ON THE EQUIVALENCE OF MEASURES ON LOOP SPACE

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Abstract. Let K be a simply-connected compact Lie Group equipped with an Ad_K-invariant inner product on the Lie Algebra $\mathfrak{K}$, of K. Given this data, there is a well known left invariant "H^1-Riemannian structure" on $L(K)$ (the infinite dimensional group of continuous based loops in K), as well as a heat kernel $\nu_T(k_0, \cdot)$ associated with the Laplace-Beltrami operator on $L(K)$. Here $T > 0$, $k_0 \in L(K)$, and $\nu_T(k_0, \cdot)$ is a certain probability measure on $L(K)$. In this paper we show that $\nu_1(e, \cdot)$ is equivalent to Pinned Wiener Measure on K on $\mathfrak{G}_{s_0} \equiv \sigma\langle x_t : t \in [0, s_0]\rangle$ (the σ-algebra generated by truncated loops up to "time" s_0)

Contents

1. Introduction 2
2. Statement of Results 3
2.1. Loop group Geometry 3
2.2. Measures on the Loop group 5
2.3. The stochastic framework 10
3. Warm-up Section: 13
3.1. Path group cases for a Lie group: 13
3.2. Semimartingale Properties of X_{ts}: 19
3.3. Abelian Loop group Examples 25
4. The Airault-Malliavin Theorem 32
5. Absolute continuity of Heat Kernel with respect to pinned Wiener measure 40
6. Semi-Martingale Properties of $g_{T,\cdot}$ 48
6.1. Preliminaries 48
6.2. The Main Theorem 52
6.3. Proof of Theorems 6.16 and 6.17 56
6.4. Propositions 6.13, 6.19, 6.20, 6.21 60
6.5. Good partitions(proof of Lemma 6.15) 71
7. $HKM \downarrow_{\mathfrak{F}_s} \backsim PWM \downarrow_{\mathfrak{F}_s}$ 77
8. Appendix 82
8.1. General Technical results 82
8.2. Brownian Sheets and bridges 84
8.3. Proof of Lemma 8.8 88
8.4. Gaussian Measures 90
9. Conjectures on Gaussian measure equivalence 97
References 100

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1. Introduction

In this paper we consider the equivalence of two measures on the Loop space of a compact Lie group. This so-called "Loop group" is the space of continuous paths in the Lie group based at the identity equipped with a certain well-known left-invariant "H^1-Riemannian structure". The study of Loop groups is motivated primarily by physics and the theory of group Representations. They have been studied extensively in both the mathematics and the physics literature. See for example [29], [19], [27],[3], [15], [16], [1], [24], [18], [12] and the references therein.

Heat Kernel and pinned Wiener measure are two natural measures that have been advocated as the "right" measure on the Loop groups. Pinned Wiener measure on a Loop group is the law of a group-valued Brownian motion that has been conditioned on loops. This measure has been extensively studied in [17], [25], [2], [26]. Heat Kernel measure has been studied in [13], [11] as another natural measure on Loop Space. In [13], Driver and Lohrenz showed that there exists a certain process that deserves to be called "Brownian motion" on the path space of a Loop group. The Heat Kernel measures on the Loop Space are the time t, $t > 0$ distributions of this Brownian motion. Thus it is a natural question to consider the equivalence of these two measures.

A further motivation comes from logarithmic Sobolev inequalities and the papers of Getzler [16], Gross [17], Driver [11], Hsu, Aida, and Elworthy. The classical Sobolev inequalities are a fundamental tool in analyzing finite-dimensional manifolds. For infinite-dimensional manifolds logarithmic Sobolev inequalities, because of their dimension-independent character, are seen to be the proper analogues of classical Sobolev inequalities. Logarithmic Sobolev inequalities have been studied extensively over infinite-dimensional linear spaces as well as finite-dimensional manifolds (see [8], [9] for surveys and [20]). If a logarithmic Sobolev inequality does hold for pinned Wiener measure, μ_0, then the Dirichlet form $\mu_0 \langle \nabla f, \nabla f \rangle$ associated with pinned Wiener measure will have a spectral gap (the so-called "Mass Gap inequality").

In [16], Getzler showed that the Bakry and Emery criteria (see [4] and [5]) for proving a logarithmic Sobolev inequality does not hold in general for loop groups when the "underlying measure" is pinned Wiener measure. In [17], using pinned Wiener measure, Gross showed that a logarithmic Sobolev inequality on Loop space does hold, but with an added potential term (a so-called "defective" logarithmic Sobolev inequality). Using Heat Kernel measure instead, Driver and Lohrenz proved in [13] that a logarithmic Sobolev inequality does hold on Loop groups, without Gross' potential. If Heat Kernel and pinned Wiener measures were equivalent with Radon-Nikodym derivatives bounded above and below then the Holley-Stroock Lemma (see [20])would tell us that pinned Wiener measure admits a classical (i.e. "non-defective") logarithmic Sobolev inequality. Even if the equivalence were not so nice, it might still be possible to use the Driver-Lohrenz result of [13] to eliminate the Gross' potential term and thereby prove a logarithmic Sobolev inequality for pinned Wiener measure.

In Section 5, using a result of Malliavin and Airault (see [26] and Theorem 4.1) as well as a maximal-inequality argument, we show that Heat Kernel measure is absolutely continuous with respect to pinned Wiener measure. Further, the relevant Radon-Nikodym derivative is bounded. We also provide a simpler and more direct proof of the result of the Malliavin-Airault Theorem in Section 4.

In Section 7 we show that pinned Wiener measure is absolutely continuous (and thus equivalent) with respect to Heat Kernel measure on $\mathfrak{F}_s$ ($\mathfrak{F}_s$ denotes the σ-algebra of functions depending on the loop up to time $s<1$). We view the Loop-Space-valued Brownian motion, developed by Driver and Lohrenz in [13], as a group-valued two-parameter process. Viewing one of the parameters fixed, the resulting process has the same distribution as Heat Kernel measure. In Section 6 we show that, in the other parameter, this process is a Brownian semimartingale on the path space of the Lie group. To do this, we use extensively the theory of two-parameter semimartingales developed by Cairoli, Walsh, Wang, and Zakai (see [7], [31]). The fact that we can pull back this process to a Lie algebra valued Brownian Semimartingale, Girsanov's Theorem, and the fact that Wiener measure and pinned Wiener measure are equivalent on $\mathfrak{F}_s$; gives us our result that on $\mathfrak{F}_s$ Heat Kernel measure and pinned Wiener measure are equivalent. In our proof, the analysis is done in a bigger space (the Wiener space of the compact Lie group) which is why we require s to be strictly less than one.

Heat Kernel measure is a time t distribution of a process on the path space of a Loop group which is started from the identity loop (i.e. the constant loop). This describes a homotopy between the endpoint of this process and the identity loop. As a consequence, Heat Kernel measure concentrates all its mass on null-homotopic loops. On the other hand pinned Wiener measure is quasi-invariant under translations by finite-energy loops. Thus Pinned Wiener measure must assign non-zero mass to all homotopy classes. Therefore if the Lie group is not simply connected, pinned Wiener measure is not equivalent to Heat Kernel measure. Thus our result showing absolute continuity on $\mathfrak{F}_s$ for $s<1$ is in a sense the best result that can be obtained in the non-simply-connected case.

In our last section, Section 9, we conjecture that pinned Wiener measure is absolutely continuous with respect to a weighted sum of Heat Kernel measures on the various homotopy classes. These Heat Kernel measures are obtained by starting the Driver-Lohrenz Loop-group-valued Brownian motion at the energy-minimizing geodesics in each homotopy class. This results in a measure that assigns non-zero mass to each homotopy class. The conjecture rests on a very informal computation done by Driver and the fact that the conjecture is true in the case that the compact Lie group is the circle S^1.

2. Statement of Results

2.1. Loop group Geometry. Let K be a connected compact Lie group, $\mathfrak{K} \equiv T_e K$ be the Lie algebra of K, and $\langle \cdot, \cdot \rangle_{\mathfrak{K}}$ be an Ad_K-invariant inner product on $\mathfrak{K}$. For $\xi \in \mathfrak{K}$, let $|\xi|_{\mathfrak{K}} \equiv \sqrt{\langle \xi, \xi \rangle_{\mathfrak{K}}}$. Let ℓ_g and ρ_g be left and right translations on K respectively. (i.e. ℓ_g and ρ_g are maps taking K to K so that $\ell_g(x) = gx$ while $\rho_g(x) = xg$). Let

$$L(K) \equiv \{\sigma \in C([0,1] \to K) \, | \sigma(0) = \sigma(1) = e\}$$

denote the based loop group on K consisting of continuous paths $\sigma : [0,1] \to K$ such that $\sigma(0) = \sigma(1) = e$, where $e \in K$, is the identity element.

Definition 2.1 (Tangent Space of $L(K)$)**.** We will need the following definitions:-

- Given a function $h : [0,1] \to \mathfrak{K}$ such that $h(0) = 0$, define $(h,h)_H = \infty$ if h is not absolutely continuous and set $(h,h)_H = \int_0^1 |h'(s)|^2 \, ds$ otherwise.

- Define
$$H(\mathfrak{K}) \equiv \{h : [0,1] \to \mathfrak{K} | h(0) = 0 \text{ and } (h,h) < \infty\}.$$
Then $H(\mathfrak{K})$ is a Hilbert space under $(\cdot,\cdot)_H$.
- Define
$$H_0(\mathfrak{K}) \equiv \{h \in H(\mathfrak{K}) | h(1) = 0\}.$$
Then $(H_0(\mathfrak{K}), (\cdot,\cdot)_H)$ is also a Hilbert space.

In order to define the tangent space $TL(K)$ of $L(K)$ let θ denote the Maurer-Cartan form. That is $\theta\langle\xi\rangle \equiv (\ell_{k^{-1}})_* \xi$ for all $\xi \in T_k K$, and $k \in K$ and where ℓ_g denotes left multiplication by $g \in K$. Let $\theta\langle X\rangle(s) \equiv \theta\langle X(s)\rangle$ and $p : TK \to K$ be the canonical projection. We now define

$$TL(K) \equiv \{X : [0,1] \to TK | \theta\langle X\rangle \in H_0 \text{ and } p \circ X \in L(K)\}.$$

By abuse of notation, use the same p to denote the canonical projection from $TL(K) \to L(K)$. As usual, define the tangent space at $k \in L(K)$ by $T_k L(K) \equiv p^{-1}\{k\}$. Using left translations, we extend the inner product $(\cdot,\cdot)_{H_0}$ on H_0 to a Riemannian metric on $TL(K)$. Explicitly set

$$(X,X)_{L(K)} \equiv (\theta\langle X\rangle, \theta\langle X\rangle)_{H_0(\mathfrak{K})} \text{ where } X \in TL(K).$$

In this way, $L(K)$ is to be thought of as an infinite-dimensional Riemannian manifold. Viewing the Lie algebra $(\mathfrak{K}, 0)$ as a Lie group in its own right with Lie algebra $\mathfrak{K}$, we obtain definitions for

$$L(\mathfrak{K}) \equiv \{\sigma \in C([0,1] \to \mathfrak{K}) | \sigma(0) = \sigma(1) = 0\}$$

as the "Lie group" with Lie algebra $H_0(\mathfrak{K})$ thought of as a commutative Lie algebra.

Definition 2.2 (Good Orthonormal basis of H_0)**.** An orthonormal basis $\{\eta_k\}_{k\in\mathbb{N}}$ of $H_0(\mathfrak{K})$ is a good orthonormal basis if the Lie Bracket $[\eta_k(s), \eta_k'(s)]$ is identically zero for all values of s and k.

Example 2.3. We will provide a couple of examples for illustration:-

1. Take $\{h_k\}$ to be an orthonormal basis of $H_0(\mathbb{R})$ and let $\{A\}$ run through an orthonormal basis of $\mathfrak{K}$. Then $\eta_{A,k} \equiv h_k A$ is a good orthonormal basis.
2. Let $\{\eta_{A,k}\}_{k\in\mathbb{N},A}$ be loops in $H_0^1(\mathfrak{K})$ where
$$\begin{aligned}\eta_{A,2k}(\tau) &\equiv \frac{A}{\pi k\sqrt{2}} \sin 2\pi k\tau \\ \eta_{A,2k-1}(\tau) &\equiv \frac{A}{\pi k\sqrt{2}} (\cos 2\pi k\tau - 1)\end{aligned}$$
and A runs through an orthonormal basis of $\mathfrak{K}$.

Definition 2.4 (The Laplacian $\triangle_{L(K)}$ and $\triangle_{L(\mathfrak{K})}$)**.** Take a good orthonormal basis of $H_0(\mathfrak{K})$. Then define an operator $\triangle_{L(K)}$ on functions f on $L(K)$ by setting

$$\triangle_{L(K)} f \equiv \sum \partial_h^2 f,$$

where

$$(\partial_h f)(\gamma) \equiv \partial_\varepsilon f(\gamma \exp \varepsilon h)|_{\varepsilon=0}.$$

Define the Laplacian $\triangle_{L(\mathfrak{K})}$ on functions f on $L(\mathfrak{K})$ in the same way above by setting

$$\triangle_{L(\mathfrak{K})} f \equiv \sum \partial_h^2 f,$$

where

$$(\partial_h f)(\gamma) \equiv \partial_\varepsilon f(\gamma + \varepsilon h)|_{\varepsilon=0}.$$

Definition 2.5 (Cylinder functions)**.** Let $(\mathcal{R}, e)$ denote either the Lie group (K, e) or the Lie algebra $(\mathfrak{K}, 0)$. Let $\widetilde{L}(\mathcal{R})$ denote either $L(\mathcal{R})$ or $W_e(\mathcal{R})$.

1. Then $f : \widetilde{L}(\mathcal{R}) \to \mathbb{R}$ is a cylinder function iff $f(\sigma) \equiv F(\sigma_{t_1}, \cdots, \sigma_{t_n})$ where $\{0 < t_1 < \cdots < t_n < 1\}$. $F \in C(\mathcal{R}^n)$.
2. f is a smooth cylinder function iff $F \in C^\infty(\mathcal{R}^n)$. $F \in C(\mathcal{R}^n)$.
3. f is a bounded cylinder function iff $F \in C_b(\mathcal{R}^n)$. Here $C_b(\mathcal{R}^n)$ are the bounded continuous functions on $\mathcal{R}^n$.
4. Let $\mathcal{F}C\left(\widetilde{L}(\mathcal{R})\right)$ denote the space of all cylindrical functions.
5. Let $\mathcal{F}C^\infty\left(\widetilde{L}(\mathcal{R})\right)$ denote the space of all smooth cylindrical functions.
6. Let $\mathcal{F}C_b^\infty\left(\widetilde{L}(\mathcal{R})\right)$ denote the space of all bounded cylindrical functions.
7. A cylinder function is $\mathfrak{F}_{\mathbb{P}}$-measurable if and only if $f(\sigma) = F(\sigma_{t_1}, \cdots, \sigma_{t_n})$ where $\{t_i\} \subset \mathbb{P}$ where $\mathbb{P}$ is some partition of $[0,1]$.

2.2. **Measures on the Loop group.**

2.2.1. *Pinned Wiener measure.* Let the Wiener space $W_e(K)$ denote the space of all continuous paths in K starting at the identity. Explicitly

$$W_e(K) \equiv \{\sigma \in C([0,1] \to K) \,|\, \sigma(0) = e\}.$$

Definition 2.6 (Heat Kernel measure on K)**.** Let $t > 0$. The Heat Kernels P_t^K on K are the unique functions so that for any smooth f on K, the function u on $[0,\infty) \times K$ defined by setting $u(t,x) \equiv \int_K f(y) P_t^K(y^{-1}x)\, dy$ is a solution to the Heat equation with initial condition f. Explicitly

$$\begin{aligned} \partial_t u &= \frac{1}{2}\triangle_K u \\ u(t,x) &\to f(x) \text{ as } t \to 0. \end{aligned}$$

It is well known that $x \to P_t^K$ are smooth function on K and that $P_t^K(x) = P_t^K(x^{-1})$.

Definition 2.7 (Wiener Measure on $W_e(K)$)**.** Wiener Measure, μ_t, on $W_e(K)$ with parameter t, is the unique measure so that for any bounded cylinder function f of the form $f(x) = F(x_{s_1}, \cdots, x_{s_n})$ we have

$$\mu_t[f] \equiv \int_{K^n} F(x_1, \cdots, x_n) \prod_{i=1}^n P^K_{t(s_i - s_{i-1})}\left(x_{i-1}^{-1} x_i\right) dx_i,$$

where $x_0 = e$ and $s_0 = 0$. [The measure μ_1 will also be denoted by μ in the sequel.]

Definition 2.8 (Brownian motion on K)**.** We will state three equivalent definitions. A process $s \to \beta(s)$ is a Brownian motion on K starting at e with parameter t iff:-

1. β is a $W_e(K)$-valued random variable distributed according to Wiener measure μ_t.
2. the process $s \to \beta(s)$ is a diffusion starting at e with generator $\frac{t}{2}\triangle_K$. This means that the process $s \to \beta(s)$ is a martingale so that $\beta(0) = e$ a.s. and

$$(\phi \circ \beta)(ds) = \phi' \circ \beta(s)\,\beta(ds) + \frac{t}{2}(\triangle_K \phi) \circ \beta(s)\,ds$$

for any smooth ϕ on K. Here $\triangle_K$ is the Laplacian on K with respect to the metric $\langle \cdot, \cdot \rangle_K$ on K.

The first definition is easier in simpler cases like $\mathbb{R}^d$ or compact Lie groups. The second definition is easier to extend to the infinite-dimensional cases and manifolds. See Definitions 2.14 and 3.6.

Definition 2.9 (Pinned Wiener Measure)**.** Pinned Wiener Measure, $\mu_{0,t}$, on $L(K)$ with parameter t is the unique measure on $L(K)$ so that for any bounded cylinder functions f of the form $f(x) = F(x_{s_1}, \cdots, x_{s_n})$ where $F \in C^\infty(K)$, then

$$\mu_{0,t}[f] \equiv \int_{K^n} F(x_1, \cdots, x_n) \frac{P^K_{t(1-s_n)}(x_n, e)}{P^K_t(e,e)} \prod_{i=1}^n P^K_{t(s_i - s_{i-1})}(x_{i-1}, x_i)\,dx_i, \tag{2.1}$$

where $x_0 = e$ and $s_0 = 0$.[We will use the notation μ_0 to denote $\mu_{0,1}$.]

Remark 2.10 (Pinned Wiener measure is really pinned!). Pinned Wiener measure is really Wiener measure pinned at e. At least on cylinder functions,

$$\mu_{0,t}[f] = \int f(x)\,\delta_e(x(1))\,\mu_t(dx) \Big/ \int \delta_e(x(1))\,\mu_t(dx).$$

As Malliavin showed, another way of looking at this measure is

$$\mu_0(f) \equiv \left(\frac{d(\pi_1)_*(f\mu_t)}{d(\pi_1)_*\mu_t}\right)(e), \tag{2.2}$$

where $\pi_1 : x_\cdot \to x_1$; μ_t is Wiener measure on K with parameter t; and $(f\mu_t)$ is that measure on $W_e(K)$ so that $(f\mu_t)(dx) = f(x)\,\mu_t(dx)$. For cylinder functions, it is trivial to check Eq. [2.2] by writing down finite-dimensional distributions.

Definition 2.11 (Brownian bridge on K)**.** $s \to \chi(s)$ is a Brownian bridge from on K with parameter t if χ is an $L(K)$-valued random variable distributed according to pinned Wiener measure $\mu_{0,t}$.

2.2.2. *Heat Kernel measure.*

Definition 2.12 (Brownian Bridge Sheet on $\mathfrak{K}$)**.** A Gaussian process $\{\chi(t)\}_{t\in[0,1]}$ is a Brownian bridge Sheet on $\mathfrak{K}$ if for (t,s) in $[0,1]^2$, $\chi(t,s)$ is a $\mathfrak{K}$-valued mean-zero Gaussian process with covariance given by

$$E\langle A, \chi(t,s)\rangle_{\mathfrak{K}} \langle B, \chi(\tau,\sigma)\rangle_{\mathfrak{K}} = \langle A, B\rangle_{\mathfrak{K}} (t \wedge \tau)\,G_0(s,\sigma),$$

where $\chi(t,s) \equiv \chi(t)(s) \in \mathfrak{K}$; $A, B \in \mathfrak{K}$; $t, \tau, s, \sigma \in [0,1]$; and $G_0(s,\sigma) \equiv s \wedge \sigma - s\sigma$.

Remark 2.13. It turns out that if β is a Brownian bridge sheet on $\mathfrak{K}$ then β_{ts} is continuous in both its parameters, $t \to \beta_{ts}$ is a Brownian motion on $\mathfrak{K}$ with parameter $G_0(s,s)$ and $s \to \beta_{ts}$ is a Brownian bridge on $\mathfrak{K}$ with parameter s.

Definition 2.14 (Brownian motion on $L(K)$)**.** A process $t \to \Sigma(t,\cdot)$ is an $L(K)$-valued Brownian motion if and only if for any smooth cylinder function $f : L(K) \to \mathbb{R}$, there is a real-valued martingale M_t so that

$$f(\Sigma(dt,\cdot)) = dM_t + \frac{1}{2}\left(\triangle_{L(K)} f\right)(\Sigma(t,\cdot))\,dt.$$

See Theorem 2.19 for the existence of this Brownian motion. So $t \to \Sigma(t,\cdot)$ is a diffusion on $L(K)$ with generator $\frac{1}{2}\triangle_{L(K)}$. [Define a Brownian motion on $L(\mathfrak{K})$ by thinking of $\mathfrak{K}$ as a Lie group and applying the above definition]

Lemma 2.15 ($\triangle_{L(K)}$ on cylinder functions, see [13])**.** *Let G_0 be as in Definition 2.12. Let $\mathbb{P}$ be the partition $\{0 < s_1 < \cdots < s_n < 1\}$. Let $\pi_{\mathbb{P}}$ be the map taking a loop σ in $L(K)$ to $(\sigma_{s_1},\cdots,\sigma_{s_n}) \in K^n$. For $F \in C^\infty(K^n)$, define*

$$\left(A^{(i)}F\right)(g_1,\cdots,g_n) \equiv \frac{d}{dt}|_{t=0} F(\cdots, g_i \exp tA, \cdots).$$

Define an elliptic operator $\triangle_{\mathbb{P}}$ on $C^\infty(K^n)$ by setting

$$\triangle_{\mathbb{P}} \equiv \sum_{i,j,A} G_0(s_i,s_j)\, A^{(i)} A^{(j)}.$$

Then letting A run through an orthonormal basis of $\mathfrak{K}$, for any smooth cylinder function $F : K^n \to \mathbb{R}$ we have

$$\triangle_{L(K)}(F \circ \pi_{\mathbb{P}}) = (\triangle_{\mathbb{P}} F) \circ \pi_{\mathbb{P}}.$$

[This Lemma can also be used on the Lie algebra $\mathfrak{K}$ by viewing $\mathfrak{K}$ itself as a Lie group, i.e. take $K = \mathfrak{K}$ and $g \exp A = g + A$.]

Proof. As in Example 2.3, take $\{h_k\}$ to be an orthonormal basis of $H_0(\mathbb{R})$ and let $\{A\}$ run through an orthonormal basis of $\mathfrak{K}$. Then $\eta_{A,k} \equiv h_k A$ is a good orthonormal basis of $H_0(\mathfrak{K})$. Then we have

$$\begin{aligned}
\left(\partial_{\eta_{A,k}} F \circ \pi_{\mathbb{P}}\right)(\gamma) &= \frac{d}{dt} F \circ \pi_{\mathbb{P}}\left(\gamma \exp t\eta_{A,k}\right) \downarrow_0 \\
&= \sum_i \eta_k(s_i) \left(A^{(i)}F\right) \circ \pi_{\mathbb{P}}(\gamma).
\end{aligned}$$

Thus

$$\begin{aligned}
\triangle_{L(K)} F \circ \pi_{\mathbb{P}} &= \sum_{k\in\mathbb{N},A} \partial_{\eta_{A,k}} \partial_{\eta_{A,k}} F \circ \pi_{\mathbb{P}} \\
&= \sum_{i=1}^{n} \sum_{k\in\mathbb{N},A} \eta_k(s_i)\, \partial_{\eta_{A,k}} \left(A^{(i)}F\right) \circ \pi_{\mathbb{P}}(\gamma) \\
&= \sum_A \sum_{i,j=1}^{n} \left[\sum_{k\in\mathbb{N}} \eta_k(s_i)\, \eta_k(s_j)\right] \left(A^{(j)} A^{(i)} F\right) \circ \pi_{\mathbb{P}}(\gamma).
\end{aligned}$$

It remains only to show that $\sum_{k\in\mathbb{N}} \eta_k(s_i)\,\eta_k(s_j) = G_0(s_i,s_j)$. Let $h \in H_0(\mathbb{R})$. Suppose we can show that

$$\langle G_0(s,\cdot), h\rangle_{H_0(\mathbb{R})} = h(s). \tag{2.3}$$

A priori we suspect that such elements $G_0(s,\cdot)$ exist because the evaluation map $h\to h_s$ is a bounded linear functional on the Hilbert Space $H_0(\mathbb{R})$. Then we will be done since we shall have

$$\begin{aligned}\sum_{k\in\mathbb{N}}\eta_k(s_i)\eta_k(s_j) &= \sum_{k\in\mathbb{N}}\langle G_0(s_i,\cdot),\eta_k\rangle\langle G_0(s_j,\cdot),\eta_k\rangle\\ &= \langle G_0(s_i,\cdot),G_0(s_j,\cdot)\rangle\\ &= G_0(s_i,s_j).\end{aligned}$$

We shall proceed to check Eq. [2.3].

$$\begin{aligned}\langle G_0(s,\cdot),h\rangle_{H_0(\mathbb{R})} &= \int_0^1\partial_t(s\wedge t-st)\,h'(t)\,dt\\ &= \int_0^1\left(1_{[0,s]}(t)-s\right)h'(t)\,dt\\ &= h(s)-(1-s)\,h(0)-sh(1)\\ &= 0\text{ since }h\text{ is a loop based at }0.\end{aligned}$$

Hence we are done. ∎

Lemma 2.16 (Brownian Motion on $L(\mathfrak{K})$ exists)**.** *If $\chi_t\equiv\chi(t,\cdot)$ is a Brownian Sheet then for any smooth cylindrical function f, there is a real-valued martingale M_t so that*

$$df(\chi_t)=dM_t+\frac{1}{2}\left(\triangle_{L(\mathfrak{K})}f\right)(\chi_t)\,dt.$$

Here $\triangle_{L(\mathfrak{K})}$ is the Laplace-Beltrami operator defined in Definition 2.4. So every Brownian bridge Sheet on $\mathfrak{K}$ is an $L(\mathfrak{K})$-valued Brownian motion.

Proof. Let G_0 be as in Definition 2.12. Let χ^A_{ts} denote as usual $\langle\chi_{ts},A\rangle_{\mathfrak{K}}$ for any $A\in\mathfrak{K}$. Then the joint quadratic variation $\chi^A_{dts}\chi^B_{dt\sigma}=\langle A,B\rangle_{\mathfrak{K}}\,G_0(s,\sigma)\,dt$. Let f be a smooth cylinder function implies (see Definition 2.5), $f(\sigma)=F(\sigma_{s_1},\cdots,\sigma_{s_n})$ where $F\in C^\infty(\mathfrak{K}^n)$ and $\mathbb{P}\equiv\{0<s_1<\cdots<s_n<1\}$. Let $\chi^{\mathbb{P}}_t$ denote $\left(\chi_{ts_1},\cdots,\chi_{ts_n}\right)$. Let $\left(A^{(i)}F\right)(g_1,\cdots,g_n)$ denote $\partial_\varepsilon F(\cdots,g_i+\varepsilon A,\cdots)\downarrow_{\varepsilon=0}$. Let $\triangle_{\mathbb{P}}$ on $\mathfrak{K}^n$ by

$$\triangle_{\mathbb{P}}\equiv\sum_{i,j,A}G_0(s_i,s_j)\,A^{(i)}A^{(j)}.$$

Thus by Ito's Lemma we have

$$\begin{aligned}df(\chi_t) &= dF\left(\chi^{\mathbb{P}}_t\right)\\ &= \sum_{i,A}\left(A^{(i)}F\right)\left(\chi^{\mathbb{P}}_t\right)\chi^A_{dts_i}\\ &\quad+\frac{1}{2}\sum_{i,j,A}\left(A^{(j)}A^{(i)}F\right)\left(\chi^{\mathbb{P}}_t\right)\chi^A_{dts_i}\chi^A_{dts_j}\\ &= dMartingale+\frac{1}{2}\sum_{i,j,A}G_0(s_i,s_j)\left(A^{(j)}A^{(i)}F\right)\left(\chi^{\mathbb{P}}_t\right)dt\\ &= dMartingale+\frac{1}{2}\left(\triangle_{\mathbb{P}}F\right)\left(\chi^{\mathbb{P}}_t\right)dt.\end{aligned}$$

By Lemma 2.15 (view $\mathfrak{K}$ as a Lie algebra in its own right with Lie algebra $\mathfrak{K}$ while applying this Lemma) this last expression is just

$$= dMartingale + \frac{1}{2}\left(\triangle_{L(K)} f\right)(\chi_t)\, dt.$$

∎

We will need the the following Theorem:

Theorem 2.17 (Malliavin)**.** *Let* $\left(\Omega_0, \mathfrak{F}^0, \{\mathfrak{F}^0_{ts}\}_{(t,s)\in[0,1]^2}, P_0\right)$ *be a filtered complete probability space where*

$$\mathfrak{F}^0_{ts} \equiv \sigma\left\langle \chi_{\tau u} : \tau \in [0,t],\, u \in [0,s]\right\rangle,$$

and $\mathfrak{F}^0 \equiv \vee_{(t,s)\in[0,1]^2}\mathfrak{F}^0_{ts}$*. Let* $k_0 \in L(K)$ *and let* χ *be a* $\mathfrak{K}$*-valued Brownian bridge sheet in the sense of Definition 2.12. Recall* $\ell_g : K \to K$ *takes* $x \to gx$*. Then there is a jointly continuous solution* $\Sigma(t,s)$ *to the stochastic differential equation*

$$\Sigma(\delta t, s) = \sum_{A\in ONB(\mathfrak{K})} \left(\ell_{\Sigma(t,s)*} A\right)\chi^A(\delta t, s) \text{ with } \Sigma(0,s) = k_0(s), \forall s \in [0,1], \tag{2.4}$$

where the A *run through an orthonormal basis of* $\mathfrak{K}$ *and where for each fixed* $s \in [0,1], \Sigma(\delta t, s)$ *and* $\chi^A(\delta t, s)$ *denote the Fisk-Stratonowicz differentials of the processes* $t \to \Sigma(t,s)$ *and* $t \to \langle \chi(t,s), A\rangle_{\mathfrak{K}}$ *respectively. Henceforth we write Eq. (2.4) more concisely as*

$$\Sigma(\delta t, s) = \left(L_{\Sigma(t,s)}\right)_* \chi(\delta t, s) \text{ with } \Sigma(0,s) = k_0(s), \forall s \in [0,1]. \tag{2.5}$$

[see Malliavin [27]*; see also Theorem 3.8 of* [11]*]*

Remark 2.18 (Explicit Matrix Representation of Eq. [2.5).] Let $\mathcal{M}_m(\mathbb{R})$ be all $m \times m$ matrices on $\mathbb{R}$ and $GL_m(\mathbb{R})$ be all invertible matrices in $\mathcal{M}_m(\mathbb{R})$. We will work with an explicit matrix representation of our Lie group K. K will be thought of as a subgroup of $GL_m(\mathbb{R}) \subset \mathcal{M}_m(\mathbb{R})$ for some m. Such a representation exists as a consequence of the Peter-Weyl Theorem. Hence Eq. (2.5) can be rewritten as

$$\Sigma(\delta t, s) = \Sigma(t,s)\,\chi(\delta t, s) \text{ with } \Sigma(0,\cdot) = k_0, \forall s \in [0,1], \tag{2.6}$$

where we have used matrix multiplication to define $\Sigma(t,s)\,\chi(\delta t, s)$. Explicitly if we let B_{ij} denote the i,j entry of the matrix B we have

$$\delta_t(\Sigma(t,s))_{ij} = \sum_k (\Sigma(t,s))_{ik}\,\delta_t(\chi(t,s))_{kj}.$$

Theorem 2.19 (Brownian motion on $L(K)$)**.** *Let* $\Sigma(t,s)$ *be the process from Theorem 2.17 and Remark 2.18. Theorem 2.17 tells us that* $s \to \Sigma(t,s)$ *is a Loop a.s. Let* Σ_t *denote this loop* $s \to \Sigma(t,s)$*. Then* $t \to \Sigma_t$ *is a Brownian motion on* $L(K)$ *in the sense of Definition 2.14.*

Proof. See Theorem 3.10 of Driver [11]. ∎

Now that we know that Brownian motion on $L(K)$ exists, we can define Heat Kernel measure on $L(K)$.

Definition 2.20 (Heat Kernel measure on $L(K)$)**.** Let $k_0 \in L(K)$ be a loop and let $t > 0$. Let $\Sigma(t,\cdot)$ be an $L(K)$-valued Brownian motion so that $\Sigma(0,\cdot) = k_0$ in

$L(K)$ a.s. Then, as in the finite-dimensional manifold case, Heat Kernel measure $\nu_t(k_0, dk)$ is defined to be the law of $\Sigma(t, \cdot)$. Explicitly

$$\int_{L(K)} f(k)\,\nu_t(k_0, dk) = Ef \circ \Sigma(t, \cdot).$$

The next Theorem shows that Heat Kernel measures behave as expected, in that they may be used to solve the Heat Equation on $L(K)$. See Remark 3.12 for motivation of Theorem 2.21.

Theorem 2.21 (Driver&Lohrenz)**.** *For each $t > 0$, for all bounded cylinder functions f on $L(K)$, the function u on $(0, \infty) \times L(K)$ given by*

$$u(t, k_0) \equiv \int_{L(K)} f(k)\,\nu_t(k_0, dk),$$

is the unique solution to the heat equation

$$\partial u(t, \cdot)/\partial t = \frac{1}{2}\triangle_{L(K)} u(t, \cdot) \ \ \textit{with} \ \lim_{t \downarrow 0} u(t, k) = f(k_0).$$

Here $\triangle_{L(K)}$ denotes the operator from Definition 2.4. See Theorem 1.1 of [13]*. See also Definitions 3.10 and 4.17 in* [13]*. [Note:- In* [13]*, results on Heat kernel measures are obtained for groups of compact type, and not merely compact Lie groups.]*

2.3. **The stochastic framework.** We shall use the results of Section 2.2.2 to obtain our probability space.

Definition 2.22 (Ambient probability space)**.** $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_{ts}\}_{(t,s) \in [0,1]^2}, P\right)$ is going to be our biparametrically-filtered probability space where

- $\Omega \equiv C([0,1] \to L(K))$ equipped with $\mathfrak{F}$, the completion of the Borel σ-algebra.
- Let Σ be the process from Theorem 2.17 so that $\Sigma_0 = e$, where e denotes the identity loop.
- P is defined to be Wiener Measure on $C([0,1] \to L(K))$. Explicitly, $P \equiv Law$ Σ.
- $g_t : C([0,1] \to L(K)) \to L(K)$ by $x \to x(t)$ for any $x \in C([0,1] \to L(K))$
- By Theorem 2.19 we see that $dLaw\ g_t = d\nu_t(e, \cdot)$.
- $g_{ts}(x) = [x(t)](s)$ in K.
- $\mathfrak{F}_{00}$ is a σ-algebra containing all the null sets of $\mathfrak{F}$.
- $\mathfrak{F}_{ts} \equiv \sigma\langle g_{\tau\sigma} : \tau \in [0,t]$ and $\sigma \in [0,s]\rangle \vee \mathfrak{F}_{00}$.

