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New Trends in Stochastic Analysis

Proceedings of a Taniguchi International Workshop

Charingworth Manor September 21 - 27 1994

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Cover picture adapted from a picture showing the garden at Charingworth Manor, Nr. Chipping Campden, Glos. England.

NEW TRENDS IN STOCHASTIC ANALYSIS
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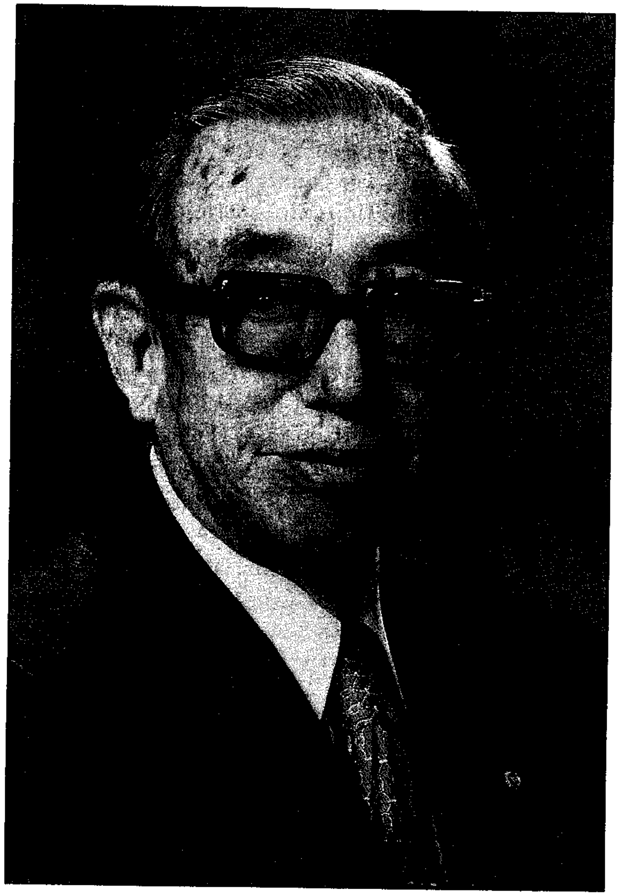
Preface

The Taniguchi International Workshop on "New Trends in Stochastic Analysis " was held in Charingworth Manor, Charingworth, Gloucestershire, England from September 21st to September 27th, 1994. Sixteen mathematicians, nine from Japan and seven from other countries, participated in the workshop to discuss several of the new directions stochastic analysis is taking, ranging from analysis on fractals to analysis on loop spaces. This volume contains contributions from the members of this workshop.

The workshop was followed by a symposium held with the Mathematics Research Centre of the University of Warwick from September 28th to October 1st. This was timed to also open the 1994-95 Stochastic Analysis Year at Warwick. In the symposium about 50 mathematicians participated, with many coming from continental Europe. There were many formal and informal talks, and many participants stayed on afterwards to continue the discussions.

It was with great sadness that we heard later in 1994 of the death of Mr Toyosaburo Taniguchi whose vision and generosity had led to the creation of these meetings with their unique character. Professor Ito, who as co-ordinator of the Taniguchi International Symposia, guided the organising committee to the success of this meeting, has kindly written a brief account for this volume of the story behind their creation. We can no longer thank Mr Taniguchi in person, but we do hope that the workshop and symposium have contributed not only mathematical advances but also to international mutual understanding, the ideal of Mr Taniguchi in supporting us. For our part we hope that we can play a role in keeping that ideal alive.

David Elworthy, Shigeo Kusuoka and Ichiro Shigekawa
September 1996



Toyosaburo Taniguchi, 1901–1994.

Toyosaburo Taniguchi and the Taniguchi Symposium

Toyosaburo Taniguchi was born in 1901. After graduating from Third High School (gymnasium) in 1922, he majored in applied mechanics in the Faculty of Engineering at the Imperial University of Tokyo. Through his work in the Toyobo Textile Company, he modernized the textile industries by introducing automatic spinning systems and market surveys, and thus became a magnate of the textile industries in Japan.

In accordance with the will of his father, he established the Taniguchi Foundation in 1929, in order to support research projects for science and technology. For example, donations by the Foundation covered all expenses connected with the construction of a cyclotron for the nuclear physicists at Osaka University.

Around 1950, Japan enthusiastically promoted industrial activities, in order to cope with the serious economic problems caused by the aftermath of the Second World War. Research projects connected with industrial plants were strongly supported, but mathematics and the theoretical sciences were mostly ignored. Responding to the request of Yasuo Akizuki, Professor of Mathematics at Kyoto University and a classmate in the Third High School, Taniguchi decided in 1956 to help young mathematicians hold nation-wide small seminars (Akizuki seminars) several times a year, each seminar consisting of about 15 members who were allowed to stay at one of the resorts of the Toyobo Company. Taniguchi was very much impressed by the young participants and their lively and friendly mathematical discussions. These Akizuki seminars continued for about twenty years.

The ideal of eternal world peace that had been eagerly pursued after the end of the Second World War was gradually fading away because of continuous international conflicts occurring at different levels. Through his bitter experiences related to international trade conflicts of the textile industries around 1970, Taniguchi became convinced that conflicts could be solved not by debate or by force but only by *international mutual understanding and friendship*, which would eventually bring about world peace.

In 1976, Taniguchi donated a large amount of his property to the Taniguchi Foundation in order to establish a new foundation, whose purpose is to contribute to the promotion of international mutual understanding and friendship by supporting international symposia on mathematics and fundamental sciences in the following way.

Every year one or two symposia are supported in each of the following sections: (1) mathematics, (2) business history, (3) biophysics, (4) medical history, (5) brain sciences, (6) ethnology, (7) neurobiology in vision, (8) theory of condensed matter, (9) philosophy, (10) art history, (11) civilization studies, (12) catalysis, (13) life sciences, (14) religious philosophy, (15) molecular and cellular biology, (16) polymer chemistry, and (17) developmental biology.

The symposia to be supported are characterized as follows:

(a) A research project in a developing field that cannot easily obtain support from other sources is preferred to one in an established field. Participation of young scholars is recommended.

(b) The number of participants is approximately 15, one half from Japan, and the other half from abroad. All participants stay in the same hotel for about 10 days, organize the symposium in whatever form they like, and promote the international mutual understanding and friendship through scientific discussions. No formal report of the achievements is required. This unique style is an international version of the Akizuki seminars of mathematics mentioned above.

In the mathematics section two symposia were organized, one for algebra and geometry and the other for analysis. In each symposium, we proposed a division into two parts.

(1) Workshop: discussions among the participants.

(2) General conference whose speakers are the participants.

Taniguchi agreed with this proposal, but asked us to keep in mind that the former is the principal part.

During the period of each symposium, Taniguchi invited the participants to a dinner party and asked them to promote international mutual understanding and friendship through exchange of scientific ideas. He also expressed his gratitude to his friend Akizuki for suggesting the symposia that fit his own aim. Participants always talked about their pleasant experiences in this unique style of symposium.

Akizuki passed away in 1984 at the age of 81, and Taniguchi in 1994 at the age of 93. The foundation is to be dissolved in 1998 in accordance with Taniguchi's wishes, when the total number of the participants in the Taniguchi Symposia will have reached approximately 6500.

It is my sincere hope that Taniguchi's dream of attaining world peace by international mutual understanding and friendship will be passed on to all participants of the Taniguchi Symposia.

Kiyosi Itô

Contents

New Trends in Stochastic Analysis

HILBERT SPACES OF HOLOMORPHIC FUNCTIONS ON COMPLEX LIE GROUPS

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An isometry is established between a Hilbert space of holomorphic, square integrable functions with respect to the heat kernel measure on a simply connected, complex Lie group and a Hilbert space of "expansion coefficients" of such functions at the identity of the group.

1 Introduction

Let G be a connected, complex Lie group. A holomorphic function $u : G \to \mathbb{C}$ is uniquely determined on G by its family of derivatives at the identity element e relative to any holomorphic coordinate system near e. If $G = \mathbb{C}^n$ then u can of course be recovered from these derivatives by Taylor's formula, which represents u as an everywhere convergent power series. Moreover in this case these "expansion coefficients" of u also determine, simply and explicitly, the norm of u in the Hilbert space $\mathcal{H}L^2(\mathbb{C}^n, Gauss)$, of holomorphic square integrable functions relative to Gauss measure. One has the identity

$$\sum_{k=0}^{\infty}(t^k/k!)|\alpha_k|^2 = \int_{C^n} |u(z)|^2 \mu_t(z)dz \tag{1.1}$$

where $\mu_t(z) = (\pi t)^{-n}\exp(-|z|^2/t)$ and α_k is the symmetric k tensor defined by

$$\langle \alpha_k, e_{i_1} \otimes \cdots \otimes e_{i_k}\rangle = (\partial_{i_1} \cdots \partial_{i_k} u)(0)$$

wherein $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n \subset \mathbb{C}^n$ and $\partial_j = \partial/\partial z_j$. Equation (1.1) reflects in part the easily verifiable fact that the monomials $z_1^{k_1} \cdots z_n^{k_n}$ are mutually orthogonal with respect to Gauss measure $\mu_t(z)dz$.

The identity (1.1) has a long history that flows from its potential usefulness in understanding the structure of quantum fields [1, 2, 3, 4, 5, 9, 17, 18, 19, 20, 23, 25, 26, 29, 30, 31, 32, 35]. It is also intimately connected with the characterization theorem for generalized functions in white noise analysis. (See [14, 21, 22, 24, 27] and their bibliographies.)

In this paper we will prove an analog of Equation (1.1) in case G is a connected, simply connected, complex Lie group with any given Hermitian inner product on its Lie algebra. The Gauss measure on the right side of (1.1) will be replaced by the heat kernel measure on G associated to a Laplacian on G. The space of symmetric tensors whose norm appears on the left side of Equation (1.1) will be replaced by a completion of the universal enveloping algebra of the Lie algebra of G. Power series will be replaced by a different global reconstruction of u from its "expansion coefficients," i.e., derivatives at the identity. (See Section 3.) The resulting (noncommutative) version of (1.1) has its origins in a technique introduced in [10] for proving ergodicity of the left action of a loop group of a compact Lie group K on the pinned K valued Brownian motion measure space. A key ingredient in that technique consisted in establishing a natural unitary map from $L^2(K, \textit{heat kernel measure})$ onto a completion $\bar{U}$ of the universal enveloping algebra of the Lie algebra of K. Motivated by this isometry B. Hall [13] extended the Segal-Bargmann transform [29, 30, 31, 32, 2, 3, 4] to obtain a natural isometry from $L^2(K, \textit{heat kernel measure})$ onto the space $\mathcal{H}L^2(K_c, \textit{another heat kernel measure})$ of holomorphic square integrable functions over the complexification, K_c, of K. Thus these three spaces, $L^2(K)$, $\mathcal{H}L^2(K_c)$, and $\bar{U}$ are all naturally isomorphic. The structure of the three isomorphisms between these three spaces has been clarified in three papers; O. Hijab [15, 16] and B. Driver [8]. The probababilistic proof in [10] has been replaced by analytical proofs. See [8] for a detailed account. However all proofs of these isomorphisms have so far required some form of centrality of the relevant Laplacian. This reflects itself, for example, in the requirement that the complex group $G = K_c$, which appears in two of the isomorphisms, have a real form (e.g. K) of compact type. It seems likely that AdK invariance of the inner product is indeed essential for the existence of those two isomorphisms which involve the space $L^2(K, \textit{heat kernel measure})$. Our main theorem, however, will show that the third isometry, that between $\mathcal{H}L^2(G, \textit{heat kernel measure})$ and a completion of the universal enveloping algebra of $Lie(G)$, does not require that the inner product on $Lie(G)$ be Ad invariant under any real form of the given complex group G. It is in this respect that we go beyond previous work in [8]. Our main results are stated in Theorems 2.5 and 2.6.

