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Logarithmic Sobolev Inequalities for Pinned Loop Groups

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Let G be a connected compact type Lie group equipped with an Ad_G-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. Given this data there is a well known left invariant "H^1-Riemannian structure" on $\mathscr{L}=\mathscr{L}(G)$—the infinite dimensional group of continuous based loops in G. Using this Riemannian structure, we define and construct a "heat kernel" $\nu_T(g_0, \cdot)$ associated to the Laplace–Beltrami operator on $\mathscr{L}(G)$. Here $T>0$, $g_0 \in \mathscr{L}(G)$, and $\nu_T(g_0, \cdot)$ is a certain probability measure on $\mathscr{L}(G)$. For fixed $g_0 \in \mathscr{L}(G)$ and $T>0$, we use the measure $\nu_T(g_0, \cdot)$ and the Riemannian structure on $\mathscr{L}(G)$ to construct a "classical" pre-Dirichlet form. The main theorem of this paper asserts that this pre-Dirichlet form admits a logarithmic Sobolev inequality.

Contents.

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1. INTRODUCTION

1.1. *Background*

In this paper we consider the existence of a logarithmic Sobolev inequalities on loop groups. The study of loop groups is motivated by physics and the theory of group representations; see, for example, [26] and the references therein. This work was motivated by the papers of Getzler [21] and Gross [23]. In [21], Getzler shows that Bakry and Emery criteria (see [8, 9]) for proving a logarithmic Sobolev inequality does not hold in general for loop groups when the "underlying" measure is pinned Wiener measure. However, Gross [23] (see also [22]) was able to show (using pinned Wiener measure) that a logarithmic Sobolev inequality *with* an added potential term does hold for loop groups. The question as to when this potential is needed is still open.

In this paper, we will change the problem slightly. Instead of using Wiener measure we will use a "heat kernel measure" on the loop group as the underlying measure. In this setting, we will show that methods of Bakry and Ledoux (see [7, 10, 11] and also [2, 4–6, 8, 18, 24, 33, 34]) may be applied to give a logarithmic Sobolev inequality *without* a potential. This result compliments the beautiful results already known about "spectral gaps" and logarithmic Sobolev inequalities for general based path spaces (see Fang [19], Hsu [24], and Aida and Elworthy [1]).

1.2. *Statement of Results*

Let G be a compact type Lie group, $\mathfrak{g} \equiv T_e G$ be the Lie algebra of G, and $\langle \cdot, \cdot \rangle$ be an Ad_G invariant inner product on $\mathfrak{g}$. For $\xi \in \mathfrak{g}$, let $|\xi| \equiv \sqrt{\langle \xi, \xi \rangle}$. Let $\mathscr{L} = \mathscr{L}(G)$ denote the based loop group on G consisting of continuous paths g: $[0, 1] \to G$ such that $g(0) = g(1) = e$, where $e \in G$ is the identity element.

Given a function h: $[0, 1] \to \mathfrak{g}$ such that $h(0) = 0$, define $(h, h) = \infty$ if h is not absolutely continuous and set $(h, h) = \int_0^1 |h'(s)|^2 \, ds$ otherwise. Let

$$H_0 \equiv \{h\colon [0, 1] \to \mathfrak{g} \mid h(0) = h(1) = 0 \text{ and } (h, h) < \infty\}.$$

We will think of H_0 as the Lie algebra of $\mathscr{L}$.

In order to define the tangent space $T\mathscr{L}$ of $\mathscr{L}$, let θ denote the Maurer Cartan form, i.e., $\theta\langle\xi\rangle = L_{g^{-1}*}\xi$ for all $\xi \in T_g G$ and $g \in G$. We now define

$$T\mathscr{L} \equiv \{X\colon [0,1] \to TG \mid \theta\langle X\rangle \in H_0 \text{ and } p \circ X \in \mathscr{L}\}, \tag{1.1}$$

where $(\theta\langle X\rangle)(s) \equiv \theta\langle X(s)\rangle$ and $p\colon TG \to G$ is the canonical projection. By abuse of notation also let $p\colon T\mathscr{L} \to \mathscr{L}$ denote the canonical projection on $T\mathscr{L}$ defined by $X \in T\mathscr{L} \to p \circ X \in \mathscr{L}$. As usual, define the tangent space at $g \in \mathscr{L}$ by $T_g\mathscr{L} \equiv p^{-1}(\{g\})$.

Using left translations, we may extend the inner product $(\cdot,\cdot)$ on H_0 to a Riemannian metric on $T\mathscr{L}$. Explicitly, for $X \in T\mathscr{L}$, set

$$(X, X) \equiv (\theta\langle X\rangle, \theta\langle X\rangle)_{H_0}.$$

In this way, $\mathscr{L}$ is to be thought of as an infinite dimensional Riemannian manifold. The following theorem is paraphrased from Corollary 6.3 of Section 6.

THEOREM 1.1. *For each $t > 0$ there is a probability kernel $(g_0 \to \nu_t(g_0, \cdot))$: $\mathscr{L}(G) \to \mathscr{M}_1(\mathscr{L}(G))$ ($\mathscr{M}_1(\mathscr{L}(G))$ is the set of probability measures on $\mathscr{L}(G)$) such that for all "bounded cylinder functions" f on $\mathscr{L}(G)$, $u(t, g_0) \equiv \int_{\mathscr{L}(G)} f(g)\,\nu_t(g_0, dg)$ is the unique solution to the heat equation*:

$$\partial u(t,\cdot)/\partial t = \tfrac{1}{2}\Delta u(t,\cdot) \qquad \text{with} \quad \lim_{t\downarrow 0} u(t,g) = f(g).$$

Here Δ denotes the "Laplace-Beltrami" operator on $\mathscr{L}(G)$.

See Definition 4.2 and Definition 4.17 below for the notion of cylinder functions and the Laplacian respectively. The main theorem of this paper is Theorem 6.4 of Section 6 which we state using the convention that $0 \log 0 \equiv 0$ and

$$\nu_T(g_0, f) \equiv \int_{\mathscr{L}(G)} f(x)\,\nu_T(g_0, dx),$$

whenever f is an integrable function on $\mathscr{L}(G)$ relative to $\nu_T(g_0, \cdot)$.

THEOREM 1.2 (Logarithmic Sobolev Inequality). *There exists a constant $C \in [0, \infty)$ (depending on G and $\langle\cdot,\cdot\rangle$) such that for all real valued bounded cylinder functions f on $\mathscr{L}$, $g_0 \in \mathscr{L}$, and $T > 0$,*

$$\nu_T(g_0, f^2 \log f^2) \leqslant \frac{2}{C}(e^{CT} - 1)\,\nu_T(g_0, \|\vec{\nabla} f\|^2)$$
$$+ \nu_T(g_0, f^2) \cdot \nu_T(g_0, \log f^2), \tag{1.2}$$

where $\|\vec{\nabla} f\|^2 = (\vec{\nabla} f, \vec{\nabla} f)$, *and* $\vec{\nabla} f$ *is the gradient of* f *related to the Riemannian structure* $(\cdot, \cdot)$ *on* $\mathscr{L}$. (*If* $C=0$ *then* $(2/C)(e^{CT}-1) \equiv 2T$.)

Remark 1.3. Given $g_0 \in \mathscr{L}(G)$ and $T>0$, let $\mathscr{E}^0_{T, g_0}$ be the symmetric quadratic form defined by

$$\mathscr{E}^0_{T, g_0}(u, v) \equiv \int_{\mathscr{L}} (\vec{\nabla} u(g), \vec{\nabla} v(g))\, v_T(g_0, dg),$$

where u and v are smooth cylinder functions on $\mathscr{L}(G)$ which are bounded and have bounded gradients. The form $\mathscr{E}^0_{T, g_0}$ is studied in [16] where it will be shown to be closable.

1.3. *Outline of Paper*

[§ 2] In the second section, the finite dimensional version of heat kernel logarithmic Sobolev inequalities is reviewed. A proof for the case of unimodular Lie groups is given here since it is needed for the main theorem. The heat kernel logarithmic Sobolev inequalities seem to have been first discovered by Bakry and Ledoux [10].

[§ 3] From Eq. (1.1), it follows that $T\mathscr{L}$ is isomorphic to $\mathscr{L} \times H_0$. Similarly, any bundle associated to $T\mathscr{L}$ is trivial. Therefore, we may consider tensor fields on $\mathscr{L}$ as functions from $\mathscr{L}$ to $H_0^{\otimes k} \otimes (H_0^*)^{\otimes l}$ for some non-negative integers k and l. This point of view is used implicitly in the sequel. In the third section we develop the geometry of the "Levi–Civita covariant" derivative restricted to constant functions from $\mathscr{L}$ to $H_0^{\otimes k} \otimes (H_0^*)^{\otimes l}$. These functions correspond to left-invariant tensor fields. In particular, we compute the curvature and the Ricci curvature tensor of $\mathscr{L}$. It is shown that the Ricci curvature tensor is bounded from below by the metric on $\mathscr{L}$. The computation of the Ricci curvature already appears in Freed [20].

[§ 4] Here the geometry of the Levi–Civita covariant derivative on $\mathscr{L}$ is developed on "cylinder" functions from $\mathscr{L}$ to $H_0^{\otimes k} \otimes (H_0^*)^{\otimes l}$. In particular we introduce the Laplacian on $\mathscr{L}$ and verify that the Bochner Wietzenböck formula still holds. This is done in more detail than necessary for the purposes of this paper in anticipation of future work.

[§ 5] In this section, we use the computations in Section 4 to verify that the Ricci curvatures associated to certain finite dimensional "cylindrical" approximations to $\mathscr{L}$ are uniformly bounded below, see Corollary 5.3. This fact in combination with the results from Section 2 enable us to prove a preliminary version of Theorem 1.2, see Theorem 5.6.

[§ 6] This section is devoted to constructing the heat kernel measures associated to the heat equation on $\mathscr{L}(G)$. This is done by writing down the finite dimensional distributions and verifying Kolmogorov's continuity criteria. After the construction of the heat kernel measure, the final form of Theorem 1.2 is an immediate consequence of Theorem 5.6, see Theorem 6.4.

There is an alternative approach to constructing the heat kernel measures, which is not pursued here. Namely, first construct an $\mathscr{L}(G)$-valued "Brownian motion" starting at $g_0 \in \mathscr{L}$ and then take $\nu_T(g_0, \cdot)$ to be the time T distribution of this Brownian motion. The construction of such a Brownian motion in the case that G is compact is indicated by Malliavin in [25].

2. LOG SOBOLEV INEQUALITIES FOR HEAT KERNELS ON UNIMODULAR LIE GROUPS

In this section we will review the method of Bakry and Ledoux for proving "heat kernel" logarithmic Sobolev inequalities. For further details the reader is referred to Bakry and Ledoux [7, 11]. The main result of this paper will be proved by applying the results of this section to finite dimensional "cylindrical" approximations of $\mathscr{L}(G)$.

For the moment let M be a connected complete Riemannian manifold without boundary of dimension N. We will write $(\cdot, \cdot)$ for the induced metric on any of the vector bundles $T^{\otimes k}M \otimes (T^*M)^{\otimes l}$ for $k, l = 0, 1, 2, \ldots$, where as usual $T^{\otimes 0}M = (T^*M)^{\otimes 0}$ is to be taken as the trivial vector bundle $M \times \mathbb{R}$. Also let $|\xi|^2 \equiv (\xi, \xi)$ for $\xi \in T^{\otimes k}M \otimes (T^*M)^{\otimes l}$. Let Δ be the Levi–Civita Laplacian on M, and Ric be the Ricci tensor of the Levi–Civita covariant derivative ∇.

By Theorem 2.4 of Strichartz [31], the Laplace–Beltrami operator Δ on $C_c^\infty(M)$ is an essentially self-adjoint non-negative densely defined operator on $L^2(M, dx)$, where dx denotes the Riemannian volume measure on M. By abuse of notation, we will continue to denote the closure of Δ by Δ. By the spectral theorem Δ generates a $L^2(M, dx)$ contraction semigroup $\{e^{t\Delta/2}\}_{t>0}$. The following theorem summarizes some well known properties of $e^{t\Delta/2}$.

THEOREM 2.1. *There is a smooth function $p_t(x, y)$ for $t > 0$ and $x, y \in M$ such that*

$$(e^{t\Delta/2} f)(x) \equiv \int_M p_t(x, y)\, f(y)\, dy \;\forall f \in C_c^\infty(M). \tag{2.1}$$

Moreover the heat kernel $(p_t(x, y))$ has the following properties.

1. $p_t(x, y) = p_t(y, x) > 0$ *for all* $t > 0$ *and* $x, y \in M$.
2. $\partial p_t(x, y)/\partial t = \frac{1}{2}\Delta_x p_t(x, y) = \frac{1}{2}\Delta_y p_t(x, y)$ *for all* $t > 0$ *and* $x, y \in M$.

3. *If there is a constant* $C \in \mathbb{R}$ *such that*

$$\mathrm{Ric}\langle v, v \rangle \geqslant -C(v, v) \ \forall v \in TM, \tag{2.2}$$

then p_t *is conservative, i.e.* $\int_M p_t(x, y)\, dy = 1$ *for all* $t > 0$ *and* $x \in M$.

Proof. See Strichartz [31] or Chavel [12] Chap. VIII for the first two items. Item 3 is proved in Yau [35], see also Bakry [3]. Q.E.D.

Given a measurable function $f\colon M \to [0, \infty]$ let

$$(P_T f)(x) \equiv \int_M p_T(x, y)\, f(y)\, dy. \tag{2.3}$$

Similarly if $f\colon M \to \mathbb{R}$ is measurable and $(P_T |f|)(x) < \infty$ we define $(P_T f)(x)$ by (2.3). We now want to consider the validity of the following "heat kernel" logarithmic Sobolev inequality:

$$P_T(f^2 \log f^2) \leqslant 2\{(e^{CT} - 1)/C\}\ P_T |\nabla f|^2 + P_T(f^2) \log P_T f^2. \tag{2.4}$$

The fact that such an inequality should hold seems to have been first discovered by D. Bakry and M. Ledoux [10]. If M is compact there are by now a number of proofs that (2.4) holds for all $f \in C^\infty(M)$, see Bakry and Ledoux [11], Bakry [7], E. Hsu [24], and F. Wang [33, 34], and Driver and Hu [18]. These proofs follow the circle of ideas introduced by Bakry and Emery [8, 9], see also Bakry [2, 4–6]. All of these proofs formally hold for noncompact manifolds as well. However, in the noncompact case, there are a number of technical details to attend to, see Bakry [7]. Since, a complete proof of (2.4) is rather difficult to find in one single source in the current literature, we will give the technical details for the special case needed in the body of this paper; namely the case when M is a unimodular Lie group.

2.1. *The Unimodular Lie Groups Case*

From now on M will be a unimodular Lie group and $(\cdot, \cdot)$ is *any* fixed inner product on $\mathfrak{m} = T_e M$—the Lie algebra of M. We will extend $(\cdot, \cdot)$ to a Riemannian metric on TM by demanding that all the left translations are isometries. Given $A \in \mathfrak{m} = T_e M = \mathrm{Lie}(M)$, let $\tilde{A}$ denote the unique left invariant vector field on M such that $\tilde{A}(e) = A$.

DEFINITION 2.2. The distance metric $d\colon M \times M \to M$ is defined by

$$d(g, h) = \inf \int_0^1 |\sigma'(s)|\, ds,$$

where the infimum is taken over all C^1-paths σ in M such that $\sigma(0)=g$ and $\sigma(1)=h$. Also set

$$|g| \doteq d(g, e) \ \forall g \in M.$$

Notice that

$$d(xg, xh) = d(g, h)$$

for all $g, h, x \in M$. Indeed, if σ is a curve joining g to h, then $x\sigma(\cdot)$ is a curve joining xg to xh which has the same length as σ. Because of the above displayed equation,

$$d(g, h) = |g^{-1}h| = |h^{-1}g|.$$

Setting $h=e$ in this equation shows that $|g| = |g^{-1}|$ for all $g \in M$.

Because d is left invariant, it is easily checked that (M, d) is a complete metric space so that $(M, (\cdot, \cdot))$ is a complete Riemannian manifold. Also, the left invariance of the metric $(\cdot, \cdot)$ implies that the curvature tensor and hence the Ricci tensor are also left invariant. In particular, this guarantees that there is a constant $C \in \mathbb{R}$ such that (2.2) holds.

Remark 2.3. Because $(\cdot, \cdot)$ is left invariant, the Riemannian volume measure dx is a left invariant Haar measure. Since M is assumed to be unimodular, dx is also a right invariant Haar measure. In this setting it is well known and easy to check that the Laplace-Beltrami operator Δ may be written as $\Delta = \sum_{i=1}^{N} \tilde{A}_i^2$, where $\{A_i\}_{i=1}^{N}$ is any orthonormal basis of $(\mathfrak{m}, (\cdot, \cdot))$, see for example Remark 2.2 of [17].

Definition 2.4. A function $f\colon M \to \mathbb{R}$ is *exponentially bounded* if there are constants B and β such that $|f(x)| \leqslant Be^{\beta |x|}$ for all $x \in M$.

We will need the following well known properties of the heat kernel $p_t(x, y)$.

Proposition 2.5. *For $t > 0$, define $v_t(x) \equiv p_t(x, e) = p_t(e, x)$, where $e \in M$ is the identity. Then*:

1. *$p_t(x, y) = v_t(y^{-1}x)$, so that*

$$(P_t f)(x) = \int_M f(y)\, v_t(y^{-1}x)\, dy = \int_M f(xy)\, v_t(y^{-1})\, dy. \tag{2.5}$$

2. *v_t is symmetric*: $v_t(x^{-1}) = v_t(x)$.

3. ν_t *is conservative*:

$$\int_G \nu_t(x)\, dx = 1. \tag{2.6}$$

4. ν_t *is an approximate δ-function, i.e.*,

$$\lim_{t \downarrow 0} \int_M f(xy)\, \nu_t(y)\, dy = f(x) \qquad \text{if} \quad f \in C_c(M). \tag{2.7}$$

5. [*Heat Kernel Bounds*] *For $T > 0$ and $\varepsilon \in (0, 1]$ there is a constant $C(T, \varepsilon) < \infty$ such that for all $t \in (0, T]$ and $x \in M$*;

$$\nu_t(x) \leqslant C(T, \varepsilon)\, t^{-N/2} \exp\{-|x|^2/2(1+\varepsilon)\, t\}. \tag{2.8}$$

Proof. Item 1 follows from the fact that left translations commute with Δ, item 2 follows from item 1 and the symmetry of p_t, item 3 is a consequence of Theorem 2.1 (item 3), and item 4 holds because $p_t(x, y)$ is a fundamental solution to the heat equation. For a further discussion on the first four items see Section III.2 of [29], Section 2.2 of [15], and Proposition 3.1 of [17]. The heat kernel bounds in (2.8) may be found on page 257 of Robinson [29], see also Section 5 of Davies [14] and Section 3 of Varopoulos [32]. Q.E.D.

The next proposition facilitates the use of heat kernel bound in the above proposition. In the proof of this proposition it is necessary to recall the volume estimates which follows from Bishop's comparison theorem.

Lemma 2.6 (Bishop's Comparison Theorem). *Let (M, g) be an N-dimensional complete Riemannian manifold, $\kappa \geqslant 0$, and assume that*

$$\operatorname{Ric}\langle \xi, \xi \rangle \geqslant -(N-1)\, \kappa g \langle \xi, \xi \rangle \ \forall \xi \in TM.$$

Let $o \in M$ and $V(r)$ denote the Riemannian volume of the ball of radius r centered at $o \in M$. Then

$$V(r) \leqslant \omega_{N-1} \int_0^r \left(\frac{\sinh \sqrt{\kappa}\, \rho}{\sqrt{\kappa}} \right)^{N-1} d\rho, \tag{2.9}$$

where ω_{N-1} is the surface area of the unit $N-1$ sphere in $\mathbb{R}^N$. Also

$$V(r) \leqslant \omega_{N-1} r^N e^{\sqrt{\kappa} r}. \tag{2.10}$$

Proof. By Bishops' comparison theorem (see Theorem 3.9, p. 123 of Chavel [13]) $V(r) \leqslant V_\kappa(r)$, where $V_\kappa(r)$ is the volume of a ball of radius r in N-dimensional hyperbolic space with constant sectional curvature $-(N-1)\,\kappa$. This proves (2.9), since $V_\kappa(r)$ is exactly the right member of (2.9), see Eq. (2.48) on page 72, the formula after Eq. (3.7) on p. 104, and the formula for the volume of a metric disk above Proposition 3.2 on p. 116 of Chavel [13].

Elementary calculus shows that $(1-e^{-x})/x \leqslant 1$, from which it easily follows that $\sinh(\sqrt{\kappa}\,\rho)/\sqrt{\kappa} \leqslant \rho e^{\sqrt{\kappa}\,\rho}$. Substituting this inequality into 2.9 shows that

$$V(r) \leqslant \omega_{N-1} \int_0^r \rho^{N-1} e^{\sqrt{\kappa}\,\rho}\, d\rho.$$

The inequality (2.10) follows from this inequality and elementary calculus. Q.E.D.

It is possible to prove an estimate of the form in (2.10) in the Lie group case just using the translation invariance of the metric, see Lemma 5.8 of [15]. However, this method does not give any control over the constants ω_{N-1} and $\sqrt{\kappa}$ in (2.10). We now return to the setting where M is a unimodular Lie group.

PROPOSITION 2.7. *There exists finite constants C_1 and C_2 such that for all bounded continuous functions $g\colon M \to [0, \infty)$,*

$$\int_M g(x)\, dx \leqslant C_1 \int_0^\infty g^*(r)\, e^{C_2 r}\, dr, \tag{2.11}$$

where $g^*(r) \equiv \sup_{|y| \geqslant r} |g(y)|$.

Proof. We will first assume that g has compact support. Then g^* has compact support in $[0, \infty)$ and $g^*(r)$ is a decreasing function on $[0, \infty)$. Let $\tilde{g}(r) \equiv \lim_{\varepsilon \downarrow 0} g^*(r+\varepsilon)$, then $-\tilde{g}$ is increasing, right continuous, and $\tilde{g} = g^*$ except on an at most a countable set. For $r > 0$ let $V(r)$ denote the Riemann volume measure (i.e., Haar measure) of $\{x \in M\colon |x| \leqslant r\}$. By Proposition 3.2 of [13], V is a continuous function. Using these observations we have

$$\begin{aligned}\int_M g(x)\, dx &\leqslant \int_M g^*(|x|)\, dx = \int_0^\infty g^*(r)\, dV(r)\\ &= \int_0^\infty \tilde{g}(r)\, dV(r) = -\int_0^\infty V(r)\, d\tilde{g}(r).\end{aligned}$$

By Lemma 2.6, there exists constants $c>0$ and $C<\infty$ such that $V(r) \leqslant Cr^N e^{cr}$. Using this in the above displayed equation gives

$$\int_M g(x)\,dx \leqslant -C \int_0^\infty r^N e^{cr}\,d\tilde{g}(r) = C \int_0^\infty \tilde{g}(r)(Nr^{N-1} + cr^N)\,e^{cr}\,dr$$

$$= C \int_0^\infty g^*(r)(Nr^{N-1} + cr^N)\,e^{cr}\,dr \leqslant C_1 \int_0^\infty g^*(r)\,e^{C_2 r}\,dr,$$

where C_2 is any constant larger than c and C_1 is sufficiently large. This proves (2.11) when g has compact support. For general g, choose $h_n \in C(M, [0, 1])$ such that $h_n \uparrow 1$ as $n \to \infty$. Since $(h_n g)^* \leqslant g^*$ it follows that

$$\int_M h_n(x)\,g(x)\,dx \leqslant C_1 \int_0^\infty g^*(r)\,e^{C_2 r}\,dr.$$

We may now use the monotone convergence theorem to take the limit as $n \to \infty$ in the above equation to get (2.11). Q.E.D.

The following corollary is an easy consequence of standard Gaussian integral estimates, the heat kernel bound in (2.8), and Proposition 2.7. See Lemma 4.3 in [17] for more details.

COROLLARY 2.8. *For all $\beta > 0$ and $T > 0$,*

$$\sup_{0<t\leqslant T} \int_M e^{\beta|y|} v_t(y)\,dy < \infty \tag{2.12}$$

and for all $\delta > 0$ and $\beta > 0$,

$$\lim_{t\downarrow 0} \int_{|y|\geqslant\delta} e^{\beta|y|} v_t(y)\,dy = 0. \tag{2.13}$$

By Eq. (2.12), $P_T|f| < \infty$ for any exponentially bounded functions. We may now state the main theorem of this section.

THEOREM 2.9 (Bakry and Ledoux). *Assume M is a unimodular Lie group given a left invariant Riemannian structure as above. Let $T>0$ and f and $|\nabla f|$ be exponentially bounded, then*

$$P_T(f^2 \log f^2) \leqslant 2\{(e^{CT} - 1)/C\}\, P_T|\nabla f|^2 + P_T(f^2) \log P_T f^2, \tag{2.14}$$

where $0 \log 0 \equiv 0$ as usual.

Our proof of Theorem 2.9 will consist of showing that the argument given in Driver and Hu [18] for the case of a compact manifold can be carried out in this case also. We will give the proof of Theorem 2.9 after a number of preparatory results.

