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ON HEAT KERNEL LOGARITHMIC SOBOLEV INEQUALITIES

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This note is devoted to a pedagogical proof of logarithmic Sobolev inequalities on compact Riemannian manifolds when the reference measure is taken to be a heat kernel measure. The inequalities explained in this paper seems to have been first discovered by D. Bakry and M. Ledoux [7].

1 Introduction

Let M be a finite dimensional connected compact[a] manifold without boundary, $(\cdot,\cdot)$ be Riemannian metric on M, Δ be the Laplacian on M, Ric be the Ricci tensor of the Levi-Civita covariant derivative ∇, Z be a smooth vector field on M, and $L \equiv \Delta + Z$. We write $P_t = e^{tL/2}$ for the associated Markov semigroup and define

$$\rho\langle v, w\rangle \equiv \mathrm{Ric}\langle v, w\rangle - (\nabla_v Z, w), \quad \forall\, v, w \in T_m M \text{ and } m \in M. \tag{1.1}$$

The purpose of this paper is to give a reasonably self contained proof of the following theorem. For applications of this theorem to path and loop spaces, see Hsu [15], Wang [19], and Driver and Lohrenz [12].

Theorem 1.1 (Bakry and Ledoux) *Choose $C \in \mathbb{R}$ such that $\rho\langle v, v\rangle \geq -C(v,v)$ for all $v \in TM$. Then for all $f \in C^\infty(M)$ and $T > 0$,*

$$P_T(f^2 \log f^2) \leq 2\{(e^{CT}-1)/C\} P_T|\vec{\nabla} f|^2 + P_T f^2 \cdot \log P_T(f^2), \tag{1.2}$$

where $\vec{\nabla} f$ denotes the gradient of f and $0 \log 0 = 0$ by definition.

[a] The proof for complete Riemannian manifolds involves a number of technicalities which we wish to avoid in this expository note.

Notation 1.2 *For each $o \in M$ and $T > 0$, let $\mu_{o,T}$ denote the probability measure on M such that*

$$\mu_{o,T}(f) \equiv \int_M f d\mu_{o,T} = (P_T f)(o) \quad \forall f \in C^\infty(M). \tag{1.3}$$

Let $\mathcal{E}_{o,T}$ denote the closure of the symmetric quadratic form $\mathcal{E}^0_{o,T}$ on $C^\infty(M)$ determined by

$$\mathcal{E}^0_{o,T}(f,f) = \int_M |\vec{\nabla} f|^2 d\mu_{o,T} = P_T(|\vec{\nabla} f|^2)(o) \quad \forall f \in C^\infty(M). \tag{1.4}$$

We also let $H_{o,T}$ denote the generator of $\mathcal{E}_{o,T}$, i.e. $H_{o,T}$ is the unique selfadjoint operator on $L^2(M, \mu_{o,T})$ such that

$$\mathcal{E}_{o,T}(f,f) = (H^{1/2}_{o,T} f, H^{1/2}_{o,T} f)_{L^2(\mu_{o,T})} \quad \forall f \in \mathcal{D}(\mathcal{E}_{o,T}) = \mathcal{D}(H^{1/2}_{o,T}), \tag{1.5}$$

where $\mathcal{D}(\mathcal{E}_{o,T})$ and $\mathcal{D}(H^{1/2}_{o,T})$ denotes the domain of $\mathcal{E}_{o,T}$ and $H^{1/2}_{o,T}$ respectively.

With this notation we may rewrite Theorem 1.1 as follows:

Theorem 1.3 *Fix $T > 0$ and $o \in M$ and choose $C \in \mathbb{R}$ such that $\rho\langle v, v\rangle \geq -C(v,v)$ for all $v \in TM$. Then for all $f \in \mathcal{D}(\mathcal{E}_{o,T})$,*

$$\mu_{o,T}(f^2 \log f^2) \leq 2\{(e^{CT} - 1)/C\}\mathcal{E}_{o,T}(f,f) + \mu_{o,T}(f^2) \cdot \log \mu_{o,T}(f^2). \tag{1.6}$$

Proof. For $f \in C^\infty(M)$, Eq. (1.6) is a special case of Eq. (1.2). Since by definition $C^\infty(M)$ is a core for $\mathcal{E}_{o,T}$, a simple limiting argument shows that (1.6) holds for all $f \in \mathcal{D}(\mathcal{E}_{o,T})$, see the last paragraph in the proof of Theorem 2 of Gross [13]. Q.E.D.