Definition 2.23. Let $\{\mathfrak{G}_s\}$ be a filtration. Then $U_\cdot$ is a K-valued $\mathfrak{G}_\cdot$-semimartingale iff for any smooth $f : K \to \mathbb{R}$ the process $t \to f(U_t)$ is an $\mathbb{R}$-valued $\mathfrak{G}_\cdot$-semimartingale.

Definition 2.24 (see Protter [30])**.** Let $\{\mathfrak{G}_s\}$ be a filtration. An $\mathbb{R}$-valued process $U_\cdot$ is called an $\mathbb{R}$-valued $\mathfrak{G}_\cdot$-semimartingale if:-

1. the paths $U_\cdot$ are continuous a.s.
2. $U_\cdot$ is adapted with respect to the filtration $\mathfrak{G}_\cdot$ (i.e. $U_t \in \mathfrak{G}_t$ for all $t \in [0,T]$)
3. Given any sequence of simple adapted processes $\{H\}$ then $\int_0^T H_t U_{dt} \downarrow 0$ in probability whenever $H \downarrow 0$ uniformly on compacts in probability. Here H is a simple adapted process if H is an $\mathbb{R}$-valued $\mathfrak{G}_t$-adapted process of the form $H(t, \omega) \equiv \sum_{i=0}^n H_i(\omega)\, 1_{(T_i, T_{i+1}]}$ with the T_i being a sequence of stopping

times with $0 \leq T_0 \leq \cdots \leq T_n \leq T$. The integral $\int_0^T H_t U_{dt}$ is defined to be the sum $\sum_{i=0}^n H_i \left(U_{T_{i+1}} - U_{T_i}\right)$ for any simple adapted process H.

Theorem 2.25 (Semimartingale properties of $g_{\cdot s}$)**.** *The process g of Definition 2.22 has the following properties:-*

1. *The process $t \to g_{ts}$ is a semimartingale.*
2. *Let $X_{ts} \equiv \int_0^t g_{\tau s}^{-1} g_{\delta \tau s}$. Then $t \to X_{t\cdot}$ is a Brownian bridge sheet on $\mathfrak{K}$ with respect to the measure P. Furthermore, X can be taken to be continuous in both its parameters.*

Remark 2.26. After the proof of Theorem 2.25 we shall never again refer to χ, Σ or the underlying abstract probability space. Also we will always use the version of X that is continuous in both parameters t and s.

Proof. **of Theorem 2.25**

First we check that $t \to g_{ts}$ is an $\mathfrak{F}_{\cdot s}$-semimartingale. For convenience we use the "good integrator" definition of a semimartingale (see Definition 2.23). Pick $f \in C^\infty(K)$. It will suffice to check that $f(g_{\cdot s})$ is a semimartingale. Let $\{H\}$ be a sequence of $\mathfrak{F}_{\cdot s}$-adapted processes which converge to zero uniformly on compacts in probability. Then we have

$$\begin{aligned} P\left(\left|\int_0^T H_t g_{dts}\right| > \varepsilon\right) &= P\left(\left|\sum_{i=0}^n H_i \left(g_{T_{i+1}s} - g_{T_i s}\right)\right| > \varepsilon\right) \\ &= P\left(\left\{\omega : \left|\sum_{i=0}^n H_i \left(\Sigma(\omega)\right)\left(\Sigma_{T_{i+1}s} - \Sigma_{T_i s}\right)\right| > \varepsilon\right\}\right) \\ &= P\left(\left\{\omega : \left|\int_0^T H_t \circ \Sigma_{\cdot s}(\omega)\, \Sigma_{dts}\right| > \varepsilon\right\}\right). \end{aligned}$$

This last term goes to zero since $t \to \Sigma_{ts}$ is an $\mathfrak{F}_{\cdot s}$-semimartingale. Thus $g_{\cdot s}$ is a semimartingale.

Now we want to show that $X_{\cdot s} \equiv \int_0^{\cdot} g_{ts}^{-1} g_{\delta ts}$ has the same law as $\chi_{\cdot s}$. Let E_{ij} denote the $m \times m$ matrix with k,l-entries $\delta_{ik}\delta_{jl}$. We can write

$$\begin{aligned} X_{\cdot s} &= \sum_{i,j,k} \int_0^{\cdot} \left(g_{ts}^{-1}\right)_{ik} \delta_t \left(g_{ts}\right)_{kj} E_{ij} \\ &= \sum_{j,k} \int_0^{\cdot} \left[\sum_i \left(g_{ts}^{-1}\right)_{ik} E_{ij}\right] \delta_t \left(g_{ts}\right)_{kj}, \end{aligned}$$

and

$$\begin{aligned} \chi_{\cdot s} &= \sum_{i,j,k} \int_0^{\cdot} \left(\Sigma_{ts}^{-1}\right)_{ik} \delta_t \left(\Sigma_{ts}\right)_{kj} E_{ij} \\ &= \sum_{j,k} \int_0^{\cdot} \left[\sum_i \left(\Sigma_{ts}^{-1}\right)_{ik} E_{ij}\right] \delta_t \left(\Sigma_{ts}\right)_{kj}. \end{aligned}$$

Thus we can write

$$X_{\cdot s} = \sum_k \int_0^{\cdot} f_k\left(g_{ts}\right) \delta_t h_k\left(g_{ts}\right),$$

and

$$\chi_{\cdot s} = \sum_k \int_0^{\cdot} f_k(\Sigma_{ts})\,\delta_t h_k(\Sigma_{ts});$$

where f_k and h_k are matrix-valued and $\mathbb{R}$-valued functions on K respectively. In particular, by the definition of the Fisk-Stratonowicz integral, X_{Ts} is the limit (in probability with respect to measure P) of the sequence

$$X_{Ts}^{\mathbb{P}} \equiv \sum_{k,\mathbb{P}} \int_0^{\cdot} \frac{1}{2}\left[f_k\left(g_{t_{i-1}s}\right) + f_k\left(g_{t_i s}\right)\right]\left[h_k\left(g_{t_i s}\right) - h_k\left(g_{t_{i-1}s}\right)\right],$$

and χ_{Ts} is the limit (in probability with respect to the measure P) of the sequence

$$\chi_{Ts}^{\mathbb{P}} \equiv \sum_{k,\mathbb{P}} \int_0^{\cdot} \frac{1}{2}\left[f_k\left(\Sigma_{t_{i-1}s}\right) + f_k\left(\Sigma_{t_i s}\right)\right]\left[h_k\left(\Sigma_{t_i s}\right) - h_k\left(\Sigma_{t_{i-1}s}\right)\right].$$

Now

$$\begin{aligned} P\left(\{\omega : |\chi_{Ts} - X_{Ts}\circ\Sigma| > \varepsilon\}\right) &= P\left(\left\{\omega : \left|\chi_{Ts}^{\mathbb{P}} - X_{Ts}\circ\Sigma\right| > \varepsilon\right\}\right) \\ &= P\left(\left\{\omega : \left|X_{Ts}^{\mathbb{P}}\circ\Sigma - X_{Ts}\circ\Sigma\right| > \varepsilon\right\}\right) \\ &= P\left(\left|X_{Ts}^{\mathbb{P}} - X_{Ts}\right| > \varepsilon\right) \to 0. \end{aligned}$$

Thus $\chi_{ts} = X_{ts}\circ\Sigma$ almost surely ω. By continuity of χ and X in both their parameters, we have $\chi = X\circ\Sigma$and therefore $t \to X_{t\cdot}$ is a Brownian bridge sheet on $\mathfrak{K}$ with respect to P.

We have only to assert that a biparametrically continuous version of X can be chosen. By Theorem 8.2 it suffices to check that

$$P\left[|X_{ts} - X_{\tau\sigma}|_{\mathfrak{K}}^{\varepsilon}\right] \le C\left[(t-\tau)^2 + (s-\sigma)^2\right]^{\frac{m+\beta}{2}},$$

for some positive ε, C, β and $m = \dim\mathfrak{K}$. Let $t > \tau$.

$$X_{ts} - X_{\tau\sigma} = (X_{ts} - X_{\tau s}) + (X_{\tau s} - X_{\tau\sigma}).$$

As in the proof of Lemma 8.3, if a martingale M has independent increments, then its quadratic variation $\int dM_t dM_t$ is given by $\int d_t EM_t^2$. The process $t \to X_{ts} - X_{\tau s}$ is a Brownian motion on $\mathfrak{K}$ with parameter $G_0(s,s)$ and so has quadratic variation $(t-\tau)\,G_0(s,s)$. The process $\tau \to X_{\tau s}^A - X_{\tau\sigma}^A$ is also a martingale with independent increments and so has quadratic variation

$$\int_0^{\tau}\left(X_{dus}^A - X_{du\sigma}^A\right)^2 = \tau\left[G_0(s,s) + G_0(\sigma,\sigma) - 2G_0(s,\sigma)\right].$$

Thus by Burkholder's inequality we see that

$$\begin{aligned} P\,|X_{ts} - X_{\tau s}|^{\varepsilon} &\le C_m \sum_A P\left|X_{ts}^A - X_{\tau s}^A\right|^{\varepsilon} \\ &\le C_{\varepsilon,m}\sum_A \left|(t-\tau)\,G_0(s,s)\right|^{\varepsilon/2} \\ &\le C_{\varepsilon,m}\left|(t-\tau)\right|^{\varepsilon/2}. \end{aligned}$$

where the constant $C_{\varepsilon,m}$ depends only on ε and m. Again by Burkholder, we have the estimate

$$\begin{aligned}
E\left|X_{\tau s}-X_{\tau\sigma}\right|^{\varepsilon} &\leq C_m \sum_A P\left|X^A_{\tau s}-X^A_{\tau\sigma}\right|^{\varepsilon} \\
&\leq C_{\varepsilon,m}\tau^{\varepsilon/2}\left|G_0\left(s,s\right)+G_0\left(\sigma,\sigma\right)-2G_0\left(s,\sigma\right)\right|^{\varepsilon/2} \\
&\leq C_{\varepsilon,m}\left[\left|s-\sigma\right|^{\varepsilon/2}+\left|s-\sigma\right|^{\varepsilon}\right]. \\
&\leq C_{\varepsilon,m}\left|s-\sigma\right|^{\varepsilon/2}.
\end{aligned}$$

Thus

$$P\left[\left|X_{ts}-X_{\tau\sigma}\right|^{\varepsilon}_{\mathfrak{K}}\right]\leq C_{\varepsilon,m}\left[\left|s-\sigma\right|^{\varepsilon/2}+\left|(t-\tau)\right|^{\varepsilon/2}\right].$$

Picking $\varepsilon > m+\beta$, we are done. ∎

We are now in a position to state the main results of this paper.

Theorem 2.27. *Let K be a compact Lie group. Then Heat Kernel measure, $\nu_1\left(e,\cdot\right)$, is absolutely continuous with respect to pinned Wiener measure, μ_0. Furthermore, the Radon-Nikodym derivative $d\nu_1\left(e,\cdot\right)/d\mu_0$ is bounded.*

Proof. This Theorem is proved as Theorem 5.1 in Section 5. ∎

Theorem 2.28. *Let $s_0<1$ and let $\mathfrak{G}_{s_0}\equiv\sigma\left\langle\pi_t : t\in[0,s_0]\right\rangle$ where $\pi_t : L\left(K\right)\rightarrow K$ is the evaluation map at time t. Then pinned Wiener measure, μ_0, is absolutely continuous with respect to Heat Kernel measure, $\nu_1\left(e,\cdot\right)$, on the σ-algebra $\mathfrak{G}_{s_0}$.*

Proof. This Theorem is proved as Theorem 7.1 in Section 7 ∎

3. Warm-up Section:

3.1. **Path group cases for a Lie group:** Let the Wiener space on K, the space of all continuous paths in K starting at the identity, be given by

$$W_e\left(K\right)\equiv\left\{\sigma\in C\left(\left[0,1\right]\rightarrow K\right)\left|\sigma\left(0\right)=e\right.\right\}.$$

The goal of this section is to assert that Heat Kernel measure on $W_e\left(K\right)$ and Wiener measure on $W_e\left(K\right)$ are the same.

Definition 3.1 (Riemannian Structure on $W_e\left(K\right)$)**.** Define $H\equiv H\left(\mathfrak{K}\right)$ to be the Sobolev space of functions with one L^2-derivative as in Definition 2.1. We will think of H as the Lie algebra of $W_e\left(K\right)$. In order to define the tangent space $TW_e\left(K\right)$ of $W_e\left(K\right)$ let θ denote the Maurer-Cartan form. That is $\theta\left\langle\xi\right\rangle\equiv\left(\ell_{k^{-1}}\right)_*\xi$ for all $\xi\in T_kK$, and $k\in K$ and where ℓ_g denotes left multiplication by $g\in K$. Let $\theta\left\langle X\right\rangle\left(s\right)\equiv\theta\left\langle X\left(s\right)\right\rangle$ and $p: TK\rightarrow K$ be the canonical projection. We now define

$$TW_e\left(K\right)\equiv\left\{X:\left[0,1\right]\rightarrow TK\left|\theta\left\langle X\right\rangle\in H\text{ and }p\circ X\in W_e\left(K\right)\right.\right\}.$$

By abuse of notation, use the same p to denote the canonical projection from $TW_e\left(K\right)\rightarrow W_e\left(K\right)$. As usual, define the tangent space at $k\in W_e\left(K\right)$ by $T_kW_e\left(K\right)\equiv p^{-1}\left\{k\right\}$. Using left translations, we extend the inner product $\left(\cdot,\cdot\right)_H$ on H to a Riemannian metric on $TW_e\left(K\right)$). Explicitly set

$$\left(X,X\right)_{W_e\left(K\right)}\equiv\left(\theta\left\langle X\right\rangle,\theta\left\langle X\right\rangle\right)_H\text{ where }X\in TW_e\left(K\right).$$

In this way, $W_e(K)$ is to be thought of as an infinite-dimensional Riemannian manifold. Viewing the Lie algebra $(\mathfrak{K}, 0)$ as a commutative Lie group in its own right with Lie algebra $\mathfrak{K}$, we obtain definitions for

$$W_0(\mathfrak{K}) \equiv \{\sigma \in C([0,1] \to \mathfrak{K}) \,|\, \sigma(0) = 0\}$$

as the "Lie group" with Lie algebra $H(\mathfrak{K})$ thought of as a commutative Lie algebra.

Definition 3.2 (Good Orthonormal basis of H). $\{\eta_k\}_{k\in\mathbb{N}}$, is a good orthonormal basis of H if $\{\eta_k\}_{k\in\mathbb{N}}$, is an orthonormal basis of H so that the Lie Bracket $[\eta_k(s), \eta_k'(s)]$ is identically zero for all values of s and k.

Example 3.3 (Good bases exist). Take $\{h_k\}$ to be an orthonormal basis of $H(\mathbb{R})$ and let $\{A\}$ run through an orthonormal basis of $\mathfrak{K}$. Then $\eta_{A,k} \equiv h_k A$ is a good orthonormal basis of H.

Definition 3.4 (**The Laplacian** $\triangle_{W_e(K)}$). Take a good orthonormal basis S of $H(\mathfrak{K})$. Define an operator $\triangle$ on functions on $W_e(K)$ by taking

$$\triangle f \equiv \sum_{h\in S} \partial_h^2 f,$$

where

$$(\partial_h f)(\gamma) \equiv \partial_\varepsilon f(\gamma \exp \varepsilon h)\,|_{\varepsilon=0}.$$

Define a Laplacian, denoted by $\triangle_{W_0(\mathfrak{K})}$, on functions on $W_0(\mathfrak{K})$ in the same way as above by taking

$$\triangle_{W_0(\mathfrak{K})} f \equiv \sum_{h\in S} \partial_h^2 f,$$

where

$$(\partial_h f)(\gamma) \equiv \partial_\varepsilon f(\gamma + \varepsilon h)\,|_{\varepsilon=0}.$$

It is well known that the operators $\triangle$ and $\triangle_{W_0(\mathfrak{K})}$ defined above are independent of the choice of good orthonormal basis (see [13]).

Definition 3.5 (Brownian Sheet on $\mathfrak{K}$). A Gaussian process $\{\beta(t)\}_{t\in[0,1]}$ is a $\mathfrak{K}$-valued Brownian sheet if for (t,s) in $[0,1]^2$, $\beta(t,s)$ is a $\mathfrak{K}$-valued mean-zero Gaussian process with covariance given by

$$E\,\langle A, \beta(t,s)\rangle_{\mathfrak{K}} \langle B, \beta(\tau,\sigma)\rangle_{\mathfrak{K}} = \langle A, B\rangle_{\mathfrak{K}} (t\wedge\tau)\, G(s,\sigma),$$

where $\beta(t,s) \equiv \beta(t)(s) \in \mathfrak{K}$; $A, B \in \mathfrak{K}$; $t, \tau, s, \sigma \in [0,1]$; and $G(s,\sigma) \equiv \min(s,\sigma)$.

Definition 3.6 (Brownian motion on $W_e(K)$). The process $t \to \Sigma(t,\cdot)$ is a $W_e(K)$-valued Brownian motion if and only if for any smooth cylinder function $f: W_e(K) \to \mathbb{R}$, there is a real-valued martingale M_t so that

$$f(\Sigma(dt,\cdot)) = dM_t + \frac{1}{2}\left(\triangle_{W_e(K)} f\right)(\Sigma(t,\cdot))\,dt. \tag{3.1}$$

We can define a Brownian motion on $W_0(\mathfrak{K})$ by thinking of $\mathfrak{K}$ as a commutative Lie group, and using $\triangle_{W_0(\mathfrak{K})}$ instead of $\triangle_{W_e(K)}$ in Eq. [3.1].

Lemma 3.7 (Effect of the Laplacian $\triangle_{W_e(K)}$ on cylinder functions, see [13]). *Let G be as in Definition3.5. Let $\mathbb{P}$ be the partition $\{0 < s_1 < \cdots < s_n < 1\}$. Define $\pi_{\mathbb{P}} : \sigma \to (\sigma_{s_1}, \cdots, \sigma_{s_n}) \in K^n$. For $F \in C^{\infty}(K^n)$, define*

$$\left(A^{(i)}F\right)(g_1, \cdots, g_n) \equiv \frac{d}{dt}|_{t=0} F(\cdots, g_i \exp tA, \cdots).$$

Define an elliptic operator $L_{\mathbb{P}}$ on $C^{\infty}(K^n)$ by setting

$$L_{\mathbb{P}} \equiv \sum_{i,j,A} G(s_i, s_j) A^{(i)} A^{(j)}.$$

Then letting A run through an orthonormal basis of $\mathfrak{K}$, for any smooth cylinder function $F : K^n \to \mathbb{R}$ we have

$$\triangle_{W_e(K)}(F \circ \pi_{\mathbb{P}}) = (L_{\mathbb{P}} F) \circ \pi_{\mathbb{P}}.$$

This Lemma can also be used on the Lie algebra $\mathfrak{K}$ by viewing $\mathfrak{K}$ itself as a commutative Lie group.

Proof. Use the same proof as that of Lemma 2.15 by replacing $H_0(\mathfrak{K})$, $H_0(\mathbb{R})$, $G_0(s, \sigma)$ by $H(\mathfrak{K})$, $H(\mathbb{R})$, $G(s, \sigma)$. ∎

Lemma 3.8 (Brownian Motion on $W_0(\mathfrak{K})$ exists). *Every Brownian Sheet on $\mathfrak{K}$ is a $W_0(\mathfrak{K})$-valued Brownian motion. More precisely, if $\beta_t \equiv \beta(t, \cdot)$ is a Brownian Sheet then for any smooth cylindrical function f, there is a real-valued martingale M_t so that*

$$df(\beta_t) = dM_t + \frac{1}{2}\left(\triangle_{W_0(\mathfrak{K})} f\right)(\beta_t)\, dt.$$

Here $\triangle_{W_0(\mathfrak{K})}$ is the Laplace-Beltrami operator defined in Definition 3.4, where $(\mathfrak{K}, 0)$ is viewed as a Lie group.

Proof. Use the proof of Lemma 2.16 with β, G in place of χ, G_0. ∎

Lemma 3.9 (Semimartingale properties of $h_{t\cdot}$). *Let b be a $\mathfrak{K}$-valued Brownian Sheet (see Definition 3.5)Let h_{ts} be the solution to*

$$h_{\delta ts} = h_{ts} b_{\delta ts} \text{ with } h_{0s} = e. \tag{3.2}$$

Then the process $s \mapsto h_{ts}$ is a K-valued Brownian motion with parameter t. Furthermore one can choose a version of h which is jointly continuous in both parameters s and t. In future, h will be taken to be this jointly continuous solution. Note:- Eq. [3.2] is to be interpreted like Eq. [2.4].

Proof. Let $s_i = i/n$. Then $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$ is a partition of $[0,T]$. For convenience, let $\triangle_i b(t) \equiv b_{ts_i} - b_{ts_{i-1}}$. We compute

$$\begin{aligned}
&\delta_t \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) \\
&\quad = h_{ts_i} b_{\delta ts_i} h_{ts_{i-1}}^{-1} - h_{ts_i} b_{\delta ts_{i-1}} h_{ts_{i-1}}^{-1} \\
&\quad = h_{ts_i} \triangle_i b(\delta t) h_{ts_{i-1}}^{-1} \\
&\quad = \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) Ad_{h_{ts_{i-1}}} \triangle_i b(\delta t) \\
&\quad = \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) Ad_{h_{ts_{i-1}}} \triangle_i b(dt) + \frac{1}{2} d_t \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) Ad_{h_{ts_{i-1}}} \triangle_i b(dt) \\
&\qquad + \frac{1}{2} Ad_{h_{ts_{i-1}}} \left[b_{dts_{i-1}}, \triangle_i b(dt) \right] \\
&\quad = \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) Ad_{h_{ts_{i-1}}} \triangle_i b(dt) + \frac{1}{2} d_t \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) Ad_{h_{ts_{i-1}}} \triangle_i b(dt) \\
&\quad = \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) \delta_t \int_0^t Ad_{h_{\tau s_{i-1}}} \triangle_i b(d\tau),
\end{aligned}$$

where we have used that fact that $b_{ts_{i-1}} \in \mathfrak{F}_{1s_{i-1}}$ and that $\triangle_i b(\cdot)$ is independent of $\mathfrak{F}_{1s_{i-1}}$. Thus

$$\delta_t \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) = \left(h_{ts_i} h_{ts_{i-1}}^{-1} \right) \delta_t \int_0^t Ad_{h_{\tau s_{i-1}}} \triangle_i b(d\tau) \text{ with } h_{0s_i} h_{0s_{i-1}}^{-1} = e.$$

It suffices to show that $\left\{ \int_0^\cdot Ad_{h_{ts_{i-1}}} \triangle_i b(dt) \right\}_{i \in \{1,\cdots,n\}}$ is a $\mathfrak{K}^n$-valued Brownian motion with parameter $1/n$, since this will imply that $t \to \left\{ h_{ts_i} h_{ts_{i-1}}^{-1} \right\}_{i \in \{1,\cdots,n\}}$ is a K^n-valued Brownian motion with the same parameter. But this is true by Levy's criterion and the following computation of quadratic variations.

Let J_t denote the joint quadratic variation

$$\int_0^t Ad_{h_{\tau s_{i-1}}} \triangle_i b(d\tau) Ad_{h_{\tau s_{j-1}}} \triangle_j b(d\tau).$$

Then

$$\begin{aligned}
dJ_t &= Ad_{h_{ts_{i-1}}} \triangle_i b(dt) Ad_{h_{ts_{j-1}}} \triangle_j b(dt) \\
&= \sum_{A,B} \left(Ad_{h_{ts_{i-1}}} A \otimes Ad_{h_{ts_{j-1}}} B \right) \triangle_i b^A(dt) \triangle_j b^B(dt) \\
&= \delta_{ij} \triangle_i s dt \sum_A \left(Ad_{h_{ts_{i-1}}} A \right)^{\otimes 2} \\
&= \frac{\delta_{ij}}{n} \sum_A \left(Ad_{h_{ts_{i-1}}} A \right)^{\otimes 2} dt \\
&= \frac{\delta_{ij}}{n} \sum_A A^{\otimes 2} dt.
\end{aligned}$$

Thus, in particular, $Law \left(h_{Ts_1} h_{Ts_0}^{-1}, \cdots, h_{Ts_n} h_{Ts_{n-1}}^{-1} \right)$ is Heat Kernel Measure on K^n at time T/n. But this implies that $s \to h_{T\cdot}$ is a K-valued Brownian motion with parameter T.

We have only to show that h_{ts} satisfies the hypothesis of Theorem 8.2. That is we must show that

$$P\left[d\left(h_{ts},h_{\tau\sigma}\right)^{p}\right]\leq C\left[\left(t-\tau\right)^{2}+\left(s-\sigma\right)^{2}\right]^{\frac{m+\beta}{2}}.$$

The proof is essentially the same as that done in Theorem 3.8 of Driver [11] with the modification that $G\left(s.\sigma\right)$ is used in place of $G_{0}\left(s,\sigma\right)$. in particular, see Eq. [3.12] of [11]. ∎

Theorem 3.10 (Brownian motion exists on $W_{e}\left(K\right)$)**.** *Let h be the jointly continuous solution of Eq. [3.2]. Let h_{t} denote the element $s\rightarrow h_{ts}$ in $W_{e}\left(K\right)$. Let k be an element of $W_{e}\left(K\right)$. Then $t\rightarrow kh_{t}$ is a $W_{e}\left(K\right)$-valued Brownian motion starting from the path k.*

Proof. Let

$$f\left(\sigma\right)\equiv F\left(\sigma_{s_{1}},\cdots,\sigma_{s_{n}}\right),$$

be a smooth cylindrical function where $\mathbb{P}$ is the partition $\{0<s_{1}<\cdots<s_{n}<1\}$. Let $A^{(i)}$, $\pi_{\mathbb{P}}$ and $L_{\mathbb{P}}\equiv\sum G\left(s_{i},s_{j}\right)A^{(i)}A^{(j)}$ be as in Lemma 3.7. Let $h^{\mathbb{P}}\left(t\right)\equiv\left(\pi_{\mathbb{P}}\circ h_{t}\right)$ and let $k^{\mathbb{P}}=\pi_{\mathbb{P}}\circ k$. Let $\ell_{k^{\mathbb{P}}}$ denote left translation by the element $k^{\mathbb{P}}\in K^{n}$ and let ℓ_{k} be left translation by the path $k\in W_{e}\left(K\right)$. Simplifying, we get

$$df\left(kh_{t}\right)=d\left(f\circ\ell_{k}\right)\left(h_{t}\right)=d\left(F\circ\pi_{\mathbb{P}}\circ\ell_{k}\right)\left(h_{t}\right)=d\left(F\circ\ell_{k^{\mathbb{P}}}\right)\left(h^{\mathbb{P}}\left(t\right)\right).$$

By Eq. [3.2] the K^{n}-valued process $h^{\mathbb{P}}$ satisfies

$$h^{\mathbb{P}}\left(\delta t\right)=h^{\mathbb{P}}\left(t\right)b^{\mathbb{P}}\left(\delta t\right)\ \text{ with }h^{\mathbb{P}}\left(0\right)=e,$$

where e denotes the identity element in K^{n}. Then by Ito's Lemma, we have

$$\begin{aligned}df\left(kh_{t}\right)&=\sum_{A,i}\left(A^{(i)}F\circ\ell_{k^{\mathbb{P}}}\right)\left(h^{\mathbb{P}}\left(t\right)\right)b^{A}\left(\delta t,s_{i}\right)\\&=\sum_{A,i}\left(A^{(i)}F\right)\circ\ell_{k^{\mathbb{P}}}\left(h^{\mathbb{P}}\left(t\right)\right)b^{A}\left(dt,s_{i}\right)\\&\quad+\frac{1}{2}\sum_{A,i}d_{t}\left[\left(A^{(i)}F\right)\circ\ell_{k^{\mathbb{P}}}\left(h^{\mathbb{P}}\left(t\right)\right)\right]b^{A}\left(dt,s_{i}\right).\end{aligned}\tag{3.3}$$

The quadratic variation

$$\begin{aligned}&d_{t}[\left(A^{(i)}F\right)\circ\ell_{k^{\mathbb{P}}}\left(h^{\mathbb{P}}\left(t\right)\right)]b^{A}\left(dt,s_{i}\right)\\&=\sum_{B,j}\left(B^{(j)}A^{(i)}F\right)\circ\ell_{k^{\mathbb{P}}}\left(h^{\mathbb{P}}\left(t\right)\right)b^{B}\left(dt,s_{j}\right)b^{A}\left(dt,s_{i}\right)\\&=\sum_{j}\left(A^{(j)}A^{(i)}F\right)\circ\ell_{k^{\mathbb{P}}}\left(h^{\mathbb{P}}\left(t\right)\right)G\left(s_{i},s_{j}\right)dt.\end{aligned}$$

Here we have used the fact (see Lemma 8.3) that the quadratic variation

$$b^{B}\left(dt,s_{j}\right)b^{A}\left(dt,s_{i}\right)=\langle A,B\rangle_{\mathfrak{K}}G\left(s_{i},s_{j}\right)dt.$$

Returning to Eq. [3.3] yields

$$df\left(kh_{t}\right)=dmartingale+\frac{1}{2}\sum_{A,i,j}G\left(s_{i},s_{j}\right)\left(A^{(i)}A^{(j)}F\right)\left(k^{\mathbb{P}}h^{\mathbb{P}}\left(t\right)\right)dt.$$

Invoking Lemma 3.7 yields

$$df\left(kh_t\right) = dmartingale + \frac{1}{2}\left(\triangle_{W_e(K)}f\right)\left(kh_t\right)dt$$

for any smooth cylinder function f. Thus $t \to h\left(t,\cdot\right)$ is a Brownian motion on $L\left(K\right)$. ∎

Definition 3.11 (Heat Kernel measure on $W_e\left(K\right)$)**.** Let k be an element of $W_e\left(K\right)$. Let $t \to h_t$ be a $W_e\left(K\right)$-valued Brownian motion so that $h_0 = k$ a.s. Then, as in the finite-dimensional manifold case, Heat Kernel measure $\nu_T^{W_e(K)}\left(k, d\gamma\right)$ is defined to be the law of $h\left(T,\cdot\right)$.

Remark 3.12 (Heat Kernel measures solve the Heat Equation). Let $\mathbb{P}$ be the partition $\{0 < s_1 < \cdots < s_n < 1\}$. Let $A^{(i)}$, $\pi_{\mathbb{P}}$ and $L_{\mathbb{P}}$ be as in Lemma 3.7. Let $f \equiv F \circ \pi_{\mathbb{P}}$ be a smooth cylinder function for some $F \in C^\infty\left(K^n\right)$. Let

$$u\left(t,k\right) \equiv \int f\left(\gamma\right)\nu_t^{W_e(K)}\left(k, d\gamma\right)$$

Let h be a $W_e\left(K\right)$-valued Brownian motion starting from $k \in W_e\left(K\right)$. Then $\nu_t^{W_e(K)}\left(k, d\gamma\right)$ is the law of h. Let G^{-1} be the $n \times n$ matrix that is inverse to $\left(G\left(s_i, s_j\right)\right)$. Endow K^n with the metric $\left\langle A^{(i)}, B^{(j)}\right\rangle = \left\langle A, B\right\rangle_{\mathfrak{K}} G_{kj}^{-1}$ so that the Laplacian on K^n (viewed as a Riemannian manifold) is the operator $L_{\mathbb{P}} = \sum_{i,j,A} G\left(s_i, s_j\right)A^{(i)}A^{(j)}$. Now $t \to h^{\mathbb{P}}\left(t\right) \equiv \pi_{\mathbb{P}} \circ h_t$ satisfies the martingale characterization of a Brownian motion on K^n with this metric since by Lemma 3.7

$$F \circ h^{\mathbb{P}}\left(dt\right) = F \circ \pi_{\mathbb{P}} \circ h_{dt} = dmartingale + \frac{1}{2}\left(L_{\mathbb{P}}F\right) \circ h^{\mathbb{P}}\left(t\right)dt.$$

Thus

$$u\left(t,k\right) = Ef \circ h_t = EF \circ h^{\mathbb{P}}\left(t\right) = \left(\exp\left(\frac{t}{2}L_{\mathbb{P}}\right)F\right) \circ \pi_{\mathbb{P}}\left(k\right). \tag{3.4}$$

By Lemma 3.7

$$\left(L_{\mathbb{P}}^n F\right) \circ \pi_{\mathbb{P}} = \triangle_{W_e(K)}\left(L_{\mathbb{P}}^{n-1}F \circ \pi_{\mathbb{P}}\right) = \triangle_{W_e(K)}^n\left(F \circ \pi_{\mathbb{P}}\right).$$

So in particular,

$$\begin{aligned}\left(\exp\left(\frac{t}{2}L_{\mathbb{P}}\right)F\right) \circ \pi_{\mathbb{P}} &= \sum_{\mathbb{N}} \frac{t^n}{2^n}\left(L_{\mathbb{P}}^n F\right) \circ \pi_{\mathbb{P}} \\ &= \sum_{\mathbb{N}} \frac{t^n}{2^n}\triangle_{W_e(K)}^n\left(F \circ \pi_{\mathbb{P}}\right) \\ &= \exp\left(\frac{t}{2}\triangle_{W_e(K)}\right)\left(F \circ \pi_{\mathbb{P}}\right).\end{aligned}$$

Returning to Eq. [3.4] yields

$$u\left(t,k\right) = \left(\exp\left(\frac{t}{2}\triangle_{W_e(K)}\right)f\right)\left(k\right).$$

Corollary 3.13. *Heat Kernel measure $\nu_T^{W_e(K)}\left(e, d\gamma\right)$ and Wiener measure with parameter t are the same measure.*

Proof. By Definition 3.11, Heat Kernel measure is the law of $h_{t\cdot}$. By Lemma 3.9 $s \to h_{ts}$ is a Brownian motion on K with parameter t. Thus heat kernel measure and Wiener measure are the same. ∎

3.2. **Semimartingale Properties of X_{ts}:** Let X_{ts} be as in Theorem 2.25. Then X is a Brownian bridge sheet on $\mathfrak{K}$. Brownian Sheets are easier to work with than Brownian bridge Sheets (they are martingales in both their parameters for instance). The goal of this section is to write $X_{t\cdot}$ as a linear functional of $b_{t\cdot}$,a Brownian sheet. To motivate this decomposition we first introduce Proposition 3.14. The Brownian bridge $\widetilde{X}_\cdot$ is supposed to play the role of $X_{t\cdot}$ but with one fewer parameter.

Proposition 3.14. *Let $\widetilde{X}$ be the canonical process on $C([0,1] \to \mathbb{R})$. That is $\widetilde{X}_s$ sends a path γ to its evaluation $\gamma(s)$ at time s. Let the*

$$P_t^{\mathbb{R}}(x) \equiv \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$$

be Heat Kernels on $\mathbb{R}$. Define a new process $\widetilde{b}$ by setting

$$\widetilde{b}_s \equiv \widetilde{X}_s - \int_0^s \nabla \ln P_{1-\sigma}^{\mathbb{R}}(\widetilde{X}_\sigma) d\sigma = \widetilde{X}_s + \int_0^s \frac{\widetilde{X}_\sigma}{1-\sigma} d\sigma.$$

Then $\widetilde{b}$ is a standard $\mathbb{R}$-valued Brownian motion.

Notation 3.15. Let $\mu_{\mathbb{R}}$ be Wiener Measure on $C([0,1] \to \mathbb{R})$. Let $\mu_0^{\mathbb{R}} = Law\ \widetilde{X}$ be pinned Wiener measure on $\mathbb{R}$ ($\mu_0^{\mathbb{R}}$ is the measure μ_0 in Definition 2.9 if K is taken to be $\mathbb{R}$). Let $\mathfrak{F}_t$ be the σ-algebra generated by the $\widetilde{X}_t$ with $t \in [0,s]$. Let $Z_s \equiv d\left(\mu_0^{\mathbb{R}} \downarrow \mathfrak{F}_s\right) / d\left(\mu_{\mathbb{R}} \downarrow \mathfrak{F}_s\right)$.

