Equivalence of heat kernel measure and pinned Wiener measure on loop groups

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Abstract.   We show that heat kernel measure and pinned Wiener measure on loop groups over simply connected compact Lie groups are equivalent. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Équivalence entre la mesure de la chaleur et la mesure du pont brownien sur des groupes de lacets

Résumé.   *On montre que la mesure de la chaleur et la mesure du pont brownien sur un groupe simplement connexe compact sont équivalentes.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soient K un groupe de Lie compact, connexe, $e \in K$ son élément neutre, K son algèbre de Lie et $\langle \cdot, \cdot \rangle$ un produit scalaire euclidien sur K qui soit invariant sous l'action adjointe.

Soient aussi $\mathcal{L}$ l'espace des lacets, μ_t^0 la mesure de Wiener du pont brownien de variance t et ν_t^0 la mesure de la chaleur sur $\mathcal{L}$, définie par l'équation (1.2). Il a été montré dans [6] que la mesure de la chaleur ν_t^0 est absolument continue relativement à la mesure μ_t^0 du pont brownien et il a été conjecturé que ces deux mesures sont équivalentes si K est simplement connexe. Dans cette Note, il est démontré que cette conjecture est vraie (théorème 2.2).

Soit $Z_t(\gamma)$ la dérivée de Radon–Nikodym $(\mathrm{d}\nu_t^0/\mathrm{d}\mu_t^0)(\gamma)$. Soit $N = \{\gamma \in \mathcal{L};\ Z_t(\gamma) = 0\}$.

Notons $\mathrm{H}^1(K)$ le sous-groupe des lacets d'énergie finie. Utilisant la quasi-invariance de la mesure μ_t^0 (*voir* [9]) et celle de ν_t^0 démontrée dans [4] sous les translations à gauche par $\mathrm{H}^1(K)$, on montre qu'essentiellement $hN = N$ pour tout $h \in \mathrm{H}^1(K)$. Utilisant le théorème d'ergodicité de Gross (*cf.* [8] et Aida [1]) on en déduit ou bien $\mu_t^0(N) = 0$ (ce qui est impossible) ou bien que $\mu_t^0(N^e) = 0$.

Note présentée par Paul MALLIAVIN.

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Une généralisation de ce résultat à des groupes qui ne sont pas simplement connexes est faite.

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1. Introduction

Let K be a connected compact Lie group, $e \in K$ be the identity, $\mathfrak{k} \equiv T_e K$ be the Lie algebra of K and $\langle \cdot, \cdot \rangle$ be an Ad_K-invariant inner product on $\mathfrak{k}$. (To simplify notation later we will, without loss of generality, assume that K is a matrix group.) The loop group on K is defined by

$$\mathcal{L} = \mathcal{L}(K) \equiv \big\{ \gamma \in \mathrm{C}\big([0,1], K\big) \, : \, \gamma(0) = \gamma(1) = e \big\}$$

and when K is not simply connected, we let $\mathcal{L}_0$ denote those loops in $\mathcal{L}$ which are homotopic to the constant loop at $e \in K$. We view $\mathcal{L}$ as a topological space equipped with the topology of uniform convergence.

Let μ_t^0 denote *pinned Wiener measure* with variance t. This measure may be described as the law of the process $\{ g_s : 0 \leqslant s \leqslant 1 \}$ pinned so that $g_1 = e$, where g is the solution to the Stratonovich stochastic differential equation

$$\mathrm{d}g_s = g_s \circ \sqrt{t}\, \mathrm{d}b_s \quad \text{with } g_0 = e.$$

Here b is a standard $\mathfrak{k}$-valued Brownian motion with covariance determined by $\langle \cdot, \cdot \rangle$. As introduced in [5] and [10], the *heat kernel measure* ν_t^0 on $\mathcal{L}$ is the time t distribution of a certain $\mathcal{L}$-valued Brownian motion which we now describe.

Let $\{ \chi(t,s) \}_{\substack{0 \leqslant s \leqslant 1 \\ 0 \leqslant t < \infty}}$ be a $\mathfrak{k}$-valued *Brownian bridge sheet*, i.e., χ is a $\mathfrak{k}$-valued centered continuous Gaussian process such that

$$\mathbb{E}\big[\langle A, \chi(t,s) \rangle \langle B, \chi(\tau,\sigma) \rangle \big] = \langle A, B \rangle \min(t,\tau)\big(\min(s,\sigma) - \sigma s \big) \tag{1.1}$$

for all $s, \sigma \in [0,1]$, $t, \tau \in [0, \infty)$ and $A, B \in \mathfrak{k}$. Let $\Sigma(t,s)$ for $0 \leqslant t < \infty$ and $0 \leqslant s \leqslant 1$ denote the solution to the Stratonovich stochastic differential equation in t (with s as a parameter),

$$\Sigma(\mathrm{d}t, s) = \Sigma(t,s) \circ \chi(\mathrm{d}t, s) \quad \text{with } \Sigma(0,s) = e, \ \forall s \in [0,1]. \tag{1.2}$$

By Malliavin [10] (*see* also Baxendale [3] or Driver [4]), we may and do choose $\Sigma(t,s)$ to be jointly continuous in (t,s). The *heat kernel measure* ν_t^0 at time $t > 0$ is the law of the process $s \to \Sigma(t,s) \in K$. Notice by construction that $\nu_t^0(\mathcal{L}_0) = 1$. For further details and motivations for these constructions the reader is referred to [4–6].