2 Statement of Results

We will consider throughout a connected, complex Lie group G of complex dimension d. We assume there is a given Hermitian inner product (,) on the complex Lie algebra $\mathfrak{g} := T_eG$.

Convention: In this paper dx will always denote a fixed right invariant Haar measure on G.

Notation 2.1 *Let $\langle\cdot,\cdot\rangle$ denote the (real) left invariant Riemannian metric on G uniquely defined by*

$$\langle\tilde{A},\tilde{B}\rangle = Re(A,B)\,\forall\, A,B\in\mathfrak{g}, \tag{2.1}$$

where for any $A\in\mathfrak{g}$, $\tilde{A}$ denotes the unique left invariant vector field on G such that $\tilde{A}(e)=A$. We will abuse notation and let $\langle\cdot,\cdot\rangle$ denote the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{g}=T_eG$. Define Δ to be the left invariant second order elliptic differential operator on $C^\infty(G)$ determined by

$$\Delta\phi=\sum_{j=1}^{2d}\tilde{V}_j^2\phi \quad (\phi\in C^\infty(G)), \tag{2.2}$$

where $V_1,\ldots,V_{2d}$ is any orthonormal basis of the **real** *inner product space $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. (Note: it is easily seen that Δ is independent of the choice of orthonormal basis.) We denote by $\mu_t(x)$ the fundamental solution to the heat equation, i.e.*

$$\partial\mu_t(x)/\partial t=(1/4)\Delta\mu_t(x) \quad t>0,\ x\in G \tag{2.3}$$

and

$$\mu_t(x)dx\to\delta_e(dx)\ \text{(weakly) as } t\to 0, \tag{2.4}$$

where δ_e is the Dirac measure at the identity $e\in G$. (The basic properties of μ_t will be explained in more detail at the beginning of Section 3.)

It is worth noting that Δ is a symmetric operator on $L^2(G,dx)$. Indeed it is easily shown that

$$\int_G(\Delta\phi)(x)\psi(x)dx=-\int_G\sum_{j=1}^{2d}\tilde{V}_j\phi(x)\tilde{V}_j\psi(x)dx=\int_G\phi(x)(\Delta\psi)(x)dx \tag{2.5}$$

for any real functions ϕ and ψ in $C^\infty(G)$ when ψ has compact support.

Remark 2.2 *If G is unimodular, it is well known and easily verified that Δ is the Laplace-Beltrami operator on the Riemannian manifold $(G,\langle\cdot,\cdot\rangle)$. To show this note that Eq. (2.5) may be written as*

$$\int_G(-\Delta\phi)(x)\psi(x)dx=\int_G\langle\nabla\phi(x),\nabla\psi(x)\rangle dx,$$

where ∇ denotes gradient operator associated with the Riemannian metric $\langle\cdot,\cdot\rangle$. Since dx is left invariant, it follows that dx is proportional to the Riemann volume measure on G. Hence the above identity shows that Δ is the Laplace-Beltrami operator for the Riemannian manifold $(G,\langle\cdot,\cdot\rangle)$.

Let $\mathcal{H}=\mathcal{H}(G)$ denote the space of complex valued holomorphic functions on G. We are going to study the Hilbert space $\mathcal{H}L^2(G,\mu_t(x)dx)\equiv L^2(G,\mu_t(x)dx)\cap\mathcal{H}(G)$ of holomorphic functions in $L^2(G,\mu_t(x)dx)$.

Notation 2.3 *Denote by T the tensor algebra over the complex vector space $\mathfrak{g}$. For each $t>0$ define a norm on T by*

$$\|\beta\|_t^2=\sum_{k=0}^{n}(k!/t^k)|\beta_k|^2 \quad \beta=\sum_{k=0}^{n}\beta_k,\ \beta_k\in\mathfrak{g}^{\otimes k},\ k=0,1,2,\ldots. \tag{2.6}$$

Here $|\beta_k|$ refers to the cross norm on $\mathfrak{g}^{\otimes k}$ arising from the inner product on $\mathfrak{g}^{\otimes k}$ determined by the given inner product on $\mathfrak{g}$. T_t will denote the completion of T in this norm. Then T_t is a complex Hilbert space with respect to the Hermitian inner product determined by polarizing (2.6).

Denote by T' the algebraic dual space of T. Writing $\mathfrak{g}^*$ for the dual space of $\mathfrak{g}$ we may identify T' with the strong direct sum ($\equiv$ direct product) $\sum_{k=0}^{\infty}(\mathfrak{g}^*)^{\otimes k}$. For α in T' and β in T we use the bilinear pairing

$$\langle\alpha,\beta\rangle=\sum_{k=0}^{\infty}\langle\alpha_k,\beta_k\rangle \quad \alpha=\sum_{k=0}^{\infty}\alpha_k,\ \beta=\sum_{k=0}^{n}\beta_k, \tag{2.7}$$
$$\alpha_k\in(\mathfrak{g}^*)^{\otimes k},\ \beta_k\in\mathfrak{g}^{\otimes k},\ k=0,1,\ldots.$$

Then the topological dual space of T_t may be identified with the subspace T_t^* of T' consisting of those $\alpha\in T'$ such that the norm

$$\|\alpha\|_t^2:=\sum_{k=0}^{\infty}(t^k/k!)|\alpha_k|^2_{(\mathfrak{g}^*)^{\otimes k}} \tag{2.8}$$

is finite. Here $|\alpha_k|_{(\mathfrak{g}^*)^{\otimes k}}$ is the cross norm over $(\mathfrak{g}^*)^{\otimes k}$ determined by the Hermitian inner product on $\mathfrak{g}^*$ dual to the given Hermitian inner product on $\mathfrak{g}$.

It is our objective to establish, for each strictly positive real number t, a natural unitary map between the space $\mathcal{H}L^2(G,\mu_t(x)dx)$ of holomorphic square integrable functions on G and a subspace of T_t^* which will be explicitly specified below. The unitary map will be given by the "Taylor" expansion as follows. Suppose that V is an open neighborhood of a point x_0 in G and $f:V\to\mathbb{C}$ is holomorphic in V. If $\xi_1,\ldots,\xi_n$ are in $\mathfrak{g}$ then $(\tilde{\xi}_1\cdots\tilde{\xi}_nf)(x_0)$ is **complex** linear in each ξ_j because f is holomorphic. Hence there is a unique vector α_n in $(\mathfrak{g}^*)^{\otimes n}$ such that $\langle\alpha_n,\xi_1\otimes\cdots\otimes\xi_n\rangle=(\tilde{\xi}_1\cdots\tilde{\xi}_nf)(x_0)$. We will denote the vector α_n by $(D^nf)(x_0)$. Thus

$$\langle(D^nf)(x_0),\xi_1\otimes\cdots\otimes\xi_n\rangle=(\tilde{\xi}_1\cdots\tilde{\xi}_nf)(x_0),\quad \xi_j\in\mathfrak{g},\quad j=1,\ldots,n. \tag{2.9}$$

Of course we will write $D^0 f(x_0) = f(x_0)$.

Remark 2.4 *If f is in $C^\infty(G)$ but not necessarily holomorphic it still makes sense to define $(D^n f)(x_0)$ by Eq. (2.9), the only difference is that $(D^n f)(x_0)$ must be viewed as an element of $(\mathfrak{g}_{\mathbb{R}}^*)^{\otimes n}$, i.e. $\mathfrak{g}^*$ is considered as a real vector space and the tensor product is now taken over $\mathbb{R}$ rather than $\mathbb{C}$.*

For f holomorphic the series

$$\sum_{n=0}^{\infty} (D^n f)(x_0)$$

always defines an element of T'. We will use the following suggestive notation, introduced in [8],

$$(1-D)_{x_0}^{-1} f = \sum_{n=0}^{\infty} (D^n f)(x_0) \in T'. \tag{2.10}$$

For later use let us note that in view of (2.9) the $(\mathfrak{g}^*)^{\otimes k}$-norm of $D^k f(x)$ is given by

$$|D^k f(x)|^2 = \sum |\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} f(x)|^2 \tag{2.11}$$

where $e_1, \ldots, e_d$ is any orthonormal basis of the complex inner product space $\mathfrak{g}$ and each i_j runs from 1 to d.

Now let J denote the 2-sided ideal in T generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]; \xi, \eta \in \mathfrak{g}\}$. Write

$$J^0 = \{\alpha \in T' : \alpha(J) = 0\}. \tag{2.12}$$

Since the map $\xi \to \tilde{\xi}$ is a Lie algebra isomorphism we have

$$\begin{aligned} \langle (1-D)_{x_0}^{-1} f, \xi_1 \otimes \cdots \otimes \xi_k \otimes (\xi \otimes \eta - \eta \otimes \xi - [\xi,\eta]) \otimes \xi_{k+1} \otimes \cdots \otimes \xi_n \rangle \\ = (\tilde{\xi}_1 \cdots \tilde{\xi}_k (\tilde{\xi}\tilde{\eta} - \tilde{\eta}\tilde{\xi} - [\xi,\eta]^\sim) \tilde{\xi}_{k+1} \cdots \tilde{\xi}_n f)(x_0) \\ = 0. \end{aligned}$$

Hence $(1-D)_{x_0}^{-1} f \in J^0$. Define now

$$J_t^0 = T_t^* \cap J^0. \tag{2.13}$$

Our main theorems are the following. They will be proved in Section 5.

Theorem 2.5 *The map $u \to (1-D)_e^{-1} u$ is an isometry from $\mathcal{H}L^2(G, \mu_t(x)dx)$* **into** *$J_t^0$.*

Theorem 2.6 *If G is simply connected then the map $u \to (1-D)_e^{-1} u$ is a unitary operator from $\mathcal{H}L^2(G, \mu_t(x)dx)$* **onto** *$J_t^0$.*

3 Bounds on Derivatives of Holomorphic Functions

In this section G will continue to denote a connected, complex Lie group of complex dimension d with a given Hermitian inner product on its Lie algebra $\mathfrak{g}$. μ_s will denote the heat kernel with respect to right invariant Haar measure dx as described in Section 2. The following proposition summarizes some of the (largely) well known properties of the heat kernel which we will need.