Proof. Now $Z_s = P_{1-s}^{\mathbb{R}}\left(\widetilde{X}_s\right) / P_1^{\mathbb{R}}(0)$. By definition, $\widetilde{X}_\cdot$ is a Brownian motion with respect to the measure $\mu_{\mathbb{R}}$. Hence by The Meyer-Girsanov Theorem , which we state as Theorem 3.16 below for convenience,

$$N_\cdot \equiv \widetilde{X}_\cdot - \int_0^\cdot \frac{d\widetilde{X}_s dZ_s}{Z_s}$$

is a local martingale. This expression has the same quadratic variation as $\widetilde{X}_\cdot$ (since the measures $\mu_{\mathbb{R}}$ and $\mu_0^{\mathbb{R}}$ are equivalent on $\mathfrak{F}_s$ when $s < 1$). Thus this expression $N_\cdot$ is a Brownian motion by Levy's criterion. Computing directly, we see that

$$\begin{aligned} d\widetilde{X}_s dZ_s &= d\widetilde{X}_s d_s \exp\left[\log P_{1-s}^{\mathbb{R}}\left(\widetilde{X}_s\right) - \log P_1^{\mathbb{R}}(0)\right] \\ &= \exp\left[\log P_{1-s}^{\mathbb{R}}\left(\widetilde{X}_s\right) - \log P_1^{\mathbb{R}}(0)\right]\left(\nabla \log P_{1-s}^{\mathbb{R}}\right)\left(\widetilde{X}_s\right) d\widetilde{X}_s d\widetilde{X}_s \\ &= Z_s \left(\nabla \log P_{1-s}^{\mathbb{R}}\right)\left(\widetilde{X}_s\right) ds \end{aligned}$$

Thus $N_\cdot = \widetilde{X}_\cdot - \int_0^\cdot \left(\nabla \log P_{1-s}^{\mathbb{R}}\right)\left(\widetilde{X}_s\right) ds = \widetilde{b}_\cdot$ and we are done. ∎

Theorem 3.16 (Meyer-Girsanov, see [30])**.** *Let P and Q be equivalent measures and let $Z_s \equiv E\left[dQ/dP|\mathfrak{F}_s\right]$. Let $\widetilde{X}$ be a semimartingale under P with decomposition $M + A$ (where M is a local martingale and A has finite variation). Then $\widetilde{X}$ is also a semimartingale under Q with decomposition $N + C$ where*

$$N_\cdot = \widetilde{X}_\cdot - \int_0^\cdot \frac{d\widetilde{X}_s dZ_s}{Z_s}$$

is a Q-local martingale and $C \equiv \widetilde{X} - N$ is a finite variation process.

Definition 3.17. Define the following linear maps:-

1. Define continuous $\mathfrak{K}$-valued linear maps on paths,
$$T_t, S_t : C([0,1] \to \mathfrak{K}) \to \mathfrak{K},$$
by setting
$$\begin{aligned} T_t(\omega) &\equiv \omega(t) - \int_0^t \omega(\tau)\frac{(1-t)}{(1-\tau)^2}d\tau \text{ if } t \in [0,1). \\ S_t(\omega) &\equiv \omega(t) + \int_0^t \frac{\omega(\tau)}{(1-\tau)}d\tau \text{ if } t \in [0,1). \end{aligned}$$
2. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the subsets of $C([0,1] \to \mathfrak{K})$ on which the limits $\lim_{t\to 1} T_t(\omega)$ and $\lim_{t\to 1} S_t(\omega)$ exist respectively. Then define maps T_1 and S_1 from $C([0,1] \to \mathfrak{K})$ to $\mathfrak{K}$ by setting
$$\begin{aligned} T_1 &\equiv 1_{\mathcal{U}_1}(\omega) \lim_{t\to 1} T_t(\omega). \\ S_1 &\equiv 1_{\mathcal{U}_2}(\omega) \lim_{t\to 1} S_t(\omega). \end{aligned}$$

Remark 3.18. Notice that in Proposition 3.14 we wrote the underlying Brownian motion $\widetilde{b}.$ as $S\left(\widetilde{X}.\right)(\cdot)$. Similarly we shall prove the process $b_{t\cdot} \equiv S(X_{t\cdot})$ is a Brownian Sheet and that $X_{t\cdot}$ can be written as $T(b_{t\cdot})$.

Theorem 3.19 (Decomposition of the Brownian bridge sheet). *Let X be the Brownian bridge sheet from Theorem 2.25. Define b by setting*
$$b_{ts} \equiv S_s(X_{t\cdot}) = X_{ts} + \int_0^s \frac{X_{t\sigma}d\sigma}{1-\sigma} \text{ for any } t, s \in [0,1].$$
Then b is a Brownian sheet on $\mathfrak{K}$ and X_{ts} can be recovered from b as:
$$X_{ts} = T_s(b_{t\cdot}) = b_{ts} - \int_0^s b_{t\sigma}\frac{(1-s)}{(1-\sigma)^2}d\sigma. \tag{3.5}$$

We shall defer the proof of Theorem 3.19 until after the Lemma 3.21 below.

Remark 3.20. For another explicit computational proof of this Theorem, see Theorem 8.5 in the Appendix 8.

Lemma 3.21 (Properties of the transformations S and T). *Define a map T from $H(\mathfrak{K})$ to $H_0(\mathfrak{K})$ by setting $T(\omega)(t) = T_t(\omega)$. Define a map S from $H_0(\mathfrak{K})$ to $H(\mathfrak{K})$ by setting $S(\omega)(t) = S_t(\omega)$. Then:-*

1. *S is well-defined and is a unitary isomorphism from $H_0(\mathfrak{K})$ to $H(\mathfrak{K})$.*
2. *T is well-defined and is the inverse of S.*

Proof. Let $\omega \in H(\mathfrak{K})$. By an integration-by-parts we can express T more concisely as
$$\begin{aligned} T(\omega)(t) &= \omega(t) - \int_0^t \omega(\tau)\frac{(1-t)}{(1-\tau)^2}d\tau \\ &= \omega(t) - (1-t)\int_{\tau=0}^{\tau=t} \omega(\tau)d\frac{1}{1-\tau} \\ &= \int_0^t \frac{(1-t)}{1-\tau}\omega'(\tau)d\tau. \end{aligned}$$

Thus we have the inequality

$$\begin{aligned}
|T(\omega)(t)| &\leq (1-t)\int_0^t \frac{1}{1-\tau}\,|\omega'(\tau)|\,d\tau \\
&\leq |\omega|_{H(\mathfrak{K})}\,(1-t)\sqrt{\int_0^t \frac{d\tau}{(1-\tau)^2}} \\
&= |\omega|_{H(\mathfrak{K})}\,\sqrt{t\,(1-t)} \\
&\to 0 \text{ as } t\to 1.
\end{aligned}$$

Thus T_t is continuous on $H\,(\mathfrak{K})$, and $\operatorname{Im} T \subset L\,(\mathfrak{K})$.

Let $\mathfrak{U}_0$ denote the subspace of functions of the form $\sigma = \int_0^{\cdot} x\,(t)\,dt$ where x is in the continuous maps from $[0,1]$ to $\mathfrak{K}$ so that its average, $\int_0^1 x\,(t)\,dt$, is 0. By the Stone-Weierstrass Theorem, $C\,([0,1]\to\mathfrak{K})$ is dense in $L^2\,([0,1]\to\mathfrak{K}, d\lambda)$ and in particular continuous functions with zero average are dense in the space of L^2 functions with zero average. Thus by the isometry provided by the map $x \to \int x dt$ from $L^2\,([0,1]\to\mathfrak{K}, d\lambda)$ to $H\,(\mathfrak{K})$ we see that $\mathfrak{U}_0$ is dense in $H_0\,(\mathfrak{K})$ in the $H_0\,(\mathfrak{K})$ norm topology.

We claim S is a norm-preserving map from $\mathfrak{U}_0$ to $H\,(\mathfrak{K})$. Let $\sigma = \int_0^{\cdot} x\,(t)\,dt$ in $\mathfrak{U}_0$. Computing, we see that

$$\begin{aligned}
|S(\sigma)|^2_{H(\mathfrak{K})} &= \int_0^1 \left|\sigma'(t) + \frac{\sigma(t)}{(1-t)}\right|^2_{\mathfrak{K}} dt \\
&= |\sigma|^2_{H_0(\mathfrak{K})} + 2\int_0^1 \left\langle \sigma'(t), \frac{\sigma(t)}{(1-t)}\right\rangle_{\mathfrak{K}} dt + \int_0^1 |\sigma(t)|^2_{\mathfrak{K}}\,\frac{dt}{(1-t)^2} \\
&= |\sigma|^2_{H_0(\mathfrak{K})} + 2\int_0^1 \left\langle \sigma'(t), \frac{\sigma(t)}{(1-t)}\right\rangle_{\mathfrak{K}} dt + \int_{t=0}^{t=1} |\sigma(t)|^2_{\mathfrak{K}}\,d\left(\frac{1}{1-t}\right) \\
&= |\sigma|^2_{H_0(\mathfrak{K})} + \lim_{t\to 1}\frac{|\sigma(t)|^2_{\mathfrak{K}}}{1-t} \\
&= |\sigma|^2_{H_0(\mathfrak{K})} - 2\,\langle \sigma(1), x\,(1)\rangle_{\mathfrak{K}} \\
&= |\sigma|^2_{H_0(\mathfrak{K})} \text{ since } \sigma\,(1) = 0.
\end{aligned}$$

By the Bounded Limit Theorem, we can extend S to a map $\overline{S}$ on all of $H_0\,(\mathfrak{K})$ by defining

$$\overline{S}\,(\omega) \equiv \lim_{n\to\infty} S\,(\omega_n) \text{ for any } \omega_n \in \mathfrak{U}_0,\ \omega_n \to \omega \text{ in } H_0\,(\mathfrak{K})\,.$$

Although $\overline{S}$ and S agree on $\mathfrak{U}_0$ they could be different maps on $H_0\,(\mathfrak{K})$. We will check that this is not the case. Notice that the evaluation map sending ω in $H\,(\mathfrak{K})$ to $\omega\,(t)$ in $\mathfrak{K}$ is a bounded linear map. Also if $s<1$, the map $S_s|_{H(\mathfrak{K})}$ is a continuous map from $H\,(\mathfrak{K})$ to $\mathfrak{K}$ in the $H\,(\mathfrak{K})$-norm. Therefore, if ω_n in $\mathfrak{U}_0$ converges to ω in the $H\,(\mathfrak{K})$-norm, we have for all $s<1$,

$$\overline{S}\,(\omega)\,(s) = \lim_{n\to\infty} S\,(\omega_n)\,(s) = \lim_{n\to\infty} S_s\,(\omega_n) = S\,(\omega)\,(s)\,.$$

Thus $S = \overline{S}$ which is a norm-preserving map from $H_0\,(\mathfrak{K})$ into $H\,(\mathfrak{K})$.

Let x in $H_0(\mathfrak{K})$. Let $y = S(x)$ in $H(\mathfrak{K})$ and $z = T(y)$ in $L(\mathfrak{K})$ so that $z = T \circ S(x)$. As before, $x(0) = z(0) = 0$. Letting $t < 1$ and computing, we have

$$\begin{aligned} z'(t) &= y'(t) - \int_0^t \frac{y'(\tau)\,d\tau}{1-\tau} \\ &= x'(t) + \frac{x(t)}{1-t} - \int_0^t \frac{x'(\tau)\,d\tau}{1-\tau} - \int_0^t \frac{x(\tau)\,d\tau}{(1-\tau)^2} \\ &= x'(t) + \frac{x(t)}{1-t} - \int_0^t \frac{dx(\tau)}{1-\tau} - \int_0^t \frac{x(\tau)\,d\tau}{(1-\tau)^2} \\ &= x'(t)\,. \end{aligned}$$

So $T \circ S$ is the identity on $H_0(\mathfrak{K})$ and so T is a surjective norm-preserving from $\operatorname{Im} S$ to $H_0(\mathfrak{K})$.

Let x in $H(\mathfrak{K})$, $y = T(x)$ in $L(\mathfrak{K})$ and $z = S_t(y)$ so that $z_t = S_t \circ T(x)$. Since $x(0) = z_0 = 0$ and

$$\begin{aligned} \frac{d}{dt} z_t &= y'(t) + \frac{y(t)}{1-t} \\ &= x'(t) - \int_0^t \frac{x'(\tau)\,d\tau}{1-\tau} + \frac{y(t)}{1-t} \\ &= x'(t) - \frac{T(x)(t)}{1-t} + \frac{y(t)}{1-t} \\ &= x'(t)\,. \end{aligned}$$

So $S_t \circ T(x) = x(t)$ for any x in $H(\mathfrak{K})$.

Now we show that T maps a dense subspace $\mathfrak{U}$ of $H(\mathfrak{K})$ into $H_0(\mathfrak{K})$. If $x \in \mathfrak{U}$, then

$$S \circ T(x)(t) = S_t \circ T(x) = x(t)\,.$$

In particular, x belongs to $\operatorname{Im} S$. Thus $\operatorname{Im} S$ contains a dense set $\mathfrak{U}$ and so is all of $H(\mathfrak{K})$. So S will be a unitary isomorphism between $H_0(\mathfrak{K})$ and $H(\mathfrak{K})$, T will be its inverse, and we shall be done.

Let $\mathfrak{U}$ denote the subspace of functions of the form $\sigma = \int_0^{\cdot} x(t)\,dt$ where x is in $C([0,1] \to \mathfrak{K})$. By the Stone-Weierstrass Theorem, $C([0,1] \to \mathfrak{K})$ is dense in $L^2([0,1] \to \mathfrak{K}, d\lambda)$. Thus by the isometry provided by the map $x \to \int x dt$ from $L^2([0,1] \to \mathfrak{K}, d\lambda)$ to $H(\mathfrak{K})$ we see that $\mathfrak{U}$ is dense in $H(\mathfrak{K})$ in the $H(\mathfrak{K})$ norm topology. Then

$$T(\sigma)'(t) = \sigma'(t) - \int_0^t \frac{\sigma'(\tau)}{1-\tau} d\tau = x(t) - \int_0^t \frac{x(\tau)}{1-\tau} d\tau.$$

$$\begin{aligned}
&\int_0^1 \left|T(\sigma)'(t)\right|^2 dt \\
&\quad = \int_0^1 \left| x(t) - \int_0^t \frac{x(\tau)}{1-\tau} d\tau \right|_{\mathfrak{K}}^2 dt \\
&\quad < 2 \sup_{[0,1]} |x(t)| \int_0^1 1 + \left| \int_0^t \frac{1}{1-\tau} d\tau \right|^2 dt \\
&\quad = 2\|x\|_\infty \int_0^1 1 + |\log(1-t)|^2 dt \\
&\quad = 2\|x\|_\infty \int_0^1 1 + |\log t|^2 dt.
\end{aligned}$$

Letting $t = -u^2/2$ we have

$$|T(\sigma)|^2_{H_0(\mathfrak{K})} \leq 2 \sup_{[0,1]} |\sigma'(t)| \int_0^\infty u\left(1 + \frac{u^4}{4}\right) \exp\left(-u^2/2\right) du < \infty.$$

∎

Proof. **of Theorem 3.19:**

First we show that

$$E\langle b_{ts}, A\rangle_{\mathfrak{K}} \langle b_{\tau\sigma}, B\rangle_{\mathfrak{K}} = (t \wedge \tau)\, G(s,\sigma) \langle B, A\rangle_{\mathfrak{K}}$$

Recall $b^A_{ts} \equiv \langle b_{ts}, A\rangle_{\mathfrak{K}}$ and $X^A_{ts} \equiv \langle X_{ts}, A\rangle_{\mathfrak{K}}$. Let

$$l_s(x) \equiv \int_0^1 \alpha_s(du)\, x(u)$$

where

$$\alpha(du) = \left[\delta_s(u) + 1_{[0,s]} \frac{1}{1-u}\right] du$$

is a positive measure on $[0,1]$. Then

$$l_s(x) = x(s) + \int_0^1 x(u) \frac{du}{1-u} = S_s(x).$$

Define $b_{ts} \equiv S_s(X_{t\cdot})$ as in Definition 3.17. So

$$Eb^A_{ts} b^B_{\tau\sigma} = E \int \alpha_s(du)\, \alpha_\sigma(d\nu)\, X^A_{tu} X^B_{\tau\nu}. \tag{3.6}$$

By Tonelli's Theorem and Hölder's inequality, we have

$$\begin{aligned}
E\int \alpha_s(du)\,\alpha_\sigma(d\nu) \left|X^A_{tu} X^B_{\tau\nu}\right| &\leq \int \alpha_s(du)\,\alpha_\sigma(d\nu) \sqrt{E\left(X^A_{tu}\right)^2 E\left(X^B_{\tau\nu}\right)^2} \\
&= \int \alpha_s(du)\,\alpha_\sigma(d\nu) \sqrt{t\tau G_0(u,u)\, G_0(\nu,\nu)} < \infty.
\end{aligned}$$

Thus applying Fubini to Eq. [3.6] we see that

$$\begin{aligned}
Eb^A_{ts} b^B_{\tau\sigma} &= \int \alpha_s(du)\,\alpha_\sigma(d\nu)\, EX^A_{tu} X^B_{\tau\nu} \\
&= (t\wedge\tau) \langle A, B\rangle_{\mathfrak{K}} \int \alpha_s(du) \int \alpha_\sigma(d\nu)\, G_0(u,\nu). \qquad (3.7)
\end{aligned}$$

Let h run through an orthonormal basis of $H_0(\mathfrak{K})$. Then

$$\begin{aligned} G_0(u,\nu) &= \langle G_0(u,\cdot), G_0(\nu,\cdot)\rangle_{H_0(\mathfrak{K})} \\ &= \sum \langle G_0(u,\cdot), h\rangle_{H_0} \langle G_0(u,\cdot), h\rangle_{H_0(\mathfrak{K})} \\ &= \sum h(u)\, h(\nu). \end{aligned}$$

Returning to Eq. [3.7] we get

$$\begin{aligned} E\langle b_{ts}, A\rangle_{\mathfrak{K}} \langle b_{\tau\sigma}, B\rangle_{\mathfrak{K}} &= (t\wedge\tau)\langle A,B\rangle_{\mathfrak{K}} \sum \int h(u)\,\alpha_s(du) \int h(\nu)\,\alpha_\sigma(d\nu) \\ &= (t\wedge\tau)\langle A,B\rangle_{\mathfrak{K}} \sum S_s(h)\, S_\sigma(h). \end{aligned} \tag{3.8}$$

Let $\mathcal{U} \equiv \left\{x|_{[0,1]} : x \in C^\infty(\mathbb{R})\right\}$. The map $S : H_0(\mathfrak{K}) \to H(\mathfrak{K})$ is a unitary isomorphism by the previous Lemma 3.21 and so the $S(h)$ run through an orthonormal basis of $H(\mathfrak{K})$. Exploiting this fact,

$$\begin{aligned} &E\langle b_{ts}, A\rangle_{\mathfrak{K}} \langle b_{\tau\sigma}, B\rangle_{\mathfrak{K}} \\ &= (t\wedge\tau)\langle A,B\rangle_{\mathfrak{K}} \sum S(h)(s)\, S(h)(\sigma) \\ &= (t\wedge\tau)\langle A,B\rangle_{\mathfrak{K}} \sum \langle G(s,\cdot), S(h)\rangle_{H(\mathfrak{K})} \langle G(\sigma,\cdot), S(h)\rangle_{H(\mathfrak{K})} \\ &= (t\wedge\tau)\langle A,B\rangle_{\mathfrak{K}} \langle G(s,\cdot), G(\sigma,\cdot)\rangle_{H(\mathfrak{K})} \\ &= (t\wedge\tau)\langle A,B\rangle_{\mathfrak{K}}\, G(s,\sigma). \end{aligned}$$

Thus b is a $\mathfrak{K}$-valued Brownian sheet.

It remains to show that $T(b_{t\cdot})(s) = X_{ts}$. Define $H^\varepsilon(\mathfrak{K})$ which is to be thought of as "$H(\mathfrak{K})|_{[0,1-\varepsilon]}$" as follows:- Given a function $h : [0,1-\varepsilon] \to \mathfrak{K}$ such that $h(0) = 0$, define $(h,h)_{H^\varepsilon(\mathfrak{K})} = \infty$ if h is not absolutely continuous and set $(h,h)_{H^\varepsilon(\mathfrak{K})} = \int_0^{1-\varepsilon} |h'(s)|^2\, ds$ otherwise. Define

$$H^\varepsilon(\mathfrak{K}) \equiv \left\{h : [0,1-\varepsilon] \to \mathfrak{K} | h(0) = 0 \text{ and } (h,h)_{H^\varepsilon(\mathfrak{K})} < \infty\right\}.$$

H^ε is dense in W_0^ε, where

$$W_0^\varepsilon(\mathfrak{K}) \equiv \{\sigma \in C([0,1-\varepsilon] \to \mathfrak{K}) \,|\, \sigma(0) = 0\}$$

is equipped with the sup-norm topology. Define bounded linear transformations T^ε and S^ε on $W_0^\varepsilon(\mathfrak{K})$ by requiring

$$\begin{aligned} T^\varepsilon(x)(t) &\equiv x(t) - \int_0^t x(\tau)\frac{(1-t)}{(1-\tau)^2}d\tau; \\ S^\varepsilon(x)(t) &\equiv x(t) + \int_0^t \frac{x(\tau)}{(1-\tau)}d\tau. \end{aligned}$$

Now for any $h \in H(\mathfrak{K})$ or $H_0(\mathfrak{K})$, $h|_{[0,1-\varepsilon]} \in H^\varepsilon(\mathfrak{K})$. Also $h \in H(\mathfrak{K})$ implies that $(T(h))|_{[0,1-\varepsilon]} = T^\varepsilon\left(h|_{[0,1-\varepsilon]}\right)$. Furthermore $h \in H_0(\mathfrak{K})$ implies that $(S(h))|_{[0,1-\varepsilon]} = S^\varepsilon\left(h|_{[0,1-\varepsilon]}\right)$. For any $x \in H^\varepsilon$ there is some $h \in H_0(\mathfrak{K})$ so that $h|_{[0,1-\varepsilon]} = x$. Using this fact and the fact that $T = S^{-1}$ from Lemma 3.21, we see that for any $x \in H^\varepsilon(\mathfrak{K})$

$$T^\varepsilon \circ S^\varepsilon(x) = T^\varepsilon \circ \left(S(h)|_{[0,1-\varepsilon]}\right) = h|_{[0,1-\varepsilon]} = x.$$

By continuity, we have $T^{\varepsilon} \circ S^{\varepsilon}(x) = x$ for any $x \in W_0^{\varepsilon}$. Thus for any $s < 1 - \varepsilon$, we have

$$T(b_{t\cdot})(s) = T^{\varepsilon}\left(b_{t\cdot}|_{[0,1-\varepsilon]}\right)(s) = T^{\varepsilon} \circ S^{\varepsilon}\left(X_{t\cdot}|_{[0,1-\varepsilon]}\right)(s) = X_{ts}.$$

Thus

$$X_{ts} = T(b_{t\cdot})(s) = b_{ts} - \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma,$$

which is exactly Eq. [3.5]. ∎

3.3. **Abelian Loop group Examples.**

3.3.1. *The Simply-Connected Lie group* $(\mathbb{R}^d, +)$*:*

Lemma 3.22. *On the Loop space of* $\mathbb{R}^d$*, Heat Kernel measure and pinned Wiener measure are the same.*

Proof. Our Lie group here is $K = (\mathbb{R}^d, +)$ with Lie algebra $\mathfrak{K} = \mathbb{R}^d$. Our probability space is $(C([0,1] \to K), Law\ \Sigma)$ as in Definition 2.22. Eq. (2.6) becomes

$$\Sigma(\delta t, s) = \chi(\delta t, s) \text{ with } \Sigma(0, s) = 0, \forall s \in [0,1].$$

In other words, $\Sigma = \chi$. This implies that Heat Kernel measure on $L(\mathbb{R}^d)$ equals *Law* $\chi(t, \cdot)$. But $\chi(t, \cdot)$ is a standard Brownian bridge from 0 to 0. Pinned Wiener measure is the law of this Brownian bridge. Hence in $\mathbb{R}^d$, Heat Kernel Measure and Pinned Wiener Measure are the **same** measure. ∎

3.3.2. *The Lie group* S^1 *with fundamental group* $\mathbb{Z}$*:* Realize the Lie group S^1 as $\{(\cos 2\pi\theta, \sin 2\pi\theta) : \theta \in [0,1]\}$, its imbedding in $\mathbb{R}^2$. Specify the left-invariant metric by setting $|\partial_\theta| = 1$. Let Heat Kernel measure $\nu_T^{S^1}(x, \cdot)$ be the family of measures in Definition 2.20. Let pinned Wiener measure $\mu_{0,T}^{S^1}$ be the measure on the loop space $L(S^1)$ as in Definition 2.9. Let Wiener measure $\mu_T^{S^1}$ on $W_e(S^1)$ be as in Definition 2.7. Let

$$W_0(\mathbb{R}) \equiv \{C([0,1] \to \mathbb{R}) : \sigma(0) = 0\}$$

be the Wiener space on $\mathbb{R}$. Let $\pi_t : L(S^1) \to \mathbb{R}$ be as usual the evaluation map. By abuse of notation, let $\pi_t : W_e(S^1) \to \mathbb{R}$ be also be the evaluation map.

We show, in the $K = S^1$ case, that Heat Kernel Measure is equivalent to Pinned Wiener Measure restricted to the null-homotopic loops. Thus Heat Kernel Measure is absolutely continuous with Pinned Wiener Measure. However, as mentioned in the Introduction, the two measures are not equivalent since S^1 is not simply connected.

We shall need to explicitly compute the Heat Kernel Measure on S^1 and this is provided for the reader's convenience in the following Lemma:

Lemma 3.23 (Heat Kernel measure on S^1)**.** *Let* ψ *be the local chart from* $\mathbb{R}$ *to* S^1 *taking* $x \mapsto (\cos 2\pi x, \sin 2\pi x)$*. Then Heat Kernel Measure on* S^1 *has the following representation:*

$$P_t^{S^1}(\psi(x)) = \sum_{\alpha \in \mathbb{Z}} P_t^{\mathbb{R}}(x + \alpha). \tag{3.9}$$

Proof. Since the right hand side of Eq. [3.9] is periodic there exist a unique map $P(t,\theta)$ from $[0,\infty)\times S^1\to\mathbb{R}$ so that $P(t,\psi(x))=\sum_{\alpha\in\mathbb{Z}}P_t^{\mathbb{R}}(x+\alpha)$.

$$\partial_x P(t,\psi(x))=[\partial_\theta P(t,\cdot)](\psi(x)).$$

Thus

$$\begin{aligned}\left(\partial_t-\frac{1}{2}\partial_\theta^2\right)P(t,\theta)|_{\theta=\psi(x)} &= \left(\partial_t-\frac{1}{2}\partial_x^2\right)P(t,\psi(x))\\ &= \sum_{\alpha\in\mathbb{Z}}\left(\partial_t-\frac{1}{2}\partial_x^2\right)P_t^{\mathbb{R}}(x+\alpha)=0.\end{aligned}$$

For any $F\in C^\infty(S^1)$, define a map $u(t,\theta)$ from $[0,\infty)\times S^1\to\mathbb{R}$ by

$$u(t,\theta)=\int_{S^1}F(\theta)P\left(t,\theta^{-1}\theta'\right)\nu ol\left(d\theta'\right).$$

Take the support of F to be less than the entire circle. The appropriate local chart here is ψ restricted to some open interval (a,b) with $|b-a|<1$. Let $\theta=\psi(y)$ for some $y\in(a,b)$. Use this local chart and the fact that $|\partial_x|_{S^1}=1$, to get

$$\begin{aligned}u(t,\psi(y)) &= \int_{(a,b)}F(\psi(x))P(t,\psi(x-y))dx\\ &= \sum_{\alpha\in\mathbb{Z}}\int_{(a,b)}F(\psi(x))P_t^{\mathbb{R}}(x-y+\alpha)dx\\ &= \int_{(a,b)+\alpha}F(\psi(x))P_t^{\mathbb{R}}(x-y)dx\\ &= \int_{\mathbb{R}}F(\psi(x))P_t^{\mathbb{R}}(x-y)dx.\end{aligned}$$

So $u(t,\psi(y))\to F(\psi(y))$ as $t\to 0$. Thus in this above sense, $P\left(t,\theta^{-1}\theta'\right)\to\delta_\theta$ as $t\to 0$. Therefore $P(t,\theta)$ must be the Heat Kernel on S^1. ∎

Remark 3.24. Recall from Definition 2.22 the following:-

1. $\Omega\equiv C\left([0,1]\to L\left(S^1\right)\right)$.
2. Let Σ be the process from Theorem 2.17 so that $\Sigma_0=e$, where e denotes the identity loop.
3. P is defined to be Wiener Measure on $C\left([0,1]\to L\left(S^1\right)\right)$. Explicitly, $P\equiv$ *Law* Σ.
4. $g_{ts}(x)\equiv x(t)(s)$, where $x\in\Omega$, $x(t)\in L(K)$, and $x(t)(s)\in K$.
5. By Theorem 2.19 we see that *Law* $g_{t\cdot}=\nu_t(e,\cdot)$, the Heat Kernel measure on $L\left(S^1\right)$ introduced in Definition 2.20.
6. $\mathfrak{F}_{ts}\equiv\sigma\left\langle g_{\tau\sigma}:\tau\in[0,t]\text{ and }\sigma\in[0,s]\right\rangle$.
7. $\mathfrak{F}\equiv\vee_{(t,s)\in[0,1]^2}\mathfrak{F}_{ts}$.

Definition 3.25 (S^1-specific definitions)**.** We will need the following:-

1. $L_0\left(S^1\right)\equiv\left\{\sigma\in L\left(S^1\right):\sigma\text{ is homotopic to }e\right\}$; the null-homotopic loops in S^1 based at e.
2. Abusing notation. let ψ also denote the map from $W_0(\mathbb{R})$ to $W_1\left(S^1\right)$ taking the $\mathbb{R}$-valued path σ to the S^1-valued path $(\cos 2\pi\sigma,\sin 2\pi\sigma)$.
3. ψ has a unique inverse $\psi^{-1}:W_1\left(S^1\right)\to W_0(\mathbb{R})$ which is the unique lift of σ starting from 0 in $\mathbb{R}$.

Let $\mu_T^{\mathbb{R}}$ be Wiener measure on $W_0(\mathbb{R})$ with parameter T. Explicitly, use Definition 2.7 with $K=\mathbb{R}$ and $P_t\left(x^{-1}y\right)$ replaced by the Heat Kernel

$$P_t^{\mathbb{R}}(y-x)=\frac{1}{\sqrt{2\pi t}}\exp\frac{(y-x)^2}{2t}.$$

Definition 3.26 (Wiener Measure conditioned on the integers)**.** Let $\mu_{\mathbb{Z},T}^{\mathbb{R}}$ be the unique measure on $W(\mathbb{R})$ such that on simple functions f of the form

$$f(x_\cdot)=F\left(x_{t_1},\cdots,x_{t_n}\right),$$

where $F\in C_b^\infty(\mathbb{R}^n)$ and $\{0=t_0<t_1<\cdots<t_n<1\}$, we have;

$$\begin{aligned}
&\mu_{\mathbb{Z},T}^{\mathbb{R}}[f]\\
&\equiv\frac{1}{P_T^{S^1}(e)}\int_{\mathbb{R}^n}F(x_1,\cdots,x_n)\sum_{\alpha\in\mathbb{Z}}P_{T(1-s_n)}^{\mathbb{R}}(x_n+\alpha)\prod_{i=1}^n P_{T\triangle_i s}^{\mathbb{R}}(\triangle_i x)\,dx_i\\
&=\frac{1}{P_T^{S^1}(e)}\int_{\mathbb{R}^n}F(x_1,\cdots,x_n)\,P_{T(1-t_n)}^{S^1}(\psi(x_n))\prod_{i=1}^n P_{T\triangle_i t}^{\mathbb{R}}(\triangle_i x)\,dx_i\\
&=\frac{1}{P_T^{S^1}(e)}\int\mu_T^{\mathbb{R}}(dx)\,f(x)\,P_{T(1-t_n)}^{S^1}(\psi(x_{t_n}))\,. \qquad (3.10)
\end{aligned}$$

We have yet to show the existence of such a measure. See remark 3.27 to see why $\mu_{\mathbb{Z},T}^{\mathbb{R}}$ deserves to be called "Wiener Measure conditioned on the integers with parameter T".

Remark 3.27 (Motivation for Definition 3.26). Our goal is to make explicit the heuristic definition

$$\mu_{\mathbb{Z},T}^{\mathbb{R}}[f]=\mu_T^{\mathbb{R}}[f(x)\,|x(1)\in\mathbb{Z}].$$

Take the function $\sum_{\alpha\in\mathbb{Z}}P_\varepsilon^{\mathbb{R}}(\pi_1+\alpha)$ which concentrates on paths which are near $\mathbb{Z}$ at time $t=1$.We would like

$$\mu_{\mathbb{Z},T}^{\mathbb{R}}[f]=\lim_{\varepsilon\downarrow 0}\mu_T^{\mathbb{R}}\left[f\sum_{\alpha\in\mathbb{Z}}P_\varepsilon^{\mathbb{R}}(\pi_1+\alpha)\right]/\mu_T^{\mathbb{R}}\left[\sum_{\alpha\in\mathbb{Z}}P_\varepsilon^{\mathbb{R}}(\pi_1+\alpha)\right]$$

to hold. Let $F\in C_b^\infty(\mathbb{R}^n)$ and

$$f(x)=F\left(x_{t_1},\cdots,x_{t_n}\right)\ \text{where}\ \{0=t_0<\cdots<t_{n+1}=1\}\,.$$

Then letting $\triangle_i t=t_i-t_{i-1}$ and $\triangle_i x=x_i-x_{i-1}$, and using the fact that $P_\varepsilon^{\mathbb{R}}(\cdot)$ goes to the delta function at 0 as $\varepsilon\to 0$; we should have

$$\begin{aligned}
&\mu_{T,\mathbb{Z}}^{\mathbb{R}}[f]\\
&=\lim_{\varepsilon\downarrow 0}\frac{\int_{\mathbb{R}^{n+1}}F(x_1,\cdots,x_n)\sum_{\alpha\in\mathbb{Z}}P_\varepsilon^{\mathbb{R}}(x_{n+1}+\alpha)\prod_{i=1}^{n+1}P_{T\triangle_i t}^{\mathbb{R}}(\triangle_i x)\,dx_i}{\int_{\mathbb{R}}\sum_{\alpha\in\mathbb{Z}}P_\varepsilon^{\mathbb{R}}(x+\alpha)\,P_T^{\mathbb{R}}(x)\,dx}\\
&=\frac{\int_{\mathbb{R}^n}F(x_1,\cdots,x_n)\sum_{\alpha\in\mathbb{Z}}P_{T(1-t_n)}^{\mathbb{R}}(x_n+\alpha)\prod_{i=1}^n P_{T\triangle_i t}^{\mathbb{R}}(\triangle_i x)\,dx_i}{\sum_{\alpha\in\mathbb{Z}}P_T^{\mathbb{R}}(\alpha)},
\end{aligned}$$

where we have replaced $x_{n+1}+\alpha$ by x_{n+1} using the change-of-variables formula. Thus for simple functions, we should have

$$\mu_{T,\mathbb{Z}}^{\mathbb{R}}[f]=\frac{1}{\sum_{\alpha\in\mathbb{Z}}P_T^{\mathbb{R}}(\alpha)}\int f(x)\sum_{\alpha\in\mathbb{Z}}P_{T(1-t_n)}^{\mathbb{R}}(x_{t_n}+\alpha)\,\mu_T^{\mathbb{R}}(dx)\,.$$

Now use Lemma 3.23 to see that $\sum_{\alpha\in\mathbb{Z}} P_T^{\mathbb{R}}(\alpha) = P_T^{S^1}(e)$.