There are by now a number of proofs of Theorem 1.1 and its generalizations, see Bakry and Ledoux [8], E. Hsu [15], and F. Wang [18,19]. All of these proofs follow the circle of ideas introduced by Bakry and Emery [5,6], see also Bakry [1,2,3,4]. The main difference is the method of proof of the important intermediate inequality:

$$|\vec{\nabla} P_T f| \leq e^{CT} P_T |\vec{\nabla} f|. \tag{1.7}$$

(Note: this inequality follows easily from Theorem 3.4 of Donnelly and Li [11] and the Bochner-Weitzenbock formulas.) The proof we give below is essentially the one explained to us by Dominique Bakry with some modifications so as to avoid using (1.7) altogether. Although avoiding (1.7) is not a virtue, we hope the reader will find the proof given below a useful introduction to the paper of Bakry and Ledoux [8].

2 Preliminaries

We will write $(\cdot,\cdot)$ and ∇ for the induced metric and covariant derivative respectively on any of the vector bundles $T^{\otimes k}M\otimes(T^*M)^{\otimes l}$ for $k,l=0,1,2,\ldots,$ where as usual $T^{\otimes 0}M=(T^*M)^{\otimes 0}$ is to be taken as the trivial vector bundle $M\times\mathbb{R}$. Also let $|\xi|^2\equiv(\xi,\xi)$ for $\xi\in T^{\otimes k}M\otimes(T^*M)^{\otimes l}$. Note that with this notation for $f\in C^\infty(M)$, $\nabla f=df$ and $|\nabla f|=|\vec{\nabla}f|$. Following Bakry and Emery [5,6] for $f\in C^\infty(M)$ let

$$\Gamma(f,f)\equiv\frac{1}{2}\{L(f^2)-2fLf\}=|\nabla f|^2, \tag{2.1}$$

and

$$\Gamma_2(f,f)\equiv\frac{1}{2}\{L(\Gamma(f,f))-2\Gamma(f,Lf)\}=\frac{1}{2}L|\nabla f|^2-(\nabla f,\nabla Lf). \tag{2.2}$$

It is well known (see Bakry and Emery [5,6]) that Γ_2 is given explicitly by:

$$\Gamma_2(f,f)=|\mathrm{Hess}f|^2+\rho\langle\vec{\nabla}f,\vec{\nabla}f\rangle, \tag{2.3}$$

where $\mathrm{Hess}f=\nabla df\in T^*M^{\otimes 2}$ is the Hessian of f.

Given $f\in C^\infty(M)$ and $T>0$, let

$$F(t,m)\equiv(P_{T-t}f)(m). \tag{2.4}$$

Then F solves the backwards heat equation:

$$\partial F/\partial t=-LF/2.$$

A simple computation shows that

$$\frac{d}{dt}(P_t(F^2(t,\cdot)))(m)=P_t(\Gamma(F(t,\cdot),F(t,\cdot)))(m).$$

To simplify notation in the sequel, we will suppress the t and the m variables from the notation and write the above equation simply as

$$\frac{d}{dt}(PF^2)=P\Gamma(F,F). \tag{2.5}$$

A similar computation using this abbreviated notation gives

$$\frac{d}{dt}(P\Gamma(F,F))=P\Gamma_2(F,F). \tag{2.6}$$

The following well known lemma shows that knowledge of Γ_2 determines ρ.