LEMMA 2.10. *Suppose* $k\colon (0, T)\times M\times M\to V$ *(V being a finite dimensional normed vector space) is continuous and assume for each closed interval* $J\subset(0, T)$ *there is a constant* $B_J<\infty$ *such that* $|k(t, x, y)|\leqslant B_J e^{B_J\{|x|+|y|\}}$ *for all* $t\in J$ *and* $x, y\in M$. *For* $t\in(0, T)$ *and* $x\in M$, *define*

$$K(t, x)=\int_M k(t, xy, y)\, \nu_t(y)\, dy.$$

Then $K\colon (0, T)\times M\to\mathbb{R}$ *is continuous and there exists* $C_J<\infty$ *such that*

$$\sup_{t\in J}|K(t, x)|\leqslant C_J e^{B_J|x|}\ \forall x\in M. \tag{2.15}$$

Moreover if $k\colon [0, T]\times M\times M\to V$ *is continuous and there exists a constant* $B<\infty$ *such that* $|k(t, x, y)|\leqslant Be^{B\{|x|+|y|\}}$ *for all* $t\in[0, T]$ *and* $x, y\in M$, *then*

$$K(t, x)\equiv\begin{cases}\int_M k(t, xy, y)\, \nu_t(y)\, dy & \text{if} \quad t>0\\ k(0, x, e) & \text{if} \quad t=0\end{cases} \tag{2.16}$$

is continuous $[0, T]\times M$ *and there is a constant* $C<\infty$ *such that*

$$\sup_{0\leqslant t\leqslant T}|K(t, x)|\leqslant Ce^{B|x|}\ \forall x\in M. \tag{2.17}$$

Proof. For $t\in J$, we have

$$|K(t, x)|\leqslant\int_M B_J e^{B_J\{|xy|+|y|\}}\nu_t(y)\, dy\leqslant B_J e^{B_J|x|}\int_M e^{2B_J|y|}\nu_t(y)\, dy.$$

Hence (2.15) holds with

$$C_J=\sup_{0<t\leqslant T}\int_M B_J e^{2B_J|y|}\nu_t(y)\, dy$$

which is finite by (2.12). Similarly one shows that (2.17) is also valid. The continuity of $K(t, x)$ for $(t, x)\in(0, T)\times M$ follows from the dominated convergence theorem using the heat kernel estimate in Eq. (2.8) and Proposition 2.7. We need only prove the continuity of $K(t, x)$ at $t=0$.

For this let $\delta > 0$, then since ν_t is a probability density we have

$$|K(t,x) - K(0,z)| = \left| \int_M [k(t, xy, y) - k(0, z, e)]\, \nu_t(y)\, dy \right|$$

$$\leqslant \left(\int_{|y| \leqslant \delta} + \int_{|y| > \delta} \right) |k(t, xy, y) - k(0, z, e)|\, \nu_t(y)\, dy$$

$$\leqslant \sup_{|y| \leqslant \delta} |k(t, xy, y) - k(0, z, e)|$$

$$+ B \int_{|y| > \delta} \{ e^{B\{|x| + 2|y|\}} + e^{B|z|} \}\, \nu_t(y)\, dy.$$

By Eq. (2.13) for any $\delta > 0$,

$$\lim_{(t,x) \to (0,z)} \int_{|y| > \delta} \{ e^{B\{|x| + 2|y|\}} + e^{B|z|} \}\, \nu_t(y)\, dy = 0.$$

By the continuity of k it follows that

$$\sup_{|y| \leqslant \delta} |k(t, xy, y) - k(0, z, e)|$$

can be made arbitrarily small by choosing (t, x) sufficiently close to $(0, z)$ and δ sufficiently close to 0. Hence $|K(t,x) - K(0,z)| \to 0$ as $(t,x) \to (0,z)$. Q.E.D.

Notation 2.11. If $k: M \to \mathbb{R}$ is a n times continuously differentiable and $m \in M$, let $\mathscr{D}^n k(m) \in (\mathfrak{m}^*)^{\otimes n}$ be defined by

$$\langle \mathscr{D}^n k(m), \beta_1 \otimes \cdots \otimes \beta_k \rangle \equiv (\tilde{\beta}_1 \cdots \tilde{\beta}_k k)(m),$$

where $\beta_1, \beta_2, ..., \beta_n \in \mathfrak{m}$ are arbitrary and $\langle \cdot, \cdot \rangle$ is used (routinely) to denote the natural pairing between a vector space and its dual. As usual we will set $\mathscr{D}^0 k \equiv k$. If $k: (0, T) \times M \to \mathbb{R}$, let $\mathscr{D}^n k(t, m) \equiv \mathscr{D}^n(k(t, \cdot))(x)$, if $k: M \times M \to \mathbb{R}$, let $\mathscr{D}_1^n k(x, y) \equiv \mathscr{D}^n(k(\cdot, y))(x)$, and if $k: (0, T) \times M \times M \to \mathbb{R}$, let $\mathscr{D}_1^n k(t, x, y) \equiv \mathscr{D}^n(k(t, \cdot, y))(x)$.

Note that $\langle dk, L_{m*}\beta \rangle = \langle \mathscr{D}k(m), \beta \rangle$ for all $\beta \in \mathfrak{m}$ and $m \in M$, where L_{m*} is the differential of left translation by m on M. Since L_{m*} is an isometry, it follows that $|dk| = |\nabla k| = |\mathscr{D}k| = |\vec{\nabla} k|$, where $\nabla k \equiv dk$ and $\vec{\nabla} k$ denotes the gradient of k where k is a function. In the sequel, we will freely use these identities.

LEMMA 2.12. *Let $k: M \times M \to \mathbb{R}$ be a C^1-function and assume there exists $B < \infty$ such that $|k(x, y)| + |\mathcal{D}_1 k(x, y)| \leqslant Be^{B(|x|+|y|)}$ for all $x, y \in M$. Then for each $t > 0$, the function $x \to K(x) \equiv \int_M k(xy, y)\, v_t(y)\, dy$ is C^1 and*

$$\langle \mathcal{D}K(x), \beta \rangle = \int_M \langle \mathcal{D}_1 k(xy, y), Ad_{y^{-1}}\beta \rangle\, v_t(y)\, dy, \tag{2.18}$$

for all $\beta \in \mathfrak{m}$ and $x \in M$. Alternatively, we may write (2.18) as

$$(\tilde{\beta}K)(x) = \int_M \tilde{\beta}_x k(xy, y)\, v_t(y)\, dy,$$

where the subscript x on the $\tilde{\beta}$ above is used to indicated that $\tilde{\beta}$ is acting only on the x variable.

Proof. First let us recall that there is a constant $c < \infty$ such that

$$\|Ad_y\| \leqslant e^{c\,|y|} \quad \forall y \in M. \tag{2.19}$$

To prove this choose a C^1-path $\sigma: [0, 1] \to G$ such that $\sigma(0) = e$ and $\sigma(1) = y$. Then

$$\frac{d}{dt} Ad_{\sigma(t)} = \frac{d}{d\varepsilon}\bigg|_0 Ad_{\sigma(t)} Ad_{\sigma(t)^{-1}\,\sigma(t+\varepsilon)} = Ad_{\sigma(t)}\, ad_{\theta\langle\dot{\sigma}(t)\rangle},$$

where $\theta\langle\dot{\sigma}(t)\rangle \equiv L_{\sigma(t)^{-1}\,*}\dot{\sigma}(t)$. Hence

$$\begin{aligned}\|Ad_{\sigma(t)}\| &= \left\| I + \int_0^t Ad_{\sigma(\tau)}\, ad_{\theta\langle\dot{\sigma}(\tau)\rangle}\, d\tau \right\| \\ &\leqslant 1 + c\int_0^t \|Ad_{\sigma(\tau)}\|\, |\theta\langle\dot{\sigma}(\tau)\rangle|\, d\tau,\end{aligned}$$

where $c = \max\{\|ad_\alpha\|: \alpha \in \mathfrak{g},\ |\alpha| = 1\}$ and $\|ad_\alpha\|$ is the operator norm of ad_α. By Gronwall's inequality,

$$\|Ad_y\| = \|ad_{\sigma(1)}\| \leqslant \exp\left(c\int_0^1 |\theta\langle\dot{\sigma}(t)\rangle|\, dt\right) = e^{cl(\sigma)},$$

where $l(\sigma)$ is the length of the curve σ relative to the left-invariant Riemannian metric on M. Minimizing this last inequality over all C^1-paths σ joining e to $y \in G$ proves (2.19).

For $\alpha, \beta \in \mathfrak{m}$, let $Q(\alpha)\,\beta \equiv (d/ds)|_0\, e^{-\alpha} e^{(\alpha + s\beta)} = L_{e^{-\alpha}\,*} \exp_*(\beta_\alpha)$, where $\beta_\alpha = (d/ds)|_0\,(\alpha + s\beta) \in T_\alpha \mathfrak{m}$. Notice that $(\alpha \to Q(\alpha)): \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$ is a smooth map, in fact $Q(\alpha) = \int_0^1 e^{(1-s)\, ad_\alpha}\, ds$ but we will not need this explicit formula. Now fix $\alpha, \beta \in \mathfrak{m}$ and $x \in M$ and define $h(s, y) \equiv k(xe^{(\alpha+s\beta)}y, y)$ and $h'(s, y) \equiv dh(s, y)/ds$. Then

$$h'(s, y) = \langle \mathscr{D}_1 k(xe^{(\alpha+s\beta)}y, y), Ad_{y^{-1}} Q(\alpha+s\beta)\,\beta \rangle$$

and hence

$$\begin{aligned} |h'(s, y)| &\leqslant |\mathscr{D}_1 k(xe^{(\alpha+s\beta)}y, y)|\; |Ad_{y^{-1}} Q(\alpha+s\beta)\,\beta| \\ &\leqslant Be^{B\{|xe^{(\alpha+s\beta)}y| + |y|\}}\; \|Ad_{y^{-1}}\| \cdot \|Q(\alpha+s\beta)\| \cdot |\beta| \\ &\leqslant B\, \|Q(\alpha+s\beta)\| \cdot |\beta|\; e^{B\{|x| + |e^{(\alpha+s\beta)}| + 2\,|y|\}} e^{c\,|y|} \\ &= B\, \|Q(\alpha+s\beta)\| \cdot |\beta|\; e^{B\{|x| + |e^{(\alpha+s\beta)}|\}} e^{(2B+c)\,|y|}. \end{aligned}$$

From this estimate and Eq. (2.12) it follows that $H(s) \equiv \int_M h(s, y)\, v_t(y)\, dy$ is differentiable, $H'(0) = \int_M h'(0, y)\, v_t(y)\, dy$. Hence we have shown for each $x \in M$ and $\alpha, \beta \in \mathfrak{m}$ that

$$\frac{d}{ds}\bigg|_0 K(xe^{(\alpha+s\beta)}) = \int_M \langle \mathscr{D}_1 k(xe^{\alpha}y, y), Ad_{y^{-1}} Q(\alpha)\,\beta \rangle\, v_t(y)\, dy. \qquad (2.20)$$

By the dominated convergence theorem, one may show that the right member of (2.20) is a continuous function of α for each $\beta \in \mathfrak{m}$. Therefore, the directional derivatives of the function $\alpha \to K(xe^{\alpha})$ exist and are continuous. Since $x \in M$ is arbitrary it follows that $x \to K(x)$ is C^1. Equation (2.18) follows from (2.20) with $\alpha = 0$. Q.E.D.

PROPOSITION 2.13. *Let $k: (0, T) \times M \to \mathbb{R}$ be a C^1 function such that $\mathscr{D}^2 k(t, y)$ exists and is continuous for $(t, y) \in (0, T) \times M$ and for each compact interval $J \subset (0, T)$ there is a constant $B = B_J$ such that*

$$|\dot{k}(t, y)| + \sum_{r=0}^{2} |\mathscr{D}^r k(t, y)| \leqslant Be^{B\{|x| + |y|\}}\ \forall t \in J,$$

where $\dot{k}(t, y) \equiv \partial k(t, y)/\partial t$. Define

$$K(t) \equiv (P_t k(t, \cdot))(e) = \int_M k(t, y)\, v_t(y)\, dy.$$

Then $K: (0, T) \to \mathbb{R}$ is differentiable and

$$\dot{K}(t) = \int_M \{\dot{k}(t, y) + \tfrac{1}{2}\Delta k(t, y)\}\, v_t(y)\, dy, \qquad (2.21)$$

i.e.,

$$\frac{d}{dt}(P_t k(t,\cdot))(e) = \left(P_t\left\{\dot{k}(t,\cdot) + \frac{1}{2}\Delta k(t,\cdot)\right\}\right)(e).$$

Proof. To simplify notation define $G(t)$ to be the right hand side of (2.21). Let $h_n \in C_c^\infty(M, [0,1])$ such that $\lim_{n\to\infty} h_n \equiv 1$, $\mathscr{D}h_n$ and Δh_n converges to zero boundedly, see for example Lemma 3.6 of [17]. Define $k_n(t,y) \equiv h_n(y)\, k(t,y)$ and $K_n(t) \equiv \int_M k_n(t,y)\, v_t(y)\, dy$. Then

$$\begin{aligned}\dot{K}_n(t) &= \int_M \frac{\partial}{\partial t}(k_n(t,y)\, v_t(y))\, dy\\ &= \int_M \left(\dot{k}_n(t,y) + \frac{1}{2}\Delta k_n(t,y)\right) v_t(y)\, dy =: G_n(t),\end{aligned}$$

where the second equality is a consequence of two integration by parts and the fact that $\partial v_t/\partial t = \frac{1}{2}\Delta v_t$. Using $\Delta k_n = \Delta h_n \cdot k + 2(\mathscr{D}h_n, \mathscr{D}k) + h_n\,\Delta k$, the assumptions on k, the properties of $\{h_n\}$, and (2.12), we see that $K_n(t)$ and $G_n(t)$ converges uniformly for t in compact subset of $(0,T)$ to $K(t)$ and $G(t)$ respectively. Therefore $K(t)$ is differentiable and $\dot{K}(t) = G(t)$. Q.E.D.

LEMMA 2.14. *Let $T>0$ and suppose that $\phi: (0,T)\times M \to \mathbb{R}$ is a smooth function such that*

$$\sup\left\{\left|\frac{\partial^l}{\partial t^l}\mathscr{D}^k\phi(t,x)\right| : (t,x)\in(0,T)\times M\right\} < \infty \tag{2.22}$$

for all $l, k \in \mathbb{N}\cup\{0\}$. Then $\Psi(t,x) \equiv (P_t\phi(t,\cdot))(x)$ is also smooth for $(t,x)\in(0,T)\times M$ and the bounds in (2.22) hold with ϕ replaced by Ψ.

Proof. Write $\mathscr{D}^k\phi(t,xy)$ for $(\mathscr{D}^k\phi(t,\cdot))(xy)$. Using (2.19) one may easily show

$$\sup\{|\Delta_y^l\{(Ad_{y^{-1}}^{tr})^{\otimes k}\,\mathscr{D}^k\phi(t,xy)\}| : (t,x)\in(0,T)\times M\} \leqslant C(k,l)\, e^{kc\,|y|}, \tag{2.23}$$

where $Ad_{y^{-1}}^{tr}$ denotes the transpose of $Ad_{y^{-1}}$, $C(k,l)$ is a finite constant, and c is the constant in (2.19).

Writing $\Psi(t,x) = \int_M \phi(t,xy)\, v_t(y)\, dy$, it follows by repeated use of Lemma 2.12 and Eq. (2.23) with $l=0$ that $\mathscr{D}^k\Psi(t,x)$ exists for all integers k and

$$\mathscr{D}^k\Psi(t,x) = \int_M v_t(y)(Ad_{y^{-1}}^{tr})^{\otimes k}\,\mathscr{D}^k\phi(t,xy)\, dy.$$

By Lemma 2.10 we know that $\mathscr{D}^k\Psi(t, x)$ is continuous in (t, x). Similarly using Eq. (2.23) and Proposition 2.13 we may show $(\partial^l/\partial t^l)\,\mathscr{D}^k\Psi(t, x)$ exists for all integers l and

$$\frac{\partial^l}{\partial t^l}\mathscr{D}^k\Psi(t, x) = 2^{-l}\int_M v_t(y)\,\Delta_y^l\{(Ad_{y^{-1}}^{tr})^{\otimes k}\,\mathscr{D}^k\phi(t, xy)\}\,dy. \qquad (2.24)$$

Again by Lemma 2.10, $(\partial^l/\partial t^l)\,\mathscr{D}^k\Psi(t, x)$ is continuous in (t, x). Since k and l are arbitrary we have shown that Ψ is smooth. Finally

$$\sup\left\{\left|\frac{\partial^l}{\partial t^l}\mathscr{D}^k\Psi(t, x)\right| : (t, x) \in (0, T)\times M\right\} \leqslant C(k, l)\sup_{0<t<T}\int_M v_t(y)\,e^{kc\,|y|}\,dy$$

which is finite by (2.12). Q.E.D.

COROLLARY 2.15. *Suppose that $f \in C^2(M)$ such that f, $\mathscr{D}f$, and $\mathscr{D}^2 f$ are bounded and set $u(t, x) \equiv (P_t f)(x)$. Then u solves the heat equation $\partial u/\partial t = \Delta u/2$ for $t > 0$.*

Proof. First recall that

$$u(t, x) = \int_M f(xy)\,v_t(y)\,dy = \int_M f(xy^{-1})\,v_t(y)\,dy$$

and hence by an application of Proposition 2.13 with $k(t, y) \equiv f(xy^{-1})$ shows that

$$\partial u(t, x)/\partial t = \tfrac{1}{2}\int_M (\Delta_y f(xy^{-1}))\,v_t(y)\,dy.$$

Now

$$\Delta_y f(xy^{-1}) = \sum_{A\in\mathfrak{g}_0}\frac{d^2}{dt^2}\bigg|_0 f(xe^{-tA}y^{-1}) = \Delta_x f(xy),$$

and hence

$$\partial u(t, x)/\partial t = \tfrac{1}{2}\int_M \Delta_x f(xy^{-1})\,v_t(y)\,dy.$$

As in the proof of Lemma 2.14, by two applications of Lemma 2.12,

$$(\tilde{A}^2 u)(t, x) = \int_M \tilde{A}_x^2 f(xy^{-1})\,v_t(y)\,dy$$

for all $A \in \mathfrak{g}$. Hence summing this equation over $A \in \mathfrak{g}_0$ shows

$$(\Delta u)(t, x) = \int_M \Delta_x f(xy^{-1})\, v_t(y)\, dy.$$

Thus u satisfies $\partial u/\partial t = \Delta u/2$. Q.E.D.

We may use the above results to easily prove the following special case of a theorem of Dodziuk, see Theorem 3, p. 183 in Chavel [12].

COROLLARY 2.16. *Suppose that* $u \in C^1([0, \infty) \times M \to \mathbb{R})$ *satisfies*

1. $\mathcal{D}^2 u(t, x)$ *exists and is continuous for* $(t, x) \in (0, \infty) \times \mathcal{M}$,
2. u *solves the heat equation* $\partial u/\partial t = \frac{1}{2}\Delta u$, *and*
3. *for all* $0 < T < \infty$, u, $\mathcal{D}u$ *and* $\mathcal{D}^2 u$ *are bounded on* $(0, T) \times M$.

Then $u(t, x) = (P_t u(0, \cdot))(x)$.

Proof. Let $T > 0$ be fixed and consider $U(t, x) \equiv (P_{T-t} u(t, \cdot))(x)$. By Proposition 2.13, we have for $(t, x) \in (0, T) \times M$ that

$$\partial U(t, x)/\partial t = (P_{T-t}[\partial u(t, \cdot)/\partial t - \tfrac{1}{2}\Delta u(t, \cdot)])(x) = 0.$$

Therefore, $U(t, x)$ is a constant for $t \in (0, T)$. But by Lemma 2.10, $\lim_{t \uparrow T} U(t, x) = u(T, x)$ and $\lim_{t \downarrow 0} U(t, x) = (P_T u(0, \cdot))(x)$. Hence $u(T, x) = (P_T u(0, \cdot))(x)$. Q.E.D.

Proof of Theorem 2.9. For the moment let $f \in C_c^\infty(M, [0, \infty))$. Let $F(T, x) \equiv f(x)$ and

$$F(t, x) \equiv (P_{(T-t)} f)(x) = \int_M f(xy)\, v_{(T-t)}(y)\, dy = \int_M f(xy^{-1})\, v_{(T-t)}(y)\, dy$$

for $t \in (-1, T)$ and $x \in M$. By Lemma 2.14, F is a smooth function on $(-1, T) \times M$ with all its derivatives bounded. It follows, either by Corollary 2.15 or by the fact that $f \in \mathcal{D}(\Delta)$ and $F(t, x) = (e^{(T-t)\,\Delta/2} f)(x)$, that $\dot{F}(t, \cdot) = -\Delta F(t, \cdot)/2$ for all $t \in (-1, T)$.

Let $\varepsilon > 0$ and define $\phi(x) \equiv (x + \varepsilon) \log(x + \varepsilon)$ and $\Psi(t, x) \equiv (P_t(\phi(F(t, \cdot))))(x)$. By Lemma 2.14, $\Psi(t, x)$ is smooth for $(t, x) \in (0, T) \times M$ and all the derivatives of Ψ are bounded. Suppressing t and x from the notation when possible, we have by Proposition 2.13 that

$$\dot{\Psi} = P\left[\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)(\phi \circ F)\right]$$

and

$$\ddot{\Psi} = P\left[\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)^2 (\phi \circ F)\right].$$

By elementary computations (see for example Lemma 3.2, Lemma 4.3, and Corollary 4.5 in [18]) one shows

$$\dot{\Psi} = P\left\{\frac{|\nabla F|^2}{2(F+\varepsilon)}\right\} \qquad \text{and} \qquad \ddot{\Psi} \geqslant -C\dot{\Psi} \text{ on } (0, T) \times M, \tag{2.25}$$

where $C \in \mathbb{R}$ such that $\mathrm{Ric}\langle \cdot, \cdot \rangle \geqslant -C(\cdot, \cdot)$. Therefore $(d/dt) \log \dot{\Psi} \geqslant -C$, and hence for $0 < t < \tau < T$,

$$\dot{\Psi}(t) \leqslant \dot{\Psi}(\tau)\, e^{C(\tau - t)}.$$

Integrating this last equation over $[\delta, \tau]$ $(0 < \delta < \tau < T)$ gives

$$\Psi(\tau) - \Psi(\delta) \leqslant \dot{\Psi}(\tau)(e^{C(\tau - \delta)} - 1)/C. \tag{2.26}$$

Using Lemma 2.10 (repeatedly) shows: F is continuous on $[0, T] \times M$, $\lim_{t \uparrow T} \Psi(t, x) = (P_T(\phi(f))(x)$, $\lim_{t \downarrow 0} \Psi(t, x) = F(0, x) = \phi((P_T f(x))$,

$$\lim_{t \uparrow T} \mathscr{D}F(t, x) = \lim_{t \uparrow T} \int_M Ad^{tr}_{y^{-1}} \mathscr{D}f(xy)\, \nu_{(T-t)}(y)\, dy = \mathscr{D}f(x),$$

and

$$\lim_{t \uparrow T} \dot{\Psi}(t, x) = \left(P_T \left\{\frac{|\nabla f|^2}{2(f+\varepsilon)}\right\}\right)(x).$$

Using the above limits in (2.26) gives:

$$P_T(\phi(f)) - \phi(P_T f) \leqslant \frac{(e^{CT} - 1)}{2C} P_T((f+\varepsilon)^{-1} |\nabla f|^2). \tag{2.27}$$

Now for arbitrary $f \in C_c^\infty(M)$, apply (2.27) to f^2 to find

$$\begin{aligned} P_T(\phi_\varepsilon(f^2)) - \phi_\varepsilon(P_T f^2) &\leqslant \frac{(e^{CT} - 1)}{2C} P_T((f^2+\varepsilon)^{-1}\, 4f^2\, |\nabla f|^2) \\ &\leqslant \frac{2(e^{CT} - 1)}{C} P_T(|\nabla f|^2), \end{aligned}$$

where $\phi_\varepsilon(s) = \phi(s) = (s+\varepsilon)\log(s+\varepsilon)$. Using the fact that $s \log s$ is bounded on $(0, K)$ for any $K > 0$, we may use the dominated convergence theorem to pass to the limit as $\varepsilon \downarrow 0$ in the above displayed equation. This proves logarithmic Sobolev inequality in (2.14) for $f \in C_c^\infty(M)$.

If $f \in C_c^1(M)$ we may convolve f with a sequence of approximate δ-functions to produce a sequence of functions $\{f_n\} \subset C_c^\infty(M)$ such that f_n and $\mathscr{D}f_n$ converges uniformly to f and $\mathscr{D}f$ respectively. Applying (2.14) to f_n and then letting n tend to infinity shows that (2.14) holds for all $f \in C_c^1(M)$. Finally, for $f \in C^1(M)$ such that f and $\mathscr{D}f$ are exponentially bounded, let $f_n \equiv h_n f$, where $\{h_n\} \subset C_c^\infty(M)$ is a sequence as in the proof of Lemma 2.12. Then again (2.14) holds with f replace by f_n for each n and one may easily pass to the limit to conclude that (2.14) holds for f also. Q.E.D.