Theorem 3.28 (Heat Kernel and pinned Wiener measures on S^1)**.** *Let ψ be as in Definition 3.25. Let $\sigma_n(s) \equiv \psi(ns)$, the minimum energy loop in the n^{th} homotopy class of S^1. Let $\nu_T^{S^1}(\sigma_n,\cdot)$ be as usual the Law of $\sigma_n(\cdot)\, g_T$.. Define a probability measure*

$$\widetilde{\nu}_T \equiv \sum_{\alpha\in\mathbb{Z}} C_{\alpha,T}\nu_T^{S^1}(\sigma_\alpha,\cdot),$$

where

$$C_{\alpha,T} \equiv P_T^{\mathbb{R}}(0)\exp\left(-\frac{1}{2T}\alpha^2\right)P_T^{S^1}(e)^{-1} = P_T^{\mathbb{R}}(\alpha)\,P_T^{S^1}(e)^{-1}.$$

Then Pinned Wiener Measure $\mu_{0,T}^{S^1} = \widetilde{\nu}_T$. Exploiting the fact that the measures $\nu_T^{S^1}(\sigma_\alpha,\cdot)$ live only on the α^{th} homotopy classes, we see that Heat Kernel Measure $\nu_T^{S^1}(e,\cdot)$ is equivalent to Pinned Wiener Measure restricted to the null-homotopic loops L_0. Furthermore, the Radon-Nikodym derivative

$$\frac{d\mu_{0,T}^{S^1}\downarrow_{L_0}}{d\nu_T^{S^1}(e,\cdot)} = C_{0,T}$$

is a constant.

The proof of this Theorem will be deferred until we have some preliminary results.

Lemma 3.29 (The measure $\mu_{T,\mathbb{Z}}^{\mathbb{R}}$exists)**.** *Wiener Measure conditioned on the integers, $\mu_{\mathbb{Z},T}^{\mathbb{R}}$, exists. Furthermore Pinned Wiener Measure on S^1 pulls back to Wiener Measure conditioned on the integers. Explicitly, $\psi_*^{-1}\mu_{0,T}^{S^1} = \mu_{\mathbb{Z},T}^{\mathbb{R}}$.*

Proof. ψ from Definition 3.25 is a continuous bijection between the Wiener spaces $W(R)$ and $W(S^1)$. Let us compute $\psi_*^{-1}\mu_{0,T}^{S^1}$ for simple functions. By Ito's Lemma it is easily seen that ψ takes $\mathbb{R}$-valued Brownian motions with parameter T to S^1-valued Brownian motions with the same parameter. So let $t\to b_t$ be an $\mathbb{R}$-valued Brownian motion with parameter t. Then $t\to\psi(b_t)$ is an S^1-valued Brownian motion. Thus

$$\int f(y)\,\mu_T^{S^1}(dy) = \int f\circ\psi(y)\,\mu_T^{\mathbb{R}}(dy).$$

Let $F\in C^\infty(S^1\times\cdots\times S^1)$ and let

$$f(y) = F(y_{s_1},\cdots,y_{s_n}) \text{ where } \{0=s_0<\cdots<s_n<1\},$$

for any path $y\in W_e(S^1)$. Then

$$\begin{aligned}
\psi_*^{-1}\mu_{0,T}^{S^1}[f] &= \mu_{0,T}^{S^1}\left[f\circ\psi^{-1}\right] \\
&= \int f\circ\psi^{-1}(y)\,\frac{P_{t(1-s_n)}^{S^1}(y_{s_n})}{P_t^{S^1}(e)}\mu_T^{S^1}(dy) \\
&= \frac{1}{P_t^{S^1}(e)}\int f(y)\,P_{t(1-s_n)}^{S^1}(\psi(y_{s_n}))\,\mu_T^{\mathbb{R}}(dy).
\end{aligned}$$

This is precisely Eq. [3.10]. Thus the measure $\mu_{\mathbb{Z},T}^{\mathbb{R}}$ and $\mu_{0,T}^{S^1}$ is pulled back to it under the map ψ. ∎

Lemma 3.30. *Let $\triangle_i u$ denote $u_i - u_{i-1}$. Then the functions*

$$J_1(x_1, \cdots, x_n) \equiv P^{\mathbb{R}}_{T(1-s_n)}(x_n - \alpha) \prod_{i=1}^{n} P^{\mathbb{R}}_{T\triangle_i s}(\triangle_i x),$$

and

$$J_2(x_1, \cdots, x_n) = \exp\left(\frac{-\alpha^2}{2T}\right) P^{\mathbb{R}}_{T(1-s_n)}(x_n - \alpha s_n) \prod_{i=1}^{n} P^{\mathbb{R}}_{T\triangle_i s}(\triangle_i x - \alpha \triangle_i s)$$

are the same. (i.e. $J_1 = J_2$).

Proof. We shall use the fact that

$$P^{\mathbb{R}}_t(x) = \frac{1}{\sqrt{2\pi t}} \exp -\frac{x^2}{2t}.$$

Letting $\triangle_{n+1}s$ denote $1 - s_n$, we have

$$J_2(x_1, \cdots, x_n) = \exp\left(-\alpha^2/2T\right) P^{\mathbb{R}}_{T(1-s_n)}(x_n - \alpha s_n) \prod_{i=1}^{n} P^{\mathbb{R}}_{T\triangle_i s}(\triangle_i x - \alpha \triangle_i s).$$

Let

$$I = -2T\left(\log J_2 + \sum_{i=1}^{n+1} \log \sqrt{2\pi T \triangle_i s}\right).$$

Then

$$\begin{aligned}
I &= \alpha^2 + \frac{(x_n - \alpha s_n)^2}{1 - s_n} + \sum_{i=1}^{n} \frac{(\triangle_i x - \alpha \triangle_i s)^2}{\triangle_i s} \\
&= \alpha^2 + \frac{x_n^2 - 2\alpha x_n s_n + \alpha^2 s_n^2}{1 - s_n} + \sum_{i=1}^{n} \left[\frac{\triangle_i x^2}{\triangle_i s} - 2\alpha \triangle_i x + \alpha^2 \triangle_i s\right] \\
&= \alpha^2 + \frac{x_n^2 - 2\alpha x_n s_n + \alpha^2 s_n^2}{1 - s_n} - 2\alpha x_n + \alpha^2 s_n + \sum_{i=1}^{n} \frac{\triangle_i x^2}{\triangle_i s} \\
&= \frac{x_n^2 + \alpha^2 s_n^2}{1 - s_n} - 2\alpha x_n \left(\frac{s_n}{1 - s_n} + 1\right) + \alpha^2 (1 + s_n) + \sum_{i=1}^{n} \frac{\triangle_i x^2}{\triangle_i s} \\
&= \frac{x_n^2 + \alpha^2 s_n^2 - 2\alpha x_n + \alpha^2 (1 - s_n^2)}{1 - s_n} + \sum_{i=1}^{n} \frac{\triangle_i x^2}{\triangle_i s} \\
&= \frac{(x_n - \alpha)^2}{1 - s_n} + \sum_{i=1}^{n} \frac{\triangle_i x^2}{\triangle_i s} \\
&= -2T\left(\log J_1 + \sum_{i=1}^{n+1} \log \sqrt{2\pi T \triangle_i s}\right).
\end{aligned}$$

Hence we are done. ∎

Proof. **of Theorem 3.28**

Let the map ψ be as in Definition 3.25. It will suffice to show $\psi_*^{-1}\mu_{0,T}^{S^1}$ is equivalent to $\psi_*^{-1}\widetilde{\nu}_T$. If this is the case then for any measurable $A \subset L\left(S^1\right)$ we have

$$\begin{aligned}
&A \text{ is a } \widetilde{\nu}_T\text{-null set} \\
&\iff \widetilde{\nu}_T\left(e, 1_A \circ \psi \circ \psi^{-1}\right) = 0 \\
&\iff \psi_*^{-1}\widetilde{\nu}_T\left(e, 1_{\psi^{-1}(A)}\right) = 0 \\
&\iff \psi_*^{-1}\mu_{0,T}^{S^1}\left(1_{\psi^{-1}(A)}\right) = 0 \\
&\iff A \text{ is a } \mu_{0,T}^{S^1}\text{-null set.}
\end{aligned}$$

Thus we would be done by the Radon-Nikodym Theorem. The rest of the proof is devoted to computing $\psi_*^{-1}\mu_{0,T}^{S^1}$ and $\psi_*^{-1}\widetilde{\nu}_T$ and showing they are equivalent .

First we compute $\psi_*^{-1}\nu_T^{S^1}(h,\cdot)$ where h is any loop in $L\left(S^1\right)$. Let $t \to X_{t\cdot}$ is an $L(\mathbb{R})$-valued Brownian motion. Let g satisfy the stochastic differential equation

$$g_{\delta ts} = \left[\left(L_{g_{ts}}\right)_* \partial_\theta\right] \cdot X_{\delta ts} \text{ with } g_{0s} = 1,$$

as in Theorem 2.25. Here, since $\theta \to (\cos 2\pi\theta, \sin 2\pi\theta)$ is our local chart,

$$(\partial_\theta F)(\cos 2\pi\theta, \sin 2\pi\theta) \equiv \partial_\theta F(\cos 2\pi\theta, \sin 2\pi\theta).$$

Then $\nu_t^{S^1}(e,\cdot) = Law\ h.g_{t\cdot}$ and thus

$$\psi_*^{-1}\nu_T^{S^1}(e,\cdot) = Law\psi^{-1}(h.g_{T\cdot}).$$

We claim $g_{T\cdot} = \psi(X_{T\cdot})$ and hence

$$\psi_*^{-1}\nu_T^{S^1}(h,\cdot) = Law\left[\psi^{-1}(h)(\cdot) + X_{T\cdot}\right].$$

To verify the claim that $g_{t\cdot} = \psi(X_{t\cdot})$, it will suffice to check that for any $F \in C^\infty\left(S^1, \mathbb{R}\right)$ we have

$$\delta_t F(\psi(X_{ts})) = (\partial_\theta F)(\psi(X_{ts}))\, X_{\delta ts}.$$

But by Ito's Lemma $\delta_t F(\psi(X_{ts})) = (F \circ \psi)'(X_{\delta ts})\, X_{\delta ts} = (\partial_\theta F)(\psi(X_{ts}))\, X_{\delta ts}$ we are done. Thus

$$\psi_*^{-1}\nu_T^{S^1}(h,\cdot) = Law\left[\psi^{-1}(h)(\cdot) + X_{T\cdot}\right].$$

Since for fixed t, $s \to X_{ts}$ is a Brownian bridge in $\mathbb{R}$ from 0 to 0 with parameter t, we have

$$\left(\psi_*^{-1}\nu_T^{S^1}(h,\cdot)\right) f = \int f\left(x + \psi^{-1}(h)\right) \mu_{0,T}^{\mathbb{R}}(dx).$$

Now we compute $\psi_*^{-1}\mu_{0,T}^{S^1}$ explicitly. By Lemma 3.29 this is just $\mu_{\mathbb{Z},T}^{\mathbb{R}}$ or Wiener Measure conditioned on the integers (see Definition 3.26 and remark 3.27).

Let $f(x_\cdot) \equiv F(x_{s_1}, \cdots, x_{s_n})$ where $F \in C^\infty(\mathbb{R}^n)$ and $0 = s_0 < \cdots < s_n < 1$. Then we have

$$\begin{aligned}
&\left(\psi_*^{-1}\mu_{0,T}^{S^1}\right)[f] \\
&\qquad = \mu_{\mathbb{Z},T}^{\mathbb{R}}[f] \\
&\qquad \equiv \int_{\mathbb{R}^n} \frac{F(x_1, \cdots, x_n)}{P_T^{S^1}(e)} \sum_{\alpha\in\mathbb{Z}} P_{T(1-s_n)}^{\mathbb{R}}(x_n + \alpha) \prod_{i=1}^n P_{T\triangle_i s}^{\mathbb{R}}(\triangle_i x)\, dx_i \\
&\qquad = \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} F(x_1, \cdots, x_n) \frac{P_{T(1-s_n)}^{\mathbb{R}}(x_n + \alpha)}{P_T^{S^1}(e)} \prod_{i=1}^n P_{T\triangle_i s}^{\mathbb{R}}(\triangle_i x)\, dx.
\end{aligned}$$

Also

$$\begin{aligned}
&\left(\psi_*^{-1}\widetilde{\nu}_T\right)[f] \\
&\qquad = \sum_{\alpha\in\mathbb{Z}} C_\alpha \left(\psi_*^{-1}\nu_T^{S^1}(\sigma_\alpha, \cdot)\right)[f] \\
&\qquad = \sum_{\alpha\in\mathbb{Z}} C_\alpha \int f\left(x + \psi^{-1}(\sigma_\alpha)\right) \mu_{0,T}^{\mathbb{R}}(dx) \\
&\qquad = \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} F(x_1 + \alpha s_1, \cdots, x_n + \alpha s_n) \\
&\qquad\qquad \times \frac{C_\alpha P_{T(1-s_n)}^{\mathbb{R}}(x_n)}{P_T^{\mathbb{R}}(0)} \prod_{i=1}^n P_{T\triangle_i s}^{\mathbb{R}}(\triangle_i x)\, dx_i \\
&\qquad = \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} F(x_1, \cdots, x_n) \frac{C_\alpha P_{T(1-s_n)}^{\mathbb{R}}(x_n - \alpha s_n)}{P_T^{\mathbb{R}}(0)} \\
&\qquad\qquad \times \prod_{i=1}^n P_{T\triangle_i s}^{\mathbb{R}}(\triangle_i x - \alpha\triangle_i s)\, dx \\
&\qquad = \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} F(x_1, \cdots, x_n) \prod_{i=1}^n P_{T\triangle_i s}^{\mathbb{R}}(\triangle_i x - \alpha\triangle_i s)\, dx \\
&\qquad\qquad \times \frac{1}{P_T^{S^1}(e)} \exp\left(-\frac{\alpha^2}{2T}\right) P_{T(1-s_n)}^{\mathbb{R}}(x_n - \alpha s_n).
\end{aligned}$$

Using Lemma 3.30 this last expression is just

$$\begin{aligned}
&= \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} F(x_1, \cdots, x_n) \frac{P_{1-s_n}^{\mathbb{R}}(x_n - \alpha) \prod_{i=1}^n P_{\triangle_i s}^{\mathbb{R}}(\triangle_i x)}{P_1^{S^1}(e)} dx \\
&= \sum_{\alpha\in\mathbb{Z}} \int_{\mathbb{R}^n} F(x_1, \cdots, x_n) \frac{P_{1-s_n}^{\mathbb{R}}(x_n + \alpha) \prod_{i=1}^n P_{\triangle_i s}^{\mathbb{R}}(\triangle_i x)}{P_1^{S^1}(e)} dx \\
&= \left(\psi_*^{-1}\mu_0^{S^1}\right) f.
\end{aligned}$$

∎

4. The Airault-Malliavin Theorem

In the next section we shall use the Airault-Malliavin Theorem (Theorem 4.1). For the reader's convenience, we give a direct (and to our mind simpler) proof of this Theorem.

Let μ_t denote Wiener Measure on $W_e(K)$ with parameter t and let $\mu_{0,t}$ be Pinned Wiener Measure as in Definition 2.9.

Theorem 4.1 (Airault & Malliavin, [26]). *Recall from Definitions 2.9 and 2.7 that μ_t denotes Wiener measure on K with variance t and $\mu_{0,t}$ denotes pinned Wiener measure. Let $\triangle_{L(K)}$ be the operator from Definition 2.4 and let $V_t : L(K) \to \mathbb{R}$ denote the function*

$$V_t(\gamma) = \frac{1}{2t^2}\left|\int_0^1 \gamma(s)^{-1}\gamma(\delta s)\right|_{\mathfrak{K}}^2 - \left[\frac{\dim \mathfrak{K}}{2t} + \partial_t \log P_t^K(e)\right],$$

where the expression

$$\int_0^1 \gamma(s)^{-1}\gamma(\delta s) \equiv \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\gamma(s)^{-1}\gamma(\delta s) \ \text{ in } L^2\left[\mu_{0,t}\right].$$

See Lemma 4.8 and Remark 4.9 Gross [17] *for the existence of such a limit. Then for any smooth cylindrical function $f : L(K) \to \mathbb{R}$ (see Definition 2.5)*

$$\partial_t \mu_{0,t}[f] = \mu_{0,t}\left[\frac{1}{2}\triangle_{L(K)} + V_t f\right]. \tag{4.1}$$

We defer the proof of Theorem 4.1 until we have developed sufficient machinery. We shall be using some results of Gross. Accordingly we will need to define a few terms so that we can state some results from [17], [19].

Definition 4.2 (Notations from [17], [19]). The following definitions hold for Lemmas 4.4, 4.3 and 4.5:-

1. $\left(\widehat{\Omega}, \widehat{P}\right)$ be an abstract probability space and let $t > 0$.
2. Let $\widehat{E}$ denote the expectation with respect to the measure $\widehat{P}$.
3. Let $s \to \mathcal{G}_s$ be an arbitrary K-valued Brownian motion with parameter t starting from e (i.e. *Law* $\mathcal{G} = \mu_t$).
4. Define a $\mathfrak{K}$-valued Brownian motion $\widetilde{\beta}$ by setting $\widetilde{\beta}_\alpha \equiv \int_0^\alpha \mathcal{G}(\delta s)\,\mathcal{G}(s)^{-1}$.
5. An element k in $C([0,1] \to K)$ is a finite energy path if
$$k'(s) \text{ exists } ds\text{-a.s. and } \int_0^1 \left|k^{-1}k'\right|_{\mathfrak{K}}^2 ds < \infty.$$
6. For any finite-energy path k define a μ_t-a.s. random variable $\widetilde{J}_k$ on $W_e(K)$ by setting
$$\widetilde{J}_k \circ \mathcal{G} = \exp\left(-\frac{1}{2t}\int_0^1 \left|k^{-1}k'\right|_{\mathfrak{K}}^2 ds - \frac{1}{t}\int_0^1 \left\langle k^{-1}k', \widetilde{\beta}_{\delta s}\right\rangle_{\mathfrak{K}}\right).$$

Lemma 4.3 (Albeverio&Hoegh-Krohn, [3]). *Let $t = 1$. Let k be a finite-energy path on K. Then for any bounded measurable $f : W_e(K) \to \mathbb{R}$ we have*

$$\widehat{E}[f(\mathcal{G})] = \widehat{E}\left[f(k\mathcal{G})\left(\widetilde{J}_k \circ \mathcal{G}\right)\right].$$

This result goes through without trouble for any $t > 0$ (see Remark 4.6).

Lemma 4.4 (Gross:[19], Corollary 3.7)**.** *Let $t = 1$. Then for any finite-energy path k*

$$\partial_\varepsilon \left(\widetilde{J}_{\exp \varepsilon h} \circ \mathcal{G} \right) |_{\varepsilon=0} = \int_0^1 \frac{1}{t} \left\langle h'(s), \widetilde{\beta}_{\delta s} \right\rangle,$$

and the limit exists in $L^p\left(\widehat{\Omega}\right)$ for any $p < \infty$. Let $\widetilde{j}_h$ denote the μ_t-a.s. random variable so that $\widetilde{j}_h \circ \mathcal{G} = \partial_\varepsilon \left(\widetilde{J}_{\exp \varepsilon h} \circ \mathcal{G} \right) |_{\varepsilon=0}$. This result goes through without trouble for any $t > 0$ (see Remark 4.6).

Lemma 4.5 (Gross:[17], Lemma 4.8 and Remark 4.9)**.** *Let $t = 1$. Then for any $p < \infty$; $\widetilde{\beta}_\alpha$ converges in $L^p\left(\mu_{t,0}\right)$ as $\alpha \uparrow 1$. Let $\widetilde{\beta}_1$ denote this limit in $L^2\left(\mu_{t,0}\right)$. [By Remark 4.6, this result goes through without trouble for any $t > 0$.*

Remark 4.6 (Lemmas 4.4, 4.3 and 4.5 go though for any $t > 0$). Take $t \neq 1$. Define a new Ad-invariant metric $\widetilde{\langle \cdot, \cdot \rangle} = \frac{1}{t} \langle \cdot, \cdot \rangle_{\mathfrak{K}}$ on $\mathfrak{K}$. Let $\left\{ \widetilde{A} \right\}$ be a $\widetilde{\langle \cdot, \cdot \rangle}$-orthonormal basis for $\mathfrak{K}$. Then $1 = \widetilde{\left\langle \widetilde{A}, \widetilde{A} \right\rangle} = \frac{1}{t} \left\langle \widetilde{A}, \widetilde{A} \right\rangle_{\mathfrak{K}}$. So $\left\{ \widetilde{A}/\sqrt{t} \right\}$ is a $\langle \cdot, \cdot \rangle_{\mathfrak{K}}$-orthonormal basis for $\mathfrak{K}$. Thus $\widetilde{\triangle}_K$, the Laplacian on $\widetilde{K} \equiv \left(K, \widetilde{\langle \cdot, \cdot \rangle} \right)$is given by

$$\widetilde{\triangle}_K = \sum_{\widetilde{A}} \partial^2_{\widetilde{A}} = t \sum_{\widetilde{A}} \partial^2_{\left(\widetilde{A}/\sqrt{t}\right)} = t \triangle_K.$$

So let $\widetilde{\mathcal{G}}$ be a standard Brownian motion on $\widetilde{K}$. So let $\widetilde{\mu}_1$ be Wiener Measure on $\widetilde{K}$ with parameter 1 (i.e. $\widetilde{\mu}_1 = Law\ \widetilde{\mathcal{G}}$). Then by the martingale characterization of a standard Brownian motion we have

$$df\left(\widetilde{\mathcal{G}}_s\right) = dMartingale + \frac{1}{2} t \left(\triangle_K f \right) \left(\widetilde{\mathcal{G}}_s \right) ds.$$

In other words on $K = (K, \langle \cdot, \cdot \rangle_{\mathfrak{K}})$, $\widetilde{\mathcal{G}}$ is a Brownian motion with parameter t. Thus $\widetilde{\mu}_1 = \mu_t$ and $\widetilde{\mu}_{1,0} = \mu_{t,0}$. So applying Lemmas 4.5, 4.4 and 4.3 to $\widetilde{K}$ we see that they extend to all $t > 0$.

Remark 4.7 (Our special case). For our purposes the space $\widehat{\Omega}$ is $W_e(K)$, the measure $\widehat{P}$ is Wiener measure μ_t, and the Brownian motion $\mathcal{G}_s$ is the map π_s^{-1}. Here $\pi_s : W_e(K) \to K$ is the map sending a path $\gamma \in W_e(K)$ to an element $\gamma_s \in K$. We let $\mathcal{G}_s$ be π_s^{-1} rather than π_s because we shall need $d\left(R_g\right)_* \mu_t / d\mu_t$ explicitly to compute derivatives whereas the theorems of Gross we cite use $d\left(\ell_g\right)_* \mu_t / d\mu_t$. Recall that $\left(\ell_g \gamma\right)(s) = g(s)\gamma(s)$, and $\left(R_g \gamma\right)(s) = \gamma(s) g(s)$.

For the rest of this section, $\widehat{\Omega}, \widehat{P}$, and $\mathcal{G}_s$ are to be interpreted as $W_e(K)$, μ_t, and π_s^{-1} respectively. For notational convenience, let π be the identity map from $W_e(K)$ to itself and let π^{-1} denote the map from $W_e(K)$ to itself taking a path γ to the path $s \to \gamma_s^{-1}$.

Lemma 4.8 (The $L^2(\mu_t)$-adjoint ∂^*)**.** *For any finite-energy path $h \in H(\mathfrak{K})$ we have the $L^2(\mu_t)$-adjoint*

$$\partial_h^* = -\partial_h + \int_0^1 \frac{1}{t} \left\langle h'(s), \gamma(s)^{-1} \gamma(\delta s) \right\rangle.$$

A more explicit statement is as follows:- Let f, g be elements of $L^{\infty-}(\mu_t)$, where

$$L^{\infty-}(\mu_t) \equiv \cap_{p<\infty} L^p(\mu_t).$$

Let $R_g : W_e(K) \to W_e(K)$ denote right multiplication by g (i.e. $R_g(\gamma)(s) = \gamma(s)g(s)$). Let $L^{\infty-}(\mu_t)$ denote $\cap_{p<\infty} L^p(\mu_t)$. Let $\mathcal{D}_h$ be the following domain:-

$$\mathcal{D}_h \equiv \left\{ u \in L^{\infty-}(\mu_t) : \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[u \circ R_{\exp \varepsilon h} - u\right] \text{ exists in } L^p(\mu_t), \forall p < \infty \right\}.$$

Let $\partial_h u$ denote $\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left[u \circ R_{\exp \varepsilon h} - u\right]$ for any $u \in \mathcal{D}_h$. Define a μ_t-a.s. random variable j_h by setting

$$j_h(\gamma) = \int_0^1 \frac{1}{t} \left\langle h'(s), \gamma(s)^{-1} \gamma(\delta s) \right\rangle.$$

Then

$$\mu_t[g \partial_h f] = -\mu_t[f \partial_h g] + \mu_t[f g j_h].$$

We will give a proof below. This result can also be obtained by using the left connection on K in Theorem 1.3 of Driver [10].

Proof. **of Lemma 4.8**
Let $\mathcal{I} : K \to K$ given by $\mathcal{I}(\kappa) = \kappa^{-1}$ for any $\kappa \in K$. Abuse notation so that is g and γ are paths in $W_e(K)$ then $\ell_g(\gamma)$ and $R_g(\gamma)$ denote the paths $s \to g(s)\gamma(s)$ and $s \to \gamma(s)g(s)$ in $W_e(K)$ respectively.

$$\begin{aligned} \mu_t[g\partial_h f] &= \mu_t\left[g \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(f \circ R_{\exp \varepsilon h} - f\right)\right] \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(\mu_t\left[gf \circ R_{\exp \varepsilon h}\right] - \mu_t[gf]\right). \end{aligned} \tag{4.2}$$

Apply Lemma 4.3 together with Remark 4.4 as well as the comments after Remark 4.7 to the finite-energy path $k_\varepsilon = \exp \varepsilon h$ and the bounded measurable function $\widetilde{f} = f \circ \mathcal{I} \circ \ell_{k^{-1}}$. We obtain

$$\widehat{E}\left[\widetilde{f} \circ \pi^{-1}\right] = \widehat{E}\left[\widetilde{f}\left(k\pi^{-1}\right) \widetilde{J}_k \circ \pi^{-1}\right].$$

Upon simplification, we obtain

$$\widehat{E}\left[f \circ R_k \circ \pi\right] = \widehat{E}\left[(f \circ \pi)\left(\widetilde{J}_k \circ \pi^{-1}\right)\right].$$

Here, as in Definition 4.2,

$$\widetilde{J}_{k_\varepsilon} \circ \pi^{-1} = \exp\left(-\frac{1}{2t}\int_0^1 \left|k_\varepsilon^{-1} k_\varepsilon'\right|_{\mathfrak{K}}^2 ds - \frac{1}{t}\int_0^1 \left\langle k_\varepsilon^{-1} k_\varepsilon', \pi_s^{-1} \pi_{\delta s}\right\rangle_{\mathfrak{K}}\right).$$

For any finite energy path k define a μ_t-a.s. random variable J_k by requiring that $J_k \circ \pi = \widetilde{J}_k \circ \pi^{-1}$. Then we see that with respect to the measure μ_t we have that

$$J_{k_\varepsilon}(\gamma) = \exp\left(-\frac{1}{2t}\int_0^1 \left|k_\varepsilon^{-1} k_\varepsilon'\right|_{\mathfrak{K}}^2 ds - \frac{1}{t}\int_0^1 \left\langle k_\varepsilon^{-1} k_\varepsilon', \gamma_s^{-1} \gamma_{\delta s}\right\rangle_{\mathfrak{K}}\right),$$

and

$$\mu_t\left[f \circ R_{k_\varepsilon}\right] = \mu_t\left[f J_{k_\varepsilon}\right]. \tag{4.3}$$

Replacing f by $gf \circ R_{\exp \varepsilon h}$ and using Eq. [4.3] yields

$$\begin{aligned} \mu_t \left[gf \circ R_{\exp \varepsilon h} \right] &= \mu_t \left[\left(g \circ R_{k_\varepsilon^{-1}} f \right) \circ R_{k_\varepsilon} \right] \\ &= \mu_t \left[\left(g \circ R_{k_\varepsilon^{-1}} \right) f J_{k_\varepsilon} \right]. \end{aligned}$$

Now we returning to Eq. [4.2] to get

$$\begin{aligned} \mu_t \left[g \partial_h f \right] &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(\mu_t \left[gf \circ R_{\exp \varepsilon h} \right] - \mu_t \left[gf \right] \right) \\ & \lim_{\varepsilon \downarrow 0} \int f \frac{1}{\varepsilon} \left[\left(g \circ R_{k_\varepsilon^{-1}} \right) J_{k_\varepsilon} - g \right] d\mu_t. \end{aligned}$$

Now by assumption $g \in \mathcal{D}_h$, and so $\partial_\varepsilon g \circ R_{k_\varepsilon^{-1}} \to \partial_h g$ in $L^p(\mu_t), \forall p < \infty$ as $\varepsilon \to 0$. By Lemma 4.4, Remark 4.6 we know that $\partial_\varepsilon \left(\widetilde{J}_{k_\varepsilon} \circ \pi^{-1} \right) \to \left(\widetilde{j}_h \circ \pi^{-1} \right)$ as $\varepsilon \to 0$ in $L^p(\mu_t), \forall p < \infty$. Notice that $j_h \circ \pi = \widetilde{j}_h \circ \pi^{-1}$ μ_t-a.s.. Thus $\partial_\varepsilon (J_{k_\varepsilon} \circ \pi) \to (j_h \circ \pi)$ in $L^p(\mu_t), \forall p < \infty$. Since *Law* $\pi = \mu_t$, we have $\partial_\varepsilon J_{k_\varepsilon} \to j_h$ in $L^p(\mu_t), \forall p < \infty$ as $\varepsilon \to 0$. Let $o(\varepsilon)$ denote a family of functions so that $\frac{1}{\varepsilon} o(\varepsilon) \to 0$ as $\varepsilon \to 0$ in $L^p(\mu_t), \forall p < \infty$. Then

$$g \circ R_{k_\varepsilon^{-1}} = g - \varepsilon \partial_h g + o(\varepsilon),$$

and

$$J_{k_\varepsilon} = 1 + \varepsilon j_h + o(\varepsilon).$$

Therefore

$$\frac{1}{\varepsilon} \left[\left(g \circ R_{k_\varepsilon^{-1}} \right) J_{k_\varepsilon} - g \right] f = -f \partial_h g + fgj_h - \varepsilon j_h f \partial_h g + R(\varepsilon),$$

where the remainder R is given by

$$R(\varepsilon) = \frac{1}{\varepsilon} o(\varepsilon) \left(1 + g - \varepsilon \partial_h g + o(\varepsilon) + \varepsilon j_h \right) f.$$

Now $j_h, f, \partial_h g$ are functions in $L^{\infty-}(\mu_t)$. By using Hölder's inequality repeatedly if necessary one can see that $j_h f \partial_h g$ and $(1 + g - \varepsilon \partial_h g + o(\varepsilon) + \varepsilon j_h) f$ are also in $L^{\infty-}(\mu_t)$. Hence $\mu_t [\varepsilon j_h f \partial_h g] \to 0$ and

$$\begin{aligned} \left[\int R(\varepsilon) \, d\mu_t \right]^2 &< \mu_t \left[\frac{o(\varepsilon)}{\varepsilon} \right]^2 \mu_t \left[\left(1 + g - \varepsilon \partial_h g + o(\varepsilon) + \varepsilon j_h \right) f \right]^2 \\ &\to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Therefore

$$\lim_{\varepsilon \downarrow 0} \int f \frac{1}{\varepsilon} \left[\left(g \circ R_{k_\varepsilon^{-1}} \right) J_{k_\varepsilon} - g \right] d\mu_t \to \mu_t \left[-f \partial_h g + fgj_h \right],$$

and so we are done. ∎

Corollary 4.9 (The $L^2(\mu_t)$-Adjoint $\left(\partial_h^2 \right)^*$)**.** *Let g, f be smooth cylinder functions (see Definition 2.5). Let $h \in H(\mathfrak{K})$ be a finite energy path. Then*

$$\mu_t[g \partial_h^2 f] = \mu_t[f \partial_h^2 g] - 2\mu_t[j_h f \partial_h g] + \mu_t[fgj_h^2] - \frac{1}{t} |h|^2_{H(\mathfrak{K})} \mu_t[gf]$$

Proof. By Lemma 4.8, we have

$$\begin{aligned} \mu_t[g\partial_h^2 f] &= -\mu_t[\partial_h g \partial_h f] + \mu_t[g j_h \partial_h f] \\ &= I_1 + I_2. \end{aligned} \tag{4.4}$$

Applying Lemma 4.8 to I_1, we see that

$$I_1 = \mu_t[f\partial_h^2 g] - \mu_t[j_h f \partial_h g]. \tag{4.5}$$

To apply Lemma 4.8 to I_2, it will be necessary to show that gj_h is in the domain of the operator ∂_h; i.e. we must show $[(gj_h)\circ R_{\exp\varepsilon h} - gj_h]$ has a limit in $L^p(\mu_t)$ for any $p<\infty$. Since g is a smooth cylinder function, we already know that $\frac{1}{\varepsilon}[g\circ R_{\exp\varepsilon h} - g]$ converges to $\partial_h g$ in $L^p(\mu_t)$. Thus, as in the proof of Lemma 4.8, if we can show that $\frac{1}{\varepsilon}[j_h\circ R_{\exp\varepsilon h} - j_h]$ converges to some $\partial_h j_h$ in $L^p(\mu_t)$ for any $p<\infty$ then we will have

$$\frac{1}{\varepsilon}[(gj_h)\circ R_{\exp\varepsilon h} - gj_h] \to [g\partial_h j_h + j_h \partial_h g] \tag{4.6}$$

as $\varepsilon\to 0$ in $L^p(\mu_t)$ for any $p<\infty$.

