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CORRIGENDUM

A Correction to the Paper "Integration by Parts and Quasi-Invariance for Heat Kernel Measures on Loop Groups"

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It is asserted in Definition 4.2 in [1] that the random operators $U(t)$ defined there are unitary. As was pointed out to the author by Shizan Fang, it is clear that $U(t)$ is an isometry but it is not obvious that $U(t)$ is surjective. The purpose of this note is to fill this gap.

1. INITIAL COMMENTS

I would first like to point out that, even without verifying the surjectivity of $U(t)$ defined in Definition 4.2 in [1], all of the results and all but one proof in [1] would still be valid. Indeed, the only place where the surjectivity of $U(t)$ was used, other than for notational simplicity, was in the first proof of Theorem 4.14 in [1]. Nevertheless, Theorem 4.14 is still valid because of Theorem 6.2; see Remark 4.15 in [1]. The only notational changes that would need to be made are: (1) replace the orthogonal group $O(H_0(\mathfrak{g}))$ on $H_0(\mathfrak{g})$ by the set $ISO(H_0(\mathfrak{g}))$ of isometries on $H_0(\mathfrak{g})$ and (2) interpret $U(t)\,\dot{H}(t)$ as

$$U(t)\,\dot{H}(t) \equiv \dot{h}(t) + \tfrac{1}{2}\,\mathrm{Ric}\;U(t)\,h(t).$$

In the next section we will give a more satisfying remedy to the gap in Definition 4.2 in [1], namely the fact that $U(t)$ is unitary.

2. A PROOF THAT $U(T)$ IS UNITARY

The reader is referred to [1] for the notation and definitions used in this corrigendum. Recall that $S_0 \subset H_0(\mathfrak{g})$ is an orthonormal basis for $H_0(\mathfrak{g})$ and

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for any $k_0 \in H_0(\mathfrak{g})$ we let $k(t)$ denote the solution to the Itô stochastic differential equation (4.2) in [1],

$$dk(t) = -D_{d\beta(t)}k(t) + \tfrac{1}{2}\Delta^{(1)}k(t)\,dt \qquad \text{with} \quad k(0) = k_0. \tag{2.1}$$

In [1], $U(t)$ was defined as $U(t)\,h := \sum_{k_0 \in S_0} (k_0, h)\,k(t)$ (Definition 4.2) and it was shown that $h(t) := U(t)\,h$ solves Eq. (2.1) with $h(0) = h$ (Lemma 4.3) and that $U(t)$ is an isometry (Theorem 4.1). The surjectivity of $U(t)$ will be an easy consequence of the next lemma.

Lemma 2.1. *Let $k_0, h_0 \in H_0(\mathfrak{g})$; then*

$$E(k_0, U(t)^*\,h_0)^2 = E(U(t)\,k_0, h_0)^2 = E(k_0, U(t)\,h_0)^2. \tag{2.2}$$

Proof. In what follows we will identify $H_0(\mathfrak{g}) \otimes H_0(\mathfrak{g})$ with the Hilbert–Schmidt operators $HS(H_0(\mathfrak{g}))$ on $H_0(\mathfrak{g})$ determined by identifying $h \otimes k \in H_0(\mathfrak{g}) \otimes H_0(\mathfrak{g})$ with the rank one operator $(h \otimes k)\,u = (k, u)\,h$ for all $u \in H_0(\mathfrak{g})$. We are using $(\cdot,\cdot)$ to denote inner product on both of the Hilbert spaces $H_0(\mathfrak{g})$ and $H_0(\mathfrak{g}) \otimes H_0(\mathfrak{g})$.

Let $k(t) = U(t)\,k_0$ and consider the random operator $k(t) \otimes k(t)$. By Itô's lemma,

$$\begin{aligned} d(k(t) \otimes k(t)) = {} & -(D_{d\beta(t)}k(t)) \otimes k(t) - k(t) \otimes D_{d\beta(t)}k(t) \\ & + \tfrac{1}{2}\Big\{\Delta^{(1)}k(t) \otimes k(t) + k(t) \otimes \Delta^{(1)}k(t) \\ & + 2 \sum_{\ell \in S_0} D_\ell k(t) \otimes D_\ell k(t)\Big\}\,dt. \end{aligned} \tag{2.3}$$

This last expression may be simplified by noticing that

$$\Delta^{(1)} \otimes I + I \otimes \Delta^{(1)} + 2 \sum_{\ell \in S_0} D_\ell \otimes D_\ell = \sum_{\ell \in S_0} (D_\ell \otimes I + I \otimes D_\ell)^2 =: \Delta^{(2)}. \tag{2.4}$$

By Theorem 3.12 and Lemma 4.21 in Driver and Lohrenz [2], the sums in Eq. (2.4) converge strongly to a bounded self-adjoint operator ($\Delta^{(2)}$) on $H_0(\mathfrak{g}) \otimes H_0(\mathfrak{g})$.

