it follows that

$$I = e^{(1/2)\varepsilon^2 C}\left(e^{\frac{1}{2}\int_{x_1}^{\beta} dV_1(t)} H\right)(e, e, 0) = e^{(1/2)\varepsilon^2 C}(e^{(1/2)y\varepsilon C_{12} - a(\beta)T_1 \cdot T_3} H)(e, e, 0).$$

Expanding out the exponential in this last expression shows

$$I = e^{(1/2)\varepsilon^2 C}\left(H + \sum_{n=1}^{\infty} -\frac{1}{n!}(\tfrac{1}{2}y\varepsilon C_{12})^{n-1}(a(\beta)T_1 \cdot T_3)H\right)(e, e, 0),$$

where it has been noted that $C_{12}H = 0$ and $T_1 \cdot T_3 C_{12}^k T_1 \cdot T_3 H = 0$ for all $k \geqq 0$. Thus

$$\begin{aligned} I &- e^{(1/2)\varepsilon^2 C}(H + (a(\beta)T_1 \cdot T_3)H)(e, e, 0) \\ &= e^{(1/2)\varepsilon^2 C}\left(\sum_{n=2}^{\infty} -\frac{1}{n!}(\tfrac{1}{2}y\varepsilon C_{12})^{n-1}(a(\beta)T_1 \cdot T_3)H\right)(e, e, 0). \end{aligned}$$

Because $H$ is a polynomial in all of its arguments, the right-hand side of this last expression may easily be estimated by

$$\|\text{r.h.s}\| \leqq Ky\varepsilon |a(\beta)| e^{Ky\varepsilon}.$$

Now using the last two equations, and the equalities $H(e, e, 0) = 0$, $T_1 \cdot T_3 H(e, e, 0) = 2C$, and $|a(\beta)| = O_g(\varepsilon^2)$ we have

$$I = e^{(1/2)\varepsilon^2 C}(-2a(\beta)C + e^{O(y\varepsilon)})O(y\varepsilon |a(\beta)|) = -2a(\beta)C + 1_{\text{supp}(g)}(x)O_g(\varepsilon^3).$$

Using this last equation and the definition of $I$ (Eq. 10.1) proves the lemma upon recalling that $a(\beta) = a(x_1 + \varepsilon) = (g, 1_{A_\varepsilon(\sigma)})$. Q.E.D.

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