Lemma 2.1 *For all $m \in M$ and $v \in T_m M$,*

$$\rho\langle v, v\rangle = \inf\{\Gamma_2(F,F)(m) : F \in C^\infty(M) \text{ s.t. } \vec{\nabla} F(m) = v\}. \qquad (2.7)$$

Proof. (Sketch) Let $Q\langle v, v\rangle$ denote the RHS of (2.7). By (2.3) it is clear that if $\vec{\nabla} F(m) = v$, then $\Gamma_2(F,F)(m) \geq \rho\langle v, v\rangle$ and hence $Q\langle v, v\rangle \geq \rho\langle v, v\rangle$. So to finish the proof it suffices to show there exists $F \in C^\infty(M)$ such that $(\text{Hess}F)(m) = 0$ and $\vec{\nabla} F = v$. For then, by Eq. (2.3), $\Gamma_2(F,F)(m) = \rho\langle v, v\rangle$ so that $Q\langle v, v\rangle \leq \rho\langle v, v\rangle$. To construct such a function, choose $F \in C^\infty(M)$ such that $F(p) \equiv \langle v, \exp_m^{-1}(p)\rangle$ for p near m, where $\exp_m : T_m M \to M$ is the exponential map at m. Q.E.D.

Before going to the proof of Theorem 1.1, it is instructive to first consider the simpler issue of spectral gaps for the operator $H_{o,T}$. This is the content of the next section.

3 Spectral gap inequalities

Remark 3.1 *The maximum principle (or the fact that $(P_t f)(x)$ may be written as an integral relative to a probability measure) shows, for any $t > 0$, that the range of $P_t f$ is contained in an interval J provided the range of f is contained in J. This comment will be used in this section without further mention.*

Lemma 3.2 *Suppose $f \in C^\infty(M)$, $\phi : J \subset \mathbb{R} \to \mathbb{R}$ is a smooth function on an open interval $J \subset \mathbb{R}$. Assume $f(M) \subset J$ so that $\phi \circ f$ is also a smooth function on M. Fix $T > 0$ and $F(t, \cdot) \equiv P_{T-t} f$ (as in (2.4)) and set*

$$\Psi(t, m) \equiv P_t(\phi(F(t, \cdot)))(m) = P_t(\phi(P_{T-t} f))(m)$$

for all $0 \leq t \leq T$ and $m \in M$. Then

$$d\Psi/dt = P\{\phi''(F)|\nabla F|^2/2\} \quad \text{for } t \in (0, T). \qquad (3.1)$$

Proof. We compute:

$$\begin{aligned} d\Psi/dt &= P\{L\phi(F)/2 + \frac{d}{dt}\phi(F)\}. \\ &= P\{L\phi(F)/2 - \phi'(F)LF/2\}. \end{aligned} \qquad (3.2)$$

Now

$$L\phi(F) = \Delta\phi(F) + Z\phi(F) = \Delta\phi(F) + \phi'(F)ZF,$$

and

$$\Delta\phi(F) = \mathrm{tr}\nabla\{\phi'(F)\nabla F\} = \{\phi''(F)|\nabla F|^2 + \phi'(F)\Delta F\},$$

so that

$$L\phi(F) = \phi''(F)|\nabla F|^2 + \phi'(F)LF. \tag{3.3}$$

Eq. (3.1) follows from (3.2) and (3.3). Q.E.D.

Theorem 3.3 (Spectral Gap) *Choose $C \in \mathbb{R}$ such that $\rho \geq -C(\cdot,\cdot)$. Then for all $f \in C^\infty(M)$ and $T > 0$*

$$P_T(f^2) - (P_T f)^2 \leq \{(e^{CT} - 1)/C\} P_T|\nabla f|^2. \tag{3.4}$$

Remark 3.4 *For $f \in L^2(\mu_{o,T})$, let $Qf = Q_{o,T}f \equiv f - (\int f d\mu)\cdot 1$ be orthogonal projection onto $\{1\}^\perp$. By an easy limiting argument (as discussed in Theorem 1.3), Theorem 3.3 implies*

$$\|Qf\|^2_{L^2(\mu_{o,T})} \leq \{(e^{CT} - 1)/C\}\mathcal{E}_{o,T}(f,f) \quad \forall f \in \mathcal{D}(\mathcal{E}_{o,T}). \tag{3.5}$$