Proposition 3.1 *The heat kernel μ_s is in $C^\infty((0,\infty) \times G)$, is strictly positive, and has the following properties:*

1. *μ_s is conservative:*

$$\int_G \mu_s(x) dx = 1, \tag{3.1}$$

2. *$\mu_s(x)dx$ is invariant under $x \to x^{-1}$, i.e. for all measurable nonnegative functions f,*

$$\int_G f(y)\mu_s(y)dy = \int_G f(y^{-1})\mu_s(y)dy, \tag{3.2}$$

3. *μ_s satisfies the semigroup property:*

$$\mu_{s+t}(x) = \int_G \mu_s(xy^{-1})\mu_t(y)dy, \tag{3.3}$$

and

4. *μ_s is an approximate δ-function, i.e.*

$$\lim_{t \downarrow 0} \int_G f(xy^{-1})\mu_t(y)dy = \lim_{t \downarrow 0} \int_G f(xy)\mu_t(y)dy = f(x) \text{ if } f \in C_c(G). \tag{3.4}$$

Proof. For item 1. see Eq. (1.3) on pg. 253 of Robinson [28]. Items 3 and 4 are discussed in great generality in Section III.2 and Theorem 2.1 of [28]. Nevertheless for the readers convenience we will sketch the proof.

The Laplacian (Δ) is symmetric on $C_c^\infty(G)$ in $L^2(G, dx)$ for right invariant Haar measure dx as already noted in (2.5). Moreover Δ is essentially self-adjoint on $C_c^\infty(G)$. (This fact is proved for the Laplace-Beltrami operator in Strichartz [33], Theorem 2.4. The same proof given there works with essentially no change for the operator Δ.) We abuse notation and continue to denote the closure of Δ on $C_c^\infty(G)$ by Δ.

Since Δ is a nonpositive self-adjoint operator, $\exp(t\Delta/4)$ for $t > 0$ is a semigroup of Hermitian contraction operators on $L^2(G, dx)$. Define convolution by

$$(f * g)(x) \doteq \int_G f(xy^{-1})g(y)dy = \int_G f(z^{-1})g(zx)dz.$$

The semigroup $\exp(t\Delta/4)$ is given in terms of the heat kernel by convolution:

$$\exp(t\Delta/4)f = f * \mu_t \text{ for } f \in L^2(G, dx)$$

Property 3 reflects the semigroup property of $\exp(t\Delta/4)$ while Property 4 reflects strong continuity at $t = 0$. Property 2 has a more special character which we will discuss in greater detail. If m is a probability measure on the Borel sets of G and we define convolution by $(C_m f)(x) = \int_G f(xy^{-1})m(dy)$ then standard techniques show that C_m is a contraction operator on $L^2(G, dx)$. Let $m^*(A) = m(A^{-1})$. Then m^* is also a probability measure on G and satisfies $\int_G f(y)m(dy) = \int_G f(y^{-1})m^*(dy)$ for every nonnegative measurable function f. A straight forward computation shows that for f and g in $C_c(G)$ one has $(C_m f, g) = (f, C_{m^*} g)$ in the $L^2(G, dx)$ inner product, from which it follows that $(C_m)^* = C_{m^*}$. In the case of interest to us the Hermitian operator $\exp(s\Delta/4)$ is given by C_m with $m(dx) = \mu_s(x)dx$. Since C_m is Hermitian we have $m^* = m$. This is property 2. Let us note, incidentally, that this implies that the heat kernel itself satisfies $\mu_s(y^{-1})\Delta(y) = \mu_s(y)$ where $\Delta(y)$ is the modular function on G, which may be defined by the identity $\int_G f(y^{-1})dy = \int_G f(y)\Delta(y)dy$ for all nonnegative measurable functions f. For indeed Eq.(3.2) gives $\int f(y)\mu_s(y)dy = \int f(y^{-1})\mu_s(y)dy = \int f(y)\mu_s(y^{-1})\Delta(y)dy$ for all nonnegative measurable functions f, which proves the assertion of the previous sentence since $\Delta(y)$ and $\mu_s(y)$ are continuous. Thus μ_s is itself invariant under the map $y \to y^{-1}$ if and only if G is unimodular. This is also discussed in Theorem 2.1 of [28]. Q.E.D.

For a function u in $\mathcal{H}L^2(\mu_t(x)dx)$ we will first derive integral bounds on the derivatives of u (cf. Lemma 3.5) and from these derive bounds on the derivatives of u at the identity (cf. Proposition 3.3). We will then use the "Taylor series" representation of u to get pointwise growth bounds for u and its derivatives (cf. Proposition 3.3 and Corollary 3.10). Aside from the general properties listed in Proposition 3.1 and the defining Equation (2.3) for μ_s no special properties of μ_s (such as the heat kernel bounds of Section 4, or parabolic Harnack inequalities of Section 8) will be used in this section.

Notation 3.2 *Given a measurable function $u : G \to \mathbb{C}$ and $s > 0$ let*

$$\|u\|_s \doteq \|u\|_{L^2(\mu_s(x)dx)}.$$

Proposition 3.3 *Let $u \in \mathcal{H}L^2(\mu_t(x)dx)$ and write $\alpha = (1 - D)_e^{-1}u$. Then*

$$\|\alpha\|_s \leq \|u\|_s < \infty \text{ for } 0 < s < t. \tag{3.5}$$

The proof will require the following four lemmas.

Lemma 3.4 *Let (M, o) be a pointed complex analytic manifold. There exists a smooth compactly supported probability measure ρ on M such that*

$$f(o) = \int_M f(x)\rho(dx)$$

for all holomorphic functions f on M.

Proof. The lemma is local, so without loss of generality we may assume that $M \doteq D_1$ and $o = 0 \in D_1$, where (for any $R > 0$,)

$$D_R \doteq \{z \in \mathbb{C}^d \,|\, |z_i| < R \;\forall i = 1, 2, \ldots, d\}. \tag{3.6}$$

Let f be a holomorphic function on D_1. By the mean value theorem for holomorphic functions;

$$f(0) = (2\pi)^{-d} \int_{[0,2\pi]^d} f(\{r_j e^{\sqrt{-1}\theta_j}\}_{j=1}^d) \prod_{j=1}^d d\theta_j, \tag{3.7}$$

where $r = (r_1, r_2, \ldots, r_d) \in \mathbb{R}^d$ such that $0 \leq r_i < 1$ for $i = 1, 2, \ldots, d$. Choose a smooth function $h : \mathbb{R} \to [0, \infty)$ such that h has support in $(-1/2, 1/2)$, h is constant near 0, and

$$\int_0^1 h(r)rdr = 1.$$

Multiply (3.7) by $r_1 \cdots r_d h(r_1) \cdots h(r_d)$ and integrate each r_i over $[0, 1)$ to find:

$$f(0) = \int_{D_1} f(z)\rho(z)\lambda(dz),$$

where $\rho(z) = (2\pi)^{-d} h(|z_1|) \cdots h(|z_d|)$ and λ is Lebesgue measure on $\mathbb{C}^d$. Q.E.D.

Lemma 3.5 *Let $\epsilon > 0$. There are constants $\{C_\epsilon(k)\}_{k=0}^\infty$ such that for $u \in \mathcal{H}L^2(\mu_t(x)dx)$, $k \in \{0, 1, 2, \ldots\}$, and $s \in (0, t - \epsilon]$,*

$$\int_G |D^k u(g)|^2 \mu_s(g)dg \leq C_\epsilon(k)\|u\|_t^2. \tag{3.8}$$

Proof. Choose a smooth probability density ρ as in Lemma 3.4 so that

$$f(e) = \int_G f(x)\rho(x)dx \text{ for all } f \in \mathcal{H}(G). \tag{3.9}$$

Let $g \in G$ and let $u \in \mathcal{H}(G)$. Put $f(x) = u(gx^{-1})$ in (3.9). Using right invariance of Haar measure we get,

$$u(g) = \int_G u(gx^{-1})\rho(x)dx = \int_G u(y^{-1})\rho(yg)dy. \tag{3.10}$$

Fix $\xi_1, \ldots, \xi_k$ in $\mathfrak{g}$ and put $\beta = \xi_1 \otimes \cdots \otimes \xi_k$. Then, since the following integrand is smooth and compactly supported, we have

$$(\tilde{\xi}_1 \cdots \tilde{\xi}_k u)(g) = \int_G u(y^{-1})(\tilde{\xi}_1 \cdots \tilde{\xi}_k \rho)(yg)dy. \tag{3.11}$$

Now ρ and all of its derivatives have compact support while $\mu_\sigma(x)$ is bounded away from zero on $\{\epsilon \le \sigma \le t\} \times \{\text{support of } \rho\}$. Hence there are constants $a(\epsilon, k)$ such that

$$|D^k \rho(x)|_{(\mathfrak{g}^*)^{\otimes k}} \le a(\epsilon, k)\mu_\sigma(x) \tag{3.12}$$

for all x in G and for all σ in $[\epsilon, t]$. Note that the subscript on the left side of (3.12) refers to $\mathfrak{g}$ and $\mathfrak{g}^*$ as real inner product spaces since ρ is not holomorphic.(See Remark 2.4.) Combining (3.11) and (3.12) we find

$$|\langle D^k u(g), \beta\rangle| \le |\beta| \cdot \int_G |u(y^{-1})| a(\epsilon, k)\mu_\sigma(yg)dy. \tag{3.13}$$

By Schwarz's inequality we have

$$|\langle D^k u(g), \beta\rangle|^2 \le a(\epsilon, k)^2 |\beta|^2 \int_G |u(y^{-1})|^2 \mu_\sigma(yg)dy. \tag{3.14}$$

Now suppose $0 < s \le t - \epsilon$. Choose $\sigma = t - s$. Then $\epsilon \le \sigma < t$. Multiplying the inequality in Eq. (3.14) by $\mu_s(g)$ and integrating we get

$$\begin{aligned}\int_G |\langle D^k u(g), \beta\rangle|^2 \mu_s(g)dg &\le a(\epsilon, k)^2|\beta|^2 \int_G \int_G |u(y^{-1})|^2 \mu_\sigma(yg)\mu_s(g)dydg \\ &= a(\epsilon, k)^2|\beta|^2 \int_G |u(y^{-1})|^2 \mu_t(y)dy \\ &= a(\epsilon, k)^2|\beta|^2 \int_G |u(z)|^2 \mu_t(z)dz \end{aligned} \tag{3.15}$$

wherein we have used the semigroup equation (3.3) and also equation (3.2) of Proposition 3.1. Letting $\xi_1, \ldots, \xi_k$ run independently over an orthonormal basis of $\mathfrak{g}$ (as a complex inner product space) we may sum the last inequalities to obtain (3.8) with $C(\epsilon, k) = a(\epsilon, k)^2 d^k$. Q.E.D.

We denote by $d(x, y)$ the Riemannian distance from x to y in G and write $|x| = d(e, x)$. We note that $d(\cdot, \cdot)$ is left invariant, i.e. $d(xk, xg) = d(k, g)$ holds for all $x, g, k \in G$. Therefore $|x^{-1}| = d(e, x^{-1}) = d(x, e) = |x|$ and $|xy| \le d(xy, x) + d(x, e) = |y| + |x|$, see Proposition 8.2 of [8] for more details.