3. GEOMETRY OF THE LOOP ALGEBRA

In this section we will be developing the geometry of the left invariant tensor fields on $\mathscr{L}(G)$. Since the left invariant tensor fields may be identified with their values at the identity loop, the geometry of these tensor fields may be developed without explicit mention of the loop group $\mathscr{L}(G)$. This explains the title of this section and the reason that $\mathscr{L}(G)$ does not appear until the next section where more general tensor fields are considered.

The Ricci tensor computed in Theorem 3.12 below has already been worked out by Freed [20]. Also see [21, 30] for closely related computations. Nevertheless, we supply full details since we will need the notation later and we also need to develop the geometry of $\mathscr{L}(G)$ a little further. The main subtlety in computing the Ricci tensor is that the curvature tensor is not trace class. This point is clearly explained in Freed [20], see also Remark 3.13 below. Since the Ricci tensor is a trace of the curvature tensor, this causes some problems. However, these problems may be overcome by using only "good" (see Definition 3.10) orthonormal bases for H_0 when computing the trace of the curvature tensor.

Throughout this section let $\mathfrak{g}$ be a real Lie algebra of compact type. A Lie algebra is said to be of compact type if there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ with the property that the adjoint operators $\{ad_A\}_{A \in \mathfrak{g}}$ are all skew-symmetric. We will fix such an inner product on $\mathfrak{g}$ in the sequel. For $A \in \mathfrak{g}$ let $|A| \equiv \sqrt{\langle A, A \rangle}$. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ denote an orthonormal basis for $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and

$$K\langle A, B \rangle \equiv -\operatorname{tr}(ad_A\, ad_B)\ \forall A, B \in \mathfrak{g}_0, \tag{3.1}$$

i.e., $-K$ is the Killing form on $\mathfrak{g}$.

Notation 3.1. Given a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$ let $H(V)$ denote the "Cameron–Martin" Hilbert space of absolutely continuous functions $h: [0, 1] \to V$ such that $h(0) = 0$ and

$$(h, h) \equiv \int_0^1 |h'(s)|^2 \, ds < \infty. \tag{3.2}$$

Let

$$H_0(V) = \{h \in H(V) \mid h(1) = 0\}.$$

If $V = \mathfrak{g}$ we will simply write H for $H(\mathfrak{g})$ and H_0 for $H_0(\mathfrak{g})$.

DEFINITION 3.2. Let $T_0^{0,0} = \mathbb{R}$, $T_0^{0,n} = H_0^{\otimes n}$ and $T_0^{m,0} = (H_0^*)^{\otimes m}$, with their natural crossed norms. For a pair of positive integers m, n let $T_0^{m,n}$ denote the Hilbert operators from $H_0^{\otimes m}$ to $H_0^{\otimes n}$, equipped with the Hilbert Schmidt norm, i.e. for $\alpha \in T_0^{m,n}$,

$$\|\alpha\|^2_{T_0^{m,n}} = \sum_{i=1}^{\infty} \|\alpha(\xi_i)\|^2_{H_0^{\otimes n}}, \tag{3.3}$$

where $\{\xi_i\}_{i=1}^{\infty}$ is any orthonormal basis for $H_0^{\otimes m}$.

For notational convenience, it is helpful to encode all of the spaces $\{T_0^{m,n}\}_{m,n=0}^{\infty}$ into one larger inner product space.

Notation 3.3. Let

$$T_0 \equiv \bigoplus_{m,n=0}^{\infty} T_0^{m,n} \text{ (algebraic direct sum)},$$

i.e., $\alpha \in T_0$ iff $\alpha = \sum_{m,n=0}^{\infty} \alpha_{m,n}$ with $\alpha_{m,n} \in T_0^{m,n}$ and $\alpha_{m,n} = 0$ for all but a finite number of pairs $(m, n) \in \mathbb{N}^2$. For $\alpha = \sum_{m,n=0}^{\infty} \alpha_{m,n}$ and $\beta = \sum_{m,n=0}^{\infty} \beta_{m,n}$ in T_0, let

$$(\alpha, \beta) \equiv \sum_{m,n=0}^{\infty} (\alpha_{m,n}, \beta_{m,n})_{T_0^{m,n}}.$$

With this definition T_0 is an inner product space such that $T_0^{m,n}$ is a subspace of T_0 for all $m, n \in \mathbb{N}$. Moreover, $(\cdot, \cdot)$ agrees with $(\cdot, \cdot)_{T_0^{m,n}}$ on $T_0^{m,n}$ for all $m, n \in \mathbb{N}$. We will write $\|\cdot\|$ for $\sqrt{(\cdot, \cdot)}$.

Remark 3.4. The Hilbert space $T_0^{m,n}$ is naturally isomorphic to $(H_0^*)^{\otimes m} \otimes H_0^{\otimes n}$. The isomorphism may be described as the continuous linear map from $(H_0^*)^{\otimes m} \otimes H_0^{\otimes n}$ to $T_0^{m,n}$ determined uniquely by requiring for all $\{h_1, ..., h_m, k_1, ..., k_n\} \subset H_0$ that

$$(h_1, \cdot) \otimes \cdots \otimes (h_m, \cdot) \otimes k_1 \otimes \cdots \otimes k_n \in (H_0^*)^{\otimes m} \otimes H_0^{\otimes n} \to \alpha \in T_0^{m,n} \quad (3.4)$$

where $\alpha \in T_0^{m,n}$ is determined by

$$\alpha(v_1 \otimes \cdots \otimes v_m) \equiv (h_1, v_1) \cdots (h_m, v_m) k_1 \otimes \cdots \otimes k_n \quad (3.5)$$

for all $v_1, ..., v_m \in H_0$. In the future, we will identify α in (3.5) with the LHS of (3.4).

The following Lemma summarizes some basic well known facts about the Hilbert Schmidt norm.

LEMMA 3.5. *Suppose that* $\alpha \in T_0^{m,n}$ *and* $\beta \in T_0^{n,k}$. *Recall that* $\alpha: H_0^{\otimes m} \to H_0^{\otimes n}$ *and* $\beta: H_0^{\otimes n} \to H_0^k$ *are Hilbert Schmidt operators*

1. *Let* α^* *denote the adjoint of* α, *then* $\|\alpha\| = \|\alpha^*\|$.
2. $\beta\alpha \in T_0^{m,k}$ *and* $\|\beta\alpha\| \leqslant \|\beta\| \, \|\alpha\|$.
3. *If* $\xi \in H_0^{\otimes m}$, *then* $\|\alpha\xi\| \leqslant \|\alpha\| \, \|\xi\|$, *i.e.,* $\|\alpha\|_{op} \leqslant \|\alpha\|$ *where* $\|\alpha\|_{op}$ *denote the operator norm of* α.

3.1. *Covariant Derivative on Left Invariant Vector Fields*

In preparation for introducing a "covariant derivative" on H_0, let $P: H \to H$ denote *orthogonal projection* of H onto H_0. It is easily checked that P is given by $Ph = h - \Lambda h(1)$, where $\Lambda(s) \equiv s$ for $s \in [0, 1]$.

DEFINITION 3.6. Let $D: H_0 \to T_0^{1,1} \cong H_0^* \otimes H_0$ denote the linear operator determined by

$$(Dk)h = P \int_0^{\cdot} [h, dk] = \int_0^{\cdot} [h, dk] - \Lambda \int_0^1 [h, dk], \quad (3.6)$$

for all $h, k \in H_0$, where $\Lambda(s) \equiv s$ and $dk(s) \equiv k'(s)\, ds$. (D is a bounded operator, see Lemma 3.9 below.) We will usually write $(Dk)h$ as $D_h k$.

In Theorems 3.15 and 3.18 below we will extend D to $T_0^{m,n}$. We now adopt the following notation throughout the remainder of this paper.

Notation 3.7. Let $\mathfrak{h} \subset H_0(\mathbb{R})$ and $\mathfrak{g}_0 \subset \mathfrak{g}$ be fixed orthonormal bases. Let S_0 denotre an (arbitrary) orthonormal basis of $H_0 = H_0(\mathfrak{g})$ and $\mathfrak{h}\mathfrak{g}_0$ denote the specific orthonormal basis of H_0 defined by

$$\mathfrak{h}\mathfrak{g}_0 \equiv \{hA \in H_0(\mathfrak{g}) : h \in \mathfrak{h} \text{ and } A \in \mathfrak{g}_0\}.$$

LEMMA 3.8. *Let $G_0(s, t) \equiv s \wedge t - st$ for $s, t \in [0, 1]$. Then*

$$\sum_{a \in \mathfrak{h}} |a(s)\, a(t)| \leqslant 1/4, \qquad \forall s, t \in [0, 1], \tag{3.7}$$

$$\sum_{h \in S_0} |h(s)|\, |h(t)| \leqslant \tfrac{1}{4} \dim \mathfrak{g}, \tag{3.8}$$

$$\sum_{a \in \mathfrak{h}} a(s)\, a(t) = G_0(s, t), \tag{3.9}$$

and

$$\sum_{h \in S_0} h(s) \otimes h(t) = G_0(s, t) \sum_{A \in \mathfrak{g}_0} A \otimes A \in \mathfrak{g} \otimes \mathfrak{g}. \tag{3.10}$$

Proof. It is well known and easy to check that G_0 is the reproducing Kernel for $H_0(\mathbb{R})$, i.e., for all $a \in H_0(\mathbb{R})$ and $s \in [0, 1]$,

$$(G_0(s, \cdot), a) \equiv \int_0^1 (\partial G_0(s, t)/\partial t)\, a'(t)\, dt = a(s). \tag{3.11}$$

Therefore,

$$\sum_{a \in \mathfrak{h}} a^2(s) = \sum_{a \in \mathfrak{h}} (a, G_0(s, \cdot))^2 = (G_0(s, \cdot), G_0(s, \cdot)) = G_0(s, s) \leqslant 1/4$$

and hence

$$\sum_{a \in \mathfrak{h}} |a(s)\, a(t)| \leqslant \left(\sum_{a \in \mathfrak{h}} a^2(s) \right)^{1/2} \left(\sum_{a \in \mathfrak{h}} a^2(t) \right)^{1/2} \leqslant 1/4.$$

This proves (3.7). By (3.11) and the assumption that $\mathfrak{h}$ is an orthonormal basis

$$G_0(s, t) = (G_0(s, \cdot), G_0(t, \cdot)) = \sum_{a \in \mathfrak{h}} (G_0(s, \cdot), a)(a, G_0(t, \cdot)) = \sum_{a \in \mathfrak{h}} a(s)\, a(t),$$

which proves (3.9). We prove (3.8) by using the Cauchy Schwarz inequality and the identity:

$$\begin{aligned} \sum_{h \in S_0} |h(s)|^2 &= \sum_{h \in S_0} \sum_{A, B \in \mathfrak{g}_0} (h, G_0(s, \cdot) A)(h, G_0(s, \cdot) B) \\ &= \sum_{A, B \in \mathfrak{g}_0} (G_0(s, \cdot) A, G_0(s, \cdot) B) \\ &= \sum_{A \in \mathfrak{g}_0} G_0(s, s) \leqslant \dim \mathfrak{g}/4. \end{aligned}$$

Finally,

$$\begin{aligned}\sum_{h\in S_0} h(s)\otimes h(t) &= \sum_{h\in S_0}\sum_{A,\,B\in\mathfrak{g}_0}(h, G_0(s,\cdot)A)(h, G_0(t,\cdot)B)A\otimes B\\ &= \sum_{A,\,B\in\mathfrak{g}_0}(G_0(s,\cdot)A, G_0(t,\cdot)B)A\otimes B\\ &= \sum_{A\in\mathfrak{g}_0} G_0(s,t)A\otimes A.\end{aligned}$$

Q.E.D.

LEMMA 3.9. *Let $k\in H_0$, then*

$$\begin{aligned}\|Dk\|^2 &= \int_0^1 G_0(s,s)\,K\langle k'(s), k'(s)\rangle\,ds\\ &\quad -\int_0^1\int_0^1 G_0(s,t)\,K\langle k'(s), k'(t)\rangle\,ds\,dt \qquad (3.12)\\ &\leqslant \int_0^1 G_0(s,s)\,K\langle k'(s), k'(s)\rangle\,ds \qquad (3.13)\end{aligned}$$

where K is defined in (3.1). *In particular, D is a bounded operator.*

Proof. Recall that $\|Dk\|^2$ is the Hilbert Schmidt norm of Dk, so

$$\begin{aligned}\|Dk\|^2 &= \sum_{h\in S_0}\|D_h k\|^2_{H_0}\\ &= \sum_{h\in S_0}\left\{\int_0^1\left|[h(s), k'(s)]-\int_0^1[h(t), k'(t)]\,dt\right|^2 ds\right\}\\ &= \sum_{h\in S_0}\left\{\int_0^1|[h(s), k'(s)]|^2\,ds\right.\\ &\quad \left.-\left\langle\int_0^1[h(t), k'(t)]\,dt, \int_0^1[h(s), k'(s)]\,ds\right\rangle\right\}.\end{aligned}$$

It now follows by Lemma 3.8 and the dominated convergence theorem that

$$\begin{aligned}\|Dk\|^2 &= \sum_{A\in\mathfrak{g}_0}\int_0^1 G_0(s,s)\langle[A, k'(s)], [A, k'(s)]\rangle\,ds\\ &\quad -\sum_{A\in\mathfrak{g}_0}\int_0^1\int_0^1 G_0(s,t)\langle[A, k'(s)], [A, k'(t)]\rangle\,ds\,dt\end{aligned}$$

$$= \sum_{A \in \mathfrak{g}_0} \int_0^1 G_0(s,s) \langle -ad_{k'(s)} ad_{k'(s)} A, A \rangle \, ds$$

$$- \sum_{A \in \mathfrak{g}_0} \int_0^1 \int_0^1 G_0(s,t) \langle -ad_{k'(s)} ad_{k'(t)} A, A \rangle \, ds \, dt$$

$$= \int_0^1 G_0(s,s) \operatorname{tr}(-ad_{k'(s)} ad_{k'(s)}) \, ds$$

$$- \int_0^1 \int_0^1 G_0(s,t) \operatorname{tr}(-ad_{k'(s)} ad_{k'(t)}) \, ds \, dt$$

$$= \int_0^1 G_0(s,s) \, K\langle k'(s), k'(s) \rangle \, ds - \int_0^1 \int_0^1 G_0(s,t) \, K\langle k'(s), k'(t) \rangle \, ds \, dt.$$

This proves (3.12). Equation (3.13) follows from (3.12) after noting that

$$\int_0^1 \int_0^1 G_0(s,t) \, K\langle k'(s), k'(t) \rangle \, ds \, dt = \sum_{h \in S_0} \left| \int_0^1 [h(t), k'(t)] \, dt \right|^2 \geqslant 0.$$

Setting $M^2 \equiv \sup\{K\langle \xi, \xi \rangle \mid \xi \in \mathfrak{g} \text{ and } |\xi| = 1\}$, it follows easily from (3.13) that

$$\|Dk\|^2 \leqslant M^2 (\sup_s G_0(s,s)) \, \|k\|^2 = (M^2/4) \, \|k\|^2.$$

Therefore D is a bounded operator with $\|D\|_{op} \leqslant M/2$. Q.E.D

In order to have the infinite sums exist in the definition of the Ricci tensor (and also in the definition of the Laplacians below), it will be necessary to choose a "good" basis of H_0.

Definition 3.10. An orthonormal basis S_0 of $H_0 = H_0(\mathfrak{g})$ is a *good* basis if for each $h \in S_0$, $[h(s), h'(s)] = 0$ for almost every $s \in [0, 1]$. Example: $S_0 = \mathfrak{h}\mathfrak{g}_0$.

Notation 3.11. For $h, k, l \in H_0$, define:

1. [Lie Bracket] $[h, k] \in H_0$ by $[h, k](s) \equiv [h(s), k(s)]$ for all $s \in [0, 1]$.
2. [Torsion Tensor] $T\langle h, k \rangle \equiv D_h k - D_k h - [h, k] \in H_0$.
3. [Curvature Tensor] $R\langle h, k \rangle l \equiv [D_h, D_k] l - D_{[h,k]} l \in H_0$.
4. [Ricci Tensor] $\operatorname{Ric}\langle h, l \rangle \equiv \sum_{k \in S_0} (R\langle h, k \rangle k, l)$, where S_0 is any good orthonormal basis of H_0. (We will see in the next theorem that this sum exists.)

THEOREM 3.12. *Let $h, k, l, p \in H_0$, then*:

1. [Metric Compatible] $(D_h k, l) + (k, D_h l) = 0$, *i.e.* D_h *is skew adjoint* $(D_h^* = -D_h)$.
2. [Zero Torsion] $T \equiv 0$.
3. [Curvature] *Let* $\Lambda(s) = s$ *for* $s \in [0, 1]$, *then*

$$R\langle h, k\rangle l = P\left(\left[\int_0^{\cdot} k\, d\Lambda, \int_0^1 [h, dl]\right] - \left[\int_0^{\cdot} h\, d\Lambda, \int_0^1 [k, dl]\right]\right) \tag{3.14}$$

and

$$(R\langle h, k\rangle l, p) = \left\langle \int_0^1 [h, dp], \int_0^1 [k, dl] \right\rangle - \left\langle \int_0^1 [k, dp], \int_0^1 [h, dl] \right\rangle. \tag{3.15}$$

4. [Ricci] (*See Freed* [20].) *The sum in the definition of the Ricci tensor is absolutely convergent and*

$$\mathrm{Ric}\langle h, p\rangle = -\int_0^1 \int_0^1 G_0(s, t)\, K\langle h'(t), p'(s)\rangle\, ds\, dt \tag{3.16}$$

$$= K\langle \bar{h}, \bar{p}\rangle - \int_0^1 K\langle h(s), p(s)\rangle\, ds \tag{3.17}$$

where

$$\bar{h} \equiv \int_0^1 h(s)\, ds \qquad \text{and} \qquad \bar{p} \equiv \int_0^1 p(s)\, ds.$$

Proof. Let $h, k, l, p \in H_0$. *Because* ad_ξ for $\xi \in \mathfrak{g}$ is skew-adjoint,

$$\langle [h, k'], l'\rangle + \langle k', [h, l']\rangle \equiv 0.$$

Integrating this equation over the interval $[0, 1]$ shows that

$$0 = \left(\int_0^{\cdot} [h, dk], l\right) + \left(k, \int_0^{\cdot} [h, dl]\right)$$

$$= \left(P\int_0^{\cdot} [h, dk], l\right) + \left(k, P\int_0^{\cdot} [h, dl]\right),$$

which is the first assertion.

For the second assertion notice that

$$T(h,k):=P\int_0^{\cdot}\{[h,dk]-[k,dh]\}-[h,k]$$

$$=P\int_0^{\cdot}d[h,k]-[h,k]=P[h,k]-[h,k]=0,$$

since $[h,k]\in H_0$.

Equation (3.15) is a simple consequence of (3.14). To prove (3.14) consider

$$D_hD_kl=D_h\left(\int_0^{\cdot}[k,dl]-\Lambda\int_0^1[k,dl]\right)$$

$$=\left(P\left\{\int_0^{\cdot}[h,[k,dl]]-\left[\int_0^{\cdot}h\,d\Lambda,\int_0^1[k,dl\right]\right\}\right).$$

By this equation and the corresponding equation with h and k interchanged,

$$[D_h,D_k]l=P\int_0^{\cdot}\{[h,[k,dl]]-[k,[h,dl]]\}$$

$$-P\left\{\left[\int_0^{\cdot}h\,d\Lambda,\int_0^1[k,dl]\right]-\left[\int_0^{\cdot}k\,d\Lambda,\int_0^1[h,dl]\right]\right\}$$

$$=P\int_0^{\cdot}[[h,k],dl]$$

$$-P\left\{\left[\int_0^{\cdot}h\,d\Lambda,\int_0^1[k,dl]\right]-\left[\int_0^{\cdot}k\,d\Lambda,\int_0^1[h,dl]\right]\right\}$$

$$=D_{[h,k]}l+P\left(\left[\int_0^{\cdot}k\,d\Lambda,\int_0^1[h,dl]\right]-\left[\int_0^{\cdot}h\,d\Lambda,\int_0^1[k,dl]\right]\right),$$

where the Jacobi identity has been used in the second equality. This last equation and the definition of R implies Eq. (3.14).

Now let S_0 be a good basis for H_0. Then by Eq. (3.15) and Definition 3.10,

$$(R\langle h,k\rangle k,p)=-\left\langle\int_0^1[k,dp],\int_0^1[k,dh]\right\rangle, \tag{3.18}$$

for all $k \in S_0$ and $h, p \in H_0$. Hence

$$\text{Ric}\langle h, p\rangle = -\sum_{k \in S_0} \left\langle \int_0^1 [k, dp], \int_0^1 [k, dh] \right\rangle. \tag{3.19}$$

Using Lemma 3.8 and the dominated convergence theorem, one easily shows for an arbitrary orthonormal basis S_0 of H_0 that

$$\begin{aligned} &\sum_{k \in S_0} \left\langle \int_0^1 [k, dp], \int_0^1 [k, dh] \right\rangle \\ &\quad = \int_0^1 \int_0^1 G_0(s, t) \sum_{A \in \mathfrak{g}_0} \langle ad_{p'(s)} A, ad_{h'(t)} A \rangle \, ds \, dt, \\ &\quad = \int_0^1 \int_0^1 G_0(s, t) \, K\langle p'(s), h'(t)\rangle \, ds \, dt. \end{aligned} \tag{3.20}$$

Equations (3.19) and (3.20) prove (3.16). Two integration by parts and the fundamental theorem of calculus yields

$$\begin{aligned} \text{Ric}\langle h, p\rangle &= \int_0^1 \int_0^1 \frac{\partial G_0(s, t)}{\partial s} K\langle p(s), h'(t)\rangle \, ds \, dt \\ &= \int_0^1 \int_0^1 \{1_{s \leqslant t} - t\} \; K\langle p(s), h'(t)\rangle \, ds \, dt \\ &= -\int_0^1 K\langle p(s), h(s)\rangle \, ds - \int_0^1 \int_0^1 tK\langle p(s), h'(t)\rangle \, ds \, dt \\ &= -\int_0^1 K\langle p(s), h(s)\rangle \, ds + \int_0^1 \int_0^1 K\langle p(s), h(t)\rangle \, ds \, dt \\ &= K\langle \bar{h}, \bar{p}\rangle - \int_0^1 K\langle h(s), p(s)\rangle \, ds, \end{aligned}$$

which proves (3.17). Q.E.D.

Remark 3.13. If $\mathfrak{g}$ is non-abelian, it is possible to find an orthonormal basis $S_0 \subset H_0$ such that $\sum_{k \in S_0} \int_0^1 [k, dk]$ is not convergent. Hence some restriction on the choice of S_0 is necessary when computing the Ricci tensor. We will see below that such a restriction is also necessary when computing the Laplacian as the trace of the Hessian, see Definition 4.17 and Proposition 4.19 below.

To verify the above remark, it suffices to construct an orthonormal sequence $\{h_n\}_{n=1}^{\infty} \subset H_0$ such that $\sum_{n=1}^{\infty} \int_0^1 [h_n, dh_n]$ is divergent. For this choose and orthonormal set $\{A, B\} \subset \mathfrak{g}$ such that $[A, B] = C \neq 0$ and define

$$h_n(s) \equiv \left(\frac{\sin(2n\pi s)}{2n\pi}\right) A + \left(\frac{\sin((2n+1)\pi s)}{(2n+1)\pi}\right) B.$$

Then one may check that $\{h_n\}_{n=1}^{\infty}$ is an orthonormal sequence in H_0 and, with the aid of one integration by parts, that

$$\int_0^1 [h_n, dh_n] = C \int_0^1 \left(\frac{\sin(2n\pi s)}{n\pi}\right) \cos((2n+1)\pi s)\, ds.$$

Hence

$$\sum_{n=1}^{\infty} \int_0^1 [h_n, dh_n] = \sum_{n=1}^{\infty} \frac{2}{\pi^2} \left(\frac{1}{n(4n+1)} - \frac{1}{n}\right) C = -\infty \cdot C.$$

LEMMA 3.14. *By abuse of notation let R denote the Linear operator from $H_0^{\otimes 3} \to H_0$ determined by*

$$R(h \otimes k \otimes l) = R\langle h, k\rangle l. \tag{3.21}$$

Then $R \in T_0^{3,1}$, i.e. R is Hilbert Schmidt.