In particular we have:

$$\|Qf\|^2_{L^2(\mu_{o,T})} \leq \{(e^{CT} - 1)/C\}(H_{o,T}f,f) \quad \forall f \in \mathcal{D}(H_{o,T}). \tag{3.6}$$

From this equation we conclude that the null space of $H_{o,T}$ is the constant functions and the spectrum of $H_{o,T}$ is contained in $\{0\} \cup [C(e^{CT} - 1)^{-1}, \infty)$. Hence Eq. (3.4) implies a spectral gap for $H_{o,T}$ which is bigger than or equal to $C(e^{CT} - 1)^{-1}$.

Proof of Theorem 3.3. Let F be as in (2.4) and take $\phi(x) = x^2$ in Lemma 3.2 to find that $\dot\Psi = P(|\nabla F|^2)$ (alternatively use (2.5)). By (2.6) we find

$$\ddot\Psi = P(\Gamma_2(F,F)) \geq -CP(|\nabla F|^2) = -C\dot\Psi, \tag{3.7}$$

where we have used the assumed bound on ρ and Eq. (2.3). It follows from (3.7) that $\frac{d}{dt}\log\dot\Psi \geq -C$ and hence $\log\dot\Psi|_t^T \geq -C(T-t)$. Thus

$$\dot\Psi(t) \leq \dot\Psi(T)e^{C(T-t)}.$$

Integrating this last equation over $[0,T]$ gives $\Psi(T) - \Psi(0) \leq \dot\Psi(T)(e^{CT} - 1)/C$, from which (3.4) follows, since $\Psi(T) = P_T f^2$, $\Psi(0) = (P_T f)^2$ and $\dot\Psi(T) = P_T(|\nabla f|^2)$. Q.E.D.

The following corollary is a weak version of Toponogov's theorem, see Theorem 9 on p. 82 of Chavel.[9] Toponogov's theorem gives an estimate on the first eigenvalue of Δ in terms of C and the dimension of the manifold M. In the original proof of this theorem one has to make use of the estimate $|\mathrm{Hess} f|^2 \geq (\Delta f)^2/n$, where $n = \dim(M)$, instead of throwing it away like in Eq. (3.7). Presumably there should be a way to improve the spectral gap inequality (3.4) by making use of $|\mathrm{Hess} f|^2 \geq (\Delta f)^2/n$ to obtain the original Toponogov's theorem.

Corollary 3.5 (Spectral Gap for the Volume Form) *Let ω denote the normalized volume measure on M. Assume $Z = 0$ and there is a constant $k > 0$ such that $\rho\langle\cdot,\cdot\rangle = \mathrm{Ric}\langle\cdot,\cdot\rangle \geq k\langle\cdot,\cdot\rangle$. Then*

$$\omega(f^2) - (\omega(f))^2 \leq \frac{1}{k}\omega(|\nabla f|^2), \tag{3.8}$$

where

$$\omega(f) \equiv \int_M f d\omega.$$

Proof. Let $o \in M$ be fixed. Since M is compact P_T is ergodic in the sense that $\omega(f) = \lim_{T\to\infty}(P_T f)(o)$ for all $f \in C^\infty(M)$. The proof is completed by letting $T \to \infty$ in Theorem 3.3 keeping in mind that $C = -k < 0$. Q.E.D.

4 Proof of the logarithmic Sobolev inequality

Before giving a proof of Theorem 1.1 let us state the analogue of Corollary 3.5 for logarithmic Sobolev inequalities. Again this corollary was explained to us by D. Bakry.

Corollary 4.1 (Bakry and Emery [5,6]) *Let ω denote the normalized volume measure on M. Assume $Z \equiv 0$ and there is a constant $k > 0$ such that $\rho\langle\cdot,\cdot\rangle = \mathrm{Ric}\langle\cdot,\cdot\rangle \geq k\langle\cdot,\cdot\rangle$. Then*

$$\omega(f^2\log(f^2)) \leq \frac{2}{k}\omega(|\nabla f|^2) + \omega(f^2)\cdot\log\omega(f^2). \tag{4.1}$$

Proof. Following the proof of Corollary 3.5, let $T \to \infty$ in Theorem 1.1 keeping in mind that $C = -k < 0$. Q.E.D.