Lemma 3.6 *There exists a sequence of functions h_n in $C_c^\infty(G)$ such that*

1. $0 \le h_n \le 1$
2. $h_n(g) = 1$ *whenever* $|g| \le n$
3. $\sup_n \sup_{g \in G} |D^k h_n(g)| < \infty$ *for* $k = 0, 1, 2, \ldots$.

Proof. Let $v \in C_c^\infty(G)$ be nonnegative with $\int_G v(y)dy = 1$ and with support in $\{y : |y| < 1\}$ where dy is right invariant Haar measure. Let $w_n(x) = 1$ if $|x| \le n + 1$ and zero otherwise. Define

$$h_n(x) = \int w_n(xy^{-1})v(y)dy = \int w_n(y^{-1})v(yx)dy.$$

Clearly condition 1 holds. If $|x| \le n$ then for $|y| \le 1$ we have $|xy^{-1}| \le |x| + |y| \le n + 1$. So when $|x| \le n$, $w_n(xy^{-1}) = 1$ on the support of v and therefore $h_n(x) = \int v(y)dy = 1$. This establishes condition 2. Now the integrand $w_n(xy^{-1})v(y)$ is zero unless $|y| \le 1$ and $|xy^{-1}| \le n + 1$, which together require $|x| \le |xy^{-1}| + |y| \le n + 2$. So $h_n(x) = 0$ if $|x| > n + 2$. Therefore h_n has compact support.[1] If $A_1, \ldots, A_k$ are left invariant vector fields then for $k = 0, 1, 2, \ldots$,

$$|A_1 \cdots A_k h_n(x)| \le \int_G |(A_1 \cdots A_k v)(yx)|dy = \int_G |(A_1 \cdots A_k v)(y)|dy,$$

which establishes condition 3 and shows that h_n is in $C_c^\infty(G)$. Q.E.D.

Remark 3.7 *We will need the following identity which has already been pointed out in [8], Section 4.2.*

$$(\Delta/4)|D^k u(x)|^2 = |D^{k+1}u(x)|^2 \text{ for } k = 0, 1, 2, \ldots \text{ and } u \in \mathcal{H}(G). \tag{3.16}$$

[1] Using the left invariance of the metric d, it is easily checked that (G, d) is a complete metric space. Hence by the Hopf-Rinow theorem (see Section 1.7 of [6]), closed bounded subsets of G are compact.

In order to prove (3.16) choose an orthonormal basis $e_1, \ldots, e_d$ of $(\mathfrak{g}, (\ ,\ ))$ as a complex inner product space. Then one easily verifies that $e_1, \ldots, e_d$, $\sqrt{-1}e_1, \ldots, \sqrt{-1}e_d$ is an orthonormal basis of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ as a real inner product space. Let $X_j = \tilde{e}_j$ and $Y_j = (\sqrt{-1}e_j)^\sim$ for $j = 1, \ldots, d$. Then by the definition of Δ in Eq. (2.2),

$$\Delta\phi = \sum_{j=1}^{d}(X_j^2 + Y_j^2)\phi \ \forall\phi \in C^\infty(G). \tag{3.17}$$

In the complexified tangent bundle $\mathbb{C} \otimes T(G)$ let $Z_j = (X_j - \sqrt{-1}Y_j)/2$ and $\overline{Z}_j = (X_j + \sqrt{-1}Y_j)/2$. Since $\mathfrak{g}$ is a complex Lie algebra X_j and Y_j commute. Hence $Z_j\overline{Z}_j = (X_j^2 + Y_j^2)/4$. Thus

$$\Delta\phi = 4\sum_{j=1}^{d} Z_j\overline{Z}_j\phi \ \forall\, \phi \in C^\infty(G).$$

But if ϕ is a holomorphic complex valued function on G then $Y_j\phi = \sqrt{-1}X_j\phi$ and therefore $\overline{Z}_j\phi = 0$. (Cauchy-Riemann equations.) Moreover $Z_j\overline{\phi} = \overline{(\overline{Z}_j\phi)} = 0$ also. Hence if $u \in \mathcal{H}(G)$ then $Z_j\overline{Z}_j|u|^2 = |Z_j u|^2 = |X_j u|^2$. Thus

$$(\Delta/4)|u|^2 = \sum_{j=1}^{d}|\tilde{e}_j u|^2 \ \ \forall\, u \in \mathcal{H}(G). \tag{3.18}$$

In view of (2.11) this proves (3.16) for $k = 0$. The general case now follows by induction.

Lemma 3.8 *Let u be in $\mathcal{H}L^2(\mu_t(x)dx)$. Define $F(s) = \|u\|_s^2$. Then F is in $C^\infty((0,t))$. Moreover*

$$F^{(k)}(s) = \int_G |D^k u(g)|^2\mu_s(g)dg,\ 0 < s < t, k = 0, 1, 2, \ldots, \tag{3.19}$$

$F^{(k)}(0) \doteq \lim_{s\downarrow 0} F^{(k)}(s)$ *exists for* $k = 0, 1, 2, \ldots$ *and*

$$|D^k u(e)|^2 \le \lim_{s\downarrow 0} F^{(k)}(s). \tag{3.20}$$

Proof. Let

$$H(k,s) = \int_G |D^k u(g)|^2\mu_s(g)dg,\ \ 0 < s < t. \tag{3.21}$$

By Lemma 3.5 this is finite for all s in $(0,t)$ and for $k = 0, 1, 2, \ldots$. Choose functions h_n as in Lemma 3.6 and define

$$H_n(k,s) = \int_G h_n(g)|D^k u(g)|^2\mu_s(g)dg,\ 0 < s < t. \tag{3.22}$$

By Lemma 3.5 and dominated convergence, $H_n(k,s)$ converges for each s in $(0,t)$ to $H(k,s)$. Since the integrand in (3.22) is in $C_c^\infty(G)$, $H_n(k,s)$ is differentiable in s and by the heat equation and integration by parts we have

$$\begin{aligned} dH_n(k,s)/ds &= \int_G h_n(g)|D^k u(g)|^2(\Delta/4)\mu_s(g)dg \\ &= \int_G [(\Delta/4)\{h_n(g)|D^k u(G)|^2\}]\mu_s(g)dg \\ &= \int_G h_n(g)|D^{k+1}u(g)|^2\mu_s(g)dg \\ &\quad + \int_G R_{n,k}(g)\mu_s(g)dg \end{aligned} \tag{3.23}$$

wherein we have used Eq. (3.16) and where

$$R_{n,k}(g) = \{(\Delta/4)h_n(g)\}|D^k u(g)|^2 + (1/2)\langle\nabla h_n(g), \nabla|D^k u(g)|^2\rangle.$$

Now the first term in (3.23) is $H_n(k+1,s)$, which, as already noted, converges to $H(k+1,s)$ for each s in $(0,t)$. In fact, for any $\epsilon > 0$ the convergence takes place boundedly in s for s in $(0, t-\epsilon]$ by Eq. (3.8) and the uniform boundedness of the functions h_n. Next, since $\Delta h_n(g)$ converges to zero boundedly as $n \to \infty$, the same argument also shows that

$$\int_G \{(\Delta/4)h_n(g)\}|D^k u(g)|^2\mu_s(g)dg$$

converges to zero as $n \to \infty$ for each s in $(0,t)$ and boundedly on each interval $(0, t-\epsilon]$. Finally, since $|\nabla h_n(g)|$ converges to zero boundedly in G while $|\nabla|D^k u(g)|^2| \le \text{const.}(|D^k u(g)|^2 + |D^{k+1}u(g)|^2)$, the same argument applies also to the second term in $\int R_{n,k}(g)\mu_s(g)dg$. Hence $\int_G R_{n,k}(g)\mu_s(g)dg \to 0$ boundedly on each interval $(0, t-\epsilon]$. We may conclude therefore that $(d/ds)H_n(k,s)$ converges to $H(k+1,s)$ for each s in $(0,t)$ and in fact boundedly on each interval $(0, t-\epsilon]$. So for $0 < a < b < t$, the equation $H_n(k,s) = H_n(k,a) + \int_a^s (d/d\sigma)H_n(k,\sigma)d\sigma$, which is valid for $a \le s < b$, goes over in the limit $n \to \infty$ to $H(k,s) = H(k,a) + \int_a^s H(k+1,\sigma)d\sigma$, by the dominated convergence theorem on $[a,s]$. This shows first, that $H(k,\cdot)$ is continuous for

$k = 0, 1, 2, \ldots$, second, that $H(k, \cdot)$ is continuously differentiable on $[a, b)$ with derivative $H(k+1, \cdot)$ and third that $F(s)$ is infinitely differentiable on $(0, t)$ with derivatives correctly given by Equation (3.19). Since $F^{(k+1)}$ is nonnegative on $(0, t)$, $F^{(k)}(s)$ is decreasing as s decreases to zero. Therefore $\lim_{s\downarrow 0} F^{(k)}(s)$ exists for each $k = 0, 1, 2, \ldots$.

In order to prove Eq. (3.20), let $h \in C_c^\infty(\mathbb{R})$ satisfy $0 \le h \le 1$ and $h(s) = 1$ for $|s| \le 1$. Let $\varphi(g) = h(|g|)$. Then $\varphi(\cdot)|D^k u(\cdot)|^2$ is in $C_c(G)$ and since μ_s is the fundamental solution to the heat equation we have

$$\lim_{s\downarrow 0} \int_G \varphi(g)|D^k u(g)|^2 \mu_s(g)dg = |D^k u(e)|^2.$$

But $\int_G (1 - \varphi(g))|D^k u(g)|^2 \mu_s(g)ds \ge 0$. Hence

$$\begin{aligned} \lim_{s\downarrow 0} F^{(k)}(s) &= \liminf_{s\downarrow 0} \Big\{ \int_G \varphi(g)|D^k u(g)|^2 \mu_s(g)dg \\ &\quad + \int_G (1 - \varphi(g))|D^k u(g)|^2 \mu_s(g)dg \Big\} \ge |D^k u(e)|^2. \end{aligned}$$

Q.E.D.

Proof of Proposition 3.3.. Let u be in $\mathcal{H}L^2(\mu_t(x)dx)$ and define $F(s) = \|u\|_s^2$. By Lemma 3.8, F is in $C^\infty((0, t))$ with all derivatives nonnegative. In particular $\|u\|_s < \infty$ for $0 < s < t$. By Taylor's formula with remainder we may write $F(s) = \sum_{k=0}^N F^{(k)}(a)(s - a)^k/(k!) + F^{(N+1)}(s_1)(s - a)^{N+1}/(N + 1)!$ for $0 < a < s < t$ and for some point s_1 in (a, s). Since the remainder term is nonnegative,

$$\sum_{k=0}^N F^{(k)}(a)(s - a)^k/(k!) \le F(s) \quad 0 < a < s < t. \tag{3.24}$$

Using Eq. (3.20), we find by letting $a \downarrow 0$ in (3.24) that

$$\sum_{k=0}^N |D^k u(e)|^2 s^k/(k!) \le F(s) = \|u\|_s^2.$$

We may now let $N \to \infty$. In view of the definition of $\|\alpha\|_s^2$ in Eq. (2.8) the resulting inequality is precisely (3.5). Q.E.D.