Proof. Since $P: H \to H_0$ is orthogonal projection, it follows by (3.14) that

$$\begin{aligned}
\|R(h \otimes k \otimes l)\|^2 &\leqslant \left\| \left[\int_0^{\cdot} k\, dA, \int_0^1 [h, dl]\right] - \left[\int_0^{\cdot} h\, dA, \int_0^1 [k, dl]\right] \right\|^2 \\
&\leqslant 2 \left\| \left[\int_0^{\cdot} k\, dA, \int_0^1 [h, dl]\right] \right\|^2 + 2 \left\| \left[\int_0^{\cdot} h\, dA, \int_0^1 [k, dl]\right] \right\|^2.
\end{aligned}$$

Let M be a constant such that $|[A, B]| \leqslant M\, |A| \cdot |B|$ for all $A, B \in \mathfrak{g}$. Then

$$\begin{aligned}
\|R\|^2 &= \sum_{h, k, l \in S_0} \|R(h \otimes k \otimes l)\|^2 \\
&\leqslant 4 \sum_{h, k, l \in S_0} \left\| \left[\int_0^{\cdot} k\, dA, \int_0^1 [h, dl]\right] \right\|^2 \\
&\leqslant 4M \sum_{h, k, l \in S_0} \left| \int_0^1 [h, dl] \right|^2 \cdot \int_0^1 |k(s)|^2\, ds, \\
&\leqslant M \cdot \dim \mathfrak{g} \cdot \sum_{h, l \in S_0} \left| \int_0^1 [h, dl] \right|^2,
\end{aligned} \tag{3.22}$$

wherein the last inequality we have used Lemma 3.8. By (3.19), (3.20), and (3.17),

$$\sum_{h,\, l \in S_0} \left| \int_0^1 [h, dl] \right|^2 = - \sum_{l \in S_0} \operatorname{Ric}\langle l, l\rangle$$
$$= \sum_{l \in S_0} \left\{ \int_0^1 K\langle l(s), l(s)\rangle \, ds - K\langle \bar{l}, \bar{l}\rangle \right\}$$
$$\leqslant \sum_{l \in S_0} \int_0^1 K\langle l(s), l(s)\rangle \, ds$$
$$= \sum_{A \in \mathfrak{g}_0} K\langle A, A\rangle \int_0^1 G_0(s, s)\, ds \leqslant \frac{1}{4} \sum_{A \in \mathfrak{g}_0} K\langle A, A\rangle < \infty,$$

where the sum on $l \in S_0$ was done using Lemma 3.8. The lemma now follows from this equation and Eq. (3.22). Q.E.D.

3.2. *Covariant Derivative on Left Invariant Tensor Fields*

Our next task is to extend D to an operator acting on $T_0^{m,n}$ for arbitrary m and n. This is the content of the next two theorems.

THEOREM 3.15. *There exists unique bounded operators* $D^{(n)}\colon T_0^{0,n} = H_0^{\otimes n} \to T_0^{1,n}$ *such that the following three conditions hold.*

1. $D^{(0)}\colon T_0^{0,0} = \mathbb{R} \to H_0^*$ *is the zero operator,*
2. $D^{(1)} = D$,
3. [Product Rule] *If* $\xi \in T_0^{0,n}$, $\eta \in T_0^{0,k}$, *and* $h \in H_0$ *then*

$$D_h^{(n+k)}(\xi \otimes \eta) = (D_h^{(n)}\xi) \otimes \eta + \xi \otimes (D_h^{(k)}\eta), \tag{3.23}$$

where $D_h^{(k)}\eta \equiv (D^{(k)}\eta)h$.

Moreover,

$$\|D^{(n)}\|_{op} \leqslant n\, \|D\|_{op} \tag{3.24}$$

and, for each $h \in H_0$, $D_h^{(n)}$ *is a skew adjoint operator on* $T^{0,n} = H_0^{\otimes n}$.

Proof. (*Uniqueness*) Assume $D^{(n)}$ exists for all $n \in \mathbb{N}$. Repeated use of the product rule shows that

$$D_h^{(n)}(h_1 \otimes \cdots \otimes h_n) = (D_h h_1) \otimes h_2 \cdots \otimes h_n + h_1 \otimes (D_h h_2) \cdots \otimes h_n + \cdots + h_1 \otimes h_2 \cdots \otimes D_n h_n, \tag{3.25}$$

for all $h, h_1, ..., h_n \in H_0$. This shows that $D_h^{(n)}$ is unique.

(*Existence*) Let $D^{(n)}$ denote the operator defined on the algebraic tensors in $T_0^{0,n} = H_0^{\otimes n}$ such that (3.25) holds for all $h, h_1, ..., h_n \in H_0$. We claim that $D^{(n)}$ is bounded on the algebraic tensors. To show this, for each $i = 1, 2, ..., n$, let $\sigma_i \colon H^{\otimes n} \to H^{\otimes n}$ be the unitary map determined by

$$\sigma_i(h_1 \otimes \cdots \otimes h_i \otimes \cdots \otimes h_n) = h_i \otimes h_1 \otimes \cdots \otimes \widehat{h_i} \otimes \cdots \otimes h_n$$

for all subsets $\{h_i\}_{i=1}^n \subset H_0$, where the hat over h_i indicates that this h_i should be omitted. Letting $\mathscr{I}$ denote the identity operator on $H^{\otimes(n-1)}$ and I the identity opeator on H_0^*, we have

$$D^{(n)} = \sum_{i=1}^{n} I \otimes \sigma_i^{-1} (D \otimes \mathscr{I}) \sigma_i. \tag{3.26}$$

Since $I \otimes \sigma_i^{-1}$ and σ_i are unitary, $\|D^{(n)}\|_{op} \leqslant n \, \|D \otimes \mathscr{I}\|_{op}$. But by Lemma 3.16 below, $\|D \otimes \mathscr{I}\|_{op} = \|D\|_{op}$ which is finite by Lemma 3.9.

It is now a simple matter to show that the bounded operators $D_h^{(n)}$ satisfying (3.25) for all $n = 1, 2, ...$ also satisfies Eq. (3.23). This is first done on decomposable tensors which implies the result for algebraic tensors. The result for general tensors then follows by continuity.

Finally, to show that $D_h^{(n)}$ is skew adjoint consider the inner product of both sides of Eq. (3.25) with $k_1 \otimes \cdots \otimes k_n$ ($\{k_i\}_{i=1}^n \subset H_0$):

$$\begin{aligned}
&(D_h^{(n)}(h_1 \otimes \cdots \otimes h_n), k_1 \otimes \cdots \otimes k_n) \\
&\quad = \sum_{i=1}^{n} (D_h h_i, k_i)(h_1, k_1) \cdots \widehat{(h_i, k_i)} \cdots (h_n, k_n) \\
&\quad = -\sum_{i=1}^{n} (h_i, D_h k_i)(h_1, k_1) \cdots \widehat{(h_i, k_i)} \cdots (h_n, k_n) \\
&\quad = -(h_1 \otimes \cdots \otimes h_n, D_h^{(n)}(k_1 \otimes \cdots \otimes k_n)),
\end{aligned}$$

where in the second equality the skew-symmetry of D_h (Theorem 3.12) was used n-times. Q.E.D.

LEMMA 3.16. *Suppose that* $A \colon H_1 \to H_2$ *and* $B \colon K_1 \to K_2$ *are bounded linear maps, where* H_i *and* K_i *for* $i = 1, 2$ *are Hilbert spaces. Then* $A \otimes B \colon H_1 \otimes K_1 \to H_2 \otimes K_2$ *is also bounded and* $\|A \otimes B\|_{op} = \|A\|_{op} \cdot \|B\|_{op}$. *If* $A_n \colon H_1 \to H_2$ *and* $B_n \colon K_1 \to K_2$ *for* $n = 1, 2, 3, ...$ *are two sequences of bounded linear maps such that* A *and* B *are the strong limits of* A_n *and* B_n *respectively, then* $A \otimes B$ *is the strong limit of the sequence* $A_n \otimes B_n$.

Proof. See the proposition on p.299 of Reed and Simon [28], for the assertion $\|A \otimes B\|_{op} = \|A\|_{op} \cdot \|B\|_{op}$. For the second assertion first note, by the uniform boundedness principle, that $\sup_n \|A_n\|_{op} < \infty$ and $\sup_n \|B_n\|_{op} < \infty$. Using this remark and the easily proved fact that

$$\lim_{n \to \infty} (A_n \otimes B_n)\xi = (A \otimes B)\xi \tag{3.27}$$

for any algebraic tensor $\xi \in H_1 \otimes K_1$, it is easy to verify that (3.27) holds for all $\xi \in H_1 \otimes K_1$. Q.E.D.

In preparation for the next theorem we need the following notation.

Notation 3.17. Let $J^{(m,n)}: T^{0,(m+n)} \equiv H_0^{\otimes (m+n)} \to T_0^{m,n}$ be the unitary isomorphism uniquely determined by

$$\begin{aligned} J^{(m,n)}&(h_1 \otimes \cdots \otimes h_m \otimes k_1 \otimes \cdots \otimes k_n) \\ &= (h_1, \cdot\,) \otimes \cdots \otimes (h_m, \cdot\,) \otimes k_1 \otimes \cdots \otimes k_n, \end{aligned}$$

where $\{h_i\}_{i=1}^m \subset H_0$ and $\{k_i\}_{i=1}^n \subset H_0$. Here we are using the identification of $T_0^{m,n}$ with $(H_0^*)^{\otimes m} \otimes H_0^{\otimes n}$ given in Remark 3.4.

THEOREM 3.18. *There exists unique bounded operators* $D^{(m,n)}: T_0^{m,n} \to T_0^{m+1,n}$ *such that*:

1. $D^{(0,n)} = D^{(n)}: T_0^{0,n} \to T_0^{1,n}$,
2. [Product Rule] *If* $\xi \in T_0^{0,m} = H_0^{\otimes n}$, $\alpha \in T_0^{m,n}$, *and* $h \in H_0$ *then*

$$D_h^{(n)}(\alpha\xi) = (D_h^{(m,n)}\alpha)\xi + \alpha D_h^{(m)}\xi \tag{3.28}$$

where $(D_h^{(m,n)}\alpha)\xi \equiv (D^{(m,n)}\alpha)(h \otimes \xi)$.

Moreover (*letting* I *denote the identity operator on* H_0^*)

$$D^{(m,n)} J^{(m,n)} = (I \otimes J^{(m,n)}) D^{(m+n)}, \tag{3.29}$$

$$\|D^{(m,n)}\|_{op} \leqslant (m+n)\, \|D^{(1)}\|_{op}, \tag{3.30}$$

and for each $h \in H_0$, $D_h^{(m,n)}$ *is a skew adjoint operator on* $T_0^{m,n}$.

Proof. First assume that $D^{(m,n)}$ exists. Then Eq. (3.28) implies that

$$(D_h^{(m,n)}\alpha)\xi = D_h^{(n)}(\alpha\xi) - \alpha D_h^{(m)}\xi, \tag{3.31}$$

which proves the uniqueness assertion. Now let $\{h_i\}_{i=1}^m$, $\{l_i\}_{i=1}^m$, and $\{k_i\}_{i=1}^n$ be subsets of H_0,

$$\begin{aligned} \tilde{\alpha} &\equiv h_1 \otimes \cdots \otimes h_m \otimes k_1 \otimes \cdots \otimes k_n, \\ \alpha &\equiv J^{(m,n)}\tilde{\alpha} = (h_1, \cdot\,) \otimes \cdots \otimes (h_m, \cdot\,) \otimes k_1 \otimes \cdots \otimes k_n, \end{aligned} \tag{3.32}$$

$$C \equiv \prod_{j=1}^m (h_j, l_j), \qquad C_i \equiv \prod_{j: j \neq i}^m (h_j, l_j),$$

and $\xi \equiv l_1 \otimes \cdots \otimes l_m$. Then using (3.31), (3.25), and Theorem 3.15,

$$\begin{aligned}(D_h^{(m,n)}\alpha)\xi &= C \cdot D_h^{(n)}(k_1 \otimes \cdots \otimes k_n) - \left(\sum_{i=1}^m C_i(h_i, D_h l_i)\right) \cdot (k_1 \otimes \cdots \otimes k_n) \\ &= C \cdot D_h^{(n)}(k_1 \otimes \cdots \otimes k_n) + \left(\sum_{i=1}^m C_i(D_h h_i, l_i)\right) \cdot (k_1 \otimes \cdots \otimes k_n) \\ &= C \cdot D_h^{(n)}(k_1 \otimes \cdots \otimes k_n) \\ &\quad + (D_h^{(m)}(h_1 \otimes \cdots \otimes h_m), \xi) \cdot (k_1 \otimes \cdots \otimes k_n) \\ &= (J^{(m,n)} D_h^{(m+n)} \tilde{\alpha})\xi.\end{aligned}$$

This computation verifies Eq. (3.29).

For existence just define $D^{(m,n)}$ by (3.29), i.e.,

$$D^{(m,n)} \equiv (I \otimes J^{(m,n)})\, D^{(m+n)} (J^{(m,n)})^{-1}.$$

Then $D^{(m,n)}$ is a bounded operator and satisfies the norm estimate in (3.30) because of Theorem 3.15 and the facts that $I \otimes J^{(m,n)}$ and $J^{(m,n)}$ are unitary. Similarly, $D_h^{(m,n)}$ is skew adjoint for all $h \in H_0$, since (by Theorem 3.15) $D_h^{(m+n)}$ is skew adjoint. Q.E.D.

So as not to have to write the superscript (m, n) in $D^{(m,n)}$ constantly we will use the following notation.

Notation 3.19. Let D denote the unique linear operator from T_0 to T_0 such that $D|_{T_0^{m,n}} = D^{(m,n)}$, where T_0 is defined in Notation 3.1. We will also write $D_h \alpha$ for $D_h^{(m,n)}\alpha$ when $\alpha \in T_0^{m,n}$. In this way, D_h is now viewed as a linear operator on T_0.

LEMMA 3.20 (Product Rule). *Suppose that $\alpha \in T_0^{m,n}$ and $\eta \in H_0^{\otimes k}$. Let $\eta \otimes \alpha \in T_0^{m,n+k}$ be defined by*

$$(\eta \otimes \alpha)\xi = \eta \otimes (\alpha\xi), \qquad \forall \xi \in H_0^{\otimes m}. \tag{3.33}$$

Then

$$D_h(\eta \otimes \alpha) = (D_h \eta) \otimes \alpha + \eta \otimes D_h \alpha, \qquad \forall h \in H_0. \tag{3.34}$$

Proof. By repeated use of the product rules in Eqs. (3.23) and (3.28),

$$\begin{aligned}(D_h(\eta \otimes \alpha))\xi &= D_h(\eta \otimes (\alpha\xi)) - (\eta \otimes \alpha)\, D_h \xi \\ &= (D_h \eta) \otimes (\alpha\xi) + \eta \otimes D_h(\alpha\xi) - (\eta \otimes \alpha)\, D_h \xi \\ &= ((D_h \eta) \otimes \alpha)\xi + \eta \otimes ((D_h \alpha)\xi).\end{aligned}$$

Q.E.D.

DEFINITION 3.21. For $k \in \mathbb{N}$, $\xi \in H_0^{\otimes k}$, and $\alpha \in T_0^{m,n}$, let $D_\xi^k \alpha \in T_0^{m,n}$ be determined by

$$(D_\xi^k \alpha)\gamma = (D^k \alpha)(\xi \otimes \gamma), \tag{3.35}$$

where

$$D^k \alpha \equiv \overbrace{D \cdots D}^{k\text{-times}} \alpha.$$

DEFINITION 3.22 (Curvature Operator). The curvature operator $\mathscr{R}$ is the linear operator from T_0 to T_0 determined by:

1. $\mathscr{R} T_0^{m,n} \subset T_0^{m+2,n}$ and $\mathscr{R}|_{T_0^{m,n}}$ is bounded from $T_0^{m,n}$ to $T_0^{m+2,n}$ and
2. for $\alpha \in T_0^{m,n}$, $h, k \in H_0$ and $\xi \in H_0^{\otimes m}$,

$$(\mathscr{R}\alpha)(h \otimes k \otimes \xi) = (D^2_{(h \wedge k)}\alpha)\xi = (D^2\alpha)((h \wedge k) \otimes \xi), \tag{3.36}$$

where

$$h \wedge k \equiv h \otimes k - k \otimes h. \tag{3.37}$$

We also denote $(\mathscr{R}\alpha)(h \otimes k \otimes \xi)$ by $(\mathscr{R}(h,k)\alpha)\xi$. With this convention for each $h, k \in H_0$, $\mathscr{R}(h,k)$ is a linear operator on T_0 such that $\mathscr{R}(h,k)$ restricted to $T_0^{m,n}$ is a bounded operator from $T_0^{m,n}$ to $T_0^{m,n}$.

Notice for $\alpha \in T_0^{m,n}$ that

$$\|\mathscr{R}\alpha\|^2 = \sum_{h,k \in S_0} \sum_{\xi \in S_0^{\otimes m}} \|(D^2\alpha)((h \wedge k) \otimes \xi)\|^2 \leqslant 2\,\|D^2\alpha\|^2.$$

Hence it follows from this equation and two applications of the bound in (3.30) that

$$\|\mathscr{R}|_{T_0^{m,n}}\|_{op} \leqslant \sqrt{2}(m+n)(m+n+1)\,\|D^{(1)}\|_{op},$$

where $D^{(1)} \equiv D|_{H_0}$. Some further properties of the curvature are summarized in the next proposition.

PROPOSITION 3.23. *Let $h, k \in H_0$. The curvature operator $\mathscr{R}$ satisfies the following properties.*

1. [Commutator Formula]

$$\mathscr{R}(h,k) \equiv [D_h, D_k] - D_{[h,k]}, \tag{3.38}$$

where D_h is thought of as an operator on T_0 as in Notation 3.19.

2. [Product Rule 1] *If* $\xi \in H_0^{\otimes m}$ *and* $\alpha \in T_0^{m,n}$, *then*

$$\mathscr{R}(h,k)(\alpha\xi) = (\mathscr{R}(h,k)\alpha)\xi + \alpha(\mathscr{R}(h,k)\xi). \tag{3.39}$$

3. [Product Rule 2] *If* $\xi \in H_0^{\otimes m}$ *and* $\eta \in H_0^{\otimes n}$, *then*

$$\mathscr{R}(h,k)(\xi \otimes \eta) = (\mathscr{R}(h,k)\xi) \otimes \eta + \xi \otimes \mathscr{R}(h,k)\eta. \tag{3.40}$$

4. [Skew Adjointness] $\mathscr{R}(h,k)$ *acts as a skew adjoint operator on each Hilbert space* $T_0^{m,n}$.

Proof. For the first item let $\xi \in H_0^{\otimes m}$, and $\alpha \in T_0^{m,n}$. Then

$$\begin{aligned}
(D^2_{h\otimes k}\alpha)\xi &= (D^2\alpha)(h \otimes k \otimes \xi) = (D_h(D\alpha))k \otimes \xi \\
&= D_h[(D\alpha)(k \otimes \xi)] - (D\alpha)(D_h(k \otimes \xi)) \\
&= D_h[(D_k\alpha)\xi] - (D\alpha)((D_h k) \otimes \xi + k \otimes D_h \xi) \\
&= (D_h D_k \alpha)\xi + (D_k\alpha)\, D_h\xi - (D\alpha)((D_h k) \otimes \xi) - (D_k \alpha)\, D_h \xi \\
&= (D_h D_k \alpha)\xi - (D\alpha)((D_h k) \otimes \xi).
\end{aligned}$$

Therefore by this equation and the zero torsion assertion in Theorem 3.12,

$$\begin{aligned}
(\mathscr{R}(h,k)\alpha)\xi \equiv (D^2_{h\wedge k}\alpha)\xi \\
&= ([D_h, D_k]\alpha)\xi - (D\alpha)((D_h k - D_k h) \otimes \xi) \\
&= ([D_h, D_k]\alpha)\xi - (D\alpha)([h,k] \otimes \xi) \\
&= ([D_h, D_k]\alpha - D_{[h,k]}\alpha)\xi,
\end{aligned}$$

which proves (3.38).

By repeated use of the product rules for D,

$$\begin{aligned}
D_h D_k(\alpha\xi) &= D_h((D_k\alpha)\xi + \alpha D_k \xi) \\
&= (D_h D_k \alpha)\xi + (D_k\alpha) D_h \xi + (D_h\alpha)\, D_k\xi + \alpha(D_h D_k \xi).
\end{aligned}$$

Also note that

$$D_{[h,k]}(\alpha\xi) = (D_{[h,k]}\alpha)\xi + \alpha D_{[h,k]}\xi.$$

Hence Eq. (3.39) follows, since

$$\begin{aligned}
\mathscr{R}(h,k)(\alpha\xi) &= ([D_h, D_k]\alpha)\xi + \alpha([D_h, D_k]\xi) - \{(D_{[h,k]}\alpha)\xi + \alpha D_{[h,k]}\xi\} \\
&= (\mathscr{R}(h,k)\alpha)\xi + \alpha(\mathscr{R}(h,k)\xi).
\end{aligned}$$

A completely analogous proof works for (3.40). The skew adjointness of $\mathscr{R}(h, k)$ follows from (3.38) and the assertion in Theorem 3.18 that D_h and D_k act as skew adjoint operators on $T_0^{m,n}$. Q.E.D.

Remark 3.24. Using (3.39) and (3.40) it is easy to verify that $\mathscr{R}$ may be expressed in terms of the curvature tensor R. For example if $u, v, h, k \in H_0$ and $\alpha v \equiv (u, v)$ then

$$\begin{aligned}(\mathscr{R}(h, k)\alpha)v &= \mathscr{R}(h, k)(\alpha v) - \alpha \mathscr{R}(h, k)v = 0 - (u, R(h, k)v) \\ &= (R(h, k)u, v).\end{aligned}$$

Using this result and the product rule for $\mathscr{R}$, it follows that

$$\mathscr{R}(h, k)\{(u, \cdot) \otimes v\} = (R(h, k)u, \cdot) \otimes v + (u, \cdot) \otimes R(h, k)v.$$

Similar formulas hold for $\alpha \in T_0^{m,n}$ given as in Eq. (3.32).

LEMMA 3.25. *Suppose $h, k, l \in H_0$, $\xi \in H_0^{\otimes m}$ and $\alpha \in T_0^{m,n}$, then*

$$(D^3_{(h \otimes (k \wedge l))}\alpha)\xi = (D_h(\mathscr{R}\alpha))(k \otimes l \otimes \xi). \tag{3.41}$$

Warning: *In general* $(D_h(\mathscr{R}\alpha))(k \otimes l \otimes \xi) \neq (D_h(\mathscr{R}(k, l)\alpha))\xi$.

Proof. Unwinding definitions along with repeated use of the product rules gives

$$\begin{aligned}(D^3_{(h \otimes (k \wedge l))}\alpha)\xi &= (D^3\alpha)(h \otimes (k \wedge l) \otimes \xi) \\ &= D_h[(D^2\alpha)((k \wedge l) \otimes \xi)] - (D^2\alpha)\, D_h((k \wedge l) \otimes \xi) \\ &= D_h[(\mathscr{R}(k, l)\alpha)\xi] - (D^2\alpha)\, D_h((k \wedge l) \otimes \xi) \\ &= D_h[(\mathscr{R}\alpha)(k \otimes l \otimes \xi)] - (\mathscr{R}\alpha)\, D_h(k \otimes l \otimes \xi) \\ &= (D_h(\mathscr{R}\alpha))(k \otimes l \otimes \xi).\end{aligned}$$

In the second to last equality, we have used

$$D_h((k \wedge l) \otimes \xi) = ((D_h k) \wedge l) \otimes \xi + (k \wedge (D_h l)) \otimes \xi + (k \wedge l) \otimes D_h \xi,$$

so that

$$\begin{aligned}(D^2\alpha)\, D_h((k \wedge l) \otimes \xi) &= (\mathscr{R}\alpha)(D_h k \otimes l \otimes \xi + k \otimes D_h l \otimes \xi + k \otimes l \otimes D_h \xi) \\ &= (\mathscr{R}\alpha)\, D_h(k \otimes l \otimes \xi).\end{aligned}$$

Q.E.D.

4. GEOMETRY OF LOOP GROUPS

For the sequel, let G be a connected Lie group of compact type and $\mathfrak{g} \equiv T_e G$ be the Lie algebra of G. Recall that G is of compact type if there exists an Ad_G-invariant inner product on $\mathfrak{g}$. This is equivalent to the statement that G is isomorphic to $K \times \mathbb{R}^d$ for some compact Lie group K and some $d \geqslant 0$. For a brief summary of the structure of compact type Lie groups, see Section 2.1 in [15].

For the remainder of this paper, $\langle \cdot, \cdot \rangle$ will be a fixed Ad_G-invariant inner product on $\mathfrak{g}$. As in Section 2, we will continue to denote the extension of $\langle \cdot, \cdot \rangle$ to a left invariant (and in this case also right invariant) Riemannian metric on G by $\langle \cdot, \cdot \rangle$. It should be noted that $\langle \cdot, \cdot \rangle$ satisfies the hypothesis in Section 3, i.e., $\langle ad_A B, C \rangle = -\langle B, ad_A C \rangle$ for all $A, B \in \mathfrak{g}$.

4.1. *Vector Fields on $\mathscr{L}(G)$*

Notation 4.1. Let $\mathscr{L} \equiv \mathscr{L}(G)$ denote the based loops on G, i.e., $g \in \mathscr{L}$ iff $g\colon [0, 1] \to G$ is a continuous path such that $g(0) = g(1) = e \in G$.

DEFINITION 4.2. Let $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ be a *partition* of $[0, 1]$. Let $G^{\mathscr{P}} \equiv G^n$ and $\pi_{\mathscr{P}}\colon \mathscr{L}(G) \to G^{\mathscr{P}}$ denote the projection

$$\pi_{\mathscr{P}}(g) \equiv (g(s_1), ..., g(s_n)). \tag{4.1}$$

We will also write $g_{\mathscr{P}}$ for $\pi_{\mathscr{P}}(g)$. A function $f\colon \mathscr{L} \to \mathbb{R}$ is said to be a *smooth cylinder function* if f has the form

$$f(g) = F \circ \pi_{\mathscr{P}}(g) = F(g_{\mathscr{P}}) \tag{4.2}$$

for some partition $\mathscr{P}$ and some $F \in C^\infty(G^{\mathscr{P}})$. Let $\mathscr{F}C^\infty$ denote the collection of all smooth cylinder functions on $\mathscr{L}$. Let $\mathscr{F}C_b^\infty$ denote those $f \in \mathscr{F}C^\infty$ such that $f = F \circ \pi_{\mathscr{P}}$ as in (4.2) where now F and all of its derivatives by left invariant differential operators on $G^{\mathscr{P}}$ are assumed to be bounded.