Remark 4.2 *For compact manifolds it is always possible to prove that (4.1) holds for some constant $k > 0$, see for example Rothaus [16]. In fact it is known from the pioneering work of L. Gross [13] that classical Sobolev inequalities imply logarithmic Sobolev inequalities. In this context see Deuschel and Stroock [10], Exercise 6.1.31. It is also possible to prove this fact using Theorem 1.1 and a Lemma of Holley and Stroock [14], see also Section II of Stroock [17].*

The proof of Theorem 1.1 will be given after some preparatory results.

Lemma 4.3 *Keeping the notation in Lemma 3.2, we have*

$$\begin{aligned}\ddot{\Psi} &= P[\frac{1}{4}\phi^{(4)}(F)|\nabla F|^4 + \frac{1}{2}\phi''(F)\Gamma_2(F,F)] \\ &\quad + P[\phi^{(3)}(F)(\nabla^2 F, \nabla F \otimes \nabla F)].\end{aligned}$$

This equation combined with (2.3) gives:

$$\begin{aligned}\ddot{\Psi} &= P(\frac{1}{4}\phi^{(4)}(F)|\nabla F \otimes \nabla F|^2 + \frac{1}{2}\phi''(F)\{|\nabla^2 F|^2 + \rho\langle\vec{\nabla} F, \vec{\nabla} F\rangle\}) \\ &\quad + P(\phi^{(3)}(F)(\nabla^2 F, \nabla F \otimes \nabla F)). \qquad (4.2)\end{aligned}$$

Proof. Because $\partial\phi(F)/\partial t = \phi'(F)LF/2$ and Eq. (3.3),

$$\frac{1}{2}L\phi(F) + \frac{d}{dt}\phi(F) = \phi''(F)|\nabla F|^2/2. \qquad (4.3)$$

We also have

$$\begin{aligned}L(fg) &= \mathrm{tr}\nabla(f\nabla g + g\nabla f) + Zf\cdot g + f\cdot Zg \\ &= \mathrm{tr}\{2\nabla f\otimes\nabla g + f\nabla^2 g + g\nabla^2 f) + Zf\cdot g + f\cdot Zg \\ &= Lf\cdot g + fLg + 2(\nabla f, \nabla g),\end{aligned}$$

and

$$\frac{1}{2}L|\nabla F|^2 + \frac{d}{dt}|\nabla F|^2 = \frac{1}{2}L|\nabla F|^2 - 2(\nabla F, \nabla(\frac{1}{2}LF)) = \Gamma_2(F,F).$$

Therefore

$$\begin{aligned}2\ddot{\Psi} &= P\{\frac{1}{2}L[\phi''(F)|\nabla F|^2] + \frac{d}{dt}[\phi''(F)|\nabla F|^2]\} \\ &= P(\{\frac{1}{2}L[\phi''(F)] + \frac{d}{dt}\phi''(F)\}|\nabla F|^2) \\ &\quad + P(\phi''(F)\{\frac{1}{2}L|\nabla F|^2 + \frac{d}{dt}|\nabla F|^2\}) \\ &\quad + P(\nabla\phi''(F), \nabla|\nabla F|^2) \\ &= P(\frac{1}{2}\phi^{(4)}(F)|\nabla F|^4) + P(\phi''(F)\Gamma_2(F,F)) \\ &\quad + P(\phi^{(3)}(F)\nabla F, \nabla|\nabla F|^2),\end{aligned}$$

wherein we have used Eq. (4.3) with ϕ replaced by ϕ''. The result follows from this equality, since

$$\begin{aligned}(\nabla F, \nabla|\nabla F|^2) &= \sum_i 2\nabla_{e_i} F \cdot (\nabla_{e_i}\nabla F, \nabla F) \\ &= \sum_{ij} 2\nabla_{e_i} F \cdot (\nabla^2 F(e_i, e_j)) \cdot \nabla_{e_j} F \\ &= 2(\nabla^2 F, \nabla F \otimes \nabla F),\end{aligned}$$

where $\{e_i\}_{i=1}^{\dim(M)}$ is any local orthonormal frame of TM. Q.E.D.