Proposition 3.9 *Let $u \in \mathcal{H}$ be such that $\alpha \equiv (1 - D)_e^{-1} u \in J_t^0$. Suppose that $r, s > 0$ are such that $r + s \le t$. Then*

$$|D^k u(g)|^2 \le k!(d/r)^k \|\alpha\|_t^2 e^{|g|^2/s}, \quad k = 0, 1, 2, \ldots, \tag{3.25}$$

where d is the complex dimension of $\mathfrak{g}$.

Proof. The global recovery of the holomorphic function u from its set, α, of "expansion coefficients" has been explained in detail in Proposition 5.1 of reference [8]. The function u and its derivatives are explicitly given in terms of α as follows. Let $\sigma : [0, 1] \to G$ be a smooth path such that $\sigma(0) = e$ and $\sigma(1) = g$. Let $c(s) = \theta\langle\sigma'(s)\rangle \doteq L_{\sigma(s)^{-1}*}\sigma'(s) \in \mathfrak{g}$ and define

$$\Psi(\sigma) \doteq \sum_{n=0}^\infty \int_{\Delta_n} c(s_1) \otimes \cdots \otimes c(s_n)\ ds. \tag{3.26}$$

where $\Delta_n = \{(s_1, \ldots, s_n) : 0 < s_1 < \cdots < s_n < 1\}$ and $ds = ds_1 ds_2 \cdots ds_n$. Then $u(g) = \langle \alpha, \Psi(\sigma)\rangle$. More generally, if $\beta \in \mathfrak{g}^{\otimes k}$ and $\tilde{\beta}$ is the corresponding left invariant k-th order differential operator on $\mathcal{H}$ then

$$(\tilde{\beta} u)(g) = \langle (1 - D)_e^{-1} \tilde{\beta} u, \Psi(\sigma)\rangle = \langle \alpha, \Psi(\sigma) \otimes \beta\rangle. \tag{3.27}$$

From (3.27) we find that

$$|(\tilde{\beta} u)(g)|^2 \le \|\alpha\|_t^2 \|\Psi(\sigma) \otimes \beta\|_t^2. \tag{3.28}$$

Let

$$\rho \doteq \int_0^1 |\sigma'(s)|ds = \int_0^1 |c(s)|ds = \ell(\sigma), \tag{3.29}$$

where $\ell(\sigma)$ denotes the length of the path σ. Then

$$\begin{aligned} \|\Psi(\sigma) \otimes \beta\|_t^2 &= \sum_{n=0}^\infty \frac{(n + k)!}{t^{n+k}} |(\int_{\Delta_n} c(s_1) \otimes \cdots \otimes c(s_n) ds) \otimes \beta|^2 \\ &\le |\beta|^2 \sum_{n=0}^\infty \frac{(n + k)!}{t^{n+k}} \int_{\Delta_n \times \Delta_n} \prod_{j=1}^n (c(s_j), c(t_j)) ds dt \\ &\le |\beta|^2 \sum_{n=0}^\infty \frac{(n + k)!}{t^{n+k}} \int_{\Delta_n \times \Delta_n} \prod_{j=1}^n |c(s_j)||c(t_j)| ds dt \\ &\le |\beta|^2 \sum_{n=0}^\infty \frac{(n + k)!}{t^{n+k}} \rho^{2n}/(n!)^2 \\ &= \frac{k!|\beta|^2}{r^k} \sum_{n=0}^\infty \frac{(n + k)!}{k! \cdot n!} (\frac{s}{t})^n (\frac{r}{t})^k (\frac{\rho^2}{s})^n / n! \end{aligned} \tag{3.30}$$

Because

$$\frac{(n + k)!}{k! \cdot n!} (\frac{s}{t})^n (\frac{r}{t})^k \le \sum_{l=0}^{n+k} \binom{n + k}{l} (\frac{s}{t})^{n+k-l} (\frac{r}{t})^l = (\frac{s + r}{t})^{n+k} \le 1,$$

it follows from (3.30) that

$$\|\Psi(\sigma)\otimes\beta\|_t^2 \le \frac{k!|\beta|^2}{r^k}\sum_{n=0}^{\infty}(\frac{\rho^2}{s})^n/n! = \frac{k!|\beta|^2}{r^k}e^{\rho^2/s} = \frac{k!|\beta|^2}{r^k}e^{\ell(\sigma)^2/s}$$

This equation and (3.28) show:

$$|(\tilde{\beta}u)(g)|^2 \le \|\alpha\|_t^2\frac{k!|\beta|^2}{r^k}e^{\ell(\sigma)^2/s}. \tag{3.31}$$

Since σ is an arbitrary path joining e to g in G, Eq. (3.31) implies

$$|(\tilde{\beta}u)(g)|^2 \le \|\alpha\|_t^2\frac{k!|\beta|^2}{r^k}e^{|g|^2/s}. \tag{3.32}$$

Hence, for any orthonormal basis $\mathfrak{g}_0$ of $\mathfrak{g}$ we have

$$\begin{aligned}|(D^k u)(g)|^2 &= \sum_{e_1,\ldots,e_k\in\mathfrak{g}_0}|(\tilde{e}_1\cdots\tilde{e}_k u)(g)|^2\\ &\le \sum_{e_1,\ldots,e_k\in\mathfrak{g}_0}\|\alpha\|_t^2\frac{k!}{r^k}e^{|g|^2/s}\\ &= d^k\|\alpha\|_t^2\frac{k!}{r^k}e^{|g|^2/s},\end{aligned}$$

which is Eq. (3.25). Q.E.D.

Corollary 3.10 *Let u be in $\mathcal{H}L^2(\mu_t(x)dx)$. Suppose that $0<s<\sigma<t$ and that $r>0$ and satisfies $s+r\le\sigma$. Then*

$$|D^k u(g)|^2 \le k!(d/r)^k\|u\|_\sigma^2 e^{|g|^2/s} < \infty \tag{3.33}$$

where d is the complex dimension of G.

Proof. If $\alpha=(1-D)_e^{-1}u$ then by Proposition 3.3 $\|\alpha\|_\sigma \le \|u\|_\sigma$. We may apply Proposition 3.9 with t replaced by σ. Then (3.33) follows from (3.5) and (3.25). Q.E.D.

Remark 3.11 *We will see in Section 5 that (3.33) holds also for $\sigma=t$.*

4 Equality of Norms Before Time t

In this section G will be assumed to be a connected complex group as in the previous section. However in Corollary 4.6 it will also be assumed to be simply connected.

Proposition 4.1 *Suppose $u\in\mathcal{H}(G)$ and $\alpha=(1-D)_e^{-1}u$. If $\|\alpha\|_t<\infty$ then $u\in\mathcal{H}L^2(\mu_s(x)dx)$ for $0<s<t$ and*

$$\|\alpha\|_s = \|u\|_s \quad 0<s<t. \tag{4.1}$$

Lemma 4.2 (Heat kernel Bounds) *There exists $\nu\in\mathbb{R}$ such that for $T>0$ and $\epsilon\in(0,1]$, there is a constant $C(T,\epsilon)$ such that*

$$\mu_s(g) \le C(T,\epsilon)s^{-d}\exp\{-(|g|-\nu s)^2/(1+\epsilon)s\},\ \forall s\in(0,T] \text{ and } g\in G. \tag{4.2}$$

Moreover, $\nu=0$ if G is unimodular.

Proof. This lemma is essentially stated in Robinson [28] on page 286. A few remarks are in order. First Robinson treats the case of right invariant differential operators and uses a left invariant Haar measure. This is of no importance since the map $x\in G\to x^{-1}\in G$ transforms right invariant differential operators to left invariant differential operators and left Haar measure to right Haar measure. Robinson shows on the top of page 286 that there are constants $0<C(T,\epsilon)<\infty$ and $\nu\in\mathbb{R}$ such that

$$\mu_s(g) \le C(T,\epsilon)s^{-d}\inf_{\rho\ge 0}\exp\{\rho^2(1+\epsilon)s/4-\rho(|g|-\nu s)\}. \tag{4.3}$$

Moreover, $\nu=0$ if G is unimodular. Choosing $\rho=\frac{2(|g|-\nu s)}{(1+\epsilon)s}$ (this ρ minimizes the exponent) in (4.3) gives (4.2). Q.E.D

Lemma 4.3 *Let $\rho\in\mathbb{R}$. Then*

1. $$\int_G e^{-\beta|g|^2}e^{\rho|g|}dg < \infty\ \forall\beta>0. \tag{4.4}$$

2. *For some $\beta_0>0$ and constant K*

$$\int_{|g|\ge 1} e^{-\beta|g|^2}e^{\rho|g|}\,dg \le Ke^{-\beta},\ \forall\beta\ge\beta_0. \tag{4.5}$$

3. *For any $s_1>0$*

$$\lim_{s\downarrow 0}\int_{|g|\ge 1} e^{|g|^2/s_1}\mu_s(g)\,dg = 0. \tag{4.6}$$

4. Let $0 < s_0 < s_1$. There is a constant K_1 depending on s_0 and s_1 such that

$$\int_G e^{|g|^2/s_1}\mu_s(g)dg \le K_1 \text{ for } 0 < s \le s_0. \tag{4.7}$$

Proof. Since for all $\beta' < \beta$ there is a constant $K = K(\beta, \beta', \rho)$ such that $e^{-\beta|g|^2}e^{\rho|g|} \le Ke^{-\beta'|g|^2}$, it suffices to prove (4.4) for the special case where $\rho = 0$. Let $V(r)$ be the right Haar measure of $\{g \in G : |g| \le r\}$. Then V is monotone increasing, $V(0) = 0$, and V is continuous. (The continuity of V is not necessary for the argument. Nevertheless it follows from Proposition 3.2, p. 116 of [6] by viewing right-Haar measure as the Riemann volume measure of a right invariant Riemannian metric on G.). If $h : [0,\infty) \to [0,\infty)$ is continuously differentiable and $0 \le a < c < \infty$ then

$$\int_{a\le|g|\le c} h(|g|)dg = \int_a^c h(r)dV(r) = h(r)V(r)|_a^c - \int_a^c V(r)h'(r)dr. \tag{4.8}$$

Let $h(r) = e^{-\beta r^2}$. Then $-h'(r) > 0$. Moreover by Lemma 5.8 in [8] there is a constant $C < \infty$ such that $V(r) \le Ce^{Cr}$. By the monotone convergence theorem we may let $c \to \infty$ to get

$$\begin{aligned}\int_{|g|\ge a} e^{-\beta|g|^2}dg &= -h(a)V(a) + \int_a^\infty V(r)2\beta re^{-\beta r^2}dr \\ &\le \int_a^\infty Ce^{Cr}2\beta re^{-\beta r^2}dr.\end{aligned} \tag{4.9}$$

Putting $a = 0$ gives (4.4).