From Lemma 4.8 recall that, for any finite-energy path h in $H(\mathfrak{K})$, the random variable j_h is given by

$$j_h(\gamma) = \int_0^1 \frac{1}{t}\left\langle h'(s), \gamma(s)^{-1}\gamma(\delta s)\right\rangle \ \mu_t\text{-a.s.}$$

Let γ_s denote the μ_t-a.s. random variable $\gamma(s)$. Thus $j_h\circ R_{\exp\varepsilon h}$ is given by

$$\begin{aligned} & j_h\circ R_{\exp\varepsilon h}(\gamma) \\ &= \int_0^1 \frac{1}{t}\left\langle h'(s), (\gamma_s \exp\varepsilon h(s))^{-1}\delta_s(\gamma_s\exp\varepsilon h(s))\right\rangle \\ &= \int_0^1 \frac{1}{t}\left\langle h'(s), \exp(-\varepsilon h(s))\gamma_s^{-1}\gamma_{\delta s}\exp\varepsilon h(s)\right\rangle \\ &\quad + \int_0^1 \frac{1}{t}\left\langle h'(s), \exp(-\varepsilon h(s))\exp'[\varepsilon h(s)]h'(s)\,ds\right\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\varepsilon}[j_h\circ R_{\exp\varepsilon h} - j_h] = & \int_0^1 \frac{1}{t\varepsilon}\left\langle h'(s), Ad_{\exp -\varepsilon h_s}\left(\gamma_s^{-1}\gamma_{\delta s}\right)\right\rangle \\ & - \int_0^1 \frac{1}{t\varepsilon}\left\langle h'(s), \gamma_s^{-1}\gamma_{\delta s}\right\rangle \\ & + \int_0^1 \frac{1}{t}\left\langle h'(s), \exp(-\varepsilon h_s)\exp'[\varepsilon h(s)]h'(s)\right\rangle ds. \end{aligned}$$

By the Ad-invariance of the metric, we see that

$$\begin{aligned} \frac{1}{\varepsilon}[j_h\circ R_{\exp\varepsilon h} - j_h] &= \int_0^1 \frac{1}{t\varepsilon}\left\langle Ad_{\exp\varepsilon h_s}h'(s) - h'(s), \gamma_s^{-1}\gamma_{\delta s}\right\rangle \\ &\quad + \int_0^1 \frac{1}{t}\left\langle h'(s), \exp(-\varepsilon h_s)\exp'[\varepsilon h(s)]h'(s)\right\rangle ds. \end{aligned}$$

However h was chosen to be good in the sense of Definition 2.2. So the matrix $h(s)$ commutes with $h'(s)$ and thus $Ad_{\exp \varepsilon h_s} h'(s) - h'(s)$ is 0. Observing this yields

$$\frac{1}{\varepsilon}\left[j_h \circ R_{\exp \varepsilon h} - j_h\right] = \int_0^1 \frac{1}{t}\left\langle h'(s), \exp(-\varepsilon h_s)\exp'(\varepsilon h_s) h'(s)\right\rangle ds.$$

which is independent of the path γ (and thus a constant random variable). Hence the above expression converges in $L^p(\mu_t)$ for all $p < \infty$ to the expression

$$\int_0^1 \frac{1}{t}\left\langle h'(s), \exp(0)\exp'(0) h'(s)\right\rangle ds = \frac{1}{t}|h|^2_{H(\mathfrak{K})}.$$

Thus $\partial_h j_h = \frac{1}{t}|h'|^2_{H(\mathfrak{K})}$. Returning to Eq. [4.6] we see that $\partial_h(j_h g)$ exists and equals $[g\partial_h j_h + j_h \partial_h g]$. Thus

$$I_2 = -\mu_t[j_h f \partial_h g] - \frac{1}{t}|h|^2_{H(\mathfrak{K})}\mu_t[gf] + \mu_t[fgj_h^2].$$

Now returning to Eqs. [4.4] and [4.5] we see that

$$\begin{aligned}\mu_t[g\partial_h^2 f] &= I_1 + I_2\\ &= \mu_t[f\partial_h^2 g] - 2\mu_t[j_h f\partial_h g] - \frac{1}{t}|h|^2_{H(\mathfrak{K})}\mu_t[gf] + \mu_t[fgj_h^2].\end{aligned}$$

∎

Definition 4.10 (Orthogonal Decomposition of $H(\mathfrak{K})$ and $H_0(\mathfrak{K})$)**.** We will need the following notions:-

1. Recall from Definition 2.1 that

   $$H(\mathfrak{K}) \equiv \{h : [0,1] \to \mathfrak{K} | h(0) = 0 \text{ and } (h,h) < \infty\}.$$

   For any unit vector $A \in \mathfrak{K}$ and α in $(0,1)$ let $\widetilde{A}$ be the unit vector in $H(\mathfrak{K})$ defined by setting

   $$\widetilde{A}(s) = \frac{1}{\sqrt{a}} A(s \wedge \alpha). \tag{4.7}$$

   Write $H(\mathfrak{K})$ as $U_\alpha^1 \bigoplus U_\alpha^2 \bigoplus U_\alpha^3$ where the U_α^i are defined by setting

   $$\begin{aligned} U_\alpha^1 &\equiv \{h \in H(\mathfrak{K}) | h = 0 \text{ on } [\alpha, 1]\};\\ U_\alpha^2 &\equiv \{h \in H(\mathfrak{K}) | h = 0 \text{ on } [0, \alpha]\};\\ U_\alpha^3 &\equiv span\left\langle \widetilde{A} : A \in \mathfrak{K}\right\rangle.\end{aligned}$$

   Let S^i be a good orthonormal basis of U_α^i. Then $S \equiv \cup_i S^i$ forms a good orthonormal basis of $H(\mathfrak{K})$. Let $\triangle_{U_\alpha^i}$ be defined as $\sum_{h \in S^i} \partial_h^2$ where the operator $(\partial_h f)(\gamma) \equiv \frac{d}{d\varepsilon} f(\gamma \exp \varepsilon h)$ [The map $h \to \partial_h$ is just the usual identification of elements of $H(\mathfrak{K})$ with left-invariant vector fields on $W_e(K)$]. Then we can see that

   $$\triangle_{W_e(K)} = \triangle_{U_\alpha^1} + \triangle_{U_\alpha^2} + \triangle_{U_\alpha^3}. \tag{4.8}$$

2. Recall from Definition 2.1 that

   $$H_0(\mathfrak{K}) \equiv \{h \in H(\mathfrak{K}) | h(1) = 0\}.$$

Decompose $H_0(\mathfrak{K})$ as $W_\alpha^1 \bigoplus W_\alpha^2 \bigoplus W_\alpha^3$. $W_\alpha^1 \equiv U_\alpha^1$ which is defined as before. W_α^2 is defined to be $U_\alpha^2 \cap H_0(\mathfrak{K})$. W_α^3 is defined to be the span of the vectors $\langle \ell_A : A \in \mathfrak{K} \rangle$ where the unit vector ℓ_A is given by setting

$$\ell_A(s) = As1_{[0,\alpha]}\sqrt{\frac{1-\alpha}{\alpha}} + A(1-s)1_{(\alpha,1]}\sqrt{\frac{\alpha}{1-\alpha}}. \tag{4.9}$$

. Let S_0^i be a good orthonormal basis of W_α^i. Then $S_0 \equiv \cup_i S_0^i$ forms a good orthonormal basis of $H_0(\mathfrak{K})$. Let $\triangle_{W_\alpha^i}$ be defined as $\sum_{h \in S_0^i} \partial_h^2$ where the operator $(\partial_h f)(\gamma) \equiv \frac{d}{d\varepsilon} f(\gamma \exp \varepsilon h)$ [The map $h \to \partial_h$ is just the usual identification of elements of $H_0(\mathfrak{K})$ with left-invariant vector fields on $L(K)$]. Then we see that

$$\triangle_{L(K)} = \triangle_{U_\alpha^1} + \triangle_{W_\alpha^2} + \triangle_{W_\alpha^3}. \tag{4.10}$$

Proof. **of Theorem 4.1:**

Fix $\alpha < 1$. Let $f(\sigma) = F(\sigma_{s_1}, \cdots, \sigma_{s_n})$ so that$\sigma_{s_n} < \alpha$. Let $S = S^1 \cup S^2 \cup S^3$ be the orthonormal basis of Definition 4.10. Then

$$\begin{aligned}
\partial_t \mu_{0,t}[f] &= \lim_{\alpha \to 1} \partial_t \mu_t \left[f \frac{P_{t(1-\alpha)}^K \circ \pi_\alpha}{P_t^K(e)} \right] \\
&= \lim_{\alpha \to 1} \mu_t \left[\frac{1}{2P_t^K(e)} \triangle_{W_e(K)} \left(f P_{t(1-\alpha)}^K \circ \pi_\alpha \right) \right] \\
&\quad + \lim_{\alpha \to 1} \mu_t \left[f \partial_t \frac{P_{t(1-\alpha)}^K \circ \pi_\alpha}{P_t^K(e)} \right] \\
&= I + J.
\end{aligned} \tag{4.11}$$

Let us work on the second term first.

$$\begin{aligned}
J &= \lim_{\alpha \to 1} \frac{1}{P_t^K(e)} \mu_t \left[f \partial_t P_{t(1-\alpha)}^K \circ \pi_\alpha \right] - \mu_{0,t}\left[f \partial_t \log P_t^K(e) \right] \\
&= J_1 - \mu_{0,t}\left[f \partial_t \log P_t^K(e) \right],
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
J_1 &= \lim_{\alpha \to 1} \frac{1}{P_t^K(e)} \mu_t \left[f \partial_t P_{t(1-\alpha)}^K \circ \pi_\alpha \right] \\
&= \lim_{\alpha \to 1} \frac{1-\alpha}{2P_t^K(e)} \mu_t \left[f \triangle_K P_{t(1-\alpha)}^K \circ \pi_\alpha \right].
\end{aligned} \tag{4.13}$$

Define

$$\begin{aligned}
&C_\alpha(x_{n+1}) \\
&\quad = \int_{K^n} F(x_1, \cdots, x_n) P_{t(\alpha - s_n)}^K \left(x_n^{-1} x_{n+1} \right) \prod_{i=1}^n P_{t\triangle_i s}^K \left(x_{i-1}^{-1} x_i \right) \lambda(dx_i).
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\alpha \to 1} \mu_t \left[f \triangle_K P_{t(1-\alpha)}^K \circ \pi_\alpha \right] &= \lim_{\alpha \to 1} \int_K C_\alpha(x) \triangle_K P_{t(1-\alpha)}^K(x) \lambda(dx) \\
&= \lim_{\alpha \to 1} \int_K \triangle_K C_\alpha(x) P_{t(1-\alpha)}^K(x) \lambda(dx) \\
&= \triangle_K C_1(e) \\
&< \infty.
\end{aligned}$$

From Eq. [4.13] we have

$$J_1 = 0.$$

Combining this fact with Eq. [4.12] gives

$$J = -\mu_{0,t}\left[f\partial_t \log P_t^K(e)\right]. \tag{4.14}$$

We proceed to work on the first term I of Eq. [4.11].

$$\begin{aligned} I &= \lim_{\alpha\to 1}\frac{1}{2}P_t^K(e)^{-1}\mu_t\left[\triangle_{W_e(K)}\left(fP_{t(1-\alpha)}^K\circ\pi_\alpha\right)\right] \\ &= \lim_{\alpha\to 1}\frac{1}{2}\mu_{0,t}\left[\triangle_{W_e(K)}f\right] + \lim_{\alpha\to 1}\frac{1}{2}P_t^K(e)^{-1}\mu_t\left[f\triangle_{W_e(K)}P_{t(1-\alpha)}^K\circ\pi_\alpha\right] \\ &\quad + \lim_{\alpha\to 1}P_t^K(e)^{-1}\sum_{h\in S}\mu_t\left[\partial_h f\partial_h P_{t(1-\alpha)}^K\circ\pi_\alpha\right] \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.15}$$

From Eq. [4.8]

$$I_1 = \lim_{\alpha\to 1}\frac{1}{2}\mu_{0,t}\left[\left(\triangle_{U_\alpha^1}+\triangle_{U_\alpha^2}+\triangle_{U_\alpha^3}\right)f\right].$$

Since f does not depend on the path on or after time α, $\triangle_{U_\alpha^2}f = 0$ and this last expression is

$$\lim_{\alpha\to 1}\frac{1}{2}\mu_{0,t}\left[\triangle_{U_\alpha^1}f\right] + \lim_{\alpha\to 1}\frac{1}{2}\mu_{0,t}\left[\triangle_{U_\alpha^3}f\right].$$

Applying Eq. [4.10] and observing that $\triangle_{W_\alpha^2}f = 0$ reduces this last to

$$I_1 = \frac{1}{2}\mu_{0,t}\left[\triangle_{L(K)}f\right] + \lim_{\alpha\to 1}\frac{1}{2}\mu_{0,t}\left[\triangle_{U_\alpha^3}f\right] - \lim_{\alpha\to 1}\frac{1}{2}\mu_{0,t}\left[\triangle_{W_\alpha^3}f\right]. \tag{4.16}$$

Now letting A run through an orthonormal basis of $\mathfrak{K}$, we see from Definition 4.10 that

$$\triangle_{W_\alpha^3}f = \sum_A \partial_{\ell_A}^2 f.$$

Since f does not depend on the path from time α onwards, we can see that from Eqs. [4.7] and [4.9] that

$$\partial_{\ell_A}f = \sqrt{1-\alpha}\,\partial_{\widetilde{A}}f \text{ and } \triangle_{W_\alpha^3}f = (1-\alpha)\triangle_{U_\alpha^3}f.$$

Thus Eq. [4.16] becomes

$$\begin{aligned} I_1 - \frac{1}{2}\mu_{0,t}\left[\triangle_{L(K)}f\right] &= \lim_{\alpha\to 1}\frac{\alpha}{2}\mu_{0,t}\left[\triangle_{U_\alpha^3}f\right] \\ &= \lim_{\alpha\to 1}\frac{1}{2P_t^K(e)}\mu_t\left[P_{t(1-\alpha)}^K\circ\pi_\alpha\sum_A\partial_{\widetilde{A}}^2 f\right]. \end{aligned}$$

Define $\beta_\alpha = \int_0^\alpha \gamma(s)^{-1}\gamma(\delta s)$, μ_t-a.s. Invoking Corollary 4.9 we obtain

$$\begin{aligned} \mu_t[P_{t(1-\alpha)}^K\circ\pi_\alpha\partial_{\widetilde{A}}^2 f] &= \mu_t[f\partial_{\widetilde{A}}^2 P_{t(1-\alpha)}^K\circ\pi_\alpha] - 2\mu_t[j_{\widetilde{A}}f\partial_{\widetilde{A}}P_{t(1-\alpha)}^K\circ\pi_\alpha] \\ &\quad + \mu_t[fP_{t(1-\alpha)}^K\circ\pi_\alpha j_{\widetilde{A}}^2] - \frac{1}{t}\left|\widetilde{A}\right|_{H(\mathfrak{K})}^2\mu_t[P_{t(1-\alpha)}^K\circ\pi_\alpha f]. \end{aligned}$$

Observing that $\widetilde{A}(s) = \alpha^{-1/2}(s \wedge \alpha) A$, we see that $\left|\widetilde{A}\right|^2_{H(\mathfrak{K})} = 1$, and $j_{\widetilde{A}}(\gamma) = \frac{1}{t}\alpha^{-1/2}\langle A, \beta_\alpha\rangle$. Thus

$$\begin{aligned} I_1 - \frac{1}{2}\mu_{0,t}\left[\triangle_{L(K)} f\right] &= I_2 + \lim_{\alpha\to 1}\frac{1}{2t^2\alpha}\mu_{t,e}[f\,|\beta_\alpha|^2_{\mathfrak{K}}] - \frac{\dim\mathfrak{K}}{2t}\mu_{t,e}[f] \\ &\quad - \lim_{\alpha\to 1}\frac{1}{tP_t^K(e)}\sum_A \mu_t[\langle A,\beta_\alpha\rangle f\partial_A P^K_{t(1-\alpha)}\circ\pi_\alpha]. \end{aligned}$$

Invoking Lemma 4.8 on I_3 and recognizing that $\partial_h P^K_{t(1-\alpha)}\circ\pi_\alpha = 0$ for any $h \in S^1 \cup S^2$ yields

$$\begin{aligned} I_3 &= \lim_{\alpha\to 1} P_t^K(e)^{-1}\sum_{h\in S}\mu_t\left[\partial_h f\partial_h P^K_{t(1-\alpha)}\circ\pi_\alpha\right] \\ &= -\lim_{\alpha\to 1}\alpha P_t^K(e)^{-1}\sum_{h\in S}\mu_t\left[f\left(\triangle_K P^K_{t(1-\alpha)}\right)\circ\pi_\alpha\right] \\ &\quad + \lim_{\alpha\to 1}\frac{1}{tP_t^K(e)}\sum_A\mu_t[\langle A,\beta_\alpha\rangle f\partial_A P^K_{t(1-\alpha)}\circ\pi_\alpha] \\ &= -2I_2 + \lim_{\alpha\to 1}\frac{1}{tP_t^K(e)}\sum_A\mu_t[\langle A,\beta_\alpha\rangle f\partial_A P^K_{t(1-\alpha)}\circ\pi_\alpha]. \end{aligned}$$

Thus,

$$I = \frac{1}{2}\mu_{0,t}\left[\triangle_{L(K)} f\right] - \frac{\dim\mathfrak{K}}{2t}\mu_{t,e}[f] + \lim_{\alpha\to 1}\frac{1}{2t^2\alpha}\mu_{t,e}[f\,|\beta_\alpha|^2_{\mathfrak{K}}].$$

The expression

$$\beta_\alpha = \beta_\alpha\circ\pi = \widetilde{\beta}_\alpha = \int_0^\alpha \pi_s^{-1}\pi_{\delta s}.$$

Combining Lemma 4.5 with Remarks 4.6 and 4.7 we have $\widetilde{\beta}_\alpha$ converges in $L^2\left(\mu_{t,0}\right)$ as $\alpha \uparrow 1$. Thus β_α converges in $L^2\left(\mu_{t,0}\right)$ as $\alpha\uparrow 1$ to a limit β_1 and so

$$I = \frac{1}{2}\mu_{0,t}\left[\triangle_{L(K)} f\right] - \frac{\dim\mathfrak{K}}{2t}\mu_{t,e}[f] + \frac{1}{2t^2}\mu_{t,e}[f\,|\beta_1|^2_{\mathfrak{K}}].$$

and so returning to Eqs. [4.11] and [4.12] yields

$$\partial_t\mu_{0,t}[f] = \frac{1}{2}\mu_{0,t}\left[\triangle_{L(K)} f\right] - \frac{\dim\mathfrak{K}}{2t}\mu_{t,e}[f] + \frac{1}{2t^2}\mu_{t,e}[f\,|\beta_1|^2_{\mathfrak{K}}] - \mu_{0,t}\left[f\partial_t\log P_t^K(e)\right].$$

∎

5. Absolute continuity of Heat Kernel with respect to pinned Wiener measure

Let γ be a generic loop in $L(K)$. Recall from Definitions 2.9 and 2.7 that $\mu_t(d\gamma)$ denotes Wiener measure on K with variance t and $\mu_{0,t}(d\gamma)$ denotes pinned Wiener measure. Recall from Definition 2.20 that $\nu_t(e, d\gamma)$ denotes Heat Kernel measure on $L(K)$. The goal of this section is to demonstrate the absolute continuity of Heat Kernel measure $\nu_t(e, d\gamma)$ with respect to pinned Wiener measure $\mu_{0,t}(d\gamma)$.

Theorem 5.1. *Heat Kernel measure $\nu_t(e,\cdot)$ on $L(K)$ is absolutely continuous with respect to pinned Wiener measure $\mu_{0,t}(d\gamma)$ and the Radon-Nikodym derivative $d\nu_t(e,\cdot)/d\mu_{0,t}$ is bounded.*

We defer the proof until some basic machinery is established.

Definition 5.2 (Basic Machinery). Let $\mathbb{P}$ be the partition $\{0 < s_1 < \cdots < s_n < 1\}$. Then:-

1. $g^{\mathbb{P}} \equiv (G_0(s_i, s_j))^{-1}$ as $n \times n$ matrices where $G_0(s,\sigma) \equiv s \wedge \sigma - s\sigma$ was introduced in Definition 2.12.
2. Let $A^{(i)}(x_1, \cdots, x_n)$ be the vector field on K^n so that
$$\left(A^{(i)} f\right)(x_1, \cdots, x_n) \equiv \frac{d}{dt} f(x_1, \cdots, x_i \exp tA, \cdots, x_n) \downarrow_{t=0}.$$
3. $\langle\cdot,\cdot\rangle_{\mathbb{P}}$ is the left invariant metric on K^n such that $\left\langle A^{(i)}, B^{(j)}\right\rangle_{\mathbb{P}} = \delta_{AB} g_{ij}$.
4. Let $K^{\mathbb{P}}$ be K^n equipped with $\langle\cdot,\cdot\rangle_{\mathbb{P}}$.
5. Let $\triangle_{\mathbb{P}}$ be the Laplacian on $K^{\mathbb{P}}$. Thus
$$\triangle_{\mathbb{P}} = \sum_{A,i,j} G_0(s_i, s_j)\, \partial_{A^{(i)}} \partial_{A^{(j)}}.$$
6. Let $\pi_s : L(K) \to K$ denote the map $\pi_s : x \to x_s$.
7. Let $\pi_{\mathbb{P}} : L(K) \to K^{\mathbb{P}}$ by $\pi_{\mathbb{P}} = (\pi_{s_1}, \cdots, \pi_{s_n})$.
8. $p_t^{\mathbb{P}} \equiv d(\pi_{\mathbb{P}})_* \nu_t(e,\cdot)/d\lambda$. Explicitly if $|\mathbb{P}| = n$ and λ denotes standard Haar-measure on K, then $p_t^{\mathbb{P}} : K^n \to \mathbb{R}$ is the function so that for any $F \in C^\infty(K^n)$ we have
$$\int_{L(K)} F \circ \pi_{\mathbb{P}}(\gamma)\, \nu_t(e, d\gamma) = \int_{x \in K^n} F(x)\, p_t^{\mathbb{P}}(x)\, \lambda^{\otimes n}(dx). \tag{5.1}$$
9. $q_t^{\mathbb{P}} \equiv d(\pi_{\mathbb{P}})_* \mu_{0,t}/d\lambda$. Explicitly if $|\mathbb{P}| = n$ and λ denotes standard Haar-measure on K, then $q_t^{\mathbb{P}} : K^n \to \mathbb{R}$ is the function so that for any $F \in C^\infty(K^n)$ we have
$$\int_{L(K)} F \circ \pi_{\mathbb{P}}(\gamma)\, \mu_{0,t}(d\gamma) = \int_{x \in K^n} F(x)\, q_t^{\mathbb{P}}(x)\, \lambda^{\otimes n}(dx). \tag{5.2}$$
10. $\mathfrak{F}^{\mathbb{P}} \equiv \sigma\langle \pi_{\mathbb{P}} \rangle$.
11. $Z_t^{\mathbb{P}} \equiv \left(p_t^{\mathbb{P}}/q_t^{\mathbb{P}}\right) \circ \pi_{\mathbb{P}} = d\left(\nu_t|_{\mathfrak{F}^{\mathbb{P}}}\right)/d\left(\mu_{0,t}|_{\mathfrak{F}^{\mathbb{P}}}\right)$. Explicitly for any $F \in C^\infty(K^n)$, combining Eqs. [5.1] and [5.2] we have
$$\int F \circ \pi_{\mathbb{P}}(\gamma)\, \nu_t(e, d\gamma) = \int F \frac{p_t^{\mathbb{P}}}{p_t^{\mathbb{P}}} \left(q_t^{\mathbb{P}} d\lambda^{\otimes n}\right) = \int F \circ \pi_{\mathbb{P}}(\gamma)\, Z_t^{\mathbb{P}}(\gamma)\, \mu_{0,t}(d\gamma).$$

The proof of Theorem 5.1 rests on Theorem 4.1, a result of Airault and Malliavin. We shall also make use of Lemma 2.15.

Lemma 5.3 (Asymptotic properties of heat Kernels on K). *Heat Kernel measure on K has the following properties:-*

1. $\varepsilon^{d/2} P_{\varepsilon}^K(e) \to (2\pi)^{-d/2}$ *as* $\varepsilon \to 0$.
2. *Let* $B^K(e,\varepsilon)$ *denote the ball of radius* ε *near* e. *Then*
$$\sup_{\varepsilon \in (0,1), u \in B^K(e,\varepsilon)^c} P^K_{\varepsilon(1-s_n)}(u) < \infty.$$

Proof. A generalization of this result is proved in Berline, Getzler, & Vergne, [6], Theorem 2.30. See also [28]. ∎

Lemma 5.4 ($\mu_{0,\varepsilon} \to \delta_e$ as $\varepsilon \to 0$). *Let $f : K^n \to \mathbb{R}$ be continuous. Abusing notation, let e denote the element $(e, \cdots, e)$ of K^n. Then*

$$\lim_{\varepsilon \to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f] = f(e).$$

Proof. Let π_s be the evaluation map as in Definition 5.2. Let $\triangle_i s$ be $s_i - s_{i-1}$ as usual. Then

$$\begin{aligned}
\lim_{\varepsilon \to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f] &= \lim_{\varepsilon \to 0} \int_M f(x_1, \cdots, x_n) q_\varepsilon^{\mathbb{P}}(x) \, d\lambda \\
&= \lim_{\varepsilon \to 0} \int_M f(x_1, \cdots, x_n) \frac{P^K_{\varepsilon(1-s_n)}(x_n)}{P^K_\varepsilon(e)} \prod_{i=1}^n P^K_{\varepsilon \triangle_i s}\left(x_{i-1}^{-1} x_i\right) dx_i \\
&= \lim_{\varepsilon \to 0} \mu_{0,\varepsilon}\left[f \circ \pi_{\mathbb{P}} \frac{P^K_{\varepsilon(1-s_n)} \circ \pi_{s_n}}{P^K_\varepsilon(e)} \right].
\end{aligned}$$

Let $B^K(e, r)$ be the open ball of all points distant less than r from e in the metric $\langle \cdot, \cdot \rangle_{\mathfrak{K}}$ on K. Then our previous expression becomes

$$\begin{aligned}
&= \lim_{\varepsilon \to 0} \mu_\varepsilon \left[\frac{f \circ \pi_{\mathbb{P}}}{P^K_\varepsilon(e)} \left(1_{B^K(e,\varepsilon)} \circ \pi_{s_n}\right) \left(P^K_{\varepsilon(1-s_n)} \circ \pi_{s_n}\right) \right] \\
&\quad + \lim_{\varepsilon \to 0} \mu_\varepsilon \left[\frac{f \circ \pi_{\mathbb{P}}}{P^K_\varepsilon(e)} \left(1_{B^K(e,\varepsilon)^c} \circ \pi_{s_n}\right) \left(P^K_{\varepsilon(1-s_n)} \circ \pi_{s_n}\right) \right] \\
&= I_1 + I_2.
\end{aligned}$$

By Lemma 5.3, we see that the expression

$$\frac{f(x_1, \cdots, x_n)}{P^K_\varepsilon(e)} 1_{B^K(e,\varepsilon)^c}(x_n) P^K_{\varepsilon(1-s_n)}(x_n)$$

is bounded and so I_2 vanishes by Dominated Convergence. Thus

$$\begin{aligned}
&\lim_{\varepsilon \to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f] \\
&\quad = \lim_{\varepsilon \to 0} \mu_\varepsilon \left[\frac{f \circ \pi_{\mathbb{P}}}{P^K_\varepsilon(e)} \left(1_{B^K(e,\varepsilon)} \circ \pi_{s_n}\right) \left(P^K_{\varepsilon(1-s_n)} \circ \pi_{s_n}\right) \right] \\
&\quad = \lim_{\varepsilon \to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f(x_1, \cdots, x_{n-1}, e)] \\
&\qquad + \lim_{\varepsilon \to 0} \left(\pi^*_{\mathbb{P}} \mu_{0,\varepsilon}\right) \left[(f(x_1, \cdots, x_n) - f(x_1, \cdots, x_{n-1}, e)) 1_{B^K(e,\varepsilon)}(x_n) \right] \\
&\quad = J_1 + J_2.
\end{aligned}$$

Now the expression

$$\sup_{K^{\mathbb{P}}} |f(x_1, \cdots, x_n) - f(x_1, \cdots, x_{n-1}, e)| 1_{B^K(e,\varepsilon)}(x_n)$$

is bounded above by

$$\sup_{x_n \in B^K(e,\varepsilon)} |f(x_1, \cdots, x_n) - f(x_1, \cdots, x_{n-1}, e)|. \tag{5.3}$$

We claim that if x_n is distance d from e in K then there is a constant $C^{\mathbb{P}}$, depending only on the partition, so that the points $(x_1, \cdots, x_n)$ and $(x_1, \cdots, x_{n-1}, e)$ are distance $dC^{\mathbb{P}}$ apart in $K^{\mathbb{P}}$. If we can verify this, then Eq.[5.3] is bounded above by

$$\sup_{\{x,y | y \in B^{\mathbb{P}}(x, \varepsilon / C^{\mathbb{P}})\}} |f(y) - f(x)|.$$

By the uniform continuity of the continuous function f on the compact topological space $K^{\mathbb{P}}$ we see that this last expression tends to 0 as $\varepsilon \to 0$. Thus J_2 can be made arbitrarily small. This, in turn, implies that

$$\lim_{\varepsilon\to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f] = \lim_{\varepsilon\to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f(x_1,\cdots,x_{n-1},e)].$$

Now replace $\mathbb{P}$ by $\mathbb{P}_1 \equiv \{0 < s_1 \cdots < s_{n-1} < 1\}$ and f by

$$f_1(x_1,\cdots,x_{n-1}) \equiv f(x_1,\cdots,x_{n-1},e).$$

f_1 is still smooth on $K^{\mathbb{P}_1}$ and the above reasoning applies inductively. Therefore

$$\lim_{\varepsilon\to 0} (\pi_{\mathbb{P}})_* \mu_{0,\varepsilon}[f] = f(e,\cdots,e),$$

and we are done once we verify the claim.

To do this, let $x(s) \equiv (c_1,\cdots,c_{n-1},x_n(s))$ be a differentiable path in $K^{\mathbb{P}}$ (with the c_i being held constant) then we have

$$x'(s) = \sum_A \langle x_n'(s), A\rangle_{\mathfrak{K}} A^{(n)}$$

for any s. This implies that

$$\begin{aligned}
\int_0^1 |x'(s)|^2_{\mathbb{P}}\, ds &= \int_0^1 \sum_A \langle x_n'(s), A\rangle^2_{\mathfrak{K}} \left|A^{(n)}(s)\right|^2_{\mathbb{P}} ds \\
&= (G_0(s_i,s_j))^{-1}_{nn} \int_0^1 |x_n'(s)|^2_{\mathfrak{K}}\, ds \\
&= \left(C^{\mathbb{P}}\right)^2 \int_0^1 |x_n'(s)|^2_{\mathfrak{K}}\, ds.
\end{aligned}$$

Thus if x_n is distance d from e in K then the points $(x_1,\cdots,x_n)$ and $(x_1,\cdots,x_{n-1},e)$ are distance $dC^{\mathbb{P}}$ apart in $K^{\mathbb{P}}$. ∎

Lemma 5.5. *Let* $C_t \equiv \log\left[t^{d/2}P_t^K(e)\right] - \log\lim_{\varepsilon\to 0}\varepsilon^{d/2}P_\varepsilon^K(e)$. C_t *is well-defined by Lemma 5.3. Then* $p_t^{\mathbb{P}}/q_t^{\mathbb{P}} \le \exp C_t$.

Proof. Given bounded smooth f, $h \ge 0$, on K^n define

$$\begin{aligned}
H(t,x) &\equiv \int h(xy)\, p_t^{\mathbb{P}}(y)\, dy \\
F(t,x) &\equiv \int f(xy)\, \tilde{q}_t^{\mathbb{P}}(y)\, dy.
\end{aligned}$$

Let $\gamma \in L(K)$ and let $t \to g_t$ be our standard Brownian motion on $L(K)$ (see Definition 2.22). Let $\ell_x : k \to xk$ denote left translation by x on K. Now $t \to \gamma g_t$ is a Brownian motion on $L(K)$ starting at γ in the sense of Definition 2.14. Heat Kernel measure, $\nu_t(\gamma,\cdot)$ is the law of γg_t.

$$\begin{aligned}
\nu_t(\gamma, h\circ\pi_{\mathbb{P}}) &= Eh\circ\pi_{\mathbb{P}}(\gamma g_t) \\
&= Eh(\pi_{\mathbb{P}}(\gamma)\,\pi_{\mathbb{P}}(g_t)) \\
&= Eh\circ\ell_{\pi_{\mathbb{P}}(\gamma)}\circ\pi_{\mathbb{P}}(g_t) \\
&= \nu_t\left(e, h\circ\ell_{\pi_{\mathbb{P}}(\gamma)}\circ\pi_{\mathbb{P}}\right) \\
&= \int h(\pi_{\mathbb{P}}(\gamma)\, y)\, p_t^{\mathbb{P}}(y)\, dy \\
&= H(t,\pi_{\mathbb{P}}(\gamma)).
\end{aligned}$$

Letting $x = \pi_{\mathbb{P}}(\gamma)$, we see from the Heat Equation, Theorem 2.21, and Lemma 2.15 that

$$
\begin{aligned}
\partial_t H(t,x) &= \frac{1}{2}\triangle_{L(K)} H(t, \pi_{\mathbb{P}}(\gamma)) = \frac{1}{2}\triangle_{\mathbb{P}} H(t,x) \\
H(t,x) &\to h \circ \pi_{\mathbb{P}}(\gamma) \ \text{ as } t \to 0.
\end{aligned} \tag{5.4}
$$

We shall now obtain a similar equation for $F(t,x)$. Let $\tilde{q}_t^{\mathbb{P}} \equiv q_t^{\mathbb{P}} \exp C_t$. Then for some smooth ϕ on K^n, we have

$$
\begin{aligned}
\int \phi(y)\left(\partial_t \widetilde{q}_t^{\mathbb{P}}\right)(y)\,dy &= \partial_t \exp(C_t)\,\mu_{0,t}[\phi \circ \pi_{\mathbb{P}}] \\
&= \exp(C_t)\,\partial_t \mu_{0,t}[\phi \circ \pi_{\mathbb{P}}] \\
&\quad + \mu_{0,t}[\phi \circ \pi_{\mathbb{P}}] \exp(C_t)\left[\frac{\dim \mathfrak{K}}{2t} + \partial_t \log p_t^K(e)\right].
\end{aligned}
$$

Applying Airault-Malliavin (Theorem 4.1) to the first term yields

$$
\begin{aligned}
= \ & \exp(C_t)\,\mu_{0,t}\left[\frac{1}{2t^2}\left|\int_0^1 x(ds)\,x(s)^{-1}\right|_{\mathfrak{K}}^2 \phi \circ \pi_{\mathbb{P}}\right] \\
& + \frac{\exp(C_t)}{2}\mu_{0,t}\left[\triangle_{L(K)}\phi \circ \pi_{\mathbb{P}}\right].
\end{aligned}
$$

Define $,_t^{\mathbb{P}} \circ \pi_{\mathbb{P}} \equiv \mu_{0,t}\left(\left|\int_0^1 \gamma_{ds}\gamma_s^{-1}\right|_{\mathfrak{K}}^2 |\mathfrak{F}^{\mathbb{P}}\right)$ where $\left|\int_0^1 \gamma_{ds}\gamma_s^{-1}\right|_{\mathfrak{K}}$ is Gross' $L^2(\mu_{0,t})$-limit of $\left|\int_0^\alpha \gamma_{ds}\gamma_s^{-1}\right|_{\mathfrak{K}}$ as $\alpha \to 1$. Then using Lemma 2.15 our last expression becomes

$$
\begin{aligned}
&= \exp(C_t)\,(\pi_{\mathbb{P}})_*\,\mu_{0,t}\left[\frac{,_t^{\mathbb{P}}}{2t^2}\phi + \frac{1}{2}\triangle_{\mathbb{P}}\phi\right] \\
&= \exp(C_t)\int\left[\frac{,_t^{\mathbb{P}}(y)}{2t^2}\phi(y) + \frac{1}{2}(\triangle_{\mathbb{P}}\phi)(y)\right] q_t^{\mathbb{P}}(y)\,dy \\
&= \int \frac{,_t^{\mathbb{P}}(y)}{2t^2}\phi(y)\,\widetilde{q}_t^{\mathbb{P}}(y)\,dy + \frac{1}{2}(\triangle_{\mathbb{P}}\phi)(y)\,\widetilde{q}_t^{\mathbb{P}}(y)\,dy.
\end{aligned}
$$

Using the smoothness of $\widetilde{q}_t^{\mathbb{P}}$, perform an integration by parts on the second term to get

$$
\begin{aligned}
&= \int \frac{,_t^{\mathbb{P}}(y)}{2t^2}\phi(y)\,\widetilde{q}_t^{\mathbb{P}}(y)\,dy + \frac{1}{2}\left(\triangle_{\mathbb{P}}\widetilde{q}_t^{\mathbb{P}}\right)(y)\,\phi(y)\,dy \\
&= \int\left[\frac{,_t^{\mathbb{P}}(y)}{2t^2}\widetilde{q}_t^{\mathbb{P}}(y) + \frac{1}{2}\left(\triangle_{\mathbb{P}}\widetilde{q}_t^{\mathbb{P}}\right)(y)\right]\phi(y)\,dy.
\end{aligned}
$$