Remark 2.2. In [2] the operator $D_\ell^{(2)} := (D_\ell \otimes I + I \otimes D_\ell)$ on $H_0(\mathfrak{g}) \otimes H_0(\mathfrak{g})$ was simply denoted by D_ℓ and $\Delta^{(1)}$ on $H_0(\mathfrak{g})$ and $\Delta^{(2)}$ on $H_0(\mathfrak{g}) \otimes H_0(\mathfrak{g})$ were both denoted by Δ.

With this notation, we may write Eq. (2.3) as

$$\begin{aligned} d(k(t) \otimes k(t)) = {} & -(D_{d\beta(t)}k(t)) \otimes k(t) - k(t) \otimes D_{d\beta(t)}k(t) \\ & + \tfrac{1}{2}\Delta^{(2)}(k(t) \otimes k(t))\,dt. \end{aligned} \tag{2.5}$$

Integrating this equation relative to t and then taking expectations of the result show that

$$E(k(t)\otimes k(t))=k_0\otimes k_0+\tfrac{1}{2}E\int_0^t \Delta^{(2)}(k(\tau)\otimes k(\tau))\,d\tau$$

$$=k_0\otimes k_0+\tfrac{1}{2}\Delta^{(2)}E\int_0^t (k(\tau)\otimes k(\tau))\,d\tau. \qquad (2.6)$$

The solution to this last equation is

$$E(k(t)\otimes k(t))=e^{t\Delta^{(2)}/2}(k_0\otimes k_0). \qquad (2.7)$$

Equation (2.6), along with the fact that $\Delta^{(2)}$ is self-adjoint, implies

$$E(U(t)\,k_0,h_0)^2=E(k(t),h_0)^2=(e^{t\Delta^{(2)}/2}(k_0\otimes k_0),h_0\otimes h_0)$$
$$=(k_0\otimes k_0,e^{t\Delta^{(2)}/2}(h_0\otimes h_0))=E(k_0,U(t)\,h_0)^2.$$

Q.E.D. (2.8)

THEOREM 2.3. *The random isometry $U(t)$ defined in Definition* 4.2 *in* [1] *is unitary a.s.*

Proof. Let $P(t):=U(t)\,U(t)^*$, a random projection operator. Our goal is to show that $P(t)=I$ a.s. Summing Eq. (2.2) on $k_0\in S_0$ and using the fact that $U(t)$ is an isometry shows that

$$E\,\|P(t)\,h_0\|^2=E\,\|U(t)^*\,h_0\|^2=E\,\|U(t)\,h_0\|^2=\|h_0\|^2$$

for all $h_0\in H_0(\mathfrak{g})$. Because $\|h_0\|^2\geqslant\|P(t)\,h_0\|^2$, it follows that $\|h_0\|^2=\|P(t)\,h_0\|^2$ a.s. or equivalently $h_0=P(t)\,h_0$ a.s. Since $H_0(\mathfrak{g})$ is separable, we may conclude that $I=P(t)$ a.s. as desired. Q.E.D

Theorem 2.3 may be strengthened as follows. Another proof of the following theorem which was discovered essentially simultaneously to the one presented here will appear in Fang [3].

THEOREM 2.4. *On a set of full measure independent of $t\geqslant 0$, the map $t\to U(t)$ is unitary. That is, the null sets implicitly appearing in Theorem* 2.3 *may be chosen to be independent of t.*

Proof. Let $h\in H_0(\mathfrak{g})$. We will start by showing that there exists a null set Ω_h such that on Ω_h^c, the map $t\to\|P(t)\,h\|^2$ is continuous. To this end let $\{S_n\}_{n=1}^{\infty}$ be a collection of finite subsets contained in S_0 such that S_n

increases to S_0 as $n \to \infty$. For $k_0 \in S_0$ set $k(t) = U(t)\, k_0$ and let $P_n(t)$ be the finite rank projection operators

$$P_n(t) := \sum_{k_0 \in S_n} U(t)\, k_0 \otimes U(t)\, k_0 = \sum_{k_0 \in S_n} k(t) \otimes k(t).$$