To prove (4.5) take $\beta_0 > |\rho| + C$ and repeat the above argument with $h(r) = e^{-\beta r^2}e^{\rho r}$ and $a = 1$ to find

$$\int_{|g|\ge 1} e^{-\beta|g|^2}e^{\rho|g|}dg \le \int_1^\infty Ce^{Cr}(2\beta r - \rho)e^{(-\beta r^2+\rho r)}dr. \tag{4.10}$$

(Note for $\beta > \rho/2$ that $-h'(r) > 0$ for $r \ge 1$.) Then for $\beta \ge \beta_0$ the right hand side of (4.10) is bounded by

$$3\beta C\int_1^\infty re^{(C+\rho-\beta)r^2}dr = \frac{3e^{-(\beta-C-\rho)}\beta C}{2(\beta - C - \rho)} \le Ke^{-\beta}.$$

To prove (4.6) choose any $s_0 < s_1$ and choose $\epsilon > 0$ such that $s_0(1+\epsilon) < s_1$. Then by (4.2) with $T = s_1$ and $s \le s_0$

$$\begin{aligned}&\int_{|g|\ge 1} e^{|g|^2/s_1}\mu_s(g)dg \\ &\le C(s_1,\epsilon)s^{-d}\int_{|g|\ge 1} e^{|g|^2/s_1}e^{-(|g|-\nu s)^2/(1+\epsilon)s}dg \\ &\le C(s_1,\epsilon)s^{-d}\int_{|g|\ge 1} \exp\left\{-\beta|g|^2 + \frac{(2\nu|g| - \nu^2 s)}{(1+\epsilon)}\right\}dg\end{aligned} \tag{4.11}$$

where $\beta = (1+\epsilon)^{-1}s^{-1} - s_1^{-1}$. By (4.5) with $\rho = 2|\nu|/(1+\epsilon)$ the right side goes to zero as $s \downarrow 0$ because $s^{-d}\exp(-a/s)$ goes to zero for any $a > 0$.

To prove (4.7) observe that if, for the given s_0 and s_1, we choose the same ϵ as in the proof of (4.6) then

$$\begin{aligned}e^{-\nu^2 s(1+\epsilon)^{-1}}&\int_{|g|\ge 1} e^{|g|^2/s_1}e^{-(|g|-\nu s)^2/(1+\epsilon)s}dg \\ &= \int_{|g|\ge 1} e^{|g|^2/s_1}\exp\left\{-\frac{|g|^2}{(1+\epsilon)s} + \frac{2\nu|g|}{1+\epsilon}\right\}dg\end{aligned}$$

decreases as s decreases from s_0. Hence the first integral on the right side of (4.11) remains bounded for $s \in (0, s_0]$ while the entire right side of (4.11) goes to zero as s goes to zero. Hence the left side of (4.11) is bounded on $(0, s_0]$. But since μ_s is a probability density we also have

$$\int_{|g|<1} e^{|g|^2/s_1}\mu_s(g)dg \le e^{1/s_1}\ \forall s > 0$$

This proves (4.7). Q.E.D.

Lemma 4.4 *Suppose $u \in \mathcal{H}(G)$ and $\alpha = (1-D)_e^{-1}u$. If $\|\alpha\|_t < \infty$ then $u \in \mathcal{H}L^2(\mu_s(x)dx)$ for $0 < s < t$ and $F(s) \doteq \|u\|_s^2$ has an analytic continuation to a neighborhood of $[0,t)$ in $\mathbb{C}$.*

Proof. By the bounds in Eq. (3.25) and Eq. (4.7) we see that $u \in \mathcal{H}L^2(\mu_s(x)dx)$ for all $s \in (0,t)$. By Lemma 3.8 $F(s) \doteq \|u\|_s^2$ is a C^∞-function on $(0,t)$ such that F and all of its derivatives have a continuous extension to $[0,t)$. As in Lemma 3.8 let $F^{(k)}(0) \doteq \lim_{s\downarrow 0} F^{(k)}(s)$. Our goal is to show that F is real analytic on $[0,t)$. More explicitly we will show that for all $t_0 \in [0,t)$ there exists $\delta > 0$ such that

$$F(s) = \sum_{k=0}^\infty F^{(k)}(t_0)(s-t_0)^k/k!,\ \forall s \in I_\delta, \tag{4.12}$$

where

$$I_\delta \doteq [0,t) \cap (t_0 - \delta, t_0 + \delta).$$

By Taylor's theorem, Eq. (4.12) may be proved by showing that

$$\lim_{k\to\infty} \sup_{\substack{0\le s<t \\ |s-t_0|\le\delta}} \frac{F^{(k)}(s)\delta^k}{k!} = 0, \tag{4.13}$$

for some $\delta > 0$.

To prove (4.13), choose $s_1 \in (t_0, t)$, set $r \doteq t - s_1$, and assume that δ is chosen sufficiently small, so that $t_0 + \delta < s_1 < t$ and $d\delta/r < 1$. Using (3.25) with $s = s_1$ and (4.7) with $s_0 = t_0 + \delta$ we find for $0 < s \le s_0$

$$\begin{aligned} F^{(k)}(s) &= \int_G |(D^k u)(g)|^2 \mu_s(g)dg \\ &\le k!(d/r)^k \|\alpha\|_t^2 \int_G e^{|g|^2/s_1}\mu_s(g)dg \\ &\le k!(d/r)^k \|\alpha\|_t^2 K_1. \end{aligned}$$

By letting $s \downarrow 0$ we see that the last inequality holds also for $s = 0$. Hence if $s \in I_\delta$ then

$$\frac{F^{(k)}(s)\delta^k}{k!} \le (\text{constant})(d\delta/r)^k,$$

which suffices to prove (4.13). Q.E.D.

Proof of Proposition 4.1. Our first goal is to show that $F^{(k)}(0) = |D^k u(e)|^2$. To do this it suffices, by looking at the last equation in the proof of Lemma 3.8, to show

$$\lim_{s\downarrow 0} \int_{|g|\ge 1} |D^k u(g)|^2 \mu_s(g)dg = 0. \tag{4.14}$$

By the bound in Eq. (3.25) we learn that for each $s_1 \in (0,t)$ there is a constant C such that $|D^k u(g)|^2 \le Ce^{|g|^2/s_1}$. Hence (4.14) follows from Eq. (4.6).

Since Lemma 4.4 guarantees that $F(s) = \|u\|_s^2$ has an analytic continuation to a neighborhood of $[0,t)$, we learn

$$F(s) = \|u\|_s^2 = \sum_{k=0}^{\infty} F^{(k)}(0)s^k/k! = \sum_{k=0}^{\infty} |D^k u(e)|^2 s^k/k! = \|\alpha\|_s^2$$

for $s \in (0,\epsilon)$ with $\epsilon > 0$ sufficiently small. Since $\|\alpha\|_s^2$ is given as a convergent power series in s it is clear that $\|\alpha\|_s^2$ has an analytic continuation to a holomorphic function in the open disc of radius t in $\mathbb{C}$. Using these facts it follows that F and $s \to \|\alpha\|_s^2$ must agree on all of $[0,t)$. Therefore, $\|u\|_s^2 = \|\alpha\|_s^2$ for all $s \in (0,t)$. Q.E.D.

Corollary 4.5 *If u is in $\mathcal{H}L^2(\mu_t(x)dx)$ and $\alpha = (1-D)_e^{-1}u$ then*

$$\|\alpha\|_s = \|u\|_s \text{ for } 0 < s < t. \tag{4.15}$$

Proof. By Proposition 3.3, $\|\alpha\|_s \le \|u\|_s < \infty$ for $0 < s < t$. Choose σ in $(0,t)$. Then $\|\alpha\|_\sigma < \infty$. We can therefore apply Proposition 4.1 with t replaced by σ to conclude that (4.15) holds for $0 < s < \sigma$. Since σ is arbitrary in $(0,t)$ (4.15) holds for all s in $(0,t)$. Q.E.D.

Corollary 4.6 *If α is in J_t^0 and G is simply connected then there exists a function u in $\mathcal{H}(G)$ such that $(1-D)_e^{-1}u = \alpha$. Moreover*

$$\|\alpha\|_s = \|u\|_s \text{ for } 0 < s < t. \tag{4.16}$$

Proof. For the existence of $u \in \mathcal{H}(G)$ such that $(1-D)_e^{-1}u = \alpha$, see Lemma 8.2 in [10] or Theorem 6.1 in [8]. The corollary now follows from Proposition 4.1. Q.E.D.

5 Equality of Norms at Time t

The following theorem proves the isometric embedding of $\mathcal{H}L^2(\mu_t(x)dx)$ into J_t^0 stated in Theorem 2.5.

Theorem 5.1 *Assume that G is a connected complex Lie group with a given Hermitian inner product on its Lie algebra. Let u be in $\mathcal{H}L^2(\mu_t(x)dx)$ and let $\alpha = (1-D)_e^{-1}u$. Then*

$$\|\alpha\|_t = \|u\|_t. \tag{5.1}$$

We will need the following lemma.

Lemma 5.2 *Assume u is in $\mathcal{H}L^2(\mu_t(x)dx)$. Then*

a) *$(|u| * \mu_s)(g) < \infty$ for all $g \in G$ and $0 < s < t$ and*

b) *$u * \mu_s = u$ on G for $0 < s < t$.*

Proof. Let $0 < s \le b < s_1 < \sigma < t$. By Corollary 3.10 with $k = 0$ and $s = s_1$ we find

$$|u(g)| \le \|u\|_\sigma e^{|g|^2/(2s_1)}. \tag{5.2}$$

Thus, using $|gx^{-1}|^2 \le (|g|+|x|)^2 \le 2|g|^2+2|x|^2$, we have

$$\begin{aligned}(|u| * \mu_s)(g) &= \int_G |u(gx^{-1})|\mu_s(x)dx \\ &\le \int_G \|u\|_\sigma e^{|gx^{-1}|^2/(2s_1)}\mu_s(x)dx \\ &\le \|u\|_\sigma e^{|g|^2/s_1}\int e^{|x|^2/s_1}\mu_s(x)dx \\ &\le Ke^{|g|^2/s_1} \text{ for } 0 < s \le b.\end{aligned}$$

by Lemma 4.3, part 4 with $s_0 = b$. This proves a) and moreover shows that for each g, $(|u| * \mu_s)(g)$ is uniformly bounded in s for each subinterval $(0,b] \subset (0,t)$. To prove b) choose functions h_n again as in Lemma 3.6 and let $v(s,g) \doteq (u * \mu_s)(g)$. Then by Eq. (3.2)

$$v(s,g) = \int_G u(gx^{-1})\mu_s(x)dx = \int_G u(gx)\mu_s(x)dx.$$

Let

$$v_n(s,g) = \int_G h_n(x)u(gx)\mu_s(x)dx.$$

Clearly $v_n(s,g)$ converges to $v(s,g)$ for each g in G and for all s in $(0,t)$. Since $h_n(x)u(gx)$ is in $C_c^\infty(G)$, $v_n(\cdot,g)$ is in $C^\infty((0,1))$. Moreover

$$\begin{aligned}(\partial/\partial s)v_n(s,g) &= \int h_n(x)u(gx)(\Delta/4)\mu_s(x)dx \\ &= \int_G (\Delta/4)\{h_n(x)u(gx)\}\mu_s(x)dx \\ &= \int_G \{[(\Delta/4)h_n(x)]u(gx) \\ &\qquad +(1/2)\langle\nabla h_n(x),(\nabla u)(gx)\rangle\}\mu_s(x)dx\end{aligned}$$

since $\Delta u = 0$. But $\Delta h_n(x)$ goes to zero pointwise and boundedly on G as does $|\nabla h_n(x)|$. The same method of estimating $|u| * \mu_s$ shows that for each g in G $\int_G |(\nabla u)(gx)|\mu_s(x)dx$ is also uniformly bounded on each subinterval $[a,b] \subset (0,t)$. One need only start with the case $k=1$ in Proposition 3.9. Hence $\partial v_n(s,g)/\partial s$ converges to zero for each s in $(0,t)$ as $n \to \infty$ and in fact converges boundedly on each subinterval $[a,b]$. Thus

$$v(b,g) - v(a,g) = \lim_{n\to\infty}\int_a^b (\partial v_n(s,g)/\partial s)ds = 0.$$

So $v(s,g)$ is constant in s on $(0,t)$. In view of Lemma 4.3 (part 3), the bound (5.2) also allows us to show (as in the first few lines of the proof of Proposition 4.1) that $\lim_{s\to 0} v(s,g) = u(g)$. This proves item b) of the lemma. Q.E.D.