We will view $H_0 = H_0(\mathfrak{g})$ as the Lie algebra of $\mathscr{L}(G)$. Given $h \in H_0$, let $\tilde{h}$ denote the left invariant vector field on $\mathscr{L}(G)$ defined by

$$\tilde{h}(g)(s) \equiv L_{g(s)*} h(s). \tag{4.3}$$

The inner product on $H_0(\mathfrak{g})$ (see Eq. (3.2) with $V = \mathfrak{g}$) extends uniquely to a left invariant Riemannian metric on $\mathscr{L}(G)$. This Riemannian metric will still be denoted by $(\cdot, \cdot)$ and satisfies $(\tilde{h}, \tilde{k}) = (h, k)$ for all $h, k \in H_0$. We will define the tangent space to $\mathscr{L}$ at g to be $T_g \mathscr{L} \equiv \{\tilde{h}(g) \mid h \in H_0\}$. With this definition, $\mathscr{L}$ is formally a Riemannian manifold and the map $(g, h) \in \mathscr{L} \times H_0 \to \tilde{h}(g) \in T\mathscr{L}$ is an isometric trivialization of the tangent bundle.

Under this trivialization, tensor fields on $\mathscr{L}$ may be identified with functions from $\mathscr{L}$ to $T_0^{k,m} \cong (H_0^*)^{\otimes k} \otimes H_0^{\otimes m}$ for appropriately chosen integers k and m.

DEFINITION 4.3 (Vector Valued Cylinder Functions). Given an inner product space T, let $\mathscr{F}C^\infty(T)$ ($\mathscr{F}C_b^\infty(T)$) denote the collection of functions $f\colon \mathscr{L} \to T$ which have the form

$$f = \sum_{i=1}^{N} f_i \alpha_i, \tag{4.4}$$

where $f_i \in \mathscr{F}C^\infty$ ($\mathscr{F}C_b^\infty$) and $\alpha_i \in T$. For $g \in \mathscr{L}$, we will often write f_g instead of $f(g)$.

Notation 4.4. If $(\cdot, \cdot)_T$ is the inner product on T (T as in Definition 4.3) and $f, k \in \mathscr{F}C^\infty(T)$, let $(f, k)_T$ denote the function in $\mathscr{F}C^\infty$ defined by $(f, k)_T(g) = (f(g), k(g))_T$.

The vector fields $\tilde{h}$ may be viewed as first order differential operators on $\mathscr{F}C^\infty(T)$ or $\mathscr{F}C_b^\infty(T)$ for any Hilbert space T. The explicit definition is:

DEFINITION 4.5. Let $h \in H_0$, $g \in \mathscr{L}$ and $f \in \mathscr{F}C^\infty(T)$, then define

$$(\tilde{h}f)(g) \equiv \frac{d}{dt}\bigg|_0 f(ge^{th}). \tag{4.5}$$

Remark 4.6. Suppose that $f \in \mathscr{F}C^\infty$ is presented as in (4.2) then

$$(\tilde{h}f)(g) = \sum_{i=1}^{n} (h(s_i)^{(i)} F)(g_{\mathscr{P}}), \tag{4.6}$$

where for any $A \in \mathfrak{g}$ and $i \in \{1, 2, ..., n\}$, $A^{(i)}$ denotes the left invariant vector field on $G^{\mathscr{P}} = G^n$ given by

$$(A^{(i)}F)(g_1, g_2, ..., g_n) \equiv \frac{d}{d\varepsilon}\bigg|_0 F(g_1, ..., g_{i-1}, g_i e^{\varepsilon A}, g_{i+1}, ..., g_n). \tag{4.7}$$

Let $\mathscr{D}_i F(g_1, g_2, ..., g_n)$ denote the unique element of $\mathfrak{g}$ such that $(A^{(i)}F)(g_1, g_2, ..., g_n) = \langle \mathscr{D}_i F(g_1, g_2, ..., g_n), A\rangle$ for all $A \in \mathfrak{g}$. Then (4.6) may be written as

$$(\tilde{h}f)(g) = \sum_{i=1}^{n} \langle (\mathscr{D}_i F)(g_{\mathscr{P}}), h(s_i)\rangle. \tag{4.8}$$

Remark 4.7. If $f \in \mathscr{F}C^\infty(T)$ is presented as in (4.4) then

$$\tilde{h}f = \sum (\tilde{h}f_i)\, \alpha_i \in \mathscr{F}C^\infty(T).$$

It is necessary to generalize the above definition from $h \in H_0$ to any $X \in \mathscr{F}C^\infty(H_0)$.

DEFINITION 4.8. Suppose that $f \in \mathscr{F}C^\infty(T)$ is presented as in (4.4) and $X \in \mathscr{F}C^\infty(H_0)$ has the form

$$X = \sum_{j=1}^{k} X_j h_j, \tag{4.9}$$

where $X_j \in \mathscr{F}C^\infty$ and $h_j \in H_0$. Let $\tilde{X}$ denote the first order differential operator on $\mathscr{F}C^\infty(T)$ given by

$$\tilde{X}f \equiv \sum X_j(\tilde{h}_j f). \tag{4.10}$$

LEMMA 4.9. *The operator $\tilde{X}$ is well defined.*

Proof. It is necessary to show: if $\sum_{j=1}^k X_j h_j = 0$ then $\sum_{j=1}^k X_j \tilde{h}_j \equiv 0$ as an operator on $\mathscr{F}C^\infty(T)$. For this, let $f \in \mathscr{F}C^\infty(T)$ and $g \in \mathscr{L}$. Let df_g denote the bounded operator from $H_0 \to T$ defined by

$$df_g h \equiv (\tilde{h}f)(g). \tag{4.11}$$

Then

$$0 = df_g\left(\sum_{j=1}^k X_j(g)\, h_j\right) = \sum_{j=1}^k X_j(g)\, df_g h_j = \left(\sum_{j=1}^k X_j \tilde{h}_j f\right)(g).$$

Q.E.D.

DEFINITION 4.10. Given $X, Y \in \mathscr{F}C^\infty(H_0)$, let

$$[X, Y]_{\mathscr{L}} \equiv \tilde{X}Y - \tilde{Y}X + [X, Y], \tag{4.12}$$

where $[X, Y] \in \mathscr{F}C^\infty(H_0)$ is the *pointwise Lie bracket* given by

$$[X, Y](g, s) = [X(g, s), Y(g, s)]_{\mathfrak{g}}. \tag{4.13}$$

LEMMA 4.11. *Suppose that $X, Y \in \mathscr{F}C^\infty(H_0)$, then*

$$\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X} = \widetilde{[X, Y]}_{\mathscr{L}}. \tag{4.14}$$

Proof. Without loss of generality, it suffices to show the equality in (4.14) holds as operators on $\mathcal{F}C^\infty$. To begin with let $h, k \in H_0$ and X and Y be the constant functions $X(g) = h$ and $Y(g) = k$ for all $g \in \mathcal{L}$. Suppose $f \in \mathcal{F}C^\infty$ is of the form in (4.2). Then using the notation in Remark 4.6, we have

$$([\tilde{h}, \tilde{k}]f)(g) = \sum_{i,j} ([h(s_i)^{(i)}, k(s_j)^{(j)}]F)(g_{\mathcal{P}}).$$

Since

$$[h(s_i)^{(i)}, k(s_j)^{(j)}] = \delta_{ij}[h(s_i), k(s_i)]^{(i)},$$

it follows that

$$([\tilde{h}, \tilde{k}]f)(g) = \sum_i ([h(s_i), k(s_i)]^{(i)} F)(g_{\mathcal{P}}) = (\widetilde{[h, k]}f)(g). \tag{4.15}$$

Because $\tilde{h}k = 0$ and $\tilde{k}h = 0$, the above displayed equation implies (4.14) when X and Y are constant.

Since both sides of (4.14) are bilinear in X and Y, to finish the proof it suffices to consider the case where $X = uh$ and $Y = vk$, with $u, v \in \mathcal{F}C^\infty$ and $h, k \in H_0$. Now

$$\tilde{X}\tilde{Y}f = \tilde{X}(v \cdot \tilde{k}f) = \tilde{X}v \cdot \tilde{k}f + vu \cdot \tilde{h}\tilde{k}f = (\tilde{X}Y)^\sim f + uv \cdot \tilde{h}\tilde{k}f. \tag{4.16}$$

Using a similar formula for $\tilde{Y}\tilde{X}f$, we find that

$$\begin{aligned}
[\tilde{X}, \tilde{Y}]f &= (\tilde{X}Y - \tilde{Y}X)^\sim f + uv \cdot [\tilde{h}, \tilde{k}]f \\
&= (\tilde{X}Y - \tilde{Y}X)^\sim f + (uv[h, k])^\sim f \\
&= (\tilde{X}Y - \tilde{Y}X + [X, Y])^\sim f. \\
&= [\widetilde{X, Y}]_{\mathcal{L}} f.
\end{aligned}$$

Q.E.D.

4.2. *Levi–Civita Covariant Derivative*

Definition 4.12 (Levi–Civita Covariant Derivative). Given $X \in \mathcal{F}C^\infty(H_0)$, let ∇_X denote the linear operator on $\mathcal{F}C^\infty(T_0)$ determined by

$$\nabla_X \alpha = \tilde{X}\alpha + D_X \alpha, \tag{4.17}$$

where

$$(D_X \alpha)(g) \equiv D_{X(g)}(\alpha_g), \tag{4.18}$$

and $D_{X(g)}$ is defined in Notation 3.19. Also let ∇ denote the linear operator on $\mathscr{F}C^\infty(T_0)$ determined on $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$ by

$$(\nabla\alpha)_g\,(h\otimes\xi) = (\nabla_h\alpha)_g\,\xi, \tag{4.19}$$

for all $h \in H_0$ and $\xi \in H_0^{\otimes m}$.

Notation 4.13. Given $\eta \in \mathscr{F}C^\infty(H_0^{\otimes k})$ and $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$, let $\nabla_\eta^k\alpha$ denote the element of $\mathscr{F}C^\infty(T_0^{m,n})$ determined by

$$(\nabla_\eta^k\alpha)_g\,\xi = (\nabla^k\alpha)_g\,(\eta_g\otimes\xi), \qquad \forall \xi \in H_0^{\otimes m}. \tag{4.20}$$

PROPOSITION 4.14. *Let $X, Y \in \mathscr{F}C^\infty(H_0)$. The operator ∇ has the following properties*:

1. [Tensorial in η] *If $\eta \in \mathscr{F}C^\infty(H_0^{\otimes k})$, $\alpha \in \mathscr{F}C^\infty(T_0)$, and $f \in \mathscr{F}C^\infty$, then*

$$\nabla_{f\eta}^k\alpha = f\nabla_\eta^k\alpha. \tag{4.21}$$

2. [Product rule 1] *If $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$ and $\gamma \in \mathscr{F}C^\infty(H_0^{\otimes k})$ then*

$$\nabla_X(\gamma\otimes\alpha) = (\nabla_X\gamma)\otimes\alpha + \gamma\otimes\nabla_X\alpha. \tag{4.22}$$

3. [Product rule 2] *If $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$ and $\gamma \in \mathscr{F}C^\infty(H_0^{\otimes m})$ then*

$$\nabla_X(\alpha\gamma) = (\nabla_X\alpha)\gamma + \alpha\nabla_X\gamma. \tag{4.23}$$

4. [Metric compatible] *If $S, T \in \mathscr{F}C^\infty(T_0^{m,n})$ and $X \in \mathscr{F}C^\infty(H_0)$, then*

$$\tilde{X}(S, T) = (\nabla_X S, T) + (S, \nabla_X T). \tag{4.24}$$

5. [Torsion free 1] *With $[X, Y]_{\mathscr{L}}$ as in* (4.12),

$$\nabla_X Y - \nabla_Y X - [X, Y]_{\mathscr{L}} = 0. \tag{4.25}$$

6. [Torsion free 2] *If $f \in \mathscr{F}C^\infty$ and $X, Y \in \mathscr{F}C^\infty(H_0)$, then*

$$\nabla_{X\wedge Y}^2 f = 0, \tag{4.26}$$

where $X\wedge Y \equiv X\otimes Y - Y\otimes X$.

Proof. The first item is clear. The product rules both have similar proofs, so we will only give a proof of the first product rule. Without loss of generality we may assume the $\gamma = u\bar{\gamma}$ and $\alpha = v\bar{\alpha}$, where $u, v \in \mathcal{F}C^\infty$, $\bar{\gamma} \in H_0^{\otimes k}$, and $\bar{\alpha} \in T_0^{m,n}$. Then

$$\begin{aligned}\nabla_X(\gamma \otimes \alpha) &= \nabla_X(uv\bar{\gamma} \otimes \bar{\alpha})\\ &\equiv \tilde{X}(uv) \cdot \bar{\gamma} \otimes \bar{\alpha} + uvD_X(\bar{\gamma} \otimes \bar{\alpha})\\ &= (v\tilde{X}u + u\tilde{X}v)\, \bar{\gamma} \otimes \bar{\alpha} + uv(D_X\bar{\gamma}) \otimes \bar{\alpha} + uv\bar{\gamma} \otimes D_X\bar{\alpha}\\ &= (\nabla_X\gamma) \otimes \alpha + \gamma \otimes \nabla_X\alpha,\end{aligned}$$

where we have used the product rules for $\tilde{X}$ and D_X in the third equality.

Metric Compatibility. It is easy to show that

$$\tilde{X}(S, T) = (\tilde{X}S, T) + (S, \tilde{X}T). \tag{4.27}$$

Since (by Theorem 3.18) D_X acts as a skew-symmetric operator,

$$0 = (D_XS, T) + (S, D_XT). \tag{4.28}$$

Thus Eq. (4.24) follows by adding Eqs. (4.27) and (4.28).

Torsion Free 1. Using Theorem 3.12 we find,

$$\begin{aligned}\nabla_X Y - \nabla_Y X &= \tilde{X}Y - \tilde{Y}X + D_X Y - D_Y X\\ &= \tilde{X}Y - \tilde{Y}X + [X, Y] = [X, Y]_{\mathcal{L}}.\end{aligned} \tag{4.29}$$

Torsion Free 2. By the product rules,

$$\begin{aligned}\nabla^2_{X\otimes Y} f &= \nabla_X((\nabla f)\, Y) - (\nabla f)\, \nabla_X Y = \nabla_X\nabla_Y f - \nabla_{\nabla_X Y} f\\ &= \tilde{X}\tilde{Y}f - \widetilde{\nabla_X Y}\, f.\end{aligned} \tag{4.30}$$

Therefore by (4.29) and Lemma 4.11,

$$\nabla^2_{X\wedge Y} f = [\tilde{X}, \tilde{Y}]\, f - \widetilde{[X, Y]}_{\mathcal{L}}\, f = 0. \qquad \text{Q.E.D.}$$

4.3. *The Curvature Tensor and Operators*

THEOREM 4.15 (Curvature). *Suppose that* $\alpha \in \mathcal{F}C^\infty(T_0^{m,n})$ *and* $X, Y \in \mathcal{F}C^\infty(H_0)$, *then*

$$\nabla^2_{X\wedge Y}\alpha = \mathcal{R}(X, Y)\alpha, \tag{4.31}$$

where $\mathcal{R}(X, Y)\alpha \in \mathcal{F}C^\infty(T_0^{m,n})$ *is defined by*

$$(\mathcal{R}(X, Y)\alpha)_g = \mathcal{R}(X_g, Y_g)\, \alpha_g, \qquad \forall g \in \mathcal{L}(G). \tag{4.32}$$

Moreover

$$\mathscr{R}(X, Y)\alpha = [\nabla_X, \nabla_Y]\alpha - \nabla_{[X, Y]_{\mathscr{L}}}\alpha. \tag{4.33}$$

Proof. We may assume that $\alpha = f\bar{\alpha}$, where $f \in \mathscr{F}C^\infty$ and $\bar{\alpha} \in T_0^{m,n}$. Then

$$\nabla\alpha = \nabla f \otimes \bar{\alpha} + f D\bar{\alpha},$$

and so by the product rules:

$$\nabla_X \nabla\alpha = (\nabla_X \nabla f) \otimes \bar{\alpha} + \nabla f \otimes D_X\bar{\alpha} + (\nabla_X f)\, D\bar{\alpha} + f D_X\, D\bar{\alpha}.$$

Therefore,

$$\nabla^2_{X\otimes Y}\alpha = (\nabla^2_{X\otimes Y} f)\bar{\alpha} + \nabla_Y f \cdot D_X\bar{\alpha} + \nabla_X f \cdot D_Y\bar{\alpha} + f D^2_{X\otimes Y}\bar{\alpha},$$

and hence

$$\nabla^2_{X\wedge Y}\alpha = (\nabla^2_{X\wedge Y} f)\bar{\alpha} + f D^2_{X\wedge Y}\bar{\alpha} = f\mathscr{R}(X, Y)\bar{\alpha} = \mathscr{R}(X, Y)\alpha,$$

where we have used (4.26) and Definition 3.22 for $\mathscr{R}$. This proves Eq. (4.31).

Let $\bar{\mathscr{R}}(X, Y)\alpha$ denote the RHS of (4.33). Elementary computations (the same as the finite dimensional case) show that $\bar{\mathscr{R}}$ is tensorial, i.e., if $u, v, w \in \mathscr{F}C^\infty$ then

$$\bar{\mathscr{R}}(uX, vY)(w\alpha) \equiv uvw\bar{\mathscr{R}}\langle X, Y\rangle\alpha. \tag{4.34}$$

Since $\mathscr{R}(X, Y)\alpha$ has the same property, it suffices to prove (4.33) in the special case where X, Y, and α are constant functions on $\mathscr{L}(G)$. But then

$$\bar{\mathscr{R}}(X, Y)\alpha = [D_X, D_Y]\alpha - D_{[X, Y]}\alpha = \mathscr{R}(X, Y)\alpha,$$

where we have used Eq. (3.38) of Proposition 3.23 and $[X, Y] = [X, Y]_{\mathscr{L}}$ for constant functions X and Y. Q.E.D.

The following Lemma is an extension of Lemma 3.25.

Lemma 4.16. *Suppose $h, k, l \in H_0$, $\xi \in H_0^{\otimes m}$ and $\alpha \in T_0^{m,n}$, then*

$$(\nabla^3_{(h\otimes(k\wedge l))}\alpha)\xi = (\nabla_h(\mathscr{R}\alpha))(k \otimes l \otimes \xi). \tag{4.35}$$

Proof. Because of Theorem 4.15, this lemma may be proved simply by replacing D_h by ∇_h everywhere in the proof of Lemma 3.25. Q.E.D.

4.4. *The Laplacian on $\mathscr{L}(G)$*

Recall that $\mathfrak{g}_0$, $\mathfrak{h}$, and S_0 are orthonormal bases for $\mathfrak{g}$, $H_0(\mathbb{R})$, and $H_0 = H_0(\mathfrak{g})$ respectively.

DEFINITION 4.17 (Laplacian). For $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$, let

$$\Delta\alpha \equiv \sum_{h \in S_0} \nabla^2_{h \otimes h} \alpha, \tag{4.36}$$

where S_0 is any good orthonormal basis of H_0 as in Definition 3.10.

The next proposition guarantees that the sum in (4.36) exists. We first need the following definition.

DEFINITION 4.18. Given an partition $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$, let $\Delta_{\mathscr{P}}$ be the second order elliptic differential operator on $G^{\mathscr{P}}$ defined by

$$\Delta_{\mathscr{P}} \equiv \sum_{A \in \mathfrak{g}_0} \sum_{i=1}^{n} \sum_{j=1}^{n} G_0(s_i, s_j)\, A^{(j)} A^{(i)}, \tag{4.37}$$

where $A^{(i)}$ is defined in (4.7).

PROPOSITION 4.19. *For any good orthonormal basis S_0, $h \in S_0$ and, $\alpha \in T_0^{m,n}$; $\nabla^2_{h\otimes h}\alpha = \nabla^2_h \alpha$, where $\nabla^2_h\alpha \equiv \nabla_h(\nabla_h \alpha)$. Now suppose that S_0 is any orthonormal basis of H_0. Then for each $g \in \mathscr{L}$ and $\alpha \in T_0^{m,n}$, the sum $\sum_{h \in S_0} (\nabla^2_h \alpha)_g$ is convergent in $T_0^{m,n}$ and the sum is independent of the choice of orthonormal basis S_0. In particular $\Delta\alpha$ defined in Eq. (4.36) is well defined. Moreover Δ has the following properties.*

1. *If $f = F \circ \pi_{\mathscr{P}} \in \mathscr{F}C^\infty$ is presented as in Eq. (4.2), then*

$$\Delta f = (\Delta_{\mathscr{P}} F) \circ \pi_{\mathscr{P}}. \tag{4.38}$$

2. *Let S_0 be* any *basis of H_0, $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$, $T \in \mathscr{F}C^\infty(H_0^{\otimes k})$, and $S \in \mathscr{F}C^\infty(H_0^{\otimes m})$ then*

$$\Delta\alpha = \sum_{h \in S_0} \nabla^2_h \alpha, \tag{4.39}$$

$$\Delta(T \otimes \alpha) = (\Delta T) \otimes \alpha + 2 \sum_{h \in S_0} \nabla_h T \otimes \nabla_h \alpha + T \otimes \Delta\alpha, \tag{4.40}$$

and

$$\Delta(\alpha S) = (\Delta\alpha)\, S + 2 \sum_{h \in S_0} (\nabla_h \alpha)(\nabla_h S) + \alpha\, \Delta S. \tag{4.41}$$

3. *If $\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$ or $\mathscr{F}C_b^\infty(T_0^{m,n})$ then $\Delta\alpha \in \mathscr{F}C^\infty(T_0^{m,n})$ or $\Delta\alpha \in \mathscr{F}C_b^\infty(T_0^{m,n})$ respectively.*

Before proving this proposition, we will first prove a couple of preparatory lemmas.

LEMMA 4.20. *Let $S_0 \subset H_0$ be an arbitrary orthonormal basis of H_0. Set $C_{ad} \equiv \sum_{A \in \mathfrak{g}_0} ad_A^2$, the Casimir operator for the adjoint representation of $\mathfrak{g}$. Then for $\alpha \in H_0$,*

$$\sum_{h \in S_0} D_h^2 \alpha = P \left\{ \int_0^{\cdot} ds\, G_0(s, s)\, C_{ad} \alpha'(s) - \int_0^{\cdot} ds \int_0^1 dt\, G_0(s, t)\, C_{ad} \alpha'(t) \right\}, \tag{4.42}$$

where $P: H \to H_0$ is orthogonal projection. Also if $\alpha \in H_0 \otimes H_0$, then the sum

$$\sum_{h \in S_0} (D_h \otimes D_h) \alpha \tag{4.43}$$

converges in $H_0 \otimes H_0$ and the sum is independent of the choice of orthonormal basis S_0. Stated briefly: $\sum_{h \in S_0} D_h^2$ and $\sum_{h \in S_0} (D_h \otimes D_h)$ are strongly convergent and basis independent.

Proof. Recall that $P \equiv k - \Lambda k(1)$, where k is in H and $\Lambda(s) = s$. Using the definition of D_h,

$$\begin{aligned} D_h^2 \alpha &= P \left\{ \int_0^{\cdot} ad_h d(D_h \alpha) \right\} \\ &= P \int_0^{\cdot} \left\{ ad_{h(s)}^2 \alpha'(s) - \int_0^1 ad_{h(s)} ad_{h(t)} \alpha'(t)\, dt \right\} ds. \end{aligned} \tag{4.44}$$

Let Γ be a finite subset of S_0, then

$$\begin{aligned} &\left\| \sum_{h \in \Gamma} \int_0^{\cdot} ad_{h(s)}^2 \alpha'(s)\, ds - \int_0^{\cdot} G_0(s, s)\, C_{ad} \alpha'(s)\, ds \right\|^2 \\ &\quad = \int_0^1 \left| \left\{ \sum_{h \in \Gamma} ad_{h(s)}^2 - G_0(s, s)\, C_{ad} \right\} \alpha'(s) \right|^2 ds \\ &\quad \leqslant \int_0^1 \left\| \sum_{h \in \Gamma} ad_{h(s)}^2 - G_0(s, s)\, C_{ad} \right\|_{op}^2 |\alpha'(s)|^2\, ds \end{aligned}$$

and

$$\begin{aligned} &\left\| \int_0^{\cdot} ds \sum_{h \in \Gamma} \int_0^1 ad_{h(s)}\, ad_{h(t)} \alpha'(t)\, dt - \int_0^{\cdot} ds \int_0^1 G_0(s, t)\, C_{ad} \alpha'(t)\, dt \right\|^2 \\ &\quad = \int_0^1 \left| \int_0^1 \left(\sum_{h \in \Gamma} ad_{h(s)}\, ad_{h(t)} - G_0(s, t)\, C_{ad} \right) \alpha'(t)\, dt \right|^2 ds \end{aligned}$$

$$\leqslant \int_0^1 \left(\int_0^1 \left\| \sum_{h \in \Gamma} ad_{h(s)}\, ad_{h(t)} - G_0(s,t)\, C_{ad} \right\|_{op} |\alpha'(t)|\, dt \right)^2 ds$$

$$\leqslant \|\alpha\|^2 \int_0^1 \int_0^1 \left\| \sum_{h \in \Gamma} ad_{h(s)}\, ad_{h(t)} - G_0(s,t)\, C_{ad} \right\|_{op}^2 ds\, dt,$$

where $\|\cdot\|_{op}$ denotes the operator norm on $End(\mathfrak{g})$. Using Lemma 3.8, it follows that the last terms in the above displayed equations tend to zero as $\Gamma \uparrow S_0$. Eq. (4.42) follows by combining these two limits with Eq. (4.44) and using the fact that P is bounded.