In order to make use of the last computation we will need the following elementary lemma.

Lemma 4.4 *Suppose that $(V, (\cdot,\cdot))$ is an inner product space of dimension larger than or equal to one and $a, b, c \in \mathbb{R}$. Then*

$$a(x,x) + b(x,y) + c(y,y) \geq 0 \quad \forall x, y \in V \tag{4.4}$$

iff

$$a \geq 0, c \geq 0, \quad \text{and} \quad b^2 \leq 4ac. \tag{4.5}$$

Proof. Assume that (4.4) holds. Choose $x \in V$ such that $|x| \equiv \sqrt{(x,x)} = 1$ and set $y = kx$, where $k \in \mathbb{R}$. Then (4.4) implies that $p(k) \equiv a + bk + ck^2 \geq 0$ for all $k \in \mathbb{R}$. This clearly implies that $a, c \geq 0$ and $b^2 - 4ac \leq 0$, since otherwise p would have two distinct real roots and hence $\min_k p(k) < 0$. Hence (4.4) implies (4.5).

Now assume that (4.5) holds, then as above the polynomial $q(k) = a - bk + ck^2$ is nonnegative. Hence if $|x| \neq 0$, then

$$\begin{aligned}a(x,x) + b(x,y) + c(y,y) &\geq a|x|^2 - |b||(x,y)| + c|y|^2 \\ &\geq a|x|^2 - |b||x||y| + c|y|^2 \\ &= |x|^2 q(|y|/|x|) \geq 0.\end{aligned}$$

Hence (4.5) implies (4.4). Q.E.D.

Corollary 4.5 *Let $f \in C^\infty(M)$ and $\phi : J \to \mathbb{R}$ be as in Lemma 3.2. Assume*

$$\phi^{(4)} \geq 0, \ \phi'' \geq 0, \ \text{and} \ 2[\phi^{(3)}]^2 \leq \phi'' \cdot \phi^{(4)}, \tag{4.6}$$

then

$$\ddot{\Psi} \geq P(\frac{1}{2}\phi''(F)\rho\langle\vec{\nabla} F, \vec{\nabla} F\rangle). \tag{4.7}$$

Proof. An application of Lemma 4.4 with $V = TM \otimes TM$, $x = \nabla F \otimes \nabla F$, $y = \nabla^2 F$, $a = \frac{1}{4}\phi^{(4)}(F)$, $b = \phi^{(3)}(F)$, and $c = \frac{1}{2}\phi''(F)$ shows that the conditions in (4.6) implies that

$$\frac{1}{4}\phi^{(4)}(F)|\nabla F \otimes \nabla F|^2 + \frac{1}{2}\phi''(F)|\nabla^2 F|^2 + \phi^{(3)}(F)(\nabla^2 F, \nabla F \otimes \nabla F) \geq 0.$$

Using this inequality in (4.2) gives (4.7). Q.E.D.

Theorem 4.6 *Choose $C \in \mathbb{R}$ such that $\rho \geq -C(\cdot,\cdot)$. Let $f \in C^\infty(M)$ and $\phi : J \to \mathbb{R}$ be functions as in Corollary 4.5. Then*

$$P_T(\phi(f)) - \phi(P_T(f)) \leq \{(e^{CT} - 1)/2C\} P_T(\phi''(f)|\nabla f|^2). \tag{4.8}$$