In [6], it was shown that ν_t^0 is absolute continuous relative to μ_t^0 and it was conjectured that these two measures are equivalent if K is simply connected. In this Note, we will prove in Theorem 2.2 that this conjecture is true. The proof is based on an elementary fact (Lemma 2.1 below) on the equivalence of two quasi-invariant measures.

2. An elementary lemma and the equivalence of ν_t^0 and μ_t^0

Let X be a topological group and Y be a subgroup. For a finite Borel measure ν and $h \in Y$, define the new measure ν^h on X by $\nu^h(A) = \nu(h \cdot A)$, where A is a Borel measurable subset and $h \cdot A = \{ hx \mid x \in A \}$.

We recall the basic notion of measures on groups. We say that ν is quasi-invariant relative to Y if ν^h is equivalent to ν for all $h \in Y$. For a quasi-invariant measure ν relative to Y, ν is said to be ergodic relative to the left action of Y if a Borel measurable subset A satisfying $\nu((A \smallsetminus h \cdot A) \cup (h \cdot A \smallsetminus A)) = 0$ for any $h \in Y$, implies that either $\nu(X \smallsetminus A) = 0$ or $\nu(A) = 0$.

LEMMA 2.1. – *Let ν_1 and ν_2 be quasi-invariant finite non-zero measures on X relative to a subgroup Y. Let $\alpha_i^h(x) = \frac{\mathrm{d}\nu_i^h}{\mathrm{d}\nu_i}(x)$ ($i = 1, 2$). Suppose that*
  1) *ν_1 is ergodic relative to Y and that*
  2) *ν_2 is absolute continuous relative to ν_1.*
Then ν_1 and ν_2 are equivalent.

Proof. – Let $Z(x) = \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}(x)$. We need only prove that $Z(x) > 0$ for ν_1-a.e. x. We will do this by showing that $\nu_1(N) = \nu_1(\{x \in X \mid Z(x) = 0\}) = 0$.

By definition of $\alpha_i^h(x)$,

$$\int_X f(h^{-1}x)\,\mathrm{d}\nu_i(x) = \int_X f(x)\alpha_i^h(x)\,\mathrm{d}\nu_i(x) \quad (i = 1, 2) \tag{2.1}$$

for all bounded measurable functions f on X and $h \in Y$. Using equation (2.1) and the definition of Z we have

$$\int_X f(h^{-1}x)\,\mathrm{d}\nu_2(x) = \int_X f(h^{-1}x)Z(x)\,\mathrm{d}\nu_1(x) = \int_X f(x)Z(hx)\alpha_1^h(x)\,\mathrm{d}\nu_1(x)$$

and

$$\int_X f(h^{-1}x)\,\mathrm{d}\nu_2(x) = \int_X f(x)\alpha_2^h(x)\,\mathrm{d}\nu_2(x) = \int_X f(x)\alpha_2^h(x)Z(x)\,\mathrm{d}\nu_1(x).$$

Since these equations hold for all bounded measurable functions, it follows

$$Z(hx)\alpha_1^h(x) = Z(x)\alpha_2^h(x) \quad \text{for } \nu_1\text{-a.e. } x, \tag{2.2}$$

for all $h \in Y$. Similar calculations shows that $\alpha_i^{h^{-1}}(hx)\alpha_i^h(x) = 1$ for ν_i-a.e. x, $i = 1, 2$, and $h \in Y$. In particular,

$$\alpha_i^h(x) > 0 \quad \text{for } \nu_i\text{-a.e. } x. \tag{2.3}$$

Thus we may write equation (2.2) as

$$Z(hx) = Z(x)\frac{\alpha_2^h(x)}{\alpha_1^h(x)}. \tag{2.4}$$

By equations (2.3) and (2.4) and the fact that $\nu_2(N) = 0$,

$$h \cdot N = N \quad \text{up to a } \nu_1\text{-null set}$$

for all $h \in Y$. Hence by the ergodicity of ν_1, either $\nu_1(X \smallsetminus N) = 0$ or $\nu_1(N) = 0$. Since ν_2 is not zero we know $\nu_1(X \smallsetminus N) > 0$ and therefore, $\nu_1(N) = 0$ which completes the proof. $\quad\square$

Now we are going to prove the equivalence of heat kernel measure and pinned Wiener measure. Let $\tilde{\mu}_t^0$ be the restriction of the pinned Wiener measure μ_t^0 to $\mathcal{L}_0$. The absolute continuity of ν_t^0 with μ_t^0 was proved in [6]. Let $Z_t(\gamma) = \frac{\mathrm{d}\nu_t^0}{\mathrm{d}\mu_t^0}(\gamma)$.