To prove (4.43) it is helpful to identify $H \otimes H$ with $\mathrm{L}^2 \equiv L^2([0,1]^2, \mathfrak{g} \otimes \mathfrak{g})$. Explicitly, let $U\colon H \otimes H \to \mathrm{L}^2$ be the unitary map determined by

$$U(k \otimes l)(s,t) \equiv k'(s) \otimes l'(t) \in \mathfrak{g} \otimes \mathfrak{g} \qquad \forall k, l \in H_0.$$

Notice that

$$D_h \otimes D_h(k \otimes l) = P \otimes P((\tilde{D}_h k) \otimes \tilde{D}_h l), \tag{4.45}$$

where $\tilde{D}_h k \equiv \int_0^{\cdot} [h, dk]$ for $k \in H$. Setting $\alpha = k \otimes l$ for $k, l \in H$, we have

$$U((\tilde{D}_h \otimes \tilde{D}_h)\alpha)(s,t) = (ad_{h(s)} \otimes ad_{h(t)})(U\alpha)(s,t). \tag{4.46}$$

Since that map $\xi(s,t) \in \mathrm{L}^2 \to (ad_{h(s)} \otimes ad_{h(t)})\, \xi(s,t) \in \mathrm{L}^2$ is bounded (because h is continuous and hence a bounded function), it follows that Eq. (4.46) holds for all $\alpha \in H \otimes H$. By arguments similar to those given above, one may use Lemma 3.8 in conjunction with the dominated convergence theorem to show, for any $\xi \in \mathrm{L}^2$, that

$$\sum_{h \in S_0} (ad_{h(s)} \otimes ad_{h(t)})\, \xi(s,t) = G_0(s,t) \sum_{A \in \mathfrak{g}_0} (ad_A \otimes ad_A)\, \xi(s,t),$$

where the left sum is $L^2(ds, dt)$-convergent. In view of Eq. (4.46), this shows that $\sum_{h \in S_0} (\tilde{D}_h \otimes \tilde{D}_h)\alpha$ is convergent in $H \otimes H$ for all $\alpha \in H \otimes H$ and the sum is independent of the choice of basis S_0. Hence, using Eq. (4.45) and the boundedness of $P \otimes P$ (Lemma 3.16) we see that the sum in (4.43) is convergent in $H_0 \otimes H_0$ and is basis independent. Q.E.D.

We now generalize the first assertion of the last Lemma to include general $\alpha \in T_0$.

Lemma 4.21. *Let S_0 be an arbitrary basis for H_0. Then for each $\alpha \in T_0$ the sum*

$$\sum_{h \in S_0} D_h^2 \alpha \tag{4.47}$$

converges in T_0, where D_h^2 is shorthand for $D_h D_h$. Moreover the sum is independent of the basis S_0.

Proof (Case 1: $m=0$ and $n \in \mathbb{N}$). By the product rule D_h^2 acting on $H_0^{\otimes n}$ may be written as

$$D_h^2 = \sum_{i=1}^{n} I^{\otimes (i-1)} \otimes (D_h^{(1)})^2 \otimes IU^{\otimes (n-i)}$$
$$+2 \sum_{i<j} I^{\otimes (i-1)} \otimes D_h^{(1)} \otimes I^{\otimes (j-i-1)} \otimes D_h^{(1)} \otimes I^{\otimes (n-j)}, \qquad (4.48)$$

where $I^{\otimes k}$ is the identity operator on $H_0^{\otimes k}$ and $D_h^{(1)} = D_h|_{H_0}$. By choosing unitary maps on $H^{\otimes n}$ which correspond to appropriate permutations of the factors, one may easily verify that the first and second summands above are unitarily equivalent to $(D_h^{(1)})^2 \otimes I^{\otimes (n-1)}$ and $D_h^{(1)} \otimes D_h^{(1)} \otimes I^{\otimes (n-2)}$ respectively. Using this remark and Lemma 3.16 it follows that $\sum_{h \in S_0} D_h^2|_{H_0^{\otimes n}}$ is strongly convergent.

(Case 2: $m, n \in \mathbb{N}$.) Let $J \equiv J^{(m,n)}$ be the isometry of $H_0^{\otimes (m+n)}$ onto $T_0^{m,n}$ defined in Notation 3.17. By Eq. (3.29), this isometry intertwines D_h on $T_0^{m,n}$ with D_h on $H_0^{\otimes (m+n)}$, i.e.,

$$JD_h|_{H_0^{\otimes (m+n)}} = D_h|_{T_0^{m,n}} J.$$

Therefore the sum

$$\sum_{h \in S_0} D_h^2|_{T_0^{m,n}} = J \sum_{hS_0} D_h^2|_{H_0^{\otimes (m+n)}} J^{-1}$$

is convergent and is basis independent by case 1 above. Q.E.D.

Definition 4.22. For $f \in \mathscr{F}C^\infty$ presented as in Eq. (4.2), let

$$(\vec{\nabla} f)_g \equiv \sum_{A \in \mathfrak{g}_0} \sum_{i=1}^{n} \langle \mathscr{D}_i F(g_{\mathscr{P}}), A \rangle \, G_0(s_i, \cdot) A, \qquad (4.49)$$

where $\mathscr{D}_i$ is defined in Remark 4.6.

Clearly, $\vec{\nabla} f \in \mathscr{F}C^\infty(H_0)$. It is also easy to check that $\vec{\nabla} f$ is the unique element of $\mathscr{F}C^\infty(H_0)$ such that

$$((\vec{\nabla} f)_g, k) = (\tilde{k} f)(g) = (\nabla_k f)_g. \qquad (4.50)$$

We are now ready for the proof of Proposition 4.19.

Proof of Proposition 4.19. If S_0 is a good basis, then $\nabla_h h = P \int_0^{\cdot} [h, dh] = 0$ for all $h \in S_0$. Combining this observation with the product rule shows, for $\xi \in H_0^{\otimes m}$, that

$$\begin{aligned}(\nabla^2_{h\otimes h}\alpha)\xi &= (\nabla_h(\nabla\alpha))(h\otimes\xi)\\ &= \nabla_h(\nabla\alpha(h\otimes\xi)) - \nabla\alpha(\nabla_h(h\otimes\xi))\\ &= (\nabla_h\nabla_h\alpha)\xi + (\nabla_h\alpha)\,\nabla_h\xi - \nabla\alpha((\nabla_h h)\otimes\xi) - \nabla\alpha(h\otimes\nabla_h\xi)\\ &= (\nabla_h\nabla_h\alpha)\xi, \end{aligned} \tag{4.51}$$

which verifies that $\nabla^2_{h\otimes h}\alpha = \nabla^2_h\alpha$. So to verify that Δ is well defined and to prove Eq. (4.39) it suffices to show that the sum in Eq. (4.39) is convergent and is independent of the choice of orthonormal basis.

For the rest of the proof, let S_0 be an arbitrary orthonormal basis of H_0. We now prove (4.38). Let $f = F\circ\pi_{\mathcal{P}} \in \mathcal{F}C^\infty$ as in Eq. (4.2), then using the notation in Remark 4.6 we find:

$$\begin{aligned}\left(\sum_{h\in S_0}\nabla^2_h f\right)(g) &= \left(\sum_{h\in S_0}\tilde{h}^2 f\right)(g)\\ &= \left(\sum_{h\in S_0}\sum_{i,j=1}^{n}\sum_{A,B\in\mathfrak{g}_0}\langle h(s_i), A\rangle\langle h(s_j), B\rangle (A^{(i)}B^{(j)}F)(g_{\mathcal{P}})\right.\\ &= \sum_{A\in\mathfrak{g}_0}\sum_{i,j=1}^{n} G_0(s_i,s_j)(A^{(i)}A^{(j)}F)(g_{\mathcal{P}})\\ &= (\Delta_{\mathcal{P}}F)\circ\pi_{\mathcal{P}}(g),\end{aligned}$$

wherein Lemma 3.8 was used to compute the sums on $h\in S_0$ and $B\in\mathfrak{g}_0$. It also follows from Lemma 3.8 that all of the sums are absolutely convergent. We have verified (4.38) and the fact that for $\alpha = f\in\mathcal{F}C^\infty$, the sum in (4.39) is independent of the choice of basis S_0. It is also clear from (4.38) that $\Delta(\mathcal{F}C^\infty)\subset\mathcal{F}C^\infty$ and $\Delta(\mathcal{F}C_b^\infty)\subset\mathcal{F}C_b^\infty$.

Now suppose that $\alpha = f\bar{\alpha}$, where $f = F\circ\pi_{\mathcal{P}}\in\mathcal{F}C^\infty$ and $\bar{\alpha}\in T_0^{m,n}$, then by the product rule:

$$\sum_{h\in S_0}\nabla^2_h\alpha = \sum_{h\in S_0}(\nabla^2_h f)\,\bar{\alpha} + 2\sum_{h\in S_0}(\nabla_h f)\,D_h\bar{\alpha} + \sum_{h\in S_0} fD^2_h\bar{\alpha}, \tag{4.52}$$

provided the sums converge. We have just shown the first sum on the RHS of (4.52) is absolutely convergent on $\mathcal{L}$ and is independent of S_0. We also know, by Lemma 4.21, that the third sum is convergent and independent of S_0. Moreover, the first and the third sum on the RHS of (4.52) are in $\mathcal{F}C^\infty(T_0^{m,n})$ or $\mathcal{F}C_b^\infty(T_0^{m,n})$ if α is in $\mathcal{F}C^\infty(T_0^{m,n})$ or $\mathcal{F}C_b^\infty(T_0^{m,n})$ respectively.

We now consider the middle sum in the RHS of (4.52). Since

$$\left(\sum_{h\in S_0}\|(\nabla_h f)\, D_h\bar{\alpha}\|\right)^2 \leqslant \sum_{h\in S_0}|\nabla_h f|^2\cdot\sum_{h\in S_0}\|D_h\bar{\alpha}\|^2=\|\nabla f\|^2\,\|D\bar{\alpha}\|^2<\infty,$$

the middle sum is absolutely convergent. Moreover

$$\begin{aligned}\sum_{h\in S_0}(\nabla_h f)\,D_h\bar{\alpha}|_g&=\sum_{h\in S_0}(\vec{\nabla}f(g),h)\,D_h\bar{\alpha}=D_{\vec{\nabla}f(g)}\bar{\alpha}\\&=\sum_{A\in\mathfrak{g}_0}\sum_{i=1}^{n}\langle(\mathcal{D}_i F)(g_{\mathcal{P}}),A\rangle\cdot D_{G_0(s_i,\,\cdot\,)A}\bar{\alpha},\end{aligned}$$

which shows the middle sum in (4.52) is basis independent and the sum is in $\mathcal{F}C^\infty(T_0^{m,n})$ or $\mathcal{F}C_b^\infty(T_0^{m,n})$ if α is in $\mathcal{F}C^\infty(T_0^{m,n})$ or $\mathcal{F}C_b^\infty(T_0^{m,n})$ respectively. Therefore, we have verified the sum in (4.39) is convergent and independent of the choice of orthonormal basis S_0. We have also proved the last assertion of the Lemma.

The proofs of the product rules in Eqs. (4.40) and (4.41) are silimar to arguments used in the above paragraph and are thus left to the reader.

Q.E.D.

4.5. *Square Field Operators*

PROPOSITION 4.23. *Let $S, T\in\mathcal{F}C^\infty(T_0^{m,n})$, then*

$$\Delta(S,T)=(\Delta S,T)+2(\nabla S,\nabla T)+(S,\Delta T). \tag{4.53}$$

Proof. Let $h\in H_0$, then by metric compatibility:

$$\begin{aligned}\nabla_h^2(S,T)&=\nabla_h\{(\nabla_h S,T)+(S,\nabla_h T)\}\\&=(\nabla_h^2 S,T)+2(\nabla_h S,\nabla_h T)+(S,\nabla_h^2 T).\end{aligned}$$

The proposition is proved by summing this last equation on $h\in S_0$.

Q.E.D.

DEFINITION 4.24 (Gamma one and two). For $u, v\in\mathcal{F}C^\infty$, let

$$\Gamma_1(u,v)\equiv\tfrac{1}{2}\{\Delta(uv)-u\Delta v-v\Delta u\} \tag{4.54}$$

and

$$\Gamma_2(u,v)\equiv\tfrac{1}{2}\{\Delta\Gamma_1(u,v)-\Gamma_1(u,\Delta v)-\Gamma_1(v,\Delta u)\}, \tag{4.55}$$

where $\Delta\Gamma_1(u,v)=\Delta(\Gamma_1(u,v))$.

PROPOSITION 4.25. *For* $u, v \in \mathscr{F}C^\infty$,

$$\Gamma_1(u, v) = (\nabla u, \nabla v) = (\vec{\nabla} u, \vec{\nabla} v) \tag{4.56}$$

and

$$\Gamma_2(u, v) = (\nabla^2 u, \nabla^2 v) + \tfrac{1}{2}\{([\Delta, \nabla] u, \nabla v) + (\nabla u, [\Delta, \nabla] v)\}. \tag{4.57}$$

Note: $\nabla^2 u$ *is often called the Hessian of u.*

Proof. For (4.56) apply Proposition 4.23 with $S = u$ and $T = v$ to find, $\Delta(uv) = \Delta u \cdot v + 2(\nabla u, \nabla v) + u\Delta v$. This implies (4.56). To prove (4.57) apply Proposition 4.23 with $S = \nabla u$ and $T = \nabla v$ to find,

$$\Delta\Gamma_1(u, v) = \Delta(\nabla u, \nabla v) = (\Delta\nabla u, \nabla v) + 2(\nabla^2 u, \nabla^2 v) + (\nabla u, \Delta\nabla v).$$

This equation and the definition of Γ_2 implies (4.57). Q.E.D.

We now wish to compute Γ_2 more explicitly. For this we will need the infinite dimensional version of the Bochner Wietzenböck formula.

THEOREM 4.26 (Bochner Wietzenböck Formula). *Let* $u \in \mathscr{F}C^\infty$ *and* $k \in H_0$, *then*

$$([\Delta, \nabla] u) k = \operatorname{Ric}\langle k, \vec{\nabla} u\rangle,$$

where $\vec{\nabla} u$ *is defined in Definition* 4.22, Ric *is the Ricci tensor defined in Notation* 3.11 *and is computed in Theorem* 3.12.

Before giving the proof of this theorem we will state and prove the main theorem of this section.

THEOREM 4.27 (Gamma-2 Formula). *Let* $f \in \mathscr{F}C^\infty$ *and* $g \in \mathscr{L}$, *then*

$$\Gamma_2(f, f)(g) = \|(\nabla^2 f)_g\|^2 + \operatorname{Ric}\langle \vec{\nabla} f_g, \vec{\nabla} f_g\rangle. \tag{4.58}$$

(*See Eqs.* (3.16) *and* (3.17) *for formulae for the Ricci tensor.*) *Moreover, setting* $C(\langle \cdot, \cdot \rangle) \equiv \max_{|\xi|=1} K\langle \xi, \xi\rangle/\pi^2 < \infty$ *where K is the negative of the Killing form on* $\mathfrak{g}$, *then*

$$\Gamma_2(f, f) \geqslant -C\Gamma_1(f, f), \qquad \forall f \in \mathscr{F}C^\infty. \tag{4.59}$$

Proof. Equation (4.58) follows from Theorem 4.26 and Proposition 4.25. To prove (4.59), it suffices to show $\operatorname{Ric}\langle h, h\rangle \geqslant -C(h, h)$ for all $h \in H_0$. By Eq. (3.17)

$$|\mathrm{Ric}\langle h, h\rangle| = \int_0^1 K\langle h(s), h(s)\rangle\, ds - K\langle \bar{h}, \bar{h}\rangle$$

$$\leqslant \int_0^1 K\langle h(s), h(s)\rangle\, ds$$

$$\leqslant \lambda_o \int_0^1 |h(s)|^2\, ds,$$

where $\lambda_0 \equiv \max_{|\xi|=1} K\langle \xi, \xi\rangle$. Now by a standard isoparametric inequality,

$$\|h\|^2 \geqslant \pi^2 \int_0^1 |h(s)|^2\, ds, \qquad \forall h \in H_0.$$

(This is easily verified by writing h in a Fourier sine series of the form $h(s) = \sum_{n=1}^{\infty} h_n \sin(n\pi s)$ with $h_n \in \mathfrak{g}$.) Combining the two above displayed equations shows

$$|\mathrm{Ric}\langle h, h\rangle| \leqslant \frac{\lambda_o}{\pi^2}\|h\|^2.$$

Therefore $\mathrm{Ric}\langle h, h\rangle \geqslant -C(h, h)$ where $C = \lambda_0/\pi^2$ and $\lambda_0 \equiv \max_{|\xi|=1} K\langle \xi, \xi\rangle$.
Q.E.D.

Proof of Theorem 4.26. Let $f = F \circ \pi_{\mathscr{P}} \in \mathscr{F}C^{\infty}$ as in (4.2). We will start by showing that for all $k \in H_0$,

$$\nabla_k \Delta f = \sum_{h \in S_0} \nabla_k \nabla_h^2 f. \tag{4.60}$$

By Eqs. (4.37), (4.38), and (4.6), for all $g \in \mathscr{L}(G)$

$$(\nabla_k \Delta f)(g) = \sum_{A \in \mathfrak{g}_0} \sum_{i,j,l=1}^{n} G_0(s_i, s_j)(k(s_l))^{(l)} A^{(i)} A^{(j)} F)(g_{\mathscr{P}}).$$

Since the map $(A, B) \in \mathfrak{g}^2 \to G_0(s_i, s_j)(k(s_l)^{(l)} A^{(i)} B^{(j)} F)(g_{\mathscr{P}}) \in \mathbb{R}$ is linear for each fixed $i, j, l \in \{1, 2, ..., n\}$ and $g \in \mathscr{L}(g)$, we may apply Lemma 3.8 to find

$$\sum_{A \in \mathfrak{g}_0} G_0(s_i, s_j)(k(s_l))^{(l)} A^{(i)} A^{(j)} F)(g_{\mathscr{P}})$$

$$= \sum_{h \in S_0} (k(s_l))^{(l)} (h(s_i))^{(i)} (h(s_j))^{(j)} F)(g_{\mathscr{P}}).$$

Hence

$$(\nabla_k \Delta f)(g) = \sum_{h \in S_0} (\tilde{k}\tilde{h}\tilde{h}f)(g) = \sum_{h \in S_0} (\nabla_k \nabla_h^2 f)(g),$$

which is (4.60).

Now let S_0 be a good orthonormal basis of H_0, $k \in H_0$, and $f \in \mathscr{F}C^\infty$, then by the definition of Δ, Eq. (4.60), and Eq. (4.51),

$$\begin{aligned}
([\Delta, \nabla] f)k &= (\Delta \nabla f)k - \nabla_k \Delta f \\
&= \sum_{h \in S_0} (\nabla^3 f)(h \otimes h \otimes k) - \sum_{h \in S_0} \nabla_k \nabla_h^2 f \\
&= \sum_{h \in S_0} (\nabla^3 f)(h \otimes h \otimes k) - \sum_{h \in S_0} \nabla_k [(\nabla^2 f)(h \otimes h)] \\
&= \sum_{h \in S_0} \{(\nabla^3 f)(h \otimes h \otimes k - k \otimes h \otimes h) - (\nabla^2 f)\, \nabla_k (h \otimes h)\}.
\end{aligned}$$

Since ∇ has zero torsion (item 6 of Proposition 4.14), we have $\mathscr{R}f = 0$. Thus it follows by Lemma 4.16 that

$$(\nabla^3 f)(h \otimes h \otimes k) = (\nabla^3 f)(h \otimes k \otimes h).$$

By the definition of the curvature operator $\mathscr{R}$ and Remark 3.24,

$$(\nabla^3 f)((h \wedge k) \otimes h) = (\mathscr{R}(h, k)\, \nabla f)h = -(\nabla f)\, R(h, k)h = (\nabla f)\, R(k, h)h.$$

Combining the above three displayed equations gives

$$\begin{aligned}
([\Delta, \nabla] f)k &= \sum_{h \in S_0} \{(\nabla f)\, R(k, h)h - (\nabla^2 f)\, \nabla_k (h \otimes h)\} \\
&= \sum_{h \in S_0} \{(R(k, h)\, h, \vec{\nabla} f) - (\nabla^2 f)\, \nabla_k (h \otimes h)\} \\
&= \mathrm{Ric}\langle k, \vec{\nabla} f\rangle - \sum_{h \in S_0} (\nabla^2 f)\, \nabla_k (h \otimes h), \qquad (4.61)
\end{aligned}$$

where in the second equality we have used the definition of $\vec{\nabla} f$ and in the third equality we used the definition of Ricci tensor, see Notation 3.11. Notice that the last sum in (4.61) is necessarily convergent and independent of the choice of good basis S_0. The proof is completed using the following Lemma. Q.E.D.

LEMMA 4.28. *Let $f \in \mathscr{F}C^\infty$, $l \in H_0$, and S_0 be a good orthonormal basis of H_0, then*

$$\sum_{h \in S_0} (\nabla^2 f)\, \nabla_l (h \otimes h) = 0. \tag{4.62}$$

Remark 4.29. Formally (4.62) should be true because $\sum_{h \in S_0} h \otimes h$ corresponds to the identity operator on H_0 under the natural isomorphism of $H_0 \otimes H_0$ with operators on H_0 determined by $h \otimes k \to (l \in H_0 \to h(k, l) \in H_0)$. Using the product rules, one easily shows formally that the identity operator should be covariantly constant. Hence we expect $\sum_{h \in S_0} h \otimes h$ also to be covariantly constant. The problem with this argument is that $\sum_{h \in S_0} h \otimes h$ does not exist in $H_0 \otimes H_0$ or equivalently the identity operator on H_0 is not Hilbert Schmidt. This argument does of course work for finite dimensional Riemannian manifolds.

Proof of Lemma 4.28. Write $f = F \circ \pi_{\mathscr{P}}$ as in Eq. (4.2). Let $[\,\cdot\,]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ denote the unique linear map on $\mathfrak{g} \otimes \mathfrak{g}$ such that $[A \otimes B] = [A, B]$ for all $A, B \in \mathfrak{g}$. Also let $\langle \cdot, \cdot \rangle_{\mathfrak{g}^{\otimes 2}}$ denote the inner product on $\mathfrak{g} \otimes \mathfrak{g}$ determined by $\langle A \otimes B, C \otimes D \rangle_{\mathfrak{g}^{\otimes 2}} = \langle A, C \rangle \langle B, D \rangle$ for all $A, B, C, D \in \mathfrak{g}$.