Since k is a continuous process for each $k_0 \in S_0$, there is a null set Ω_1 such that

$$t \to \|P_n(t)\, h\|^2 = \sum_{k_0 \in S_n} (k(t), h)^2 = \sum_{k_0 \in S_n} (k(t) \otimes k(t), h \otimes h)$$

is continuous on Ω_1^c for all $h \in H_0(\mathfrak{g})$ and $n = 1, 2, 3, \ldots$. Let $P_{m,n}(t) := P_m(t) - P_n(t)$ and suppose for concreteness that $m > n$. Then by Eq. (2.5), the skew symmetry of $D_\ell^{(2)}$, and the symmetry of $\Delta^{(2)}$,

$$\|P_{m,n}(t)\, h\|^2 - \|P_{m,n}(0)\, h\|^2 = M_t^{m,n} + A_t^{m,n},$$

where

$$M_t^{m,n} := \int_0^t (P_{m,n}(\tau), D^{(2)}_{d\beta(\tau)}(h \otimes h))$$

and

$$A_t^{m,n} := \tfrac{1}{2} \int_0^t (P_{m,n}(\tau), \Delta^{(2)}(h \otimes h))\, d\tau.$$

Because $M_t^{m,n}$ is a square integrable martingale,

$$\begin{aligned} E\Big(\sup_{0 \leqslant t \leqslant T} |M_t^{m,n}|^2\Big) &\leqslant CE\, |M_T^{m,n}|^2 \\ &= C \int_0^T \sum_{\ell \in S_0} E(P_{m,n}(t), D_\ell^{(2)}(h \otimes h))^2\, dt \\ &= 4C \int_0^T E\left(\sum_{\ell \in S_0} \sum_{k_0 \in S_m \setminus S_n} (k(t), D_\ell h)^2\, (k(t), h)^2\right) dt \\ &\leqslant 4C\, \|h\|^2 \int_0^T E\left(\sum_{\ell \in S_0} \|P_{m,n}(t)\, D_\ell h\|^2\right) dt, \end{aligned}$$

which converges to zero as $m, n \to \infty$ by the dominated convergence theorem along with the facts: (1) $\|P_{m,n}(t)\, D_\ell h\|^2 \leqslant \|D_\ell h\|^2$, (2) $\sum_{\ell \in S_0} \|D_\ell h\|^2 = (-\Delta h, h) \leqslant \|\Delta\|_{op}\, \|h\|^2$, and (3) $\lim_{m,n \to \infty} \|P_{m,n}(t)\, D_\ell h\|^2 = 0$. Similarly,

$$\sup_{0 \leqslant t \leqslant T} |A_t^{m,n}| = \sup_{0 \leqslant t \leqslant T} \left| \int_0^t \sum_{k_0 \in S_m \setminus S_n} (k(\tau), \Delta h)(k(\tau), h)\, d\tau \right.$$
$$\left. + \int_0^t \sum_{\ell \in S_0} \sum_{k_0 \in S_m \setminus S_n} (D_\ell h, k(\tau))^2\, d\tau \right|$$
$$\leqslant \int_0^T \left[|(\Delta h, P_{m,n}(\tau)\, h)| + \sum_{\ell \in S_0} \|P_{m,n}(\tau)\, D_\ell h\|^2 \right] d\tau$$

which converges to zero boundedly as $m, n \to \infty$. Combining the above estimates shows that

$$E \sup_{0 \leqslant t \leqslant T} | \|P_m(t)\|^2 - \|P_n(t)\, h\|^2|^2 = E \sup_{0 \leqslant t \leqslant T} \|P_{m,n}(t)\, h\|^4 \to 0 \qquad m, n \to \infty.$$

Therefore there exists a null set Ω_h such that on Ω_h^c, $t \in [0, T] \to \|P(t)\, h\|^2$ is the uniform limit of the continuous functions and hence is continuous.

Since $H_0(\mathfrak{g})$ is separable, we may choose a null set Ω_2 independent of $h \in H_0(\mathfrak{g})$ and $T > 0$ such that $t \in [0, T] \to \|P(t)\, h\|^2$ is continuous on Ω_2^c. By Theorem 2.3, given a countable dense subset $D \subset [0, T]$, there exists a null set Ω_D such that $P(t) = I$ on Ω_D^c; i.e., $\|P(t)\, h\| = \|h\|$ for all $h \in H_0(\mathfrak{g})$ and $t \in D$. Let Ω_0 be the null set, $\Omega_0 = \Omega_2 \cup \Omega_D$. Then on Ω_0^c, $\|P(t)\, h\| = \|h\|$ for $t \in [0, T]$ and $h \in H_0(\mathfrak{g})$ or equivalently $P(t) = I$ for all $t \in [0, T]$. Q.E.D

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