Therefore we have the dy-a.s. equality

$$
\left(\partial_t \widetilde{q}_t^{\mathbb{P}}\right)(y) = \frac{,_t^{\mathbb{P}}(y)}{2t^2}\widetilde{q}_t^{\mathbb{P}}(y) + \frac{1}{2}\left(\triangle_{\mathbb{P}}\widetilde{q}_t^{\mathbb{P}}\right)(y). \tag{5.5}
$$

Let $s \to \beta_s$ be a standard Brownian motion on K with parameter t. Then for any continuous ϕ on K^n we have

$$\begin{aligned}
\int \phi(y)\, \tilde{q}_t^{\mathbb{P}}(y)\, dy &= \exp C_t \mu_{0,t}\left[\phi \circ \pi_{\mathbb{P}}\right] \\
&= \exp C_t \mu_t \left[\phi \circ \pi_{\mathbb{P}} \frac{P_{t(1-s_n)}^K \circ \pi_{s_n}}{P_t^K(e)}\right] \\
&= \exp C_t E\left[\phi\left(\beta_{s_1} \cdots \beta_{s_n}\right) \frac{P_{t(1-s_n)}^K\left(\beta_{s_n}\right)}{P_t^K(e)}\right].
\end{aligned}$$

Notice that $s \to \beta_s^{-1}$ is also a standard Brownian motion on K. This means that our previous expression becomes

$$= \exp C_t E\left[\phi\left(\beta_{s_1}^{-1} \cdots \beta_{s_n}^{-1}\right) \frac{P_{t(1-s_n)}^K\left(\beta_{s_n}^{-1}\right)}{P_t^K(e)}\right].$$

Now using the fact that $P_t^K(x) = P_t^K\left(x^{-1}\right)$ on K yields

$$\begin{aligned}
&= \exp C_t E\left[\phi\left(\beta_{s_1}^{-1} \cdots \beta_{s_n}^{-1}\right) \frac{P_{t(1-s_n)}^K\left(\beta_{s_n}\right)}{P_t^K(e)}\right] \\
&= \exp C_t \int \phi\left(\gamma_{s_1}^{-1} \cdots \gamma_{s_n}^{-1}\right) \mu_{0,t}(d\gamma) \\
&= \int \phi\left(y^{-1}\right) \tilde{q}_t^{\mathbb{P}}(y)\, dy \\
&= \int \phi(y)\, \tilde{q}_t^{\mathbb{P}}\left(y^{-1}\right) dy.
\end{aligned}$$

Thus for any continuous ϕ on K^n we have

$$\int \phi(y)\, \tilde{q}_t^{\mathbb{P}}(y)\, dy = \int \phi(y)\, \tilde{q}_t^{\mathbb{P}}\left(y^{-1}\right) dy. \tag{5.6}$$

Using this yields

$$\begin{aligned}
F(t,x) &= \int f(xy)\, \tilde{q}_t^{\mathbb{P}}(y)\, dy \\
&= \int f(xy)\, \tilde{q}_t^{\mathbb{P}}\left(y^{-1}\right) dy \\
&= \int f(y)\, \tilde{q}_t^{\mathbb{P}}\left(y^{-1}x\right) dy.
\end{aligned}$$

Applying Eq. [5.5] to compute the derivative $\partial_t F(t,x)$ yields

$$\begin{aligned}
\partial_t F(t,x) &= \int f(y)\, \partial_t \tilde{q}_t^{\mathbb{P}}\left(y^{-1}x\right) dy \\
&= \int f(y) \frac{{,}_t^{\mathbb{P}}\left(y^{-1}x\right)}{2t^2} \tilde{q}_t^{\mathbb{P}}\left(y^{-1}x\right) dy + \frac{1}{2}\int f(y)\left(\triangle_{\mathbb{P}} \tilde{q}_t^{\mathbb{P}}\right)\left(y^{-1}x\right) dy \\
&= I(t,x) + \frac{1}{2}\int f(y)\left(\triangle_{\mathbb{P}} \tilde{q}_t^{\mathbb{P}}\right)\left(y^{-1}x\right) dy.
\end{aligned}$$

By the left-invariance of the Laplacian, $\triangle_{\mathbb{P}}$ this last expression is

$$\begin{aligned} &= I(t,x) + \frac{1}{2}\int f(y)\left(\triangle_{\mathbb{P}}\widetilde{q}_t^{\mathbb{P}} \circ \ell_{y^{-1}}\right)(x)\,dy \\ &= I(t,x) + \frac{\triangle_{\mathbb{P}}}{2}\int f(y)\,\widetilde{q}_t^{\mathbb{P}}\left(y^{-1}x\right)dy \\ &= I(t,x) + \frac{\triangle_{\mathbb{P}}}{2}\int f(xy)\,\widetilde{q}_t^{\mathbb{P}}\left(y^{-1}\right)dy. \end{aligned}$$

Using Eq. [5.6] a second time yields

$$\begin{aligned} &= I(t,x) + \frac{\triangle_{\mathbb{P}}}{2}\int f(xy)\,\widetilde{q}_t^{\mathbb{P}}(y)\,dy \\ &= I(t,x) + \frac{\triangle_{\mathbb{P}}}{2}F(t,x). \end{aligned}$$

Therefore

$$\partial_t F(t,x) = I(t,x) + \frac{\triangle_{\mathbb{P}}}{2}F(t,x).$$

As t gets small, we have

$$\begin{aligned} \lim_{t\to 0} F(t,x) &= \lim_{t\to 0}\int f(xy)\,\widetilde{q}_t^{\mathbb{P}}(y)\,dy \\ &= \lim_{t\to 0}\exp C_t\int f(xy)\,q_t^{\mathbb{P}}(y)\,dy \\ &= \lim_{t\to 0}\left(\pi_{\mathbb{P}}\right)_*\mu_{0,t}\left[f\circ\ell_x\right] \\ &= f(x) \text{ by Lemma 5.4.} \end{aligned}$$

Thus we have

$$\begin{aligned} \partial_t F(t,x) &= I(t,x) + \frac{\triangle_{\mathbb{P}}}{2}F(t,x) \\ F(t,x) &\to f(x) \text{ as } t\to 0. \end{aligned} \tag{5.7}$$

We are now ready to apply the Duhammel's principle. For any $\eta > 0$, pick $f(x) = \eta + h(x) \geq 0$. Since $f \geq 0$ a.s., $I \geq 0$, since , ${}^{\mathbb{P}}_{t} \circ \pi_{\mathbb{P}}$ is the conditional expectation of the positive function $\left|\int_0^1 \gamma_{ds}\gamma_s^{-1}\right|_{\mathfrak{K}}^2$. Let

$$U(t,x) \equiv (F-H)(t,x).$$

Using the fact that $f(x) = \eta + h(x)$ implies that

$$U(t,x) = \eta\exp C_t + \int h(xy)\left(\tilde{q}_t^{\mathbb{P}} - p_t^{\mathbb{P}}\right)(y)\,dy. \tag{5.8}$$

By Eqs. [5.4] and [5.7] we see that

$$\partial_t U(t,x) = \frac{1}{2}\triangle_{\mathbb{P}}U(t,x) + I(t,x). \tag{5.9}$$

Formally guessing a solution by Duhammel's principle let us define

$$\widetilde{U}(t,x) \equiv \exp\left(\frac{t-\varepsilon}{2}\triangle_{\mathbb{P}}\right)U(\varepsilon,x) + \int_{\varepsilon}^{t}\exp\left(\frac{t-\tau}{2}\triangle_{\mathbb{P}}\right)I(\tau,x)\,d\tau.$$

Then $\widetilde{U}(\varepsilon, x) = U(\varepsilon, x)$ and

$$\begin{aligned}\partial_t \widetilde{U}(t,x) &= \frac{1}{2}\triangle_{\mathbb{P}} \exp\left(\frac{t-\varepsilon}{2}\triangle_{\mathbb{P}}\right) U(\varepsilon, x) + I(t,x) \\ &\quad + \int_{\varepsilon}^{t} \frac{\triangle_{\mathbb{P}}}{2} \exp\left(\frac{t-\tau}{2}\triangle_{\mathbb{P}}\right) I(\tau, x)\, d\tau \\ &= \frac{1}{2}\triangle_{\mathbb{P}} \widetilde{U}(t,x) + I(t,x).\end{aligned}$$

Therefore $U = \widetilde{U}$ and so

$$U(t,x) = \exp\left(\frac{t-\varepsilon}{2}\triangle_{\mathbb{P}}\right) U(\varepsilon, x) + \int_{\varepsilon}^{t} \exp\left(\frac{t-\tau}{2}\triangle_{\mathbb{P}}\right) I(\tau, x)\, d\tau. \tag{5.10}$$

Let t_0 be "the first time that U goes below zero". Explicitly

$$t_0 \equiv \inf\left\{t > 0 | \inf_x U(t,x) < 0\right\}. \tag{5.11}$$

If we can show that $t_0 > 0$ then we can take ε in Eq.[5.10] equal to $t_0/2$. Then since $U(\varepsilon, \cdot)$ and I are non-negative we must have $U(t,x) \geq 0$ for any t and x. Then we can let $\eta \to 0$ in Eq. [5.8] to obtain

$$\int h(xy) \left(\tilde{q}_t^{\mathbb{P}} - p_t^{\mathbb{P}}\right)(y)\, dy \geq 0.$$

Then $\tilde{q}_t^{\mathbb{P}} - p_t^{\mathbb{P}}$ will be non-negative almost surely and so $p_t^{\mathbb{P}} \leq q_t^{\mathbb{P}} \exp C_t$ and we shall be done.

Thus the problem reduces to showing $t_0 > 0$. Suppose $t_0 = 0$. There exist times τ_i, $\tau_i > 0$, $\tau_i \downarrow 0$ as $i \to \infty$ so that $\inf_x U(\tau_i, x) < -1/i$. By the compactness of K^n, for each τ_i there must exist an x_i so that

$$U(\tau_i, x_i) = \inf_x U(\tau_i, x)(\tau_i, x) < -1/i.$$

Thus by compactness, there exist a convergent subsequence of the x_i So without losing generality, suppose $x_i \to x_\infty$ in K^n. Then $(\tau_i, x_i) \to (0, x_\infty)$ in $[0,\infty) \times K^n$ so that $U(\tau_i, x_i) \to (f-h)(x_\infty) = \eta > 0$. But $U(\tau_i, x_i) \leq 0$ for all i which implies that $(f-h)(x_\infty) \leq 0$ giving us our contradiction. Thus $t_0 > 0$ and we are done. ∎

We are now able to return to the proof of Theorem 5.1.

Proof. **of Theorem 5.1**

Let $\{\mathbb{P}_n\}$ be a refining sequence of partitions of $(0,1)$ (i.e. one is not allowed to include the endpoints $0, 1$ in the partition) such that $|\mathbb{P}_n| \to 0$. Since $\mathbb{P}_n$ a refining sequence, $Z_t^{\mathbb{P}_n}$ is a non-negative discrete $\mathfrak{F}^{\mathbb{P}_n}$ martingale, where $\mathfrak{F}^{\mathbb{P}_n} \equiv \sigma\langle \pi_{\mathbb{P}_n}\rangle$.

To make this clear, let $n > m$ and $f \in \mathfrak{F}^{\mathbb{P}_m}$. Then can find smooth functions $F_1 : K^{\mathbb{P}_m} \to \mathbb{R}$ and $F_2 : K^{\mathbb{P}_n} \to \mathbb{R}$ so that $f = F_1 \circ \pi_{\mathbb{P}_m} = F_2 \circ \pi_{\mathbb{P}_n}$. Now

$$\mu_{0,t} Z_t^{\mathbb{P}_n} f = \left(\pi_{\mathbb{P}_n}^* \mu_{0,t}\right) F_2 p_t^{\mathbb{P}_n} / q_t^{\mathbb{P}_n} = \left(\pi_{\mathbb{P}_m}^* \nu_t\right) F_1 = \mu_{0,t} Z_t^{\mathbb{P}_m} f,$$

which shows that $Z_t^{\mathbb{P}_n}$ is a discrete $\mathfrak{F}^{\mathbb{P}_n}$ martingale.

Suppose we can show

$$\sup_n \left\| Z_t^{\mathbb{P}_n} \right\|_{L^2(\mu_{0,t})} < \infty. \tag{5.12}$$

Then we have $\left\{Z_t^{\mathbb{P}_n}\right\}$ is uniformly integrable since

$$\lim_{M\to\infty}\sup_n \mu_{0,t}\left[Z_t^{\mathbb{P}_n}1_{\left\{Z_t^{\mathbb{P}_n}>M\right\}}\right] \le \lim_{M\to\infty}\frac{1}{M}\sup_n \mu_{0,t}\left(Z_t^{\mathbb{P}_n}\right)^2 = 0.$$

The fact that $\left\{Z_t^{\mathbb{P}_n}\right\}$ a uniformly-integrable discrete L^1-martingale implies the following (see Durrett [14]):-

1. $Z_t^{\mathbb{P}_n}$ converges in L^1.
2. If $Z_t \equiv \lim_{n\to\infty} Z_t^{\mathbb{P}_n}$ then $\mu_{0,t}\left(Z_t|\mathfrak{F}^{\mathbb{P}_n}\right) = Z_t^{\mathbb{P}_n}$.

But now $\mu_{0,t}\left[Z_t f\circ\pi_{\mathbb{P}_n}\right] = \nu_t f\circ\pi_{\mathbb{P}_n}$ for any $n\in N$. Hence Z_t must be $d\nu_t/d\mu_{0,t}$ on the desired σ-algebra $\langle x_t : \gamma_\cdot \to \gamma_t : t\in[0,1]\rangle$.

So the problem reduces to proving Eq. (5.12).

To this end, pick an arbitrary partition $\mathbb{P}\equiv\{0<s_1<\cdots<s_n<1\}$ and let f be smooth on $K^{\mathbb{P}}$.We want to compute

$$\left\|Z_t^{\mathbb{P}}\right\|^2_{L^2(\mu_{0,t})} = \pi^*_{\mathbb{P}_n}\mu_{0,t}\left(p_t^{\mathbb{P}}/q_t^{\mathbb{P}}\right)^2.$$

But by Lemma 5.5, $p_t^{\mathbb{P}}/q_t^{\mathbb{P}} \le \exp C_t$ where C_t is finite and defined by Lemma 5.5. We have a stronger condition than Eq. (5.12). Hence we are done. Moreover $Z_t < \exp C_t$. ∎

6. Semi-Martingale Properties of $g_{T,\cdot}$

Let $\Omega = C([0,1]\to L(K))$ be our probability space, let P be the law of a Brownian motion on $L(K)$, and let $g_t:\Omega\to L(K)$ be the evaluation map at t as in Definition 2.22. Then $t\to g_t$ is an $L(K)$-valued Brownian motion and thus *Law* g_t equals Heat Kernel measure $\nu_t(e,\cdot)$.

Remark 6.1 ($g_{t\cdot}$ is a semimartingale). In Section 5 we showed that Heat kernel measure $\nu_t(e,\cdot)$ is absolutely continuous with respect to pinned Wiener measure $\mu_{0,t}$. Let $\gamma_s : L(K)\to K$ be the evaluation map at time s. Then equipping $L(K)$ with pinned Wiener measure $\mu_{0,t}$, we see that $s\to\gamma_s$ is a Brownian bridge and thus a semimartingale. Since $\nu_t(e,\cdot)\ll\mu_{0,t}$ and $s\to\gamma_s$ is a $\mu_{0,t}$-semimartingale, we know (see Theorem 2, page 45 of [30]) that $s\to\gamma_s$ is a $\nu_t(e,\cdot)$-semimartingale. Now the random variables $(\gamma,\nu_t(e,\cdot))$ and (g_t,P) share the same law. Therefore, by Definition 2.24 we see that $s\to g_{ts}$ is an $\mathfrak{F}_{t\cdot}$-semimartingale.

In this section we provide an explicit decomposition for the $\mathfrak{F}_{ts}$-semimartingale $s\mapsto g_{ts}$ and compute its pullback $\int_0^s g_{t\delta\sigma}g_{t\sigma}^{-1}$. We do this by approximating $g_{t\cdot}$ by the piecewise C^1 functions $g_{t\cdot}^{\mathbb{P}}$ (described in Definition 6.9). Then we compute the approximate pullback $\int_0^\cdot\left(\partial_s g_{ts}^{\mathbb{P}}\right)\left(g_{ts}^{\mathbb{P}}\right)^{-1}ds$ (which is a semimartingale since it is piecewise C^1). Then as a result of Propositions 6.13, 6.19, 6.20, and 6.21 we show that these approximations converge to a process Y_{ts}. In Theorem 6.11 we then show that g satisfies $g_{t\cdot} = 1+\int_0^\cdot Y_{t\delta s}g_{ts}$.

We care about the pullback $Y_{t\cdot}$ because we will show in Section 7 that it has a law to equivalent to that of a Brownian motion on a restricted σ-algebra. This will then imply that Pinned Wiener measure is absolutely continuous with Heat Kernel measure on $\sigma\langle\gamma_s|s\in[0,1-\varepsilon)\rangle$ for any $\varepsilon>0$ (Recall γ_s is evaluation at time s).

6.1. Preliminaries.

6.1.1. *Two parameter processes.* We will need to introduce two-parameter stochastic integration in order to state our main result, Theorem 6.11.

Remark 6.2 (The Brownian sheet b generates the filtration). From Definition 2.22 we see that $\mathfrak{F}_{ts} = \sigma \langle g_{\tau\sigma} | \tau \leq t \text{ or } \sigma \leq s\rangle \vee \mathfrak{F}_{00}$. From Theorem 2.25 we see that

$$g \text{ satisfies } g_{\delta ts} = g_{ts} X_{\delta ts} \text{ with } g_{0s} = e \text{ and } X_{ts} = \int g_{\delta ts} g_{ts}^{-1}.$$

Therefore g_{ts} is in the σ-algebra generated by the random variables

$$\langle X_{\tau\sigma} : (\tau, \sigma) \in [0, t] \times [0, s]\rangle,$$

and X_{ts} is in the σ-algebra generated by the variables

$$\langle g_{\tau\sigma} : (\tau, \sigma) \in [0, t] \times [0, s]\rangle.$$

Again by Theorem 3.19, b_{ts} is in the σ-algebra generated by the

$$\langle X_{\tau\sigma} : (\tau, \sigma) \in [0, t] \times [0, s]\rangle,$$

while X_{ts} is in the σ-algebra generated by

$$\langle b_{\tau\sigma} : (\tau, \sigma) \in [0, t] \times [0, s]\rangle.$$

Therefore $\mathfrak{F}_{ts} = \sigma \langle b_{\tau\sigma} | \tau \leq t \text{ or } \sigma \leq s\rangle \vee \mathfrak{F}_{00}$. This observation is important to use the results of Cairoli and Walsh in [7].

Definition 6.3 (Cairoli & Walsh [7])**.** We will use the following notions from Cairoli & Walsh:-

1. Let $(\Omega, \{\mathfrak{F}_{ts}\})$ be our probability space from Definition 2.22.
2. $\mathfrak{F}^1_{ts} \equiv \mathfrak{F}_{t1} \vee \mathfrak{F}_{1s}$. This is the σ-algebra generated by $\langle b_{\tau\sigma} | \tau \leq t \text{ or } \sigma \leq s\rangle \vee \mathfrak{F}_{00}$ from Remark 6.2.
3. Let b be the Brownian sheet from Theorem 3.19 and let $b^A \equiv \langle b, A\rangle_{\mathfrak{K}}$ for any $A \in \mathfrak{K}$.
4. For $t_1 s_1$ $t_1 < t_2$ and $s_1 < s_2$, let $(t_1 s_1, t_2 s_2]$ denote the rectangle $(t_1, t_2] \times (s_1, s_2]$.
5. $R_{ts} \equiv (0, t] \times (0, s] = (00, ts]$.
6. Let $\mathcal{L}$ be the set of all $\mathbb{R}$-valued processes $\phi : [0,1] \times [0,1] \times \Omega$, so that $\phi_{ts}(\cdot) \equiv \phi(t, s, \cdot)$ is $\mathfrak{F}_{ts}$-measurable and the expectation $E \int_0^1 \int_0^1 \phi^2_{ts}(g)\, dsdt < \infty$.
7. For any $\phi \in \mathcal{L}$, define $\phi((t_1 s_1, t_2 s_2]) \equiv \phi_{t_2 s_2} - \phi_{t_2 s_1} - \phi_{t_1 s_2} + \phi_{t_1 s_1}$.
8. A two-parameter process $M \in \mathcal{L}$ such that M vanishes on the axes (i.e. $M_{0t} = M_{s0} = 0$ P-a.s..) is a strong martingale if $E\left(M(t_1 s_1, t_2 s_2] | \mathfrak{F}^1_{t_1 s_1}\right) = 0$ for any $(t_1 s_1, t_2 s_2] \subset R_{11}$.
9. For $t_1 < t_2$ and $s_1 < s_2$, let $\phi \in \mathcal{L}$ be characteristic if $\phi(t, s) = f(\omega) 1_{(t_1 s_1, t_2 s_2]}$ with $f \in \mathfrak{F}^1_{t_1 s_1}$.
10. Let $\phi \in \mathcal{L}$ be simple if it is a linear combination of characteristic functions.
11. For characteristic processes $\phi_{ts}(g) = f(g) 1_{(t_1 s_1, t_2 s_2]}$, define the integral
$$\int b^A_{dtds} \phi_{ts} \equiv f(g)\, b((t_1 s_1, t_2 s_2]).$$
We can do this because $(t_1 s_1, t_2 s_2] \cap R_{11}$ is a rectangle of the form $(\tau\sigma, ts]$.
12. Extend the definition of $\int b^A_{dtd\sigma}$ to simple functions by linearity.

We shall need the following results of Cairoli & Walsh which we state without proof. They are to be found in [7].

Theorem 6.4 (Cairoli & Walsh [7]). 1. *For any simple $\phi \in \mathcal{L}$,*

$$E\left|\int b^A_{dtds}\phi_{ts}(g)\right|^2 = E\left|\int_0^1\int_0^1 \phi_{ts}(g)\,dtds\right|^2 |A|^2_{\mathfrak{K}}.$$

2. *$\int b^A_{dtds}$ provides an isometry between $\mathcal{L}$ and $L^2(\Omega)$ and we can extend the definition of $\int b^A_{dtds}$ to all of $\mathcal{L}$ via this isometry.*
3. *For any $\phi \in \mathcal{L}$, the process*

$$M_{ts} \equiv \int_{R_{ts}} b^A_{d\tau d\sigma}\phi_{\tau\sigma} \equiv \int b^A_{d\tau d\sigma}\phi_{\tau\sigma}1_{R_{ts}}(\tau,\sigma)$$

is a strong martingale.

4. *For any strong martingale M and for any $p \geq 1$, we have*

$$E\sup_{\tau\leq t}\sup_{\sigma\leq s}|M_{\tau\sigma}|^p \leq \left(\frac{p}{p-1}\right)^{2p}\sup_{\tau\leq t}\sup_{\sigma\leq s}E|M_{\tau\sigma}|^p.$$

6.1.2. *The Norm $\mathcal{H}^2(s)$.*

Definition 6.5 (The Hilbert-Schmidt norm $\|\cdot\|_{HS}$ on $\mathcal{M}_m(\mathbb{R})$). Let $\mathcal{M}_m(\mathbb{R})$ and $GL_m(\mathbb{R})$ be as in Remark 2.18. The Hilbert-Schmidt norm $\|\cdot\|_{HS}$ of a matrix $k \in \mathcal{M}_m(\mathbb{R})$ with i,j-entries k_{ij} is given by

$$\|k\|_{HS} = \left(\sum_{i=1}^m\sum_{j=1}^m k_{ij}^2\right)^{1/2}.$$

[Note:- By the equivalence of norms on a linear space in finite dimensions, $\|\cdot\|_{HS}$-convergence on $\mathfrak{K}$ is the same as $\langle\cdot,\cdot\rangle_{\mathfrak{K}}$-convergence on $\mathfrak{K}$.]

The next couple of definitions have been adapted for our purposes from Chap. IV of Protter, [30].

Definition 6.6 (Very Special Semimartingales). A semimartingale in the sense of Definition 2.24 is very special if it can be written as $M + \int \nu(\sigma)d\sigma$ where M is a mean-zero martingale and ν is an adapted process. [Such a decomposition is always unique see Theorem 18, Chap III of [30]]. An $\mathbb{R}^d$-valued semimartingale is very special if its individual coordinates are very special. In particular, viewing $\mathcal{M}_m(\mathbb{R})$ as $\mathbb{R}^{m\times m}$ we have a definition for very special $\mathcal{M}_m(\mathbb{R})$-valued semimartingales.

Definition 6.7 (The norm $\mathcal{H}^2(s)$ on $\mathcal{M}_m(\mathbb{R})$-semimartingales). Let $\mathcal{M}_m(\mathbb{R})$ denote $m\times m$ matrices as in Definition 6.5. By Remark 2.18 $K \subset GL_m(\mathbb{R}) \subset \mathcal{M}_m(\mathbb{R})$. Let R be a very special $\mathcal{M}_m(\mathbb{R})$-semimartingale which is Doob-decomposable as $M + \int \nu(\sigma)d\sigma$.

$$\begin{aligned}\|R\|_{\mathcal{H}^2(s)} &\equiv \|M_s\|_{L^2} + \left\|\int_0^s \|\nu(\sigma)\|_{HS}\,d\sigma\right\|_{L^2} \\ &= \left(E\|M_s\|^2_{HS}\right)^{\frac{1}{2}} + \left(E\left(\int_0^s\|\nu(\sigma)\|_{HS}\,d\sigma\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

Theorem 6.8 (Closure of very special semimartingales under $\|\cdot\|_{\mathcal{H}^2(s)}$). *Let $\mathcal{H}^2(s)$ be the space of all very special $\mathcal{M}_m(\mathbb{R})$-semimartingales R so that $\|R\|_{\mathcal{H}^2(s)} < \infty$. Then $\mathcal{H}^2(s)$ is a Banach space.*

Proof. Clearly $\left(\mathcal{H}^2(s), \|\cdot\|_{\mathcal{H}^2(s)}\right)$ is a normed vector space. It will suffice to check completeness. Let $R_n = M_n + \int \nu_n(u)\, du$ be Cauchy. Then M_n is Cauchy in L^2 and so converges to some martingale M. $\int \nu_n(u)\, du$ is also Cauchy in $\|\cdot\|_{\mathcal{H}^2(s)}$. To show it converges, it will suffice to show a subsequence converges and so without losing generality we assume that

$$\begin{aligned} \infty &> \sum_n \sqrt{E\left(\int_0^s \|\nu_{n+1}(u) - \nu_n(u)\|_{HS}\, du\right)^2} \\ &> E\int_0^s \sum_n \|\nu_{n+1}(u) - \nu_n(u)\|_{HS}\, du \end{aligned}$$

Thus on a set $\mathcal{A} \subset \Omega$ of measure 1, we see that the expression

$$\int_0^s \sum_n \|\nu_{n+1}(u) - \nu_n(u)\|_{HS}\, du < \infty.$$

Thus on $\mathcal{A}$, $\sum_n \|\nu_{n+1}(u) - \nu_n(u)\|_{HS} < \infty$ a.s.-du. Thus on $\mathcal{A}$ there exists some random variable $\nu(u)$ so that, $\nu_n(u) \to \nu(u)$ a.s.-du. ν_n is adapted so ν is adapted as well. hence we are done. ∎

6.1.3. *The approximation scheme.*

Definition 6.9. We will proceed to define some basic terms we will need to state the main Theorem of this section. Let g be the $L(K)$-valued Brownian motion from Definition 2.22. Let X be the $\mathfrak{K}$-valued Brownian bridge sheet given by the Fisk-Stratonowicz integral $X_{ts} = \int_0^t g_{\tau s}^{-1} g_{\delta\tau s}$ in Theorem 2.25. Then:-

1. $\mathbb{P}$ a partition $\{0 = t_0 < \ldots < t_n = T\}$.
2. $$X^{\mathbb{P}}(t,s) \equiv X_{t_{i-1}s}\frac{t_i - t}{t_i - t_{i-1}} + X_{t_i s}\frac{t - t_{i-1}}{t_i - t_{i-1}} \forall t \in (t_{i-1}, t_i].$$
3. $\Delta_j X(s) \equiv X_{t_j s} - X_{t_{j-1}s}$ and $\Delta_j X^A(s) \equiv \langle \Delta_j X(s), A\rangle_{\mathfrak{K}}$ for any $A \in \mathfrak{K}$.
4. $g^{\mathbb{P}}(t,s)$ be defined to be the solution of
$$\partial_t g^{\mathbb{P}}(t,s) = g^{\mathbb{P}}(t,s)\partial_t X^{\mathbb{P}}(t,s) \text{ and } g^{\mathbb{P}}(0,s) = 1.$$
Observe that for any $t \in (t_{i-1}, t_i]$ the this equation reduces to
$$\partial_t g^{\mathbb{P}}(t,s) = g^{\mathbb{P}}(t,s)\frac{\triangle_i X(s)}{\triangle_i t}.$$
5. $\mathcal{M}_m(\mathbb{R})$ be the set of all $m \times m$ matrices (see Remark 2.18).
6. Define $G : GL_m(\mathbb{R}) \to GL_m(\mathbb{R})$ by setting $G(A) \equiv A^{-1}$. Recall from Remark 2.18 that $GL_m(\mathbb{R})$ denotes invertible $m \times m$ matrices.
7. $F : \mathcal{M}_m(\mathbb{R}) \to \mathcal{M}_m(\mathbb{R})$ be the exponential map
$$F : A \to \sum A^n/n!.$$
Notice that for any $A \in \mathfrak{K}$, we have $\exp A = F(A)$ where $\exp : \mathfrak{K} \to K$ is the intrinsic exponential map on K.
8. $y_i \equiv g^{\mathbb{P}}(t_i, s) = F(\triangle_1 X(s)) \cdots F(\triangle_i X(s))$.
9. $B^{\mathbb{P}}(T,s) \equiv \int_0^s g^{\mathbb{P}}(T, \delta s) g^{\mathbb{P}}(T,s)^{-1}$.

Lemma 6.10. *Recall that $K \subset GL_m(\mathbb{R})$ as in Remark 2.18. Let F and G be transformations of $GL_m(\mathbb{R})$ as in Definition 6.9. Then the following relations hold where $A \in U$ and $B, C \in \mathfrak{K}$:-*

1.
$$F'(A)B = \int_0^1 F\left[(1-\tau)A\right] BF\left[\tau A\right] d\tau. \tag{6.1}$$

2.
$$\begin{aligned} &F''(A)B \otimes C \\ &\quad = \int_0^1 d\tau \int_0^1 (1-u) F\left[(1-\tau)(1-u)A\right] \\ &\qquad \times CF\left[\tau(1-u)A\right] BF\left[uA\right] du \\ &\qquad + \int_0^1 d\tau \int_0^1 uF\left[(1-u)A\right] \\ &\qquad \times BF\left[(1-\tau)uA\right] CF\left[\tau uA\right] du. \end{aligned} \tag{6.2}$$

3.
$$G'(A)B = -A^{-1}BA^{-1}. \tag{6.3}$$

4.
$$G''(A)B \otimes C = A^{-1}BA^{-1}CA^{-1} + A^{-1}CA^{-1}BA^{-1}. \tag{6.4}$$

5.
$$\sup_{A \in \mathfrak{K}} \left\| F'(A)B \right\|_{HS} \le Const \left\| B \right\|_{HS}. \tag{6.5}$$

6.
$$\sup_{A \in \mathfrak{K}} \left\| F^{(n)}(A) B_1 \otimes \cdots \otimes B_n \right\|_{HS} \le Const \left\| B_1 \right\|_{HS} \cdots \left\| B_n \right\|_{HS}. \tag{6.6}$$

Proof. See Lemma 8.8 in the appendix. ∎

6.2. **The Main Theorem .**

Theorem 6.11 (Semimartingale properties of $g_{T\cdot}$)**.** *Let g be a $L(K)$-valued Brownian motion. Then:-*

1. *$s \to g_{Ts}$ is a K-valued $\mathfrak{F}_{Ts}$-semimartingale.[Note:-In Remark 6.1 we have already reached the conclusion that $g_{T\cdot}$ was a semimartingale. We provide another independent proof since this fact is an easy consequence of our computation.]*
2.
$$\int_0^s g_{T\delta\sigma} g_{T\sigma}^{-1} = \int_{R_{T_s}} Ad_{g_{t\sigma}} b_{dtd\sigma} - \int_0^s \frac{d\sigma}{1-\sigma} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma},$$

   where the expression $\int_{R_{T_s}} Ad_{g_{t\sigma}} b_{dtd\sigma}$ is defined as in Theorem 6.4.

The proof of this Theorem will be given after the proof of Theorem 6.17.

Remark 6.12. **(Theorem 6.11 is reasonable)**g satisfies

$$g_{\delta ts} = g_{ts} X_{\delta ts} \text{ with } g_{0s} = e, \tag{6.7}$$

where $X_{\cdot s}$ is the Brownian bridge sheet from Theorem 2.25. By Theorem 3.19, there is a Brownian sheet b on $\mathfrak{K}$ so that

$$X_{ts} = b_{ts} - \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma. \tag{6.8}$$

If we replace X by b in Eq. (6.7), then Lemma 3.9 shows that $s \to g_{Ts}$ would be a K-valued Brownian motion with variance T and hence $\int_0^\cdot g_{T\delta s} g_{Ts}^{-1}$ would be a $\mathfrak{K}$-valued Brownian motion with variance T. In reality, because $X_{t\cdot}$ contains an extra finite-variation term, it turns out that the law of $Y_{T\cdot}$ is equivalent (but not equal) to the law of a Brownian motion on $\mathfrak{K}$.

Define

$$Y_{Ts} \equiv \int_{R_{Ts}} Ad_{g_{t\sigma}} b_{dtd\sigma} - \int_0^s \frac{d\sigma}{1-\sigma} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma}. \tag{6.9}$$

In the proof of Theorem 6.11 we will show that $g_{T\delta s} = Y_{T\delta s} g_{Ts}$ with $Y_{T0} = 0$. Before we do that we shall need to state a few results.

Proposition 6.13 (Semimartingale decomposition of $B^{\mathbb{P}}_{T\cdot}$)**.** *As in Definition 6.9 let $\mathbb{P}$ be a partition of $[0.T]$ and let*

$$B^{\mathbb{P}}_T(s) \equiv B^{\mathbb{P}}(T,s) = \int_0^s g^{\mathbb{P}}(T,\delta s) g^{\mathbb{P}}(T,s)^{-1}.$$

Define

$$M^{\mathbb{P}}_T(s) \equiv \int_0^s \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}(\sigma) \left(F'(\Delta_i X) d\Delta_i b(\sigma)\right) y_i^{-1}(\sigma), \tag{6.10}$$

$$\nu^{\mathbb{P}}_1(T,s) \equiv -\frac{1}{1-s} \sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}} \Delta_i X(s), \tag{6.11}$$

$$\begin{aligned} \nu^{\mathbb{P}}_2(T,s) \equiv & \sum_{i=1}^{n_{\mathbb{P}}} \sum_A \frac{(\Delta_i t)}{2} y_{i-1} \left(F''(\Delta_i X) A^{\otimes 2}\right) y_i^{-1} \\ & - \sum_{i=1}^{n_{\mathbb{P}}} \sum_A \frac{(\Delta_i t)}{2} \left(y_{i-1} \left(F'(\Delta_i X) A\right) y_i^{-1}\right)^2. \end{aligned} \tag{6.12}$$

Then

$$B^{\mathbb{P}}_T(s) = M^{\mathbb{P}}_T(s) + \int_0^s \nu^{\mathbb{P}}_1(T,\sigma) d\sigma + \int_0^s \nu^{\mathbb{P}}_2(T,\sigma) d\sigma. \tag{6.13}$$

This Proposition is proved in subsection 6.4.