Now for $p, q \in H_0$,

$$\begin{aligned}
(\nabla^2 f)_g\, (p \otimes q) &= (\tilde{p}\tilde{q} f)(g) - (\widetilde{\nabla_p q}\; f)(g) \\
&= \sum_{A, B \in \mathfrak{g}_0} \sum_{i, j=1}^{n} (A^{(i)} B^{(j)} F)(g_{\mathscr{P}}) \langle A \otimes B, p(s_i) \otimes q(s_j) \rangle_{\mathfrak{g}^{\otimes 2}} \\
&\quad - \sum_{A \in \mathfrak{g}_0} \sum_{i=1}^{n} (A^{(i)} F)(g_{\mathscr{P}}) \langle A, (\nabla_p q)(s_i) \rangle \\
&= \sum_{A, B \in \mathfrak{g}_0} \sum_{i, j=1}^{n} (A^{(i)} B^{(j)} F)(g_{\mathscr{P}}) \langle A \otimes B, \xi(s_i, s_j) \rangle_{\mathfrak{g}^{\otimes 2}} \\
&\quad - \sum_{A \in \mathfrak{g}_0} \sum_{i=1}^{n} (A^{(i)} F)(g_{\mathscr{P}}) \\
&\quad \times \left\langle A, \int_0^{s_i} [\dot{\xi}(t, t)]\, dt - s_i \int_0^1 [\dot{\xi}(t, t)]\, dt \right\rangle,
\end{aligned}$$

where $\xi(s, t) \equiv p(s) \otimes q(t)$, and $\dot{\xi}(s, t) \equiv \partial \xi(s, t)/\partial t = p(s) \otimes \dot{q}(t)$. By linearity, the above equation extends to arbitrary algebraic tensors $\xi \in H_0^{\otimes 2}$, i.e.,

$$(\nabla^2 f)_g \,\xi = \sum_{A,\,B \in \mathfrak{g}_0} \sum_{i,j=1}^{n} (A^{(i)} B^{(j)} F)(g_{\mathscr{P}}) \langle A \otimes B, \xi(s_i, s_j) \rangle_{\mathfrak{g}^{\otimes 2}}$$

$$- \sum_{A \in \mathfrak{g}_0} \sum_{i=1}^{n} (A^{(i)} F)(g_{\mathscr{P}})$$

$$\times \left\langle A, \int_0^{s_i} [\dot{\xi}(t,t)]\, dt - s_i \int_0^1 [\dot{\xi}(t,t)]\, dt \right\rangle. \tag{4.63}$$

By the observation at the end of the proof of Theorem 4.26, we may compute the sum in (4.62) using the good basis $S_0 = \mathfrak{h}\mathfrak{g}_0$, where $\mathfrak{h}$ is an orthonormal basis to $H_0(\mathbb{R})$ and $\mathfrak{g}_0$ is an orthonormal basis for $\mathfrak{g}$. Let $\mathfrak{h}_n \subset \mathfrak{h}$ be an increasing sequence of finite subsets of $\mathfrak{h}$ such that $\bigcup \mathfrak{h}_n = \mathfrak{h}$, $\Gamma_n \equiv \{aA \in S_0 \mid a \in \mathfrak{h}_n,\ A \in \mathfrak{g}_0\}$, and $\xi_n \equiv \sum_{h \in \Gamma_n} \nabla_k (h \otimes h) \in H_0^{\otimes 2}$. Our goal is to show that $\lim_{n \to \infty} (\nabla^2 f)_g \,\xi_n = 0$. According to (4.63), it suffices to show that

$$\lim_{n \to \infty} \xi_n(s,t) = 0 \text{ and } \lim_{n \to \infty} \int_0^t [\dot{\xi}_n(\tau,\tau)]\, d\tau = 0, \qquad \forall s, t \in [0,1]. \tag{4.64}$$

Let $K(s) \equiv \int_0^s k(r)\, dr$ and for $a \in H_0(\mathbb{R})$ set

$$J_a(s) \equiv \int_0^s a'(r)\, k(r)\, dr - s \int_0^1 a'(r)\, k(r)\, dr$$

$$= a(s)\, k(s) - \int_0^s a\, dk + s \int_0^1 a\, dk \qquad \text{(Integrate by Parts)} \tag{4.65}$$

then

$$\xi_n(s,t) \equiv \sum_{h \in \Gamma_n} \nabla_k (h \otimes h)|_{(s,t)}$$

$$= \sum_{a \in \mathfrak{h}_n} \sum_{A \in \mathfrak{g}_0} \{ [J_a(s), A] \otimes (a(t) A) + a(s) A \otimes [J_a(t), A] \}. \tag{4.66}$$

By Lemma 3.8, the sum $\sum_{a \in \mathfrak{h}} a(t)\, J_a(s)$ converges absolutely and satisfies

$$\sum_{a \in \mathfrak{h}} a(t)\, J_a(s) = G_0(t,s)\, k(s) - \int_0^s G_0(r,t)\, k'(r)\, dr + s \int_0^1 G_0(r,t)\, k'(r)\, dr$$

$$= \int_0^s \partial G_0(r,t)/\partial r \cdot k(r)\, dr - s \int_0^1 \partial G_0(r,t)/\partial r \cdot k(r)\, dr$$

$$= \int_0^s \{1_{r \leqslant t} - t\}\, k(r)\, dr - s \int_0^1 \{1_{r \leqslant t} - t\} \cdot k(r)\, dr$$

$$= \hat{K}(s,t),$$

where

$$\hat{K}(s,t) \equiv K(s \wedge t) - (tK(s) + sK(t)) + stK(1).$$

This shows

$$\lim_{n\to\infty} \xi_n(s,t) = \sum_{A\in\mathfrak{g}_0} \{[\hat{K}(s,t), A] \otimes A + A \otimes [\hat{K}(s,t), A]\} = 0 \tag{4.67}$$

where we have used Lemma 4.30 below to get the last equality.

For the second limit in (4.64), let $Q_n(t) \equiv \int_0^t [\dot{\xi}_n(\tau,\tau)]\, d\tau$ and $C_{ad} \equiv \sum_{A\in\mathfrak{g}_0} ad_A^2$. Then

$$\begin{aligned}
Q_n(t) &= \sum_{a\in\mathfrak{h}_n} \sum_{A\in\mathfrak{g}_0} \int_0^t \{a'(s)[[J_a(s), A], A] + a(s)[A, [J'_a(s), A]\}\, ds \\
&= \sum_{a\in\mathfrak{h}_n} \int_0^t \{a'(s)\, C_{ad} J_a(s) - a(s)\, C_{ad} J'_a(s)\}\, ds \\
&= C_{ad} \sum_{a\in\mathfrak{h}_n} \left\{a(t)\, J_a(t) - 2\int_0^t a(s)\, J'_a(s)\, ds\right\} \qquad \text{(integrate by parts)},
\end{aligned}$$

so that

$$\lim_{n\to\infty} Q_n(t) = C_{ad}\left\{\hat{K}(t,t) - 2\sum_{a\in\mathfrak{h}} \int_0^t a(s)\left(a'(s)\, k(s) - \int_0^1 a'(\tau)\, k(\tau)\, d\tau\right) ds\right\},$$

provided the sum converges. Now

$$\begin{aligned}
2\sum_{a\in\mathfrak{h}} \int_0^t a(s)\, a'(s)\, k(s)\, ds &= \sum_{a\in\mathfrak{h}} \int_0^t k(s)\, da^2(s) \\
&= \sum_{a\in\mathfrak{h}} \left\{k(t)\, a^2(t) - \int_0^t a^2(s)\, dk(s)\right\} \\
&= k(t)\, G_0(t,t) - \int_0^t G_0(s,s)\, dk(s) \\
&= \int_0^t k(s) \frac{d}{ds} G_0(s,s)\, ds \\
&= \int_0^t k(s)(1-2s)\, ds \\
&= K(t) - 2\int_0^t k(s)\, s\, ds,
\end{aligned}$$

where we have again used Lemma 3.8 in the third equality. Similarly we compute

$$2\sum_{a\in\mathfrak{h}}\int_0^t ds\, a(s)\int_0^1 d\tau\, a'(\tau)\,k(\tau) = -2\sum_{a\in\mathfrak{h}}\int_0^t ds\, a(s)\int_0^1 d\tau\, a(\tau)\,k'(\tau)$$

$$= -2\int_0^t ds\int_0^1 d\tau\, G_0(s,\tau)\,k'(\tau)$$

$$= 2\int_0^t ds\int_0^1 d\tau\, \partial G_0(s,\tau)/\partial\tau\cdot k(\tau)$$

$$= 2\int_0^t ds\int_0^1 d\tau\,\{1_{\tau\leqslant s} - s\}\cdot k(\tau)$$

$$= 2\int_0^t (t-\tau)\,k(\tau)\,d\tau - 2\int_0^t sK(1)\,ds$$

$$= 2tK(t) - 2\int_0^t sk(s)\,ds - t^2K(1).$$

Assembling the three above displayed equations shows

$$\lim_{n\to\infty} Q_n(t) = C_{ad}\left\{\hat{K}(t,t) - \left(K(t) - 2\int_0^t k(s)\,s\,ds\right)\right.$$

$$\left. + 2tK(t) - 2\int_0^t sk(s)\,ds - t^2K(1)\right\}$$

$$= C_{ad}\,\{(K(t) - 2tK(t) + t^2K(1)) - K(t) + 2tK(t) - t^2K(1)\} = 0.$$

Q.E.D.

LEMMA 4.30. *For all $B\in\mathfrak{g}$,*

$$\sum_{A\in\mathfrak{g}_0}\{ad_B A\otimes A + A\otimes ad_B A\}\equiv 0. \tag{4.68}$$

Proof. It is easily checked that $\sum_{A\in\mathfrak{g}_0} A\otimes A$ is independent of the choice of orthonormal basis $\mathfrak{g}_0$ of $\mathfrak{g}$. (Indeed, under the isomorphism $\mathfrak{g}\otimes\mathfrak{g}\to \mathrm{End}(\mathfrak{g})$ determined by $A\otimes B\to A\langle B,\cdot\rangle$, the sum $\sum_{A\in\mathfrak{g}_0} A\otimes A$ corresponds to the identity operator on $\mathfrak{g}$.) Since e^{tad_B} is an orthogonal transformation for all $B\in\mathfrak{g}$,

$$\sum_{A\in\mathfrak{g}_0}(e^{tad_B}A)\otimes(e^{tad_B}A) = \sum_{A\in\mathfrak{g}_0} A\otimes A, \qquad \forall t\in\mathbb{R}.$$

Equation (4.68) is proved by differentiating this least equation in t at $t=0$. Q.E.D.

5. LOGARITHMIC SOBOLEV INEQUALITY

In this section we will prove the first version of the logarithmic Sobolev inequality on $\mathscr{L}(G)$. We will first need some more notation.

DEFINITION 5.1. Let $\mathscr{P}=\{0<s_1<s_2<\cdots<s_n<1\}$ be a partition of $[0,1]$. Define:

1. $(\cdot,\cdot)_{\mathscr{P}}$ to be the unique left-invariant Riemannian metric on $G^{\mathscr{P}}$ such that

$$(A^{(i)},B^{(j)})_{\mathscr{P}}=\langle A,B\rangle\, Q_{ij} \qquad \forall A,B\in \mathfrak{g}, \tag{5.1}$$

where Q is the inverse of the matrix $\{G_0(s_i,s_j)\}_{i,j=1}^{n}$ and $A^{(i)}$ and $B^{(j)}$ are the vector fields on $G^{\mathscr{P}}$ defined in Eq. (4.7).

2. $\vec{\nabla}_{\mathscr{P}}$ to be the gradient operator on $G^{\mathscr{P}}$ relative to Riemannian metric $(\cdot,\cdot)_{\mathscr{P}}$.

3. For $F\in C^\infty(G^{\mathscr{P}})$, let

$$\Gamma^{\mathscr{P}}(F,F)\equiv \tfrac{1}{2}\{\Delta_{\mathscr{P}}(F^2)-2F\Delta_{\mathscr{P}}F\} \tag{5.2}$$

and

$$\Gamma_2^{\mathscr{P}}(F,F)\equiv \tfrac{1}{2}\{\Delta_{\mathscr{P}}\Gamma^{\mathscr{P}}(F,F)-2\Gamma^{\mathscr{P}}(F,\Delta_{\mathscr{P}}F)\}, \tag{5.3}$$

where $\Delta_{\mathscr{P}}$ was defined in Definition 4.18.

4. Let $\mathrm{Ric}_{\mathscr{P}}$ denote the Ricci tensor on $G^{\mathscr{P}}$ relative to the Riemannian metric $(\cdot,\cdot)_{\mathscr{P}}$.

In the sequel, $G^{\mathscr{P}}$ will be used to mean the Riemannian manifold $(G^n,(\cdot,\cdot)_{\mathscr{P}})$, where the metric, $(\cdot,\cdot)_{\mathscr{P}}$, is defined in (5.1). The next two results relate the geometry of $G^{\mathscr{P}}$ with that of $\mathscr{L}(G)$.

LEMMA 5.2. *Let $F,H\in C^\infty(G^{\mathscr{P}})$, then*

$$(\vec{\nabla}_{\mathscr{P}}F,\vec{\nabla}_{\mathscr{P}}H)_{\mathscr{P}}=\sum_{A\in\mathfrak{g}_0}\sum_{i=1}^{n}\sum_{j=1}^{n}G_0(s_i,s_j)\,A^{(j)}F\cdot A^{(i)}H, \tag{5.4}$$

$\Delta_{\mathscr{P}}$ is the Levi–Civita Laplacian on $G^{\mathscr{P}}$ relative to the Riemannian metric $(\cdot,\cdot)_{\mathscr{P}}$, and

$$\Gamma^{\mathscr{P}}(F,F)=(\vec{\nabla}_{\mathscr{P}}F,\vec{\nabla}_{\mathscr{P}}H)_{\mathscr{P}}. \tag{5.5}$$

Now suppose $g \in \mathcal{L}(G)$, $f = F \circ \pi_{\mathcal{P}} \in \mathcal{F}C^\infty$, *and* $g_{\mathcal{P}} = \pi_{\mathcal{P}}(g)$, *then*

$$(\Delta_{\mathcal{P}} F)(g_{\mathcal{P}}) = (\Delta f)(g), \tag{5.6}$$

$$\Gamma^{\mathcal{P}}(F, F)(g_{\mathcal{P}}) = \Gamma(f, f)(g) \tag{5.7}$$

(*or equivalently* $\|\vec{\nabla}_{\mathcal{P}} F(g_{\mathcal{P}})\|^2_{\mathcal{P}} = \|\vec{\nabla} f(g)\|^2$), *and*

$$\Gamma_2^{\mathcal{P}}(F, F)(g_{\mathcal{P}}) = \Gamma_2(f, f)(g). \tag{5.8}$$

Proof. The verification of Eq. (5.4) is an exercise in linear algebra that will be left to the reader. Recall that the Levi–Civita Laplacian may be characterized as the unique operator ($\Delta_{\mathcal{P}}$) on $C^\infty(G^{\mathcal{P}})$ satisfying

$$\int_{G^{\mathcal{P}}} (\Delta_{\mathcal{P}} F) \cdot H \, d\lambda_{\mathcal{P}} = -\int_{G^{\mathcal{P}}} (\vec{\nabla}_{\mathcal{P}} F, \vec{\nabla}_{\mathcal{P}} H)_{\mathcal{P}} \, d\lambda_{\mathcal{P}} \qquad \forall h \in C_c^\infty(G^{\mathcal{P}}), \tag{5.9}$$

where $\lambda_{\mathcal{P}}$ denotes the Riemannian volume element on $G^{\mathcal{P}}$ which is also a biinvariant Haar measure on $G^{\mathcal{P}}$. By the right-invariance of $\lambda_{\mathcal{P}}$, it follows that the first order differential operators $A^{(i)}$ are skew adjoint. Hence (5.9) is easy to verify using (4.37) and (5.4). It is straight forward to verify (5.5) using the definitions.

Equation (5.6) is the same as Eq. (4.38) in Proposition 4.19. Combining (5.6) and (5.2) gives

$$\begin{aligned}
\Gamma^{\mathcal{P}}(F, F)(g_{\mathcal{P}}) &\equiv \tfrac{1}{2}\{\Delta_{\mathcal{P}}(F^2) - 2F\Delta_{\mathcal{P}} F\}(g_{\mathcal{P}}) \\
&= \tfrac{1}{2}\{\Delta(f^2) - 2f\Delta f\}(g) \\
&= \Gamma(f, f)(g),
\end{aligned}$$

which verifies (5.7). The proof of (5.8) is completely analogous. Q.E.D.

COROLLARY 5.3. *Let* $C = C(\langle \cdot, \cdot \rangle) < \infty$ *be the constant in Theorem* 4.27. *Then for all partitions* $\mathcal{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ *of* $[0, 1]$, $\mathrm{Ric}_{\mathcal{P}} \geqslant -C(\cdot, \cdot)_{\mathcal{P}}$.

Proof. For $F \in C^\infty(G^{\mathcal{P}})$ let $F_{\mathcal{P}} \equiv F \circ \pi_{\mathcal{P}} \in \mathcal{F}C^\infty$. Let $\vec{g} \in G^{\mathcal{P}}$ and $v \in T_{g_{\mathcal{P}}} G^{\mathcal{P}}$. Choose $g \in \mathcal{L}(G)$ such that $g_{\mathcal{P}} = \vec{g}$. Then by Lemma 2.1 in [18]

$$\mathrm{Ric}_{\mathcal{P}}\langle v, v\rangle = \inf\{\Gamma_2^{\mathcal{P}}(F, F)(g_{\mathcal{P}}) : F \in C^\infty(G^{\mathcal{P}}) \ni \vec{\nabla}_{\mathcal{P}} F(\vec{g}) = v\}.$$

Therefore, using Eq. (5.2), Theorem 4.27, Eq. (5.11), and Eq. (5.5),

$$\begin{aligned}
\mathrm{Ric}_{\mathcal{P}}\langle v, v\rangle &= \inf\{\Gamma_2^{\mathcal{P}}(F, F)(g_{\mathcal{P}}) : F \in C^\infty(G^{\mathcal{P}}) \ni \vec{\nabla}_{\mathcal{P}} F(\vec{g}) = v\} \\
&= \inf\{\Gamma_2(F_{\mathcal{P}}, F_{\mathcal{P}})(g) : F \in C^\infty(G^{\mathcal{P}}) \ni \vec{\nabla}_{\mathcal{P}} F(\vec{g}) = v\}
\end{aligned}$$

$$\geqslant \inf\{-C\Gamma(F_{\mathscr{P}}, F_{\mathscr{P}})(g) \colon F \in C^\infty(G^{\mathscr{P}}) \ni \vec{\nabla}_{\mathscr{P}} F(\vec{g}) = v\}$$
$$= \inf\{-C\Gamma^{\mathscr{P}}(F, F)(g_{\mathscr{P}}) \colon F \in C^\infty(G^{\mathscr{P}}) \ni \vec{\nabla}_{\mathscr{P}} F(\vec{g}) = v\}$$
$$= -C(v, v)_{\mathscr{P}}. \qquad \text{Q.E.D.}$$

DEFINITION 5.4. Let $f = F \circ \pi_{\mathscr{P}}$ where $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ is a partition of $[0, 1]$ and $F \in C^\infty(G^{\mathscr{P}})$ is an exponentially bounded function, see Definition 2.4. For $g \in \mathscr{L}(G)$, define

$$(e^{t\Delta} f)(g) \equiv \int_{G^{\mathscr{P}}} F(g_{\mathscr{P}} y)\, v_t^{\mathscr{P}}(y)\, d\lambda_{\mathscr{P}}(y), \tag{5.10}$$

where $v_t^{\mathscr{P}}$ is the convolution heat kernel density described in Proposition 2.5 when $M = G^{\mathscr{P}}$ with metric $(\cdot, \cdot)_{\mathscr{P}}$ and $\lambda_{\mathscr{P}}$ is the Riemannian volume measure on $G^{\mathscr{P}}$.

PROPOSITION 5.5. *The definition of $e^{t\Delta}$ in Definition 5.4 is well defined*

Proof. Suppose $\mathscr{P} \subset \tilde{\mathscr{P}}$ are two partitions of $[0, 1]$ and that $f = F \circ \pi_{\mathscr{P}}$, where F is a smooth bounded function on $G^{\mathscr{P}}$ with bounded first and second derivatives. Let $\pi\colon G^{\tilde{\mathscr{P}}} \to G^{\mathscr{P}}$ denote the canonical projection, then $\pi_{\mathscr{P}} = \pi_{\tilde{\mathscr{P}}} \circ \pi$ and $f = \tilde{F} \circ \pi_{\tilde{\mathscr{P}}}$ where $\tilde{F} = F \circ \pi$. Since $f = \tilde{F} \circ \pi_{\tilde{\mathscr{P}}} = F \circ \pi_{\mathscr{P}}$, it follows from (4.38) that $\Delta f = (\Delta_{\tilde{\mathscr{P}}} \tilde{F}) \circ \pi_{\tilde{\mathscr{P}}} = (\Delta_{\mathscr{P}} F) \circ \pi$. In particular,

$$\Delta_{\tilde{\mathscr{P}}}(F \circ \pi) = \Delta_{\mathscr{P}} F. \tag{5.11}$$

This identity may also be checked directly.

Set $u(t, x) \equiv \int_{G^{\mathscr{P}}} F(xy)\, v_t^{\mathscr{P}}(y)\, d\lambda_{\mathscr{P}}(y)$ for $t \geqslant 0$ and $x \in G^{\mathscr{P}}$. Then by Corollary 2.15, u solves the heat equation

$$\partial u/\partial t = \Delta_{\mathscr{P}} u \qquad \text{with} \quad u(0, x) = F(x).$$

Let $\tilde{u}(t, \cdot) \equiv u(t, \cdot) \circ \pi$ for all $t \geqslant 0$. Using (5.11), $\tilde{u}$ is seen to solve the heat equation:

$$\partial \tilde{u}/\partial t = \Delta_{\tilde{\mathscr{P}}} \tilde{u} \qquad \text{with} \quad \tilde{u}(0, \cdot) = \tilde{F}(\cdot).$$

Let $\tilde{x} \in G^{\tilde{\mathscr{P}}}$, then by Corollary 2.16 of the Section 2,

$$u(t, \pi(\tilde{x})) = \tilde{u}(t, \tilde{x}) = \int_{G^{\tilde{\mathscr{P}}}} \tilde{F}(\tilde{x}\tilde{y})\, v_t^{\tilde{\mathscr{P}}}(\tilde{y})\, d\lambda_{\tilde{\mathscr{P}}}(\tilde{y}).$$

Hence

$$\int_{G^{\mathscr{P}}} F(\pi(\tilde{x}) y)\, v_t^{\mathscr{P}}(y)\, d\lambda_{\mathscr{P}}(y) = \int_{G^{\tilde{\mathscr{P}}}} \tilde{F}(\tilde{x}\tilde{y})\, v_t^{\tilde{\mathscr{P}}}(\tilde{y})\, d\lambda_{\tilde{\mathscr{P}}}(\tilde{y}).$$

Taking $\tilde{x}=\pi_{\tilde{\mathscr{P}}}(g)=g_{\tilde{\mathscr{P}}}$ so that $\pi(\tilde{x})=\pi_{\mathscr{P}}(g)=g_{\mathscr{P}}$ shows that

$$\int_{G^{\mathscr{P}}} F(g_{\mathscr{P}}y)\, \nu_t^{\mathscr{P}}(y)\, d\lambda_{\mathscr{P}}(y)=\int_{G^{\tilde{\mathscr{P}}}} \tilde{F}(g_{\tilde{\mathscr{P}}}\,\tilde{y})\, \nu_t^{\tilde{\mathscr{P}}}(y)\, d\lambda_{\tilde{\mathscr{P}}}(\tilde{y}),$$

and hence Definition (5.4) is well defined independent of how f is represented. Q.E.D.

We are now ready to state the first version of the logarithmic Sobolev inequality on $\mathscr{L}(G)$ describe in Theorem 1.2.

THEOREM 5.6 (Logarithmic Sobolev). *Let $C=C(\langle\cdot,\cdot\rangle)$ be the constant in Theorem 4.27. Then for all $f\in\mathscr{F}C_b^\infty$ and $T\in(0,\infty)$,*

$$e^{t\Delta/2}(f^2\log f^2)\leqslant\frac{2}{C}(e^{CT}-1)\, e^{t\Delta/2}(\|\vec{\nabla} f\|^2)+e^{t\Delta/2}(f^2)\cdot e^{t\Delta/2}(\log f^2), \tag{5.12}$$

where $(2/C)(e^{CT}-1)\equiv 2T$ if $C=0$.

Proof. Let $\mathscr{P}$ be a partition of $[0,1]$ and $f=F\circ\pi_{\mathscr{P}}\in\mathscr{F}C_b^\infty$. In view of Eqs. (5.10) and (5.7), Eq. (5.12) follows by applying Theorem 2.9 to Riemannian manifold $(G^{\mathscr{P}},(\cdot,\cdot)_{\mathscr{P}})$ and using Corollary 5.3 to check that $\mathrm{Ric}_{\mathscr{P}}\geqslant -C(\cdot,\cdot)_{\mathscr{P}}$. Q.E.D.

6. HEAT KERNEL MEASURE

In this section, we will show that $e^{t\Delta/2}$ may be represented as a probability kernel on $\mathscr{L}(G)$. Once this is done, Theorem 5.6 may be written in the form described in Theorem 1.2 of the introduction, see Theorem 6.4 below.

DEFINITION 6.1. Let $\mathscr{G}$ denote the smallest σ-algebra on $\mathscr{L}(G)$ such that $\pi_{\mathscr{P}}\colon\mathscr{L}(G)\to G^{\mathscr{P}}$ is measurable for all finite partitions $\mathscr{P}$ on $[0,1]$. (The σ-algebra on $G^{\mathscr{P}}$ is taken to be the Borel σ-algebra.)

THEOREM 6.2 (Heat Kernel Measure). *For each $T>0$ there exists a unique probability measure ν_T on $(\mathscr{L}(G),\mathscr{G})$ such that*

$$(e^{t\Delta/2}f)(\mathbf{e})=\int_{\mathscr{L}(G)} f(g)\, d\nu_T(g) \tag{6.1}$$

for all bounded $f\in\mathscr{F}C^\infty$.

The proof of this theorem will be given after Theorem 6.4 below.

Corollary 6.3. *For each $T>0$ and $g_0 \in \mathscr{L}(G)$ there exists a unique probability measure $\nu_T(g_0, \cdot)$ on $(\mathscr{L}(G), \mathscr{G})$ such that*

$$(e^{t\Delta/2}f)(g_0) = \int_{\mathscr{L}(G)} f(g)\, \nu_T(g_0, dg) \tag{6.2}$$

for all bounded cylinder functions f on $\mathscr{L}$. Moreover, $\nu_T(g_0, \cdot) = L_{g_0} * \nu_T = \nu_T \circ L_{g_0}^{-1}$, where $L_{g_0}: \mathscr{L}(G) \to \mathscr{L}(G)$ denotes left translation by g_0.