THEOREM 2.2. – *The Radon–Nikodým derivative $Z_t(\gamma)$ is positive for $\tilde{\mu}_t^0$-a.e. γ. Namely, μ_t^0 and ν_t^0 are equivalent on $\mathcal{L}_0$.*

Proof. – Let $\mathrm{H}^1(K)$ and $\mathrm{H}^1(K)_0$ denote the subgroups of finite energy loops in $\mathcal{L}$ and $\mathcal{L}_0$ respectively. The quasi-invariance of the *un*-pinned measure relative to $\mathrm{H}^1(K)$ is standard (*see* [2,7,11]). However, the quasi-invariance of pinned Wiener measure relative to $\mathrm{H}^1(K)$ is much more delicate and is due to M.-P. Malliavin and P. Malliavin [9]. The quasi-invariance of heat kernel measure ν_t^0 relative to $\mathrm{H}^1(K)_0$ was

proved in [4]. Furthermore the ergodicity of $\tilde{\mu}_t^0$ relative to $\mathrm{H}^1(K)_0$ was proved by Gross [8] (*see* also Aida [1] for this result and its generalizations). Hence the theorem is a consequence of Lemma 2.1 after setting $X = \mathcal{L}_0$, $Y = \mathrm{H}^1(K)_0$, $\nu_1 = \tilde{\mu}_t^0$ and $\nu_2 = \nu_t^0$. $\quad\square$

COROLLARY 2.3. – *For each $h \in \mathrm{H}^1(K)$, let $\nu_t^0(h, A) := \nu_t^0(h^{-1}A)-$ heat kernel measure on $\mathcal{L}$ starting at h. Let $\Pi \subset \mathrm{H}^1(K)$ be chosen so that to each homotopy class in $\mathcal{L}$, there is a unique representative in Π. Then μ_t^0 and $\sum_{h \in \Pi} \nu_t^0(h, \cdot)$ are equivalent measures on $\mathcal{L}$.*

Proof. – For $h \in \mathrm{H}^1(K)$, let $\mathcal{L}_h$ denote those loops in $\mathcal{L}$ which are homotopic to h. By Theorem 2.2, $\nu_t^0(h, \cdot)$ is equivalent to $\tilde{\mu}_t^0(h^{-1}(\cdot))$ on $\mathcal{L}_{h^{-1}}$. By the quasi-invariance of μ_t^0 relative to $\mathrm{H}^1(K)$, it follows that $\tilde{\mu}_t^0(h^{-1}(\cdot))$ is equivalent to $\mu_t^0|_{\mathcal{L}_{h^{-1}}}$ on $\mathcal{L}_{h^{-1}}$. Therefore, $\nu_t^0(h, \cdot)$ is equivalent to $\mu_t^0|_{\mathcal{L}_{h^{-1}}}$ on $\mathcal{L}_{h^{-1}}$ and hence $\sum_{h \in \Pi} \nu_t^0(h, \cdot)$ is equivalent to $\sum_{h \in \Pi} \mu_t^0|_{\mathcal{L}_{h^{-1}}} = \mu_t^0$. $\quad\square$

References

[1] Aida S., Differential calculus on path and loop spaces, II. Irreducibility of Dirichlet forms on loop spaces, Bull. Sci. Math. 122 (8) (1998) 635–666.

[2] Albeverio S., Høegh-Krohn R., The energy representation of Sobolev–Lie groups, Compos. Math. 36 (1) (1978) 37–51.

[3] Baxendale P., Wiener processes on manifolds of maps, Proc. Roy. Soc. Edinburgh Sect. A 87 (1–2) (1980/81) 127–152.

[4] Driver B.K., Integration by parts and quasi-invariance for heat kernel measures on loop groups, J. Funct. Anal. 149 (2) (1997) 470–547.

[5] Driver B.K., Lohrenz T., Logarithmic Sobolev inequalities for pinned loop groups, J. Funct. Anal. 140 (2) (1996) 381–448.

[6] Driver B.K., Srimurthy V., Absolute continuity of heat kernel measure with respect to pinned Wiener measure on loop groups, UCSD Preprint, 2000; The Annale of Probab. (to appear).

[7] Gross L., Logarithmic Sobolev inequalities on loop groups, J. Funct. Anal. 102 (2) (1991) 268–313.

[8] Gross L., Uniqueness of ground states for Schrödinger operators over loop groups, J. Funct. Anal. 112 (2) (1993) 373–441.

[9] Malliavin M.-P., Malliavin P., Integration on loop groups, I. Quasi-invariant measures, J. Funct. Anal. 93 (1) (1990) 207–237.

[10] Malliavin P., Hypoellipticity in infinite dimensions, Diffusion processes and related problems in analysis, Vol. I (Evanston, IL, 1989), Birkhäuser Boston, Boston, MA, 1990, pp. 17–31.

[11] Shigekawa I., Transformations of the Brownian motion on the Lie group, Stochastic analysis (Katata/Kyoto, 1982), North-Holland, Amsterdam, 1984, pp. 409–422.

[12] Srimurthy V., On the equivalence of measures on loop spaces, Probab. Theory Relat. Fields (2000) (to appear).