Remark 6.14 (Idea of the proof of Theorem 6.11). Given Theorem 6.13 we can indicate the idea of the proof of Theorem 6.11. Roughly speaking we have the following approximations

$$\begin{aligned} \nu^{\mathbb{P}}_2(T,s) &\cong \sum_{i=1}^{n_{\mathbb{P}}} \frac{(\Delta_i t)}{2} y_{i-1} \left(F''(\Delta_i X) \sum_A A^{\otimes 2}\right) y_{i-1}^{-1} - \sum_{i=1}^{n_{\mathbb{P}}} \sum_A \frac{(\Delta_i t)}{2} \left(y_{i-1} \left(F'(\Delta_i X) A\right) y_{i-1}^{-1}\right)^2 \\ &= \sum_{i=1}^{n_{\mathbb{P}}} \sum_A \frac{(\Delta_i t)}{2} y_{i-1} \left(F''(\Delta_i X) A^{\otimes 2} - \left(F'(\Delta_i X) A\right)^2\right) y_{i-1}^{-1} \\ &\cong \sum_{i=1}^{n_{\mathbb{P}}} \sum_A \frac{(\Delta_i t)}{2} y_{i-1} \left(F''(0) A^{\otimes 2} - \left(F'(0) A\right)^2 + O(\Delta_i X)\right) y_{i-1}^{-1} \cong 0. \end{aligned}$$

Also one expects

$$M_T^{\mathbb{P}}(s) \equiv \int_0^s \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}(\sigma)\left(F'(\Delta_i X) d\Delta_i b(\sigma)\right) y_i^{-1}(\sigma) \to \int_0^s \int_0^T Ad_{g(t,\sigma)} b(dt, d\sigma)$$

and

$$\nu_1^{\mathbb{P}}(T,s) \equiv -\frac{1}{1-s} \sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}} \Delta_i X(s) \to -\frac{1}{1-s} \int_0^T Ad_{g(t,s)} X(dt, s)$$

as $|\mathbb{P}| \to 0$.

Lemma 6.15. *Let $z < 1$. Let $\alpha \in [-1,1]$. Then there exists a sequence of partitions $\{\mathbb{P}_r^z\}$ of $[0,T]$, depending only on T and z with the following properties:-*

1. $|\mathbb{P}_r^z| \downarrow 0$ *as* $r \to \infty$
2. $$\sup_{\{\mathbb{P}_r^z\}} \sup_{\sigma \in [0,z]} \sum_{i=1}^{n_{\mathbb{P}_r^z}} \| F(\alpha \Delta_i X(\sigma)) - 1 \|_{HS}^2 < \infty \ P\text{-}a.s.$$
3. *As* $r \to \infty$,
$$\left\| \sup_{t \in [0,T]} \left\| g^{\mathbb{P}_r^z}(t,s) - g_{ts} \right\|_{HS} \right\|_{L^p} \to 0 \ \forall p \in [1,\infty), T < \infty.$$

This result is proved in subsection 6.5.

Theorem 6.16. *Let $B_{T\cdot}^{\mathbb{P}_r^z}$ be the approximation to $\int_0^{\cdot} g_{T\delta s} g_{Ts}^{-1}$ as in Definition 6.9. Let $M_{T\cdot}^{\mathbb{P}_r^z}$ be the martingale part of $B_{T\cdot}^{\mathbb{P}_r^z}$ as in Proposition 6.13. Let $M_{T\cdot} \equiv \int_{R_{T\cdot}} Ad_{g_{ts}} b_{dtds}$ be the martingale part of $Y_{T\cdot}$ as in the proof of Theorem 6.11. Then as $r \to \infty$ the expression*

$$\int_0^{\cdot} M_{Td\sigma}^{\mathbb{P}_r^z} M_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} \to \int_0^{\cdot} M_{Td\sigma} M_{Td\sigma} g_{T\sigma} \text{ in } \mathcal{H}^2(z).$$

Theorem 6.17. *Let $t \mapsto g_{t\cdot}$ be the canonical $L(K)$-valued Brownian motion from Definition 2.22. Let $Y_{T\cdot}$ be as in the proof of Theorem 6.11. Let $B_{T\cdot}^{\mathbb{P}_r^z}$ be the approximation to $\int_0^{\cdot} g_{T\delta s} g_{Ts}^{-1}$ as in Definition 6.9. Then as $r \to \infty$, we have*

$$\int_0^{\cdot} B_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} \to \int_0^{\cdot} Y_{Td\sigma} g_{T\sigma} \text{ in } \mathcal{H}^2(z).$$

Proof of Theorem 6.11. Since $\sigma \to g_{T\sigma}$ is bounded and continuous, the integral

$$\int_0^{\cdot} Y_{Td\sigma} g_{T\sigma} + \frac{1}{2} Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma}$$

is well-defined. We have only to show that for any $z < 1$,

$$g_{Ts}^{\mathbb{P}_r^z} \to 1 + \int_0^s Y_{Td\sigma} g_{T\sigma} + \frac{1}{2} Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma} \text{ in } \mathcal{H}^2(z) \text{ as } r \to \infty.$$

If this is done, Lemma 6.15 will imply that

$$g_{Ts} = 1 + \int_0^s Y_{Td\sigma} g_{T\sigma} + \frac{1}{2} Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma}.$$

This will mean that $g_{T\cdot}$ is a semimartingale and that

$$\begin{aligned} Y_{T\delta s} g_{Ts} &= Y_{Tds} g_{Ts} + \frac{1}{2} Y_{Tds} g_{Tds} \\ &= Y_{Tds} g_{Ts} + \frac{1}{2} Y_{Tds} Y_{Tds} g_{Ts} \\ &= g_{T\delta s}. \end{aligned}$$

This will prove the Theorem completely.

So will suffice to prove, for any $z < 1$,

$$g_{Ts}^{\mathbb{P}_r^z} \to 1 + \int_0^s Y_{Td\sigma} g_{T\sigma} + \frac{1}{2} Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma} \text{ in } \mathcal{H}^2(z) \text{ as } r \to \infty.$$

Doob-decompose Y_{Ts} as $M_{Ts}^{\mathbb{P}_r^z} + \int_0^s \nu^{\mathbb{P}_r^z}(T,\sigma) d\sigma$. From Definition 6.9, notice that $g_{T\delta s}^{\mathbb{P}_r^z}$ solves

$$g_{T\delta s}^{\mathbb{P}_r^z} = B_{T\delta s}^{\mathbb{P}_r^z} g_{Ts}^{\mathbb{P}_r^z} \text{ with } g_{T0}^{\mathbb{P}_r^z} = 1.$$

Now let $M_{T\cdot}^{\mathbb{P}_r^z}$ be the martingale part of $M_{T\cdot}^{\mathbb{P}_r^z}$ and let J denote

$$\left\| g_{T\cdot}^{\mathbb{P}_r^z} - \left(1 + \int_0^{\cdot} Y_{Td\sigma} g_{T\sigma} + \frac{1}{2} Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma} \right) \right\|_{\mathcal{H}^2(z)}.$$

We see that

$$\begin{aligned} J &= \left\| \int_0^{\cdot} \left(B_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} - Y_{Td\sigma} g_{T\sigma} \right) + \frac{1}{2} \left(B_{Td\sigma}^{\mathbb{P}_r^z} B_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} - Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma} \right) \right\|_{\mathcal{H}^2(z)} \\ &= \left\| \int_0^{\cdot} \left(B_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} - Y_{Td\sigma} g_{T\sigma} \right) + \frac{1}{2} \left(M_{Td\sigma}^{\mathbb{P}_r^z} M_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} - Y_{Td\sigma} Y_{Td\sigma} g_{T\sigma} \right) \right\|_{\mathcal{H}^2(z)} \end{aligned}$$

By Theorem 6.16,

$$\int_0^{\cdot} M_{Td\sigma}^{\mathbb{P}_r^z} M_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} \to \int_0^{\cdot} M_{Td\sigma} M_{Td\sigma} g_{T\sigma} \text{ in } \mathcal{H}^2(z).$$

By Theorem 6.17,

$$\int_0^{\cdot} B_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} \to \int_0^{\cdot} Y_{Td\sigma} g_{T\sigma} \text{ in } \mathcal{H}^2(z).$$

Hence we are done. ∎

Remark 6.18. By Eq. (3.5),

$$X_{ts} = b_{ts} - \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma.$$

Computing informally with Ito's Lemma and using Eq. (6.9),

$$X_{ts} = b_{ts} - \int_0^s b_{t\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma.$$

$$X_{dts} = b_{dts} - \int_0^s b_{dt\sigma} \frac{(1-s)}{(1-\sigma)^2} d\sigma.$$

$$\begin{aligned} X_{dtds} &= b_{dtds} - \frac{b_{dts} ds}{(1-s)} + \int_0^s b_{dt\sigma} \frac{1}{(1-\sigma)^2} d\sigma ds \\ &= b_{dtds} - \frac{X_{dts} ds}{(1-s)}. \end{aligned}$$

$$\int_{R_{Ts}} Ad_{g_{t\sigma}} X_{dtd\sigma} = \int_{R_{Ts}} Ad_{g_{t\sigma}} b_{dtd\sigma} - \int_0^s \frac{d\sigma}{(1-\sigma)} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma}.$$

$$\int_{R_{Ts}} Ad_{g_{t\sigma}} X_{dtd\sigma} = Y_{Ts}.$$

Thus we see that in spirit, Y_{Ts} equals $\int_{R_{Ts}} Ad_{g_{t\sigma}} X_{dtd\sigma}$.

$$\text{In future, } \textbf{define} \int_{R_{Ts}} Ad_{g_{t\sigma}} X_{dtd\sigma} = Y_{Ts}.$$

6.3. **Proof of Theorems 6.16 and 6.17.** We will need the following three Propositions (in addition to Proposition 6.13) in the proof of Theorems 6.16 and 6.17.

Proposition 6.19. *Let $z < 1$ and let $\{\mathbb{P}_r^z\}$ be the sequence of partitions from Lemma 6.15 and let $M_T^{\mathbb{P}_r^z}(\cdot)$ be the martingale part of $B_{T\cdot}^{\mathbb{P}_r^z} = \int_0^s g_{T\delta\sigma}^{\mathbb{P}_r^z} \left(g_{T\sigma}^{\mathbb{P}_r^z}\right)^{-1}$. Then $M_T^{\mathbb{P}_r^z}(s)$ converges in L^2 as $r \to \infty$ to $\int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma}$. Furthermore the process $s \to \int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma}$ is a Brownian motion on $\mathfrak{K}$ with variance T.*

Proposition 6.20. *Let $z < 1$. Let $\{\mathbb{P}_r^z\}$ be the sequence of partitions in Lemma 6.15. Let $\nu_2^{\mathbb{P}_r^z}(T,\cdot)$ be as in Proposition 6.13. Let convergence in $\mathcal{H}^2(z)$ be defined as in Definition 6.7. Then*

$$\int_0^{\cdot} \nu_2^{\mathbb{P}_r^z}(T,\sigma)d\sigma \to 0 \text{ as } |\mathbb{P}_r^z| \downarrow 0 \text{ in } \mathcal{H}^2(z).$$

Proposition 6.21. *Let $z < 1$ and let convergence in $\mathcal{H}^2(z)$ be as in Definition 6.7. $\nu_1^{\mathbb{P}_r^z}(T,\cdot)$ be as in Proposition 6.13. Then as $r \to \infty$ we have.*

$$\int_0^{\cdot} \nu_1^{\mathbb{P}_r^z}(T,\sigma)d\sigma \to -\int_0^{\cdot} \frac{d\sigma}{1-\sigma} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma} \text{ in } \mathcal{H}^2(z).$$

Proof of Theorem 6.16. By Eq. (6.10)

$$M_T^{\mathbb{P}_r^z}(s) \equiv \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}(\sigma)\left(F'(\Delta_i X) d\Delta_i b(\sigma)\right) y_i^{-1}(\sigma).$$

which implies

$$\begin{aligned} &M_{Tds}^{\mathbb{P}_r^z} M_{Tds}^{\mathbb{P}_r^z} \\ &\quad= \sum_{i,j=1}^{n_{\mathbb{P}_r^z}} y_{i-1}(s)\left(F'(\Delta_i X)\Delta_i b(ds)\right) y_i^{-1}(s)\, y_{j-1}(s)\left(F'(\Delta_j X)\Delta_j b(ds)\right) y_j^{-1}(s) \\ &\quad= \sum_A \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}(s)\left(F'(\Delta_i X)A\right)\left(y_i^{-1} y_{i-1}\right)(s)\left(F'(\Delta_i X)A\right) y_i^{-1}(s)\,\Delta_i t ds. \end{aligned}$$

By Definition 6.9, $y_i = y_{i-1}F(\triangle_i X(s))$, so we have $y_i^{-1}y_{i-1} = F(-\triangle_i X(s))$. Thus

$$M_{Tds}^{\mathbb{P}_r^z} M_{Tds}^{\mathbb{P}_r^z} = \sum_A \sum_{i=1} Ad_{y_{i-1}}\left[\left(F'(\Delta_i X)A\right)F(-\triangle_i X)\right]^2 \Delta_i t ds. \tag{6.14}$$

Also, by Theorem 6.19,

$$M_{Tds} M_{Tds} = T ds \sum_A A^2 = ds \sum_A \sum_{i=1}^{n_{\mathbb{P}_r^z}} \left(Ad_{y_{i-1}} A^2\right)\Delta_i t. \tag{6.15}$$

Then using Equations (6.14) and (6.15) we have

$$\begin{aligned}
&\left\| \int_0^{\cdot} M^{\mathbb{P}^z_r}_{Td\sigma} M^{\mathbb{P}^z_r}_{Td\sigma} g^{\mathbb{P}^z_r}_{T\sigma} - M_{Td\sigma} M_{Td\sigma} g_{T\sigma} \right\|_{\mathcal{H}^2(z)} \\
&\quad = \left\| \begin{array}{c} \int_0^{\cdot} \left(M^{\mathbb{P}^z_r}_{Td\sigma} M^{\mathbb{P}^z_r}_{Td\sigma} - M_{Td\sigma} M_{Td\sigma} \right) g^{\mathbb{P}^z_r}_{T\sigma} \\ + \int_0^{\cdot} M_{Td\sigma} M_{Td\sigma} \left(g^{\mathbb{P}^z_r}_{T\sigma} - g_{T\sigma} \right) \end{array} \right\|_{\mathcal{H}^2(z)} \\
&\quad \le \left\| \int_0^{\cdot} \left(M^{\mathbb{P}^z_r}_{Td\sigma} M^{\mathbb{P}^z_r}_{Td\sigma} - M_{Td\sigma} M_{Td\sigma} \right) g^{\mathbb{P}^z_r}_{T\sigma} \right\|_{\mathcal{H}^2(z)} \\
&\qquad + \left\| \int_0^{\cdot} M_{Td\sigma} M_{Td\sigma} \left(g^{\mathbb{P}^z_r}_{T\sigma} - g_{T\sigma} \right) \right\|_{\mathcal{H}^2(z)} \\
&\quad = J_1 + J_2.
\end{aligned}$$

Using the definition of $\|\cdot\|_{\mathcal{H}^2(z)}$ (see Definition 6.7):

$$\begin{aligned}
J_2^2 &= \left\| \int_0^{\cdot} M_{Td\sigma} M_{Td\sigma} \left(g^{\mathbb{P}^z_r}_{T\sigma} - g_{T\sigma} \right) \right\|^2_{\mathcal{H}^2(z)} \\
&= \left\| \int_0^{\cdot} Td\sigma \left(\sum_A A^2 \right) \left(g^{\mathbb{P}^z_r}_{T\sigma} - g_{T\sigma} \right) \right\|^2_{\mathcal{H}^2(z)} \\
&= E\left(\int_0^z Td\sigma \left\| \left(\sum_A A^2 \right) \left(g^{\mathbb{P}^z_r}_{T\sigma} - g_{T\sigma} \right) \right\|_{HS} \right)^2 \\
&\le CE\left(\int_0^z Td\sigma \left\| g^{\mathbb{P}^z_r}_{T\sigma} - g_{T\sigma} \right\|_{HS} \right)^2,
\end{aligned}$$

which vanishes by dominated convergence and Lemma 6.15.

Again using Definition 6.7:

$$\begin{aligned}
J_1^2 &= \left\| \int_0^{\cdot} \left(M^{\mathbb{P}^z_r}_{Td\sigma} M^{\mathbb{P}^z_r}_{Td\sigma} - M_{Td\sigma} M_{Td\sigma} \right) g^{\mathbb{P}^z_r}_{T\sigma} \right\|^2_{\mathcal{H}^2(z)} \\
&= \left\| \int_0^{\cdot} d\sigma \sum_A \sum_{i=1}^{n_{\mathbb{P}^z_r}} \Delta_i t A d_{y_{i-1}} \left(\left[(F'(\Delta_i X) A)\, F(-\triangle_i X) \right]^2 - A^2 \right) g^{\mathbb{P}^z_r}_{T\sigma} \right\|^2_{\mathcal{H}^2(z)} \\
&= E\left[\int_0^z d\sigma \left\| \sum_A \sum_{i=1}^{n_{\mathbb{P}^z_r}} \Delta_i t A d_{y_{i-1}} \left(\left[(F'(\Delta_i X) A)\, F(-\triangle_i X) \right]^2 - A^2 \right) g^{\mathbb{P}^z_r}_{T\sigma} \right\| \right]^2 \\
&\le CE\left(\int_0^z d\sigma \sum_A \sum_{i=1}^{n_{\mathbb{P}^z_r}} \Delta_i t \left\| \left[(F'(\Delta_i X) A)\, F(-\triangle_i X) \right]^2 - A^2 \right\|_{HS} \right)^2.
\end{aligned}$$

By Lemma 6.10, $F'(\Delta_i X)A$ is bounded in $\|\cdot\|_{HS}$. Thus $\|(F'(\Delta_i X)A)\, F(-\triangle_i X)\|_{HS}$ and $\|A\|_{HS}$ are bounded. Observing this and decomposing

$$\begin{aligned}
&[(F'(\Delta_i X)A)F(-\triangle_i X)]^2 - A^2 \\
&\quad = (F'(\Delta_i X)A)\, F(-\triangle_i X) \left[(F'(\Delta_i X)A)\, F(-\triangle_i X) - A \right] \\
&\qquad + \left[(F'(\Delta_i X)A)\, F(-\triangle_i X) - A \right] A,
\end{aligned}$$

we have

$$J_1^2 \leq CE\left(\int_0^z d\sigma \sum_{i,A} \Delta_i t \left\|(F'(\Delta_i X)A)\exp\left(-\triangle_i X\right) - A\right\|_{HS}\right)^2.$$

The integrand in this last expression is bounded and hence J_1^2 vanishes by the dominated convergence Theorem. ∎

Proof of Theorem 6.17. Doob decompose $Y_{T\cdot}$ as the sum of its martingale part $M_{T\cdot}$ and its bounded variation part $\int_0^\cdot \nu(\sigma)\,ds$. Then, from Theorem 6.11 we have $M_{T\cdot} \equiv \int_{R_{T\cdot}} Ad_{g_{ts}} b_{dtds}$ and $\nu(\sigma) = -\frac{1}{1-\sigma}\int_0^T Ad_{g_{t\sigma}} X_{dt\sigma}$.

By the definition of $\|\cdot\|_{\mathcal{H}^2(z)}$ (see Definition 6.7) and the expression for $B_{Ts}^{\mathbb{P}_r^z}$ in Eq. 6.13

$$\begin{aligned}
&\left\|\int_0^\cdot B_{Td\sigma}^{\mathbb{P}_r^z} g_{T\sigma}^{\mathbb{P}_r^z} - Y_{Td\sigma} g_{T\sigma}\right\|_{\mathcal{H}^2(z)} \\
&\quad\leq \left\|\int_0^\cdot \left(B_{Td\sigma}^{\mathbb{P}_r^z} - Y_{Td\sigma}\right) g_{T\sigma}^{\mathbb{P}_r^z}\right\|_{\mathcal{H}^2(z)} + \left\|\int_0^\cdot Y_{Td\sigma}\left(g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma}\right)\right\|_{\mathcal{H}^2(z)} \\
&\quad= \left\|\int_0^z \left(M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}\right) g_{T\sigma}^{\mathbb{P}_r^z}\right\|_{L^2} \\
&\qquad+ \left\|\int_0^\cdot d\sigma\left(\nu_1^{\mathbb{P}_r^z}(T,\sigma) + \nu_2^{\mathbb{P}_r^z}(T,\sigma) - \nu_{T\sigma}\right) g_{T\sigma}^{\mathbb{P}_r^z}\right\|_{\mathcal{H}^2(z)} \\
&\qquad\quad+ \left\|\int_0^z M_{Td\sigma}\left(g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma}\right)\right\|_{L^2} + \left\|\int_0^\cdot \nu(T,\sigma)\left(g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma}\right) d\sigma\right\|_{\mathcal{H}^2(z)} \\
&\quad= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

$$\begin{aligned}
I_1^2 &= E\left\|\int_0^z \left(M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}\right) g_{T\sigma}^{\mathbb{P}_r^z}\right\|_{HS}^2 \\
&= E\left\|\sum_A \int_0^s \left\langle M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}, A\right\rangle_{\mathfrak{K}} A g_{T\sigma}^{\mathbb{P}_r^z}\right\|_{HS}^2 \\
&\leq \dim\mathfrak{K}\sum_A E\left\|\int_0^s A g_{T\sigma}^{\mathbb{P}_r^z}\left\langle M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}, A\right\rangle_{\mathfrak{K}}\right\|_{HS}^2 \\
&= \dim\mathfrak{K}\sum_A E\int_0^s \left\|A g_{T\sigma}^{\mathbb{P}_r^z}\right\|_{HS}^2 \left\langle M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}, A\right\rangle_{\mathfrak{K}}^2 \\
&\leq C\sum_A E\int_0^s \left\langle M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}, A\right\rangle_{\mathfrak{K}}^2 \\
&= CE\left\|M_{Td\sigma}^{\mathbb{P}_r^z} - M_{Td\sigma}\right\|_{\mathfrak{K}}^2 \to 0 \text{ as } r \to \infty,
\end{aligned}$$

by Theorem 6.19 and the equivalence of norms on finite-dimensional spaces.

$$\begin{aligned}
I_2^2 &= \left\| \int_0^{\cdot} d\sigma \left(\nu_1^{\mathbb{P}_r^z}(T,\sigma) + \nu_2^{\mathbb{P}_r^z}(T,\sigma) - \nu_{T\sigma} \right) g_{T\sigma}^{\mathbb{P}_r^z} \right\|_{\mathcal{H}^2(z)}^2 \\
&= E\left(\int_0^z \left\| \nu_1^{\mathbb{P}_r^z}(T,\sigma) - \nu(T,\sigma) + \nu_2^{\mathbb{P}_r^z}(T,\sigma) \right\|_{HS} \left\| g_{T\sigma}^{\mathbb{P}_r^z} \right\|_{HS} d\sigma \right)^2 \\
&\leq C \left\| \int_0^{\cdot} \nu_1^{\mathbb{P}_r^z}(T,\sigma) - \nu(T,\sigma) \right\|_{\mathcal{H}^2(z)}^2 + C \left\| \int_0^{\cdot} \nu_2^{\mathbb{P}_r^z}(T,\sigma) \right\|_{\mathcal{H}^2(z)}^2 \\
&\to 0 \text{ as } r \to \infty \text{ by Theorems 6.20 and 6.21.}
\end{aligned}$$

$$\begin{aligned}
I_3^2 &= E\left\| \int_0^z M_{Td\sigma} \left(g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma} \right) \right\|_{L^2}^2 \\
&= E\left\| \int_0^z \sum_A \left(A g_{T\sigma}^{\mathbb{P}_r^z} - A g_{T\sigma} \right) M_{Td\sigma}^A \right\|_{HS}^2 \\
&= \dim \mathfrak{K} \sum_A E\left\| \int_0^z \left(A g_{T\sigma}^{\mathbb{P}_r^z} - A g_{T\sigma} \right) M_{Td\sigma}^A \right\|_{HS}^2 \\
&= \dim \mathfrak{K} \sum_A E \int_0^z \left\| A g_{T\sigma}^{\mathbb{P}_r^z} - A g_{T\sigma} \right\|_{HS}^2 M_{Td\sigma}^A M_{Td\sigma}^A \\
&= \dim \mathfrak{K} \sum_A E \int_0^z \left\| A g_{T\sigma}^{\mathbb{P}_r^z} - A g_{T\sigma} \right\|_{HS}^2 T d\sigma \\
&\to 0 \text{ by Dominated Convergence.}
\end{aligned}$$

$$\begin{aligned}
I_4^2 &= E\left(\int_0^z \|\nu(T,\sigma)\|_{HS} \left\| g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma} \right\|_{HS} d\sigma \right)^2 \\
&\leq E \int_0^z \|\nu(T,\sigma)\|_{HS}^2 d\sigma \int_0^z \left\| g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma} \right\|_{HS} d\sigma \\
&\leq \sqrt{E\left(\int_0^z \|\nu(T,\sigma)\|_{HS}^2 d\sigma \right)^2} \sqrt{E\left(\int_0^z \left\| g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma} \right\|_{HS} d\sigma \right)^2} \\
&\leq \sqrt{E \int_0^z \|\nu(T,\sigma)\|_{HS}^4 d\sigma} \sqrt{E\left(\int_0^z \left\| g_{T\sigma}^{\mathbb{P}_r^z} - g_{T\sigma} \right\|_{HS} d\sigma \right)^2}.
\end{aligned}$$

We will be done by dominated convergence if we can only show that

$$E \int_0^z \|\nu(T,\sigma)\|_{HS}^4 d\sigma < \infty.$$

But $\nu(T,\sigma) = -\frac{1}{1-\sigma} \widetilde{X}_{T\sigma}$ where $\widetilde{X}_{\cdot\sigma} \equiv \int_0^{\cdot} Ad_{g_{u\sigma}} X_{du\sigma}$. Thus

$$\begin{aligned}
E \int_0^z \|\nu(T,\sigma)\|_{HS}^4 d\sigma &\leq \frac{1}{(1-z)^4} \sup_{[0,z]} E \left\| \widetilde{X}_{T\sigma} \right\|_{HS}^4 \\
&< \frac{1}{(1-z)^4} \sup_{[0,z]} 3 \left(T\sigma - \sigma^2 \right)^2,
\end{aligned}$$

since $\widetilde{X}_{\cdot\sigma}$ is a Brownian motion with parameter $\left(\sigma-\sigma^{2}\right)$ by Lemma 7.10. Hence we are done. ∎

6.4. **Propositions 6.13, 6.19, 6.20, 6.21.**

6.4.1. *Proof of Proposition 6.13.*

.

$$\begin{aligned}
y_{n}(s) &= F(\Delta_{1}X(s))\cdots F(\Delta_{i}X(s))\\
\delta y_{n}(s) &= \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}(s)\left(\delta F(\triangle_{i}X(s))\right)y_{i}^{-1}(s)\,y_{n}(s)\\
\delta y_{n}(s)\,y_{n}^{-1}(s) &= \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}(s)\left(\delta F(\triangle_{i}X(s))\right)y_{i}^{-1}(s)\\
&= \sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}}\left(\left(\delta F(\triangle_{i}X)\right)F(-\triangle_{i}X)\right)\\
&= \sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}}\left(\left(dF(\triangle_{i}X)\right)F(-\triangle_{i}X)\right)\\
&\quad +\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}}\left(dF(\triangle_{i}X)\,dF(-\triangle_{i}X)\right)\\
&\quad +\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\left(dAd_{y_{i-1}}\right)\left(dF(\triangle_{i}X)\,F(-\triangle_{i}X)\right)\\
&= I+J+K.
\end{aligned}$$

Letting A run through an orthonormal basis of $\mathfrak{K}$ we can write $dF(\triangle_{i}X(s))$ as

$$dF(\triangle_{i}X)=\sum_{A} d\triangle_{i}b^{A}(s)\left(F'(\triangle_{i}X(s))\,A\right)+\text{finite variation terms.} \tag{6.16}$$

From Lemma 8.3 we can see that

$$d\triangle_{i}b^{A}(s)\,d\triangle_{j}b^{B}(s)=\delta_{ij}\left\langle A,B\right\rangle_{\mathfrak{K}}(\triangle_{i}t)\,ds. \tag{6.17}$$

From Eq. [6.16] and Eq. [6.17] above, we can conclude that

$$K=0$$

and that

$$\begin{aligned}
J &= \frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}}\left(dF(\triangle_{i}X)\,d\left(F(\triangle_{i}X)^{-1}\right)\right)\\
&= -\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left(dF(\triangle_{i}X)\right)F(\triangle_{i}X)^{-1}\left(dF(\triangle_{i}X)\right)F(\triangle_{i}X)^{-1}\,y_{i-1}^{-1}\\
&= -\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\left(y_{i-1}\left(dF(\triangle_{i}X)\right)y_{i}^{-1}\right)^{2}\\
&= -\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\sum_{A}\left(y_{i-1}\left(F'(\triangle_{i}X)\,A\right)y_{i}^{-1}\right)^{2}(\triangle_{i}t)\,ds.
\end{aligned}$$

By Ito's Lemma, I can be computed as

$$I=\sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left(F'(\Delta_i X)\,(d\Delta_i X)+\frac{1}{2}F''(\Delta_i X)\,(d\Delta_i X)^{\otimes 2}\right)y_i^{-1}.$$

Thus

$$\begin{aligned}\delta y_n\,(s)\,y_n^{-1}\,(s) &= \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left(F'(\Delta_i X)\,(d\Delta_i X)+\frac{1}{2}F''(\Delta_i X)\,(d\Delta_i X)^{\otimes 2}\right)y_i^{-1}\\ &\qquad -\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\sum_A \left(y_{i-1}\left(F'\left(\triangle_i X\right)A\right)y_i^{-1}\right)^2\left(\triangle_i t\right)ds.\end{aligned}$$

Using this result , Ito's Lemma, and Eq. (3.5),

$$\begin{aligned}B_T^{\mathbb{P}}(ds) &= \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left(F'(\Delta_i X)d\Delta_i X\right)y_i^{-1}\\ &\quad +\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\sum_A (\Delta_i t)\,y_{i-1}\left(F''(\Delta_i X)A^{\otimes 2}\right)y_i^{-1}ds\\ &\quad -\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\sum_A \Delta_i t\left(y_{i-1}\left(F'(\Delta_i X)A\right)y_i^{-1}\right)^2 ds\\ &= \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\,(s)\left(F'(\Delta_i X)d_s\left(\Delta_i b\,(s)-\int_0^s \Delta_i b\,(\sigma)\,\frac{(1-s)}{(1-\sigma)^2}d\sigma\right)\right)y_i^{-1}\,(s)\\ &\quad +\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\sum_A (\Delta_i t)\left[\begin{array}{c} y_{i-1}\left(F''(\Delta_i X)A^{\otimes 2}\right)y_i^{-1}\\ -\left(y_{i-1}\left(F'(\Delta_i X)A\right)y_i^{-1}\right)^2\end{array}\right]ds.\end{aligned}$$

Therefore,

$$\begin{aligned}dB_T^{\mathbb{P}}(s) &= \sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left(F'(\Delta_i X)\Delta_i b(ds)\right)y_i^{-1}\\ &\quad -\sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left[F'(\Delta_i X)\left(\frac{\Delta_i b(s)}{(1-s)}-\int_0^s \frac{\Delta_i b(\sigma)}{(1-\sigma)^2}d\sigma\right)\right]y_i^{-1}ds\\ &\quad +\frac{1}{2}\sum_{i=1}^{n_{\mathbb{P}}}\sum_A (\Delta_i t)\left[\begin{array}{c} y_{i-1}\left(F''(\Delta_i X)A^{\otimes 2}\right)y_i^{-1}\\ -\left(y_{i-1}\left(F'(\Delta_i X)A\right)y_i^{-1}\right)^2\end{array}\right]ds.\\ &= dM_T^{\mathbb{P}}(s)+\widetilde{\nu}_1^{\mathbb{P}}(T,s)ds+\nu_2^{\mathbb{P}}(T,s)ds;\end{aligned}$$

where we have defined

$$\widetilde{\nu}_1^{\mathbb{P}}(T,s)\equiv-\sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left[F'(\Delta_i X)\left(\frac{\Delta_i b(s)}{(1-s)}-\int_0^s \frac{\Delta_i b(\sigma)}{(1-\sigma)^2}d\sigma\right)\right]y_i^{-1}.$$

So to be done, we only need to show that $\nu_1^{\mathbb{P}}\,(T,s)=\widetilde{\nu}_1^{\mathbb{P}}(T,s)$.
By Eq. (3.5) of Theorem 3.19

$$\frac{X_{ts}}{1-s}=\frac{b_{ts}}{1-s}-\int_0^s \frac{b_{t\sigma}d\sigma}{(1-\sigma)^2}.$$

So simplifying, we see that

$$\widetilde{\nu}_1^{\mathbb{P}}(T,s) = -\frac{1}{1-s}\sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left[F'(\Delta_i X)\Delta_i X(s)\right] y_i^{-1}.$$

But again we must make the observation that $F|_{\mathfrak{K}}$ is the Lie group exponential map. Therefore we have $t \mapsto F(tA)$ satisfies $F'(tA)\,A = F(tA)\,A$. Therefore we see that

$$\widetilde{\nu}_1^{\mathbb{P}}(T,s) = -\frac{1}{1-s}\sum_{i=1}^{n_{\mathbb{P}}} y_{i-1}\left[F(\Delta_i X)\Delta_i X(s)\right] y_i^{-1}.$$

Now from Definition 6.9 observe that $y_i(s) = y_{i-1}(s)\,F(\triangle_i X(s))$. Thus $y_i^{-1}y_{i-1} = F(-\triangle_i X(s))$.