Proof. As in Lemma 3.2, set $F(t, m) \equiv (P_{T-t} f)(m)$ and $\Psi(t) \equiv P_t(\phi \circ F(t, \cdot))$. Then by Corollary 4.5, Eq. (3.1), and the assumption that $\rho \geq -C(\cdot,\cdot)$, it follows that

$$\ddot{\Psi} \geq -C\dot{\Psi}. \tag{4.9}$$

As in the proof of Theorem 3.3, this implies

$$\Psi(T) - \Psi(0) \leq \dot{\Psi}(T)(e^{CT} - 1)/C.$$

This last equation implies Eq. (4.8), since $\Psi(T) = P_T(\phi(f))$, $\Psi(0) = \phi(P_T f)$ and $\dot{\Psi}(T) = P_T(\phi''(f)|\nabla f|^2)/2$ by Eq. (3.1). Q.E.D.

Lemma 4.7 *The solutions to the differential equation*

$$2[\phi^{(3)}]^2 = \phi'' \cdot \phi^{(4)} \tag{4.10}$$

are

$$\phi(x) = Bx^2 + Cx + D, \tag{4.11}$$

or

$$\phi(x) = A^{-2}(Ax + B)\log|Ax + B| + Cx + D \tag{4.12}$$

for some constants A, B, C, and D, such that $A \neq 0$.

Proof. Suppose that ϕ is a solution to (4.10) and set $k(x) \equiv \phi''(x)$. Then k solves the ODE:

$$2[k']^2 = k \cdot k''. \tag{4.13}$$

One way to solve this equation is to have $k' \equiv 0$, in which case $k(x)$ has a constant value, i.e. $\phi''(x)$ is a constant. Hence $\phi(x) = Bx^2 + Cx + D$ for some constants B, C, and D.

Now assume that $k' \neq 0$ and we are on an interval in $\mathbb{R}$ where $k \neq 0$. In this case Eq. (4.13) may be written as:

$$2k'/k = k''/k'. \tag{4.14}$$

Integrating this last equation gives:

$$\log|k'| = \log|k|^2 + \text{const.}$$

and hence $k' = \tilde{A}k^2$, for some constant $\tilde{A}$. From this differential equation we learn that $k(x) = (Ax + B)^{-1}$ for some constants A and B. We may and do assume that $A \neq 0$, otherwise we will be back in the case where $k' \equiv 0$. Integrating the equation

$$\phi''(x) = k(x) = (Ax + B)^{-1}$$

gives (4.12). Q.E.D.

Remark 4.8 *Let $\phi(x) = x^2$, then ϕ verifies (4.6). Hence Theorem 3.3 easily follows from Theorem 4.6.*

Proof of Theorem 1.1. Let $B > 0$, and define $\phi_B(x) \equiv (x + B)\log(x + B)$, for $x > -B$. Then ϕ_B verifies (4.6) on $(-B, \infty)$ and $\phi_B''(x) = (x+B)^{-1}$. Therefore by Theorem 4.6, for a non-negative function $f \in C^\infty(M)$,

$$P_T(\phi_B(f)) - \phi_B(P_T(f)) \leq \frac{1}{2}\{(e^{CT} - 1)/C\}P_T((f + B)^{-1}|\nabla f|^2). \tag{4.15}$$

Now suppose that $f \in C^\infty(M)$ is arbitrary. Apply (4.15) with f replaced by f^2 to find:

$$\begin{aligned} P_T(\phi_B(f^2)) - \phi_B(P_T(f^2)) &\leq \frac{1}{2}\{(e^{CT} - 1)/C\}P_T(4(f^2 + B)^{-1}f^2|\nabla f|^2) \\ &\leq 2\{(e^{CT} - 1)/C\}P_T(|\nabla f|^2). \end{aligned} \tag{4.16}$$

By the dominated convergence theorem,

$$\lim_{B\downarrow 0} P_T(\phi_B(f^2)) = P_T(f^2 \log f^2),$$

where $0\log 0 \equiv 0$ since $\lim_{B\downarrow 0} \phi_B(0) = 0$. Hence letting $B \downarrow 0$ in Eq. (4.16) shows that Eq. (1.2) holds which proves Theorem 1.1. Q.E.D.

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