Proof. The proof of uniqueness is routine and will be left to the reader. To prove existence of the measure $\nu_T(g_0, \cdot)$, notice by the left translation invariance of Δ (more precisely the left translation invariance of $\Delta_{\mathscr{P}}$ for all partitions $\mathscr{P}$), it follows that

$$(e^{t\Delta/2}f)(g_0) = (e^{t\Delta/2}f)(L_{g_0}(\mathbf{e})) = (e^{t\Delta/2}(f \circ L_{g_0}))(\mathbf{e}),$$

where $\mathbf{e}$ is the identity in $\mathscr{L}$, i.e. the constant loop at the identity in G. Therefore by the definition of ν_T,

$$(e^{t\Delta/2}f)(g_0) = \int_{\mathscr{L}(G)} f(L_{g_0}(g))\, d\nu_T(g) = \int_{\mathscr{L}(G)} f(g)(L_{g_0} * \nu_T)(dg).$$

Hence $\nu_T(g_0, \cdot) \equiv L_{g_0} * \nu_T$ is the desired measure. Q.E.D.

For completeness we now restate Theorem 5.6 of the last section in the form of Theorem 1.2.

Theorem 6.4 (Logarithmic Sobolev). *Let $C = C(\langle \cdot, \cdot \rangle)$ be the constant in Theorem 4.27, $g_0 \in \mathscr{L} = \mathscr{L}(G)$, $f \in \mathscr{F}C^\infty$ be such that f and $\|\vec{\nabla} f\|$ are bounded, and $T \in (0, \infty)$. Then*

$$\int_{\mathscr{L}} f^2 \log f^2\, d\mu \leqslant \frac{2}{C}(e^{CT} - 1) \int_{\mathscr{L}} \|\vec{\nabla} f\|^2\, d\mu + \int_{\mathscr{L}} f^2\, d\mu \cdot \int_{\mathscr{L}} \log f^2\, d\mu, \tag{6.3}$$

where μ is the measure $\nu_T(g_0, \cdot)$ and $(2/C)(e^{CT} - 1) \equiv 2T$ if $C = 0$. As usual $0 \log 0 \equiv 0$.

Proof of Theorem 6.2. Using Proposition 5.5, one may apply Komogorov's extension theorem to show that there is a measure $\bar{\nu}_T$ on $\bar{G}^{(0,1)}$ (with the product σ-algebra) such that

$$(e^{t\Delta/2}f)(e) = \int_{\bar{G}^{(0,1)}} f(g)\, d\bar{\nu}_T(g) \tag{6.4}$$

for all bounded $f \in \mathscr{F}C^\infty$, where $\bar{G} = G \cup \{*\}$ denotes the one point compactification of G. As usual, for any $F: G^{\mathscr{P}} \to \mathbb{R}$ we extend F to $\bar{G}^{\mathscr{P}}$ by setting $F(g) \equiv 0$ if $g(s_i) = *$ for some $s_i \in \mathscr{P}$. In this way we may consider $f \in \mathscr{F}C^\infty$ as functions on $\bar{G}^{(0,1)}$.

To finish the proof we must now show that $\bar{\nu}_T$ "restricts" to a measure on $\mathscr{L}(G)$. Let d: $G \times G \to [0, \infty)$ be the metric on G induced by the biinvariant Riemannian metric on G determined by $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, see Definition 2.2. By Lemmas 6.6 and 6.9 below, for any $p \geqslant 1$, there is a finite constant K_p such that for all $u, v \in (0, 1)$,

$$\int_{\bar{G}^{(0,1)}} d(g(v), e)^p \, d\bar{\nu}_T \leqslant K_p (v(1-v))^{p/2}, \tag{6.5}$$

and

$$\int_{\bar{G}^{(0,1)}} d(g(u), g(v))^p \, d\bar{\nu}_T \leqslant K_p \, |u-v|^{p/2}, \tag{6.6}$$

where $d(*, *) \equiv 0$ and $d(x, *) \equiv \infty$ if $x \in G$. For $u \in (0, 1)$ define $\tilde{\Sigma}_u$: $\bar{G}^{(0,1)} \to G$ by

$$\tilde{\Sigma}_u(g) = \begin{cases} g(u) & \text{if} \quad g(u) \neq * \\ e & \text{if} \quad g(u) = * \end{cases}$$

and set $\tilde{\Sigma}_0(g) = \tilde{\Sigma}_1(g) = e$. Then (6.5) and (6.6) may be combined to show that there is a constant $K_p < \infty$ such that

$$\int_{\bar{G}^{(0,1)}} d(\tilde{\Sigma}_u(g), \tilde{\Sigma}_v(g))^p \, d\bar{\nu}_T \leqslant K_p \, |u-v|^{p/2} \qquad \forall u, v \in [0, 1].$$

Kolmogorov's continuity criteria (see Protter [27] Theorem 53, p. 171 for example) shows that the process $\tilde{\Sigma}$ has a continuous version Σ. Notice that $\Sigma(g) \in \mathscr{L}(G)$ for $\bar{\nu}_T$ almost every g. The desired measure ν_T on $\mathscr{L}(G)$ is the law of Σ, i.e. ν_T is uniquely determined by

$$\int_{\mathscr{L}(G)} f \, d\nu_T = \int_{\bar{G}^{(0,1)}} f(\Sigma) \, d\bar{\nu}_T,$$

where f is an arbitrary bounded cylinder function. Q.E.D.

An alternative proof Theorem 6.2 in the case that $G = K$ is compact, may be given using Proposition 1.1 in Malliavin [25] which asserts the existence of a "Brownian motion" on $\mathscr{L}(G)$ starting at the identity loop. The time T distribution of this process is ν_T. In the case that $G = Z$ is abelian, one may easily construct a "Brownian" motion on $\mathscr{L}(G)$ by taking a Brownian motion on $\mathscr{L}(\mathfrak{g})$ with variance determined by the $H_0(\mathfrak{g})$-norm and composing this Brownian motion with the exponential map on G. Again ν_T is the time T distribution of this Brownian motion. For general compact Lie groups, one may decompose G as $G=KZ$, where K and Z are Lie subgroups of G, K is compact, and Z is in the center of G. The product of an $\mathscr{L}(K)$-valued Brownian motion with an $\mathscr{L}(Z)$-valued Brownian motion gives an $\mathscr{L}(G)$-valued Brownian motion. The time T distribution of this $\mathscr{L}(G)$-valued Brownian motion is the desired measure ν_T. Complete details of this construction may be found in Driver [16], where the $\mathscr{L}(G)$-valued Brownian motion is used to prove Remark 1.3 in Section 1.

Remark 6.5. The Brownian motion construction of ν_T has the added benefit of showing that ν_T is concentrated on the homotopy class containing the constant loop at the identity. Since, pinned Wiener measure charges all homotopy classes, it follows, for non-simply connected groups, that ν_T is different then any pinned Wiener measure. The exact relationship between ν_T and pinned Wiener measures on $\mathscr{L}(G)$ is still an open question.

LEMMA 6.6. *For each $p\in(0,\infty)$ there is a constant K_p such that*

$$\int_{\bar{G}(0,1)} d(g(v),e)^p \, d\bar{\nu}_T \leqslant K_p(v(1-v))^{p/2}, \qquad \forall v\in(0,1). \tag{6.7}$$

Proof. Let $v\in(0,1)$ and $\mathscr{P}=\{0<v<1\}$ then for any non-negative measurable function f on G we have

$$\int_{\bar{G}(0,1)} f(g(v))\, d\bar{\nu}_T = (e^{T\Delta_{\mathscr{P}}/2} f)(e). \tag{6.8}$$

By Definition 4.8

$$\Delta_{\mathscr{P}} = G_0(v,v)\,\Delta_G = v(v-1)\,\Delta_G,$$

where $\Delta_G \equiv \sum_{A\in\mathfrak{g}_0} \tilde{A}^2$. Hence Eq. (6.8) may be written as

$$\int_{\bar{G}(0,1)} f(g(v))\, d\bar{\nu}_T(g) = \int_G f(x)\,\rho_t(x)\,dx, \tag{6.9}$$

where $t\equiv G_0(v,v)\,T$ and ρ_t is the convolution heat kernel density for $e^{t\Delta_{\mathscr{P}}/2}$. Let $V(r)$ be the volume of the ball of radius r centered at $e\in G$, relative to the metric $\langle\cdot,\cdot\rangle$ on G. By Lemma 2.6, there is a constant $\gamma\in(0,\infty)$ such that $V(r)\leqslant \gamma r^n e^{\gamma r}$. We may use (6.9) and the heat kernel estimate in Proposition 2.5 to find (for any $\varepsilon>0$) that

$$\int_{\bar{G}(0,1)} d(g(v), e)^p \, d\bar{v}_T \leqslant C(T, \varepsilon) \, t^{-n/2} \int_0^\infty r^p \exp\left\{\frac{-r^2}{2(1+\varepsilon)\,t}\right\} dV(r)$$

$$= -C(T, \varepsilon)\, t^{-n/2} \int_0^\infty \{pr^{p-1} - r^{p+1}/(1+\varepsilon)\,t\}$$

$$\times \exp\left\{\frac{-r^2}{2(1+\varepsilon)\,t}\right\} V(r)\, dr$$

$$\leqslant Kt^{-n/2} \int_0^\infty \frac{r^{p+1}}{(1+\varepsilon)\,t} r^n \exp\left\{\frac{-r^2}{2(1+\varepsilon)\,t}\right\} e^{\gamma r}\, dr,$$

where $K \equiv C(T, \varepsilon)\gamma$. Now an easy scaling argument shows that

$$\int_0^\infty r^k e^{-r^2/\alpha} e^{\gamma r}\, dr = \alpha^{(k+1)/2} \int_0^\infty r^k e^{-r^2} e^{\sqrt{\alpha}\,\gamma r}\, dr.$$

Hence it follows from the two above displayed equations that there is a constant $K_p = K_p(T, \varepsilon)$ such that

$$\int_{\bar{G}(0,1)} d(g(v), e)^p \, d\bar{v}_T \geqslant \leqslant K_p t^{p/2} = K_p(v(1-v))^{p/2}. \qquad \text{Q.E.D.}$$

The next two Lemmas will be used in the proof of Lemma 6.9 which was the key to the proof of Theorem 6.2. The following notation will be used in the next three Lemmas. Let $u, v \in (0, 1)$ with $u < v$ and let $\mathscr{P} = \{0 < u < v < 1\}$. We will let $\|\cdot\|^2 = (\cdot, \cdot)_{\mathscr{P}}$ so that for $\{A, B\} \in \mathfrak{g} \times \mathfrak{g}$,

$$\|\{A, B\}\|^2 = a\,|A|^2 - 2b\langle A, B\rangle + c\,|B|^2 \qquad \forall A, B \in \mathfrak{g},$$

where $a, b, c \in \mathbb{R}$ are determined by

$$\begin{bmatrix} a & -b \\ -b & c \end{bmatrix} = \begin{bmatrix} G_0(u, u) & G_0(u, v) \\ G_0(u, v) & G_0(v, v) \end{bmatrix}^{-1},$$

see Definition 5.1. Thus $a = G_0(v, v)/\delta$, $b = G_0(u, v)/\delta$, and $c = G_0(u, u)/\delta$, where

$$\delta \equiv G_0(u, u)\, G_0(v, v) - G_0^2(u, v) = u(1-v)(v-u).$$

LEMMA 6.7. *Let* $u, v \in (0, 1)$ *with* $u < v$, $\beta \equiv G_0(u, v)/G_0(v, v) < 1$, $A, B \in \mathfrak{g}$, *and* $\|\cdot\| = \sqrt{(\cdot, \cdot)_{\mathscr{P}}}$. *Then*

$$|A - \beta B| \leqslant \sqrt{u(v-u)/v}\, \|\{A, B\}\|, \qquad (6.10)$$

and

$$|B| \leqslant \sqrt{v(1-v)}\ \|\{A, B\}\| \leqslant \tfrac{1}{2}\|\{A, B\}\|. \tag{6.11}$$

Proof. By completing the squares we have

$$\begin{aligned}
\|\{A, B\}\|^2 &= a\left(\left|A - \frac{b}{a}B\right|^2 + \left(\frac{c}{a} - \frac{b^2}{a^2}\right)|B|^2\right) \\
&= a\left|A - \frac{b}{a}B\right|^2 + \frac{ac-b^2}{a}|B|^2 \\
&= \frac{G_0(v, v)}{\delta}\left|A - \frac{G_0(u, v)}{G_0(v, v)}B\right|^2 + \frac{1}{G_0(v, v)}|B|^2. \\
&= \frac{\alpha}{\delta}|A - \beta B|^2 + \frac{1}{\alpha}|B|^2,
\end{aligned} \tag{6.12}$$

where $\alpha = G_0(v, v) = v(1-v)$. Now $\alpha/\delta = (v/u)/(v-u)$, and $\beta = u/v < 1$. Hence it follows from (6.12) that

$$|A - \beta B| \leqslant \sqrt{u(v-u)/v}\ \|\{A, B\}\|$$

and

$$|B| \leqslant \sqrt{v(1-v)}\ \|\{A, B\}\| \leqslant \tfrac{1}{2}\|\{A, B\}\|. \qquad \text{Q.E.D.}$$

LEMMA 6.8. *Keep the same notation as in Lemma 6.7. Let d respectively $d_{\mathscr{P}}$ be the Riemannian distance function on G respectively $G^{\mathscr{P}} = G^2$ relative to $\langle\cdot,\cdot\rangle$ respectively $(\cdot,\cdot)_{\mathscr{P}}$. We will also set $|x| \equiv d(x, e)$ and $|\{x, y\}| \equiv d_{\mathscr{P}}(\{x, y\}, \{e, e\})$ for all $x, y \in G$. Then for $x, y \in G$,*

$$d(x, y) = |x^{-1}y| \leqslant 2\sqrt{|v-u|}\cdot|\{x, y\}|. \tag{6.13}$$

Proof. Let $x, y \in G$, σ: $[0, 1] \to G$ and τ: $[0, 1] \to G$ be two smooth paths such that $\sigma(0) = \tau(0) = e$, $\sigma(1) = x$, and $\tau(1) = y$. Since $\sigma\tau^{-1}$: $[0, 1] \to G$ is a path joining e to xy^{-1}, it follows that $|xy^{-1}| \leqslant \int_0^1 |(\sigma\tau^{-1})'(s)|\, ds$. Define $A \equiv \theta\langle\sigma'\rangle$ and $B \equiv \theta\langle\tau'\rangle$, then

$$(\tau^{-1})' = \iota_* L_{\tau *} B = \frac{d}{dt}\Big|_0 \iota(\tau e^{tB}) = \frac{d}{dt}\Big|_0 e^{-tB}\tau^{-1} = -R_{\tau^{-1}*}B,$$

where $\iota(g) \equiv g^{-1}$. Therefore

$$\begin{aligned}\theta\langle(\sigma\tau^{-1})'\rangle &= L_{\tau\sigma^{-1}*}\{R_{\tau^{-1}*}\sigma' + L_{\sigma*}(\tau^{-1})'\}\\ &= L_{\tau\sigma^{-1}*}\{R_{\tau^{-1}*}L_{\sigma*}A - L_{\sigma*}R_{\tau^{-1}*}B\}\\ &= Ad_\tau(A-B).\end{aligned}$$

To simplify notation, let $\beta = u/v \in (0, 1)$ and $l(\sigma, \tau)$ denote the length of the curve (σ, τ) in the Riemannian manifold $G^{\mathscr{P}}$. Using the orthogonality of Ad_τ, Eq. (6.10) and Eq. (6.11), it follows that

$$\begin{aligned}|xy^{-1}| &\leqslant \int_0^1 |Ad_\tau(A-B)|\, ds = \int_0^1 |A-B|\, ds\\ &\leqslant \int_0^1 [\,|A-\beta B| + (1-\beta)|B|\,]\, ds\\ &\leqslant \int_0^1 [\,|A-\beta B| + (1-\beta)|B|\,]\, ds\\ &\leqslant \int_0^1 (\sqrt{(u/v)(v-u)} + (1-\beta)\sqrt{v(1-v)})\|\{A, B\}\|\, ds\\ &\leqslant \left(\sqrt{(u/v)(v-u)} + \frac{(v-u)}{\sqrt{v}}\right) l(\sigma, \tau)\\ &\leqslant 2\sqrt{v-u}\, l(\sigma, \tau),\end{aligned}$$

wherein we have used $1-\beta = 1-u/v = (v-u)/v$. Minimizing this last inequality over all σ joining e to x and all τ from e to y shows that

$$|xy^{-1}| \leqslant 2\sqrt{v-u}\, |\{x, y\}|.$$

Eq. (6.13) follows by replacing x by x^{-1} and y by y^{-1} in the above inequality and using $|\{x^{-1}, y^{-1}\}| = |\{x, y\}|$. Q.E.D.

LEMMA 6.9. *Let $u, v \in (0, 1)$ with $u < v$. Then for each $p \in [1, \infty)$ there is a constant K_p such that*

$$\int_{\bar{G}(0,1)} d(g(u), g(v))^p\, d\bar{v}_T(g) \leqslant K_p\, |u-v|^{p/2} \qquad \forall u, v \in (0, 1). \quad (6.14)$$

Proof. Let $\mathscr{P}=\{0<u<v<1\}$, $\|\cdot\|=\|\cdot\|_{\mathscr{P}}$ be as above. Let $\lambda_{\mathscr{P}}$ denote the Riemannian volume element on $G^{\mathscr{P}}$ and $v_T^{\mathscr{P}}(x, y)$ denote the convolution heat kernel on $G^{\mathscr{P}}=G^2$ determined by

$$\int_{G^2} f(x, y)\, v_T^{\mathscr{P}}(x, y)\, d\lambda_{\mathscr{P}}(x, y) := (e^{T\Delta_{\mathscr{P}}/2} f)(e, e),$$

for all $f \in C_c^\infty(G^2)$. Then for any non-negative measurable function f on G^2 we have

$$\int_{\bar{G}(0,1)} f(g(u), g(v))\, d\bar{v}_T(g) = \int_{G^2} f(x, y)\, v_T^{\mathscr{P}}(x, y)\, d\lambda_{\mathscr{P}}(x, y).$$

Taking $f(x, y)=d^p(x, y)$, using Eq. (6.13) and the heat kernel estimate in Eq. (2.8), there is a constant $K_p=K_p(T, \varepsilon)<\infty$ such that

$$\begin{aligned}
&\int_{\bar{G}(0,1)} d^p(g(u), g(v))\, d\bar{v}_T(g) \\
&\quad\leqslant (2\sqrt{v-u})^p \int_{G^{\mathscr{P}}} |\{x, y\}|^p\, v_T^{\mathscr{P}}(\{x, y\})\, d\lambda_{\mathscr{P}}(\{x, y\}) \\
&\quad\leqslant K_p\, |v-u|^{p/2} \int_0^\infty r^p \exp\left\{\frac{-r^2}{2(1+\varepsilon)T}\right\} dV(r),
\end{aligned} \tag{6.15}$$

where $V(r)=\lambda_{\mathscr{P}}(\{\{x, y\} \in G^2 : |\{x, y\}| \leqslant r\})$. Now by Corollary 5.3 and Lemma 2.6 there are constant $c>0$ and $K<\infty$ independent of u and v such that $V(r) \leqslant K r^{2n} e^{cr}$. Using the same methods as in the proof of Lemma 6.6, it is easily seen that

$$\int_0^\infty r^p \exp\left\{\frac{-r^2}{2(1+\varepsilon)T}\right\} dV(r) \leqslant A_p < \infty, \tag{6.16}$$

where A_p is a constant independent of $u, v \in (0, 1)$. Combining (6.15) and (6.16) proves (6.14). Q.E.D.

Remark 6.10. With some extra effort, it is possible to use similar methods to generalize Theorem 6.2 to the case where G is an arbitrary Lie group and $\langle \cdot, \cdot \rangle$ is an arbitrary left invariant Riemannian metric on G.

REFERENCES

1. Shigeki Aida and David Elworthy, Differential calculus on path and loop spaces, 1. Logarithmic Sobolev inequalities on path and loop spaces, *C.R. Acad. Sci. Paris Ser. I* **321** (1995), 97–102.
2. D. Bakry, Transformations de Riesz pour les semigroupes symétriques, *in* "Séminaire de Probabilités XIX, Lecture Notes in Math.," Vol. 1123, pp. 130–174, Springer-Verlag, Berlin/New York, 1985.
3. D. Bakry, Un critère de non-explosion pour certaines diffusions sur une variété riemannienne complète, *Compte-Rendus* **303** (1986), 23–26.
4. D. Bakry, Etude des transformations de Riesz sur les variétés riemanniennes à courbure de Ricci minorée, *in* "Séminaire de Probabilités XXI, Lecture Notes in Math.," Vol. 1247, pp. 137–172, Springer-Verlag, Berlin/New York, 1987.
5. D. Bakry, La propriété de sous-harmonicité des diffusions dans les variétés, *in* "Séminaire de Probabilités XXII, Lecture Notes in Math.," Vol. 1321, pp. 1–50, Springer-Verlag, Berlin/New York, 1988.
6. D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes, *in* "St-Flour Lecture notes in Lectures on Probability Theory, Lecture Notes in Math.," Vol. 1581, Springer-Verlag, Berlin/New York, 1994.
7. D. Bakry, On Sobolev and logarithmic Sobolev inequalities for Markov semigroups, preprint, 1995.
8. D. Bakry and M. Emery, Hypercontractivitie de semi-groups de diffusion, *Compte-Rendus* **299** (1984), 775–778.
9. D. Bakry and M. Emery, Diffusions hypercontractivites, *in* "Séminaire de Probabilites XIX, Lecture Notes in Math." (J. Azema and M. Yor, Eds.), Vol. 1123, pp. 177–207, Springer-Verlag, New York, 1985.
10. D. Bakry and M. Ledoux, Private communication in November of 1994 at the 1994 Taniguchi Symposium, held in England.
11. D. Bakry and M. Ledoux, Lévy–Gromov isoperimetric inequality for an infinite dimensional diffusion generator, preprint, Summer 1995.
12. Isac Chavel, "Eigenvalues in Riemannian Geometry," Academic Press, Orlando/San Diego, 1984.
13. Isac Chavel, "Riemannian Geometry—A Modern Introduction," Cambridge Tract in Mathematics, Cambridge Univ. Press, New York/Melbourne, 1993.
14. E. B. Davies, "Heat Kernels and Spectral Theory," Cambridge Univ. Press, Cambridge, UK, 1990.
15. B. K. Driver, On the Kakutani–Itô–Segal–Gross and the Segal–Bargmann–Hall isomorphisms, *J. Funct. Anal.* **133** (1995), 69–128.
16. B. K. Driver, An integration by parts formula on the path space of a loop group, in preparation.
17. B. K. Driver and L. Gross, Hilbert spaces of holomorphic functions on complex Lie groups, *in* "Proceedings of the 1994 Taniguchi Symposium," in press.
18. B. K. Driver and Yaozhong Hu, On the heat kernel logarithmic Sobolev inequalities, *in* "Proceedings of the Fifth Gregynog Symposium" (I. M. Davies, A. Truman, and K. D. Elworthy, Eds.), Stochastic Analysis and Applications, pp. 189–200, World Scientific, N.J., 1996.
19. S. Fang, Inégalité du type de Poincaré sur un espace de chemins, Univ. de Paris VI preprint, October 1993.
20. D. S. Freed, The geometry of loop groups, *J. Diff. Geom.* **28** (1988), 223–276.
21. E. Getzler, Dirichlet forms on loop space, *Bull. Soc. Math. (2)* **113** (1989), 151–174.
22. E. Getzler, An extension of Gross's log-Sobolev inequality for the loop space of a compact Lie group, *in* "Proc. Conf. on Probability Models in Mathematical Physics, Colorado

Springs, 1990" (G. J. Morrow and W-S. Yang, Eds.), pp. 73–97, World Scientific, N.J., 1991.

23. L. Gross, Logarithmic Sobolev inequalities on loop groups, *J. Funct. Anal.* **102** (1992), 268–313.
24. Elton P. Hsu, Inégalités de Sobolev logarithmic sur un space de chemins, *C.R. Acad. Sci. Paris Sér. I Math.* **320**, No. 8 (1995), 1009–1012.
25. P. Malliavin, Hypoellipticity in infinite dimension, *in* "Diffusion Process and Related Problems in Analysis" (Mark A. Pinsky, Ed.), Vol. I, Birkhäuser, Basel, 1989/1991.
26. A. N. Pressley and G. Segal, "Loop Groups," Oxford Univ. Press, London, 1986.
27. P. Protter, "Stochastic Integration and Differential Equations; A New Approach," Springer-Verlag, Berlin, 1990.
28. Michale Reed and Barry Simon, "Methods of Modern Mathematical Physics. I: Functional Analysis," Academic Press, New York, 1980.
29. Derek W. Robinson, "Elliptic Operators and Lie Groups," Clarendon Press, Oxford, 1991.
30. Ichiro Shigekawa, Differential calculus on a based loop group, preprint, 1994.
31. R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, *J. Funct. Anal.* **52** (1983), 48–79.
32. N. Th. Varopoulos, Small time Gaussian estimates of heat diffusion kernels. Part I: The semigroup technique, *Bull. Soc. Math. 2* **113** (1989), 253–277.
33. Feng-Yu Wang, On estimation of logarithmic Sobolev constant and gradient estimates of heat semigroups, preprint, 1995.
34. Feng-Yu Wang, Logarithmic Sobolev inequalities for diffusion processes with application to path space, preprint, 1995.
35. S. T. Yau, On the heat kernel of a complete Riemannian manifold, *J. Math. Pures Appl.* **57** (1978), 191–201.