Proof of Theorem 5.1. Let $g(\cdot)$ denote the G valued Brownian motion beginning at e and with transition semigroup $e^{t\Delta/4}$. Let u be in $\mathcal{H}L^2(\mu_t(x)dx)$. By Lemma 5.2 and the Markov property of $g(\cdot)$, the process $M(s) = u(g(s))$ is a martingale on $[0,t]$. Moreover $E(|M(s)|^2) = \|u\|_s^2$. It follows, by viewing conditional expectations as orthogonal projections, that $M(s) \to M(t)$ in L^2 as $s \uparrow t$ and in particular that

$$\lim_{s\uparrow t}\|u\|_s^2 = \|u\|_t^2.$$

But from the monotone convergence theorem and the definition in Eq. 2.8 we see that $\|\alpha\|_t^2 = \lim_{s\uparrow t}\|\alpha\|_s^2$. Combining these limits with Corollary 4.5, Theorem 5.1 follows. Q.E.D.

Remark 5.3 *Another proof of Theorem 5.1 will be given in the Appendix. It avoids Lemma 5.2 and the associated martingale argument, but uses instead the Li-Yau parabolic Harnack inequalities.*

Corollary 5.4 *Let $u \in \mathcal{H}L^2(\mu_t(x)dx)$. Suppose that $0 < s < t$, $r > 0$, and $s + r \le t$. Then*

$$|D^k u(g)|^2 \le k!(d/r)^k\|u\|_t^2 e^{|g|^2/s},\ k = 0,1,2,\ldots, \tag{5.3}$$

Proof. By Theorem 5.1 we may apply Proposition 3.9. Q.E.D.

Remark 5.5 *If we put $k = 0$ in Eq. (5.3) we obtain an inequality in which r does not appear. We may therefore take the limit $s \uparrow t$ to find:*

$$|u(g)|^2 \le \|u\|_t^2 e^{|g|^2/t}. \tag{5.4}$$

This inequality reduces exactly to Bargmann's pointwise bound (Eq. (1.7) in [2]) in case $G = \mathbb{C}^n$.

Theorem 5.6 *Assume G is simply connected and α is in J_t^0. Then there exists a function u in $\mathcal{H}L^2(\mu_t(x)dx)$ such that $(1-D)_e^{-1}u = \alpha$. Moreover*

$$\|\alpha\|_t = \|u\|_t. \tag{5.5}$$

Proof. Assume α is in J_t^0. By Corollary 4.6 there is a function u in $\mathcal{H}(G)$ such that $(1-D)_e^{-1}u = \alpha$ and moreover (4.16) holds. By Fatou's Lemma in the integral and the monotone convergence theorem in the sum we may take the limit $s \uparrow t$ in (4.16) and find $\|\alpha\|_t \geq \|u\|_t$. Hence u is in $\mathcal{H}L^2(\mu_t(x)dx)$. By Theorem 5.1, Equation (5.5) holds. Q.E.D.

This completes the proof of Theorem 2.6 .

6 Reproducing Kernels

The next lemma asserts that $\mathcal{H}L^2(\mu_t(x)dx)$ has a reproducing kernel. G need not be simply connected. But in Proposition 6.2 we will give a "power series" representation for the reproducing kernel which is valid in case G is simply connected.

Lemma 6.1 *Let $t > 0$. For all $g \in G$, there exists a unique holomorphic function $K_t(g, \cdot) \in \mathcal{H}L^2(G, \mu_t(x)dx)$ such that*

$$u(g) = (u, K_t(g, \cdot))_t, \forall u \in \mathcal{H}L^2(G, \mu_t(x)dx).$$

Proof. By Lemma 3.4, the map $(u \to u(g)) : \mathcal{H}L^2(G, \mu_t(x)dx) \to \mathbb{C}$ is a bounded linear functional on $\mathcal{H}L^2(G, \mu_t(x)dx)$. Hence, the Riesz representation theorem guarantees the existence of $K_t(g, \cdot)$. Q.E.D.

Proposition 6.2 *Keep the same assumptions and notation as in Lemma 6.1. Assume further that G is simply connected. Let $J_t^\perp$ denote the orthogonal complement of J in T_t. Then*

$$K_t(g, x) = (\Psi(\sigma), P_t\Psi(\tau))_{T_t}, \tag{6.1}$$

where P_t is the orthogonal projection of T_t onto $J_t^\perp$, and σ and τ are any two smooth curves ($\sigma, \tau : [0,1] \to G$) such that $\sigma(0) = \tau(0) = e$, $\sigma(1) = x$, and $\tau(1) = g$.

Proof. Let $u \in \mathcal{H}L^2(G, \mu_t(x)dx)$ and set $\alpha = (1-D)_e^{-1}u$ and $\beta \doteq (1-D)_e^{-1}K_t(g, \cdot)$. Then as in the proof of Proposition 3.9,

$$u(g) = \langle \alpha, \Psi(\tau)\rangle = \langle \alpha, P_t\Psi(\tau)\rangle = (\alpha, (\cdot, P_t\Psi(\tau))_{T_t})_t.$$

While by Lemma 6.1

$$u(g) = (u, K_t(g, \cdot))_t = (\alpha, \beta)_t.$$

Since $u \in \mathcal{H}L^2(G, \mu_t(x)dx)$ is arbitrary so is α by Theorem 2.6. Hence comparing the above displayed equations shows that

$$\beta = (\cdot, P_t\Psi(\tau))_{T_t}.$$

Finally

$$K_t(g, x) = \langle \beta, \Psi(\sigma)\rangle = (\Psi(\sigma), P_t\Psi(\tau))_{T_t}.$$

Q.E.D.

7 Differential Operators, Annihilation Operators, and Uniqueness of Isomorphisms

We continue the notation of Section 2. We will take G to be simply connected as well as connected and complex in this section.

Definition 7.1 *Let $\xi \in \mathfrak{g}$. Denote by R_ξ the operation of right multiplication by ξ on T. That is, $R_\xi\beta = \beta \otimes \xi$. Let $A_\xi = R_\xi^* : T' \to T'$.*

Note that $A_\xi J^0 \subset J^0$ because $R_\xi J \subset J$. Now let $f \in \mathcal{H}(G)$ and write $\beta = \xi_1 \otimes \cdots \otimes \xi_n \in \mathfrak{g}^{\otimes n}$. Fix $x \in G$. By Eq. (2.9) we have

$$\langle A_\xi(1-D)_x^{-1}f, \beta\rangle = \langle (1-D)_x^{-1}f, \beta \otimes \xi\rangle = \tilde{\beta}\tilde{\xi}f(x) = \langle (1-D)_x^{-1}\tilde{\xi}f, \beta\rangle,$$

and hence

$$A_\xi(1-D)_x^{-1}f = (1-D)_x^{-1}\tilde{\xi}f. \tag{7.1}$$

This equation is purely algebraic; no norm restrictions on either side are required for its validity. But we wish to consider now $\tilde{\xi}$ and A_ξ as operators in $\mathcal{H}L^2(\mu_t(x)dx)$ and J_t^0 respectively. Define $\tilde{\xi}_t f = \tilde{\xi} f$ with domain $\mathcal{D}(\tilde{\xi}_t) = \{f \in \mathcal{H}L^2(\mu_t(x)dx) : \tilde{\xi}f \in \mathcal{H}L^2(\mu_t(x)dx)\}$. Similarly write $A_{\xi,t}\alpha = A_\xi\alpha$ with $\mathcal{D}(A_{\xi,t}) = \{\alpha \in J_t^0 : A_\xi\alpha \in J_t^0\}$. Then we have

Lemma 7.2

$$A_{\xi,t}(1-D)_e^{-1} = (1-D)_e^{-1}\tilde{\xi}_t \tag{7.2}$$

Proof. In view of Theorem 2.6 and the definition of the domains of these operators this is merely a restatement of Eq. (7.1) but with domains asserted to match up correctly under the unitary operator $(1-D)_e^{-1}$. Q.E.D.

Now it is not immediately clear that, for a general complex (connected, simply connected) Lie group G, $\mathcal{D}(\tilde{\xi}_t)$ is dense in $\mathcal{H}L^2(\mu_t(x)dx)$ or even contains any nonzero functions. Equivalently, the domain of $A_{\xi,t}$ is not manifestly nontrivial. More generally, denoting by $C^\infty(\mu_t)$ the set of all functions $f \in \mathcal{H}L^2(\mu_t(x)dx)$ which are in the domain of all finite products of the operators $\{\tilde{\xi}_t : \xi \in \mathfrak{g}\}$ and by $C^\infty(J_t^0)$ the similarly defined subspace of J_t^0 for the $A_{\xi,t}$, one may ask whether $C^\infty(\mu_t)$ is dense in $\mathcal{H}L^2(\mu_t(x)dx)$ or, equivalently, whether $C^\infty(J_t^0)$ is dense in J_t^0. Since $R_\xi\mathfrak{g}^{\otimes n} \subset \mathfrak{g}^{\otimes(n+1)}$ and $\mathfrak{g}$ is finite dimensional, A_ξ is defined on all of $(\mathfrak{g}^*)^{\otimes(n+1)}$ and takes this subspace of T' into

$(\mathfrak{g}^*)^{\otimes n}$. That is, A_ξ lowers rank by one. Consequently any finite rank tensor $\alpha \in J^0$ is in $C^\infty(J_t^0)$. However it is known [11] that if $\mathfrak{g}$ is semisimple then J^0 contains no finite rank tensors except the zero rank tensor. Nevertheless it is true that if G has a real form of compact type and the inner product on $\mathfrak{g}$ is Ad invariant under the real form then $C^\infty(J_t^0)$ is dense in J_t^0. (See [10], Section 6, or [8], Section 7.) In particular if G is semisimple and the given inner product on $\mathfrak{g}$ is Ad invariant under some compact real form then $C^\infty(J_t^0)$ is dense in J_t^0. We conjecture that if G is solvable then J_t^0 has a dense set of finite rank tensors. If this conjecture is correct then $C^\infty(J_t^0)$ is again dense in J_t^0. Finally, we mention that in the simplest case, that in which $\mathfrak{g}$ is abelian, J_t^0 is just the space of symmetric tensors in T_t^* and therefore $C^\infty(J_t^0)$ is dense in J_t^0. But for a general complex Lie algebra we have only the following limited, though suggestive, information.

Proposition 7.3 *Let $\mathfrak{g}$ be a complex Lie algebra with a Hermitian inner product. Let $t > 0$ and $\epsilon > 0$. Then*

$$J_{t+\epsilon}^0 \subset C^\infty(J_t^0). \tag{7.3}$$

Proof. For any $\beta \in \mathfrak{g}^{\otimes k}, \xi \in \mathfrak{g}$, and $\alpha = \sum_{k=o}^\infty \alpha_k \in T'$ we have

$$\begin{aligned} |\langle (A_\xi \alpha)_k, \beta\rangle| &= |\langle A_\xi \alpha, \beta\rangle| = |\langle \alpha, \beta \otimes \xi\rangle| = |\langle \alpha_{k+1}, \beta \otimes \xi\rangle| \\ &\leq |\alpha_{k+1}|\ |\beta \otimes \xi| = |\alpha_{k+1}||\beta||\xi|. \end{aligned}$$

Hence $|(A_\xi\alpha)_k|_{(\mathfrak{g}*)^{\otimes k}} \leq |\xi||\alpha_{k+1}|_{(\mathfrak{g}*)^{\otimes(k+1)}}$ Thus if $0 < r < s$ then

$$\begin{aligned} ||A_\xi \alpha||_r^2 &= \sum_{k=0}^\infty (r^k/k!)|(A_\xi\alpha)_k|^2_{(\mathfrak{g}^*)^{\otimes k}} \\ &\leq \sum_{k=0}^\infty (r^k(k+1)/s^{k+1})(s^{k+1}/(k+1)!)|\xi|^2|\alpha_{k+1}|^2_{(\mathfrak{g}*)^{\otimes(k+1)}} \\ &\leq C(r,s)|\xi|^2||\alpha||_s^2 \end{aligned}$$

where $C(r,s) := \sup_{k\geq 0} s^{-1}(r/s)^k(k+1) < \infty$. Hence $A_\xi J_s^0 \subset J_r^0$ whenever $r < s$. By dividing the interval $[t, t+\epsilon]$ into n equal subintervals it now follows that $A_{\xi_1}\cdots A_{\xi_n} J_{t+\epsilon}^0 \subset J_t^0$. Q.E.D.

Remark 7.4 *$\cup_{\epsilon>0} J_{t+\epsilon}^0$ is dense in J_t^0 if G is semisimple (and the inner product is Ad invariant under some real form) or commutative. But we don't know whether such density holds in general. Nevertheless we will show in the next proposition how the identity (7.1) determines the unitary map $(1-D)_e^{-1}$ uniquely when $C^\infty(J_t^0)$ is dense in J_t^0.*

Let us write $\omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in T'$. Since no tensor in J has a non-zero component of rank zero it follows that $\omega \in J_t^0$ for all $t > 0$. Moreover for any $\alpha \in J_t^0$ we have $(\alpha, \omega)_t = \alpha_0 = \langle \alpha, \overline{\omega}\rangle$ where $\overline{\omega}$ is defined just as ω but regarded as an element of T.

Proposition 7.5 *Let $t > 0$. Let G be a connected, simply connected, complex Lie group. Assume that $C^\infty(\mu_t)$ is dense in $\mathcal{H}L^2(\mu_t(x)dx)$. Suppose that $U : \mathcal{H}L^2(\mu_t(x)dx) \to J_t^0$ is a unitary operator such that*

1. *$U\hat{1} = \omega$*
2. *$U\tilde{\xi}_t = A_{\xi,t}U$ for all $\xi \in \mathfrak{g}$.*

Then $U = (1-D)_e^{-1}$.

Proof. We assert that

$$(f,1)_{L^2(\mu_t(x)dx)} = f(e) \text{ for all } f \in \mathcal{H}L^2(\mu_t(x)dx) \tag{7.4}$$

For in fact $(1-D)_e^{-1}1 = \omega$, so that

$$(f,1)_{L^2(\mu_t(x)dx)} = ((1-D)_e^{-1}f, \omega)_t = D^0 f(e) = f(e).$$

Let $f \in C^\infty(\mu_t)$. Then $\alpha := Uf \in C^\infty(J_t^0)$ by repeated application of condition 2. on U. Let $\beta = \xi_1 \otimes \cdots \otimes \xi_n \in \mathfrak{g}^{\otimes n}$ and let $\tilde{\beta} = \tilde{\xi}_1 \cdots \tilde{\xi}_n$. Then

$$\langle \alpha, \beta\rangle = \langle A_{\xi_n}\alpha, \xi_1 \otimes \cdots \otimes \xi_{n-1}\rangle = \langle U\tilde{\xi}_n f, \xi_1 \otimes \cdots \otimes \xi_{n-1}\rangle$$

by 2. again. So by induction we have

$$\begin{aligned} \langle \alpha, \beta\rangle &= \langle U\tilde{\beta}f, \overline{\omega}\rangle = (U\tilde{\beta}f, \omega)_t = (U\tilde{\beta}f, U1)_t \\ &= (\tilde{\beta}f, 1)_{L^2(\mu_t(x)dx)} = (\tilde{\beta}f)(e) = \langle (1-D)_e^{-1}f, \beta\rangle. \end{aligned}$$

Hence $Uf = (1-D)_e^{-1}f$. Since $C^\infty(\mu_t)$ is dense in $\mathcal{H}L^2(\mu_t(x)dx)$ and U and $(1-D)_e^{-1}$ are both unitary they are both equal. Q.E.D.

8 Appendix: Two Applications of Parabolic Harnack Inequalities

Our proof of Theorem 5.1 is elementary but relies on Lemma 5.2 for the transition from equality of norms at time $s < t$ to equality at time $s = t$. A shorter proof of this transition can be given which depends, however, on the Li-Yau parabolic Harnack inequalities. We will state them and apply them in case G is unimodular. The reason for restricting our attention to a unimodular group is that in this case the operator Δ in Eq. (2.2) is the Laplace-Beltrami operator for $(G, \langle \cdot, \cdot \rangle)$ as was pointed out in Remark 2.2. In the nonunimodular case Δ differs from the Laplace-Beltrami operator by a left invariant vector field.[2]

Lemma 8.1 (Li-Yau Harnack Inequality) *Assume that G is unimodular. Let $T > 0$ be given. For each $\gamma > 1$, there is a constant K depending on γ and T such that any positive solution, u, to the heat equation*

$$\partial u/\partial t = \Delta u/4,$$

on $(0, T) \times G$ satisfies the Harnack inequality:

$$u(s,x) \le K(\frac{t}{s})^{d\gamma} u(t,y) \exp\{\frac{\gamma d^2(x,y)}{t-s}\} \tag{8.1}$$

for all x and y in G, and $0 < s < t < T$.

For a proof of this lemma for the Laplace-Beltrami operator the reader is referred to Theorem 5.3.5, p. 162 in Davies [7], which is applicable in our case because the Ricci curvature is left translation invariant and has therefore a uniform lower bound.

Alternate proof of Theorem 5.1 for a unimodular group. Assume $u \in \mathcal{H}L^2(\mu_t(x)dx)$ and $\alpha = (1-D)_e^{-1}u$. By Corollary 4.5 we have $\|\alpha\|_s = \|u\|_s$ for $0 < s < t$. By the monotone convergence theorem we have $\lim_{s\uparrow t} \|\alpha\|_s = \|\alpha\|_t$. It therefore suffices to show that

$$\lim_{s\uparrow t} \|u\|_s = \|u\|_t. \tag{8.2}$$

But by Lemma 8.1 with $x = y$, we have $\mu_s(x) \le K(\frac{t}{s})^{d\gamma}\mu_t(x)$ for $0 < s < t$. Since $\int |u(x)|^2 \mu_t(x)dx < \infty$ and $\lim_{s\uparrow t} \mu_s(x) = \mu_t(x)$ pointwise, the parabolic Harnack inequality allows us to prove (8.2) by the dominated convergence theorem. Q.E.D.

It is interesting that the Li-Yau Harnack inequality also gives pointwise bounds on the derivatives of holomorphic functions similar to those of Corollary 5.4. The proof avoids the combinatorial method on which Proposition 3.9 (and therefore Corollary 5.4) is based. But the result is less precise than that of Proposition 3.9 and in particular inadequate for proving the key Lemma 4.4. However the proof is very short, given steps that we have already used elsewhere.

[2] In a recent preprint, Feng-Yu Wang [34] (see Theorem 2.1) has given an extension of the Li-Yau Harnack inequality which could be used in this appendix to remove the unimodular restriction. All of the proofs would remain unchanged.

Proposition 8.2 *Assume that G is unimodular and that $u \in \mathcal{H}L^2(\mu_t(x)dx)$. Let $0 < s < t$. There are constants $C(s,k) < \infty$ such that*

$$|D^k u(g)|^2 \le C(s,k)\|u\|_t^2 \exp(|g|^2/s), \quad k = 0,1,2,\ldots. \tag{8.3}$$

Proof. Given $0 < s < t$ choose $\epsilon > 0$ and $\gamma > 1$ such that $(t-\epsilon)/\gamma = s$. Because of the left invariance of the Riemannian distance d, $d(yg, y) = |g|$ for all $y, g \in G$. In (8.1) replace s by ϵ, x by yg and u by the heat kernel μ_t to find:

$$\mu_\epsilon(yg) \le K(\frac{t}{\epsilon})^{d\gamma}\mu_t(y)\exp\{\frac{\gamma|g|^2}{t-\epsilon}\} = K(\frac{t}{\epsilon})^{d\gamma}\mu_t(y)e^{|g|^2/s}, \tag{8.4}$$

where $K < \infty$ is a constant depending on $\gamma = (t-\epsilon)/s$. In Eq. (3.14), σ is at our disposal in the interval $[\epsilon, t]$. Choose $\sigma = \epsilon$ in (3.14) and use Eq. (8.4) to get

$$\begin{aligned} |\langle D^k u(g), \beta\rangle|^2 &\le a(\epsilon,k)^2|\beta|^2 K(\frac{t}{\epsilon})^{d\gamma} e^{|g|^2/s}\int_G |u(y^{-1})|^2\mu_t(y)dy \\ &= a(\epsilon,k)^2|\beta|^2 K(\frac{t}{\epsilon})^{d\gamma} e^{|g|^2/s}\|u\|_t^2, \end{aligned}$$

where we have used Eq. (3.2) in the last equality. Summing this last equation over β running through an orthonormal basis of $\mathfrak{g}^{\otimes k}$ as in the proof of Lemma 3.5 gives (8.3) with $C(s,k) = a(\epsilon,k)^2 d^k K(\frac{t}{\epsilon})^{d\gamma}$. Q.E.D.

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