Thus we are done since

$$\begin{aligned}
\widetilde{\nu}_1^{\mathbb{P}}(T,s) &= -\frac{1}{1-s}\sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}}\left[F(\Delta_i X)\Delta_i X(s) F(-\triangle_i X(s))\right] \\
&= -\frac{1}{1-s}\sum_{i=1}^{n_{\mathbb{P}}} Ad_{y_{i-1}}\Delta_i X(s) \\
&= \nu_1^{\mathbb{P}}(T,s).
\end{aligned}$$

∎

6.4.2. *Proof of Proposition 6.19.*

Lemma 6.22. *Recall that $M_T^{\mathbb{P}_r^z}(s)$ is the martingale part of $B_T^{\mathbb{P}_r^z}(s)$. Then*

$$M_T^{\mathbb{P}_r^z}(s) \ \text{“approximates”}\ \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\,(d\Delta_i b)\,y_i^{-1}.$$

Specifically, for any $s \in [0,z]$,

$$\left\| M_T^{\mathbb{P}_r^z}(s) - \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\,(d\Delta_i b)\,y_i^{-1} \right\|_{L^2} \to 0 \ \text{as}\ r \to \infty.$$

Proof.

$$\text{Let } J \equiv \left\| M_T^{\mathbb{P}_r^z}(s) - \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\,(d\Delta_i b)\,y_i^{-1} \right\|_{L^2}.$$

Using Eq.(6.10) yields and Lemma 8.7,

$$\begin{aligned}
J^2 &= E\left\| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\,(d\Delta_i b)\,y_i^{-1} - \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\,(F'(\Delta_i X) d\Delta_i b)\,y_i^{-1} \right\|_{HS}^2 \\
&= E\left\| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} \sum_A y_{i-1}\,(A - F'(\Delta_i X)A)\,y_i^{-1} d\Delta_i b^A \right\|_{HS}^2.
\end{aligned}$$

Applying Lemma 8.7 to this last term shows that

$$\begin{aligned} J^2 &= E\sum_{i,A}\Delta_i t\int_0^s \left\| y_{i-1}\left(A - F'(\Delta_i X)A\right) y_i^{-1}\right\|_{HS}^2 d\sigma \\ &= C\sum_{i,A}\Delta_i t\int_0^s d\sigma E\left\|A - F'(\Delta_i X)A\right\|_{HS}^2 . \end{aligned}$$

Appealing to Eq. (6.1), the expression

$$\left\|A - F'(\Delta_i X)A\right\|_{HS}^2 = \left\|\int_0^1 \left\{A - F\left((1-\tau)\Delta_i X\right) AF\left(\tau\Delta_i X\right)\right\} d\tau\right\|_{HS}^2$$

is bounded because $e^{\alpha\Delta_i X}$ is group valued (and hence bounded). Thus $J^2 \to 0$ as $r \to \infty$ by Dominated Convergence. ∎

Lemma 6.23. *The expression*

$$\int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\left(d\Delta_i b\right) y_i^{-1} \text{ “approximates” } \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} \left(Ad_{y_{i-1}} d\Delta_i b\right).$$

Specifically, for any $s \in [0,z]$ we have as $r \to \infty$

$$\left\| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\left(d\Delta_i b\right) y_i^{-1} - \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} \left(Ad_{y_{i-1}} d\Delta_i b\right)\right\|_{L^2} \to 0.$$

Proof. Using Lemma 6.15,and Dominated Convergence,

$$\begin{aligned} &\lim_{r\to\infty}\left\| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} \left(Ad_{y_{i-1}} d\Delta_i b\right) - \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}} y_{i-1}\left(d\Delta_i b\right) y_i^{-1}\right\|_{L^2}^2 \\ &= \lim_{r\to\infty} E\left\|\int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^z}}\sum_A Ad_{y_{i-1}}\left(A - AF\left(-\Delta_i X\right)\right) d\Delta_i b^A\right\|_{HS}^2 \\ &\le \lim_{r\to\infty} E\sum_{i,A}\Delta_i t\int_0^s \left\|Ad_{y_{i-1}}\left(A - AF\left(-\Delta_i X\right)\right)\right\|_{HS}^2 d\sigma \\ &\le \lim_{r\to\infty} CE\int_0^s \sum_i \Delta_i t\left\|1 - F\left(-\Delta_i X\right)\right\|_{HS}^2 d\sigma \\ &= CE\int_0^s \lim_{r\to\infty}\sum_i \Delta_i t\left\|1 - F\left(-\Delta_i X\right)\right\|_{HS}^2 d\sigma = 0. \end{aligned}$$

The expression $\sum_i \Delta_i t\left\|1 - F\left(-\Delta_i X\right)\right\|_{HS}^2$ is bounded since $F\left(-\Delta_i X\right)$ is K-valued and $\sum_i \Delta_i t = 1$. So by Dominated convergence, this last term becomes

$$= CE\int_0^s \lim_{r\to\infty}\sum_i \Delta_i t\left\|1 - F\left(-\Delta_i X\right)\right\|_{HS}^2 d\sigma.$$

Now

$$\begin{aligned} &\lim_{r\to\infty}\sum_i \Delta_i t\left\|1 - F\left(-\Delta_i X\right)\right\|_{HS}^2 \\ &\qquad \le \lim_{r\to\infty}\left|\mathbb{P}_r^z\right|\sup_r \sum_i \left\|1 - F\left(-\Delta_i X\right)\right\|_{HS}^2 . \end{aligned}$$

By Lemma 6.15 this last expression goes to 0 in the limit as $r \to \infty$. ∎

Lemma 6.24. *Let $\mathbb{P}$ be a partition of $[0,T]$, $g^{\mathbb{P}}_{t\sigma}$ the approximation to $g_{t\sigma}$, and $y_i(\sigma) = g^{\mathbb{P}}_{t_i\sigma}$ as in Definition 6.9. Then for any $s \in [0,z]$,*

$$\int_0^s \sum_{i=1}^{n_{\mathbb{P}}} \left(Ad_{y_{i-1}(\sigma)} d\Delta_i b(\sigma) \right) = \int_{R_{Ts}} Ad_{g^{\mathbb{P}}_{[\tau]\sigma}} b_{d\tau d\sigma}.$$

where $[\tau] \equiv t_{i-1} \forall \tau \in (t_{i-1}, t_i]$.

Proof. It will suffice to show

$$\int_0^s Ad_{y_{i-1}(\sigma)} \triangle_i b(d\sigma) = \int_{(t_{i-1}0, t_i s]} Ad_{g^{\mathbb{P}}_{t_{i-1}\sigma}} b_{d\tau d\sigma}.$$

Letting $\mathbb{Q}_r$ be a refining sequence of partitions of $[0,s]$, we have, by Theorem 6.4

$$\int_{(t_{i-1}0, t_i s]} Ad_{g^{\mathbb{P}}_{t_{i-1}\sigma}} b_{d\tau d\sigma} = \lim_{r\to\infty} \sum_{s_j \in \mathbb{Q}_r} Ad_{g^{\mathbb{P}}_{t_{i-1}s_{j-1}}} b(t_{i-1}s_{j-1}, t_i s_j];$$

where the limit is taken in L^2. However,

$$\begin{aligned} &\lim_{r\to\infty} \sum_{s_j \in \mathbb{Q}_r} Ad_{g^{\mathbb{P}}_{t_{i-1}s_{j-1}}} b(t_{i-1}s_{j-1}, t_i s_j] \\ &\qquad = \int_0^s Ad_{g^{\mathbb{P}}_{t_{i-1}\sigma}} \left(b_{t_i d\sigma} - b_{t_{i-1} d\sigma} \right) \\ &\qquad = \int_0^s Ad_{y_{i-1}(\sigma)} \triangle_i b(d\sigma). \end{aligned}$$

As usual, we use the fact that $g^{\mathbb{P}}$ is bounded and the metric is Ad-invariant. So dominated convergence goes through and L^2 convergence is justified. ∎

Lemma 6.25. *Let $g^{\mathbb{P}^z_r}_{\tau\sigma}$ be the approximation to $g_{\tau\sigma}$ from Definition 6.9. Then for any $s \in [0,z]$,*

$$\int_{R_{Ts}} Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}} b_{d\tau d\sigma} \to \int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma} \text{ in } L^2 \text{ as } r \to \infty.$$

Proof. Let

$$J \equiv \left\| \int_{R_{Ts}} Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}} b_{d\tau d\sigma} - \int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma} \right\|^2_{L^2}.$$

By Theorem 6.4,

$$J = E \left\| \sum_A \int_{R_{Ts}} \left(Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}} A - Ad_{g_{\tau\sigma}} A \right) b^A_{d\tau d\sigma} \right\|^2_{HS}.$$

$$\begin{aligned}
J &\leq C\sum_A\sum_{pq} E\left(\int_{R_{Ts}}\left(Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}}A - Ad_{g_{\tau\sigma}}A\right)_{pq} b^A_{d\tau d\sigma}\right)^2 \\
&= C\sum_A\sum_{pq} E\int_0^T d\tau\int_0^s d\sigma\left(\left(Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}}A - Ad_{g_{\tau\sigma}}A\right)_{pq}\right)^2 \\
&= C\sum_A E\int_0^T d\tau\int_0^s d\sigma\left\|\left(Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}}A - Ad_{g_{\tau\sigma}}A\right)_{pq}\right\|^2_{HS} \\
&\leq C\sum_A E\int_0^T d\tau\int_0^s d\sigma\left\|g^{\mathbb{P}^z_r}_{[\tau]\sigma} - g_{\tau\sigma}\right\|^2_{HS} + \left\|\left(g^{\mathbb{P}^z_r}_{[\tau]\sigma}\right)^{-1} - (g_{\tau\sigma})^{-1}\right\|^2_{HS} \\
&\to 0 \text{ as } r\to\infty \text{ by Dominated Convergence and Lemma 6.15.}
\end{aligned}$$

∎

Proof of Proposition 6.19. For the purposes of this proof, define the symbol '$\backsim$' to mean "has the same limit in L^2 as $|\mathbb{P}^z_r|\to 0$"). Defining $[\tau]\equiv t_{i-1}\ \forall\tau\in(t_{i-1},t_i]$, for any $s\in[0,z]$,

$$\begin{aligned}
M_T^{\mathbb{P}^z_r}(s) &\backsim \int_0^s\sum_{i=1}^{n_{\mathbb{P}^z_r}} y_{i-1}\,(d\Delta_i b)\,y_i^{-1} \text{ by Lemma 6.22;} \\
\int_0^s\sum_{i=1}^{n_{\mathbb{P}^z_r}} y_{i-1}\,(d\Delta_i b)\,y_i^{-1} &\backsim \int_0^s\sum_{i=1}^{n_{\mathbb{P}^z_r}}\left(Ad_{y_{i-1}}d\Delta_i b\right) \text{ by Lemma 6.23;} \\
\int_0^s\sum_{i=1}^{n_{\mathbb{P}^z_r}}\left(Ad_{y_{i-1}}d\Delta_i b\right) &= \int_{R_{1s}} Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}} b_{d\tau d\sigma} \text{ by Lemma 6.24;} \\
\int_{R_{Ts}} Ad_{g^{\mathbb{P}^z_r}_{[\tau]\sigma}} b_{d\tau d\sigma} &\backsim \int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma} \text{ by Lemma 6.25.}
\end{aligned}$$

Putting all this together yields

$$M_T^{\mathbb{P}^z_r}(s)\backsim\int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma}.$$

We have still to show that $\int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma}$ is a $\mathfrak{K}$-valued Brownian motion with parameter T. Let J denote the process

$$J\equiv\int_0^{\cdot}\sum_{i=1}^{n_{\mathbb{P}^z_r}}\left(Ad_{y_{i-1}}d\Delta_i b\right).$$

By Lemmas 6.24 and 6.25

$$J_s\backsim\int_{R_{Ts}} Ad_{g_{\tau\sigma}} b_{d\tau d\sigma}.$$

Thus, since L^2 limits of Brownian motions are Brownian motions, it suffices to show the process $s\to J_s$ is a Brownian motion.

The rest of the proof is devoted to showing that $s\to J_s$ is a Brownian motion. with parameter T. We shall use the notation $[N]_\cdot$ to denote the quadratic variation of a martingale N. If N is an $\mathbb{R}^d$-valued martingale then

$$[N]_\cdot = \sum_{i,j}\left[N^{(i)},N^{(j)}\right] e_i\otimes e_j,$$

where e_i is a basis for $\mathbb{R}^d$ and $\left[N^{(i)},N^{(j)}\right]$ are the joint quadratic variations of the $\mathbb{R}$-valued martingales $N^{(i)}$ and $N^{(j)}$. Let E_{pq} be the matrix with ij-entry $\delta_{ip}\delta_{jq}$. Letting $\{A\}$ run through an orthonormal basis of $\mathfrak{K}$, we have

$$\begin{aligned}
[J]_s &= \left[\sum_{i=1}^{n_{\mathbb{P}_r^z}}\sum_A\int_0^{\cdot}\left(Ad_{y_{i-1}}A\right)d\Delta_i b^A\right]_s\\
&= \left[\sum_{pq}\sum_{i=1}^{n_{\mathbb{P}_r^z}}\sum_A\left(\int_0^{\cdot}\left(Ad_{y_{i-1}}A\right)_{pq}d\Delta_i b^A\right)E_{pq}\right]_s\\
&= \sum_{i=1}^{n_{\mathbb{P}_r^z}}\sum_A\sum_{pq}\sum_{p'q'}\left(E_{pq}\otimes E_{p'q'}\right)\Delta_i t\int_0^s\left(Ad_{y_{i-1}}A\right)_{pq}\left(Ad_{y_{i-1}}A\right)_{p'q'}d\sigma\\
&= \sum_{i=1}^{n_{\mathbb{P}_r^z}}\sum_A\Delta_i t\int_0^s\left(Ad_{y_{i-1}}A\right)\otimes\left(Ad_{y_{i-1}}A\right)d\sigma\\
&= \sum_{i=1}^{n_{\mathbb{P}_r^z}}\Delta_i t\int_0^s\left(\sum_A A\otimes A\right)d\sigma\\
&= \left(\sum_A A^{\otimes 2}\right)Ts.
\end{aligned}$$

Thus by Levy's Theorem $s\to J_s$ is a Brownian motion with parameter t. ∎

6.4.3. *Proof of Proposition 6.20.*

Lemma 6.26. *The expression* $\sum_{\mathbb{P}_r^z}(\triangle_i t)\,y_{i-1}\left(F''(\triangle_i X)\,A^{\otimes 2}\right)y_i^{-1}$ *is approximately the same as the expression* $\sum_{\mathbb{P}_r^z}(\triangle_i t)\,Ad_{y_{i-1}}F''(\triangle_i X)\,A^{\otimes 2}$. *Specifically, P-a.s. as $r\to\infty$,*

$$J_r\equiv\left\|\sum_{\mathbb{P}_r^z}(\triangle_i t)\left[y_{i-1}\left(F''(\triangle_i X)\,A^{\otimes 2}\right)y_i^{-1}-Ad_{y_{i-1}}F''(\triangle_i X)\,A^{\otimes 2}\right]\right\|_{HS}\to 0.$$

Proof. Recall $y_i=y_{i-1}F(\triangle_i X)$ as in Definition 6.9. Using the boundedness of the Adjoint operator (i.e. the fact that $\sup_{k\in K}|Ad_k|<\infty$) gives us

$$\begin{aligned}
J_r &= \left\|\sum_{\mathbb{P}_r^z}(\triangle_i t)\left[Ad_{y_{i-1}}\left(F''(\triangle_i X)\,A^{\otimes 2}\right)\right]\left(y_{i-1}y_i^{-1}-1\right)\right\|_{HS}\\
&\le C\sum_{\mathbb{P}_r^z}(\triangle_i t)\left\|F(-\triangle_i X)-1\right\|_{HS}\\
&\le C\sum_{\mathbb{P}_r^z}(\triangle_i t)^2\sum_{\mathbb{P}_r^z}\left\|F(-\triangle_i X)-1\right\|_{HS}^2\\
&\le C\left|\mathbb{P}_r^z\right|\sup_r\sum_{\mathbb{P}_r^z}\left\|F(-\triangle_i X)-1\right\|_{HS}^2.
\end{aligned}$$

By Lemma 6.15,

$$\sup_r \sum_{\mathbb{P}_r^z} \left\| F\left(-\triangle_i X\right) - 1 \right\|_{HS}^2 < \infty, \ P\text{-a.s.}$$

Hence we are done. ∎

Lemma 6.27. *The expression* $\sum_{\mathbb{P}_r^z} (\triangle_i t) \left\{ y_{i-1} \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} \right\}^2$ *is approximately the same as the expression* $\sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left(F'\left(\triangle_i X\right) A\right)^2$. *Specifically almost surely as* $r \to \infty$, *we have,* P*-a.s., that*

$$J_r \equiv \left\| \sum_{\mathbb{P}_r^z} (\triangle_i t) \left[\left\{ y_{i-1} \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} \right\}^2 - Ad_{y_{i-1}} \left(F'\left(\triangle_i X\right) A\right)^2 \right] \right\|_{HS} \to 0.$$

Proof.

$$\begin{aligned}
&J_r \\
&= \sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left\{ \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} y_{i-1} \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} y_{i-1} - \left(F'\left(\triangle_i X\right) A\right)^2 \right\} \\
&= \sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left(F'\left(\triangle_i X\right) A\right) \\
&\qquad \times Ad_{y_{i-1}} \left[y_i^{-1} y_{i-1} \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} y_{i-1} - \left(F'\left(\triangle_i X\right) A\right) \right] \\
&= \sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left(F'\left(\triangle_i X\right) A\right) \\
&\qquad \times Ad_{y_{i-1}} \left\{ y_i^{-1} y_{i-1} \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} y_{i-1} - y_i^{-1} y_{i-1} \left(F'\left(\triangle_i X\right) A\right) \right\} \\
&\quad + \sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left(F'\left(\triangle_i X\right) A\right) \\
&\qquad \times Ad_{y_{i-1}} \left\{ y_i^{-1} y_{i-1} \left(F'\left(\triangle_i X\right) A\right) - \left(F'\left(\triangle_i X\right) A\right) \right\} \\
&= \sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left\{ \left(F'\left(\triangle_i X\right) A\right) y_i^{-1} y_{i-1} \left(F'\left(\triangle_i X\right) A\right) \left(y_i^{-1} y_{i-1} - 1 \right) \right\} \\
&\quad + \sum_{\mathbb{P}_r^z} (\triangle_i t)\, Ad_{y_{i-1}} \left\{ \left(F'\left(\triangle_i X\right) A\right) \left(y_i^{-1} y_{i-1} - 1 \right) \left(F'\left(\triangle_i X\right) A\right) \right\}.
\end{aligned}$$

Bringing the Hilbert-Schmidt norm within the sum, exploiting the boundedness of the Adjoint operator, and using Eq. (6.6) of Lemma 6.10; we see that the norm of this last expression is bounded above by

$$\begin{aligned}
&Const \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| y_i^{-1} y_{i-1} - 1 \right\|_{HS} \\
&\qquad \le Const \left(\sum_{\mathbb{P}_r^z} (\triangle_i t)^2 \sum_{\mathbb{P}_r^z} \left\| y_i^{-1} y_{i-1} - 1 \right\|_{HS}^2 \right)^{1/2}.
\end{aligned}$$

Now from Definition 6.9 we see that $y_i^{-1} y_{i-1} = F\left(-\triangle_i X\right)$. Now invoking Lemma 6.15 we see that this last expression vanishes in the limit. ∎

Lemma 6.28. *As usual, let F be the exponential map as in Definition 6.9. Then as $r \to \infty$ we have, P-a.s., that*

$$\sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| F''(\triangle_i X) A^{\otimes 2} - (F'(\triangle_i X) A)^2 \right\|_{HS} \to 0.$$

Proof. Let I_r,J_r be defined as follows:

$$\begin{aligned} I_r &\equiv \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| F''(\triangle_i X) A^{\otimes 2} - A^2 \right\|_{HS}; \\ J_r &\equiv \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| A^2 - (F'(\triangle_i X) A)^2 \right\|_{HS}. \end{aligned}$$

It will suffice to show that the random variables I_r and J_r vanish almost surely as $r \to \infty$. To do this, first notice that $F'(0) A = A$ and that $F''(0) A^{\otimes 2} = A^2$.

The expression

$$\begin{aligned} I_r &= \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| F''(\triangle_i X) A^{\otimes 2} - F''(0) A^{\otimes 2} \right\|_{HS} \\ &= \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| \int_0^1 (F'''(\varepsilon \triangle_i X) A \otimes A \otimes \triangle_i X) \, d\varepsilon \right\|_{HS} \\ &\leq \sum_{\mathbb{P}_r^z} (\triangle_i t) \int_0^1 \left\| F'''(\varepsilon \triangle_i X) A \otimes A \otimes \triangle_i X \right\|_{HS} d\varepsilon. \\ &\leq C \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| \triangle_i X \right\|_{HS}, \text{ by Lemma 6.10.} \end{aligned}$$

However, the expression

$$C \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| \triangle_i X \right\|_{HS} \leq C \left[\sup_r \sum_{\mathbb{P}_r^z} \left\| \triangle_i X \right\|_{HS}^2 \right] \sum_{\mathbb{P}_r^z} (\triangle_i t)^2. \tag{6.18}$$

Invoking Eq. [6.24] in the proof of Lemma 6.15, we see that the right hand side of Eq. [6.18] goes to zero as $r \to \infty$. Thus $I_r \to 0$, P-a.s., as $r \to \infty$.

Turning now to J_r, we see that

$$\begin{aligned} J_r &= \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| (F'(\triangle_i X) A)^2 - (F'(0) A)^2 \right\|_{HS} \\ &= \sum_{\mathbb{P}_r^z} (\triangle_i t) \, 2 \int_0^1 \left\| F'(\varepsilon \triangle_i X) A \right\|_{HS} \left\| F''(\varepsilon \triangle_i X) A \otimes \triangle_i X \right\|_{HS} \\ &\leq C \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| \triangle_i X \right\|_{HS} \text{ by Lemma 6.10.} \end{aligned}$$

Therefore,

$$J_r \leq C \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| \triangle_i X \right\|_{HS} \to 0 \text{ by Eqs. [6.18] and [6.24].}$$

∎

Proof of Proposition 6.20. Let $y_i(\cdot)$ denote $g_{t_i\cdot}^{\mathbb{P}_r^z}$ as in Definition 6.9. Let F be the exponential map acting on $m\times m$ matrices as in Definition 6.9. Let A run through an orthonormal basis of $\mathfrak{K}$. Let

$$J_r \equiv \left\| \int_0^{\cdot} \nu_2^{\mathbb{P}_r^z}(T,\sigma)d\sigma \right\|_{\mathcal{H}^2(z)}^2 .$$

Then using Hölder's inequality and the fact that $s\leq 1$ yields

(6.19)

$$J_r$$

$$\leq E\int_0^z d\sigma \left\| \sum_{\mathbb{P}_r^z,A} \frac{\triangle_i t}{4}\left[y_{i-1}\left(F''\left(\triangle_i X\right)A^{\otimes 2}\right)y_i^{-1} - \left(y_{i-1}\left(F'\left(\triangle_i X\right)A\right)y_i^{-1}\right)^2\right]\right\|_{HS}^2 .$$

Notice that by invoking Eq. (6.6) of Lemma 6.10 as well as the boundedness of K, we see that the Hilbert-Schmidt norm in Eq. (6.19) is bounded. Thus we can invoke dominated convergence and so it suffices to show for fixed A that as $r\to\infty$, we have, P-a.s., that the expression

$$\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)\left[y_{i-1}\left(F''\left(\triangle_i X\right)A^{\otimes 2}\right)y_i^{-1} - \left(y_{i-1}\left(F'\left(\triangle_i X\right)A\right)y_i^{-1}\right)^2\right]\to 0.$$

For simplicity, we shall use "$\backsim$" to mean "has the same limit in $\mathfrak{K}$". Explicitly $\left\{f_{\mathbb{P}_r^z}\right\}\backsim\left\{g_{\mathbb{P}_r^z}\right\}$ iff $\left\|f_{\mathbb{P}_r^z}-g_{\mathbb{P}_r^z}\right\|_{HS}\to 0$, P-a.s., as $r\to\infty$. So the problem reduces to showing

$$\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)y_{i-1}\left(F''\left(\triangle_i X\right)A^{\otimes 2}\right)y_i^{-1}\backsim\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)\left(y_{i-1}\left(F'\left(\triangle_i X\right)A\right)y_i^{-1}\right)^2 .$$

By Lemma 6.26 we have

$$\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)y_{i-1}\left(F''\left(\triangle_i X\right)A^{\otimes 2}\right)y_i^{-1}\backsim\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)Ad_{y_{i-1}}F''\left(\triangle_i X\right)A^{\otimes 2}.$$

By Lemma 6.27 we have

$$\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)\left(y_{i-1}\left(F'\left(\triangle_i X\right)A\right)y_i^{-1}\right)^2\backsim\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)Ad_{y_{i-1}}\left(F'\left(\triangle_i X\right)A\right)^2 .$$

Thus the problem reduces to showing

$$\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)Ad_{y_{i-1}}F''\left(\triangle_i X\right)A^{\otimes 2}\backsim\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)Ad_{y_{i-1}}\left(F'\left(\triangle_i X\right)A\right)^2 .$$

Invoking the boundedness of the Adjoint operator on $(K,\|\cdot\|_{HS})$, and letting J denote the expression

$$\left\|\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)Ad_{y_{i-1}}F''\left(\triangle_i X\right)A^{\otimes 2}-\sum_{\mathbb{P}_r^z}\left(\triangle_i t\right)Ad_{y_{i-1}}\left(F'\left(\triangle_i X\right)A\right)^2\right\|_{HS},$$

we have

$$
\begin{aligned}
J &\leq \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| Ad_{y_{i-1}} \left[F''(\triangle_i X) A^{\otimes 2} - \left(F'(\triangle_i X) A\right)^2 \right] \right\|_{HS} \\
&\leq C \sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| F''(\triangle_i X) A^{\otimes 2} - \left(F'(\triangle_i X) A\right)^2 \right\|_{HS}.
\end{aligned}
$$

However by Lemma 6.28 as $r \to \infty$ we have, P-a.s., that

$$
\sum_{\mathbb{P}_r^z} (\triangle_i t) \left\| F''(\triangle_i X) A^{\otimes 2} - \left(F'(\triangle_i X) A\right)^2 \right\|_{HS} \to 0.
$$

Hence we are done. ∎

6.4.4. *Proof of Proposition 6.21.*

. Let $s \in [0, z]$ and let $y_i(\cdot)$ denote $g_{t_i \cdot}^{\mathbb{P}_r^z}$ as in Definition 6.9. Let F be the matrix-exponential map as in Definition 6.9. Let A run through an orthonormal basis of $\mathfrak{K}$.

By Eq. (6.11)

$$
\nu_1^{\mathbb{P}_r^z}(T, s) \equiv -\frac{1}{1-z} \sum_{i=1}^{n_{\mathbb{P}_r^z}} Ad_{y_{i-1}(s)} \Delta_i X(s).
$$

By the definition of $\mathcal{H}^2(z)$, it will suffice to show that the expression

$$
J_r \equiv E\left[\int_0^{\cdot} d\sigma \left\| \nu_1^{\mathbb{P}_r^z}(T, \sigma) + \frac{1}{1-\sigma} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma} \right\|_{HS} \right]^2
$$

vanishes in the limit. Thus we have:

$$
\begin{aligned}
J_r &= E\left[\int_0^{\cdot} \frac{d\sigma}{1-\sigma} \left\| \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma} - \sum_{i=1}^{n_{\mathbb{P}_r^z}} Ad_{y_{i-1}} \Delta_i X(\sigma) \right\|_{HS} \right]^2 \\
&\leq \int_0^{\cdot} \frac{d\sigma}{(1-\sigma)^2} \int_0^{\cdot} d\sigma E \left\| \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma} - \sum_{i=1}^{n_{\mathbb{P}_r^z}} Ad_{y_{i-1}} \Delta_i X(\sigma) \right\|_{HS}^2.
\end{aligned}
$$

Thus since $z < 1$, the problem reduces to showing

$$
E \left\| \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma} - \sum_{i=1}^{n_{\mathbb{P}_r^z}} Ad_{y_{i-1}} \Delta_i X(\sigma) \right\|_{HS}^2
$$

vanishes. Defining

$$
\gamma_{t\sigma}^{\mathbb{P}_r^z} \equiv \sum_{i=1}^{n_{\mathbb{P}_r^z}} g_{t_{i-1}\sigma}^{\mathbb{P}_r^z} 1_{(t_{i-1}, t_i]},
$$

we have only to show that the expression

$$
K_r \equiv E \left\| \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma} - Ad_{\gamma_{t\sigma}^{\mathbb{P}_r^z}} X_{dt\sigma} \right\|_{HS}^2
$$

vanishes in the limit. By Lemma 8.3 $X^A_{dt\sigma}X^B_{dt\sigma} = \delta_{AB}\left(\sigma-\sigma^2\right)dt$. Evaluate our last expression yields

$$\begin{aligned} K_r &= \sum_{p,q} E\left[\sum_A \int_0^T \left(Ad_{g_{t\sigma}}A - Ad_{\gamma^{\mathbb{P}^z_r}_{t\sigma}}A\right)_{pq} X^A_{dt\sigma}\right]^2 \\ &= \sum_{p,q}\sum_A E\int_0^T \left(Ad_{g_{t\sigma}}A - Ad_{\gamma^{\mathbb{P}^z_r}_{t\sigma}}A\right)^2_{pq}\left(\sigma-\sigma^2\right)dt \\ &= \sum_A \left(\sigma-\sigma^2\right) E\int_0^T \left\|Ad_{g_{t\sigma}}A - Ad_{\gamma^{\mathbb{P}^z_r}_{t\sigma}}A\right\|^2_{HS} dt \\ &\leq C\sum_A \left(\sigma-\sigma^2\right)\int_0^T E\left\|g_{t\sigma} - \gamma^{\mathbb{P}^z_r}_{t\sigma}\right\|^2_{HS} + E\left\|g^{-1}_{t\sigma} - \left(\gamma^{\mathbb{P}^z_r}_{t\sigma}\right)^{-1}\right\|^2_{HS} dt. \end{aligned}$$

Thus by Dominated convergence and the continuity of inverses, it will suffice to show $\left\|g_{t\sigma} - \gamma^{\mathbb{P}^z_r}_{t\sigma}\right\|_{HS} \to 0$ P-a.s. Define $[t] \equiv t_{i-1}$ for any $t \in [t_{i-1}, t_i)$. Then $\gamma^{\mathbb{P}^z_r}_{t\sigma} = g^{\mathbb{P}^z_r}_{[t]\sigma}$.

$$\left\|g_{t\sigma} - g^{\mathbb{P}^z_r}_{[t]\sigma}\right\|_{HS} \leq \left\|g_{t\sigma} - g_{[t]\sigma}\right\|_{HS} + \left\|g_{[t]\sigma} - g^{\mathbb{P}^z_r}_{[t]\sigma}\right\|_{HS}.$$

By Lemma 6.15,

$$\sup_{t\in[0,T]}\left\|g^{\mathbb{P}^z_r}(t,\sigma) - g_{t\sigma}\right\|_{HS} \to 0,\ P\text{-a.s.}$$

Thus, for each ω, pick a partition $\mathbb{P}^z_r$ so that $\left\|g_{\cdot\sigma} - g^{\mathbb{P}^z_r}_{\cdot\sigma}\right\|_\infty < \varepsilon$ for all $m \geq n$. Pick $m > n$ so that $\left\|g_{t\sigma} - g_{[t]\sigma}\right\|_{HS} < \varepsilon$. Then we see that $\left\|g_{t\sigma} - g^{\mathbb{P}^z_r}_{[t]\sigma}\right\|_{HS} \to 0$ almost surely and so we are done. ∎

6.5. **Good partitions(proof of Lemma 6.15).**

Theorem 6.29. *For any $r \in \mathbb{N}$, let $t_i \equiv \frac{iT}{2^r}$, and let*

$$\widetilde{\mathbb{P}}_r \equiv \{0 = t_0 < \cdots < t_{2^r} = T\}$$

be a partition of $[0,T]$. Let $t \mapsto g_{t\cdot}$ be an $L(K)$-valued Brownian motion and let $g^{\widetilde{\mathbb{P}}_r}_{ts}$ be the approximation from Definition 6.9. Then $g^{\widetilde{\mathbb{P}}_r}(\cdot,s)$ converges to $g_{\cdot s}$ in L^p for any $p \in [1,\infty)$ as $r \to \infty$. Specifically, we have

$$\left\|\sup_{t\in[0,T]}\left\|g^{\widetilde{\mathbb{P}}_r}(t,s) - g_{ts}\right\|_{HS}\right\|_{L^p} \to 0\ \forall p \in [1,\infty), T < \infty.$$

This result is a direct consequence of Theorem 7.2 in [21]. See also Wong and Zakai [31].

Proof of Theorem 6.29. View $GL_m(\mathbb{R})$ as $\mathbb{R}^{m^2}$. Then the path $t \to X_{ts}$ is an element of the Wiener space

$$W_0\left(\mathbb{R}^{m^2}\right) \equiv \left\{\sigma \in C\left([0,1] \to \mathbb{R}^{m^2}\right) | \sigma(0) = 0\right\}.$$

Let $\delta_r = T/r$. Then $X_{ts}^{\widetilde{\mathbb{P}}_r}$ is an approximation to the Wiener process $t \to X_{ts}$ in the sense of Definition 7.1 of Ikeda and Watanabe [21]. Now apply Theorem 7.2 of [21] to show that as $r \to \infty$,

$$\sup_{t\in[0,T]} \left\| g_{ts}^{\widetilde{\mathbb{P}}_r} - g_{ts} \right\|_{HS} \to 0 \text{ in } L^2(\Omega).$$

Both processes $g_{ts}^{\widetilde{\mathbb{P}}_r}$, and g_{ts} are K valued and hence bounded in the Hilbert Schmidt norm. Conclude that

$$\sup_{t\in[0,T]} \left\| g_{ts}^{\widetilde{\mathbb{P}}_r} - g_{ts} \right\|_{HS} \to 0 \text{ in } L^p(\Omega).$$

∎

Lemma 6.30. *Let $\{\mathbb{P}_r\}$ be a sequence of partitions of the interval $[0,T]$ so that $|\mathbb{P}_r| \to 0$ as $r \to \infty$. Then there exists a subsequence of partitions $\{\mathbb{P}'_r\}$ so that*

$$\sup_{\{\mathbb{P}'_r\}} \sup_{1\geq s\geq\sigma\geq 0} \left| \sum_{i=1}^{n_{\mathbb{P}'_r}} \Delta_i b^A(s) \Delta_i b^A(\sigma) \right| < \infty, \; P\text{-a.s.}$$

Proof of Lemma 6.15. Let $\left\{\widetilde{\mathbb{P}}_r\right\}$ be the sequence of partitions chosen in Theorem 6.29. We will show that there exists a subsequence $\{\mathbb{P}_r^z\}$ of partitions of the $\left\{\widetilde{\mathbb{P}}_r\right\}$ depending only on T and z, with $|\mathbb{P}_r^z| \downarrow 0$ as $r \to \infty$ and

$$\sup_{\{\mathbb{P}_r^z\}} \sup_{\sigma\in[0,z]} \sum_{i=1}^{n_{\mathbb{P}_r^z}} \|F(\alpha\Delta_i X(\sigma)) - 1\|_{HS}^2 < \infty \; P\text{-a.s.}$$

Since $\{\mathbb{P}_r^z\}$ is a subsequence of the $\left\{\widetilde{\mathbb{P}}_r\right\}$ from Theorem 6.29, we will still have

$$\left\| \sup_{t\in[0,T]} \left\| g^{\mathbb{P}_r^z}(t,s) - g_{ts} \right\|_{HS} \right\|_{L^p} \to 0 \; \forall p \in [1,\infty), T < \infty,$$

as $r \to \infty$ and so we shall be done.

Let $\left\{\widetilde{\mathbb{P}}_r\right\}$ be a refining sequence of partitions, so that $\left|\widetilde{\mathbb{P}}_r\right| \downarrow 0$ as $r \to \infty$.

$$\begin{aligned}
\|F(\alpha\Delta_i X(s)) - 1\|_{HS} &\leq \sum_{j=1}^{\infty} \frac{1}{j!} \|\alpha\Delta_i X(s)\|_{HS}^j \\
&\leq \|\alpha\Delta_i X(s)\|_{HS} \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \|\alpha\Delta_i X(s)\|_{HS}^j \\
&\leq \|\alpha\Delta_i X(s)\|_{HS} \sum_{j=0}^{\infty} \frac{1}{j!} \|\alpha\Delta_i X(s)\|_{HS}^j .
\end{aligned}$$

By the equivalence of norms in finite dimensions there is some finite constant so that for any $A \in \mathfrak{K}$, we have $\|A\|_{HS} \leq C\|A\|_{\mathfrak{K}}$. Thus

$$\|F(\alpha\Delta_i X(s)) - 1\|_{HS} \leq \|\alpha\Delta_i X(s)\|_{HS} \exp\{C|\alpha| \|\Delta_i X(s)\|_{\mathfrak{K}}\}. \tag{6.20}$$

But picking a particular orthonormal basis $\{A\}$ of $\mathfrak{K}$, we have

$$\|\Delta_i X(s)\|_{\mathfrak{K}}^2 = \sum_A \left(\Delta_i X^A(s)\right)^2.$$

By Eq. [3.5] we see that

$$\begin{aligned}\|\Delta_i X(s)\|_{\mathfrak{K}}^2 &= \sum_A \left[\Delta_i b^A(s) - \int_0^s \Delta_i b^A(\sigma)\frac{(1-s)}{(1-\sigma)^2}d\sigma\right]^2 \\ &\leq \sum_A \left[\left|\Delta_i b^A(s)\right| + \int_0^s \left|\Delta_i b^A(\sigma)\right| \frac{(1-s)}{(1-\sigma)^2}d\sigma\right]^2 \\ &\leq \sum_A \left[\sup_{(\tau,\sigma)\in R_{Ts}} \left|b^A_{\tau\sigma}\right| \left(1 + \int_0^s \frac{(1-s)}{(1-\sigma)^2}d\sigma\right)\right]^2,\end{aligned}$$

where we have let R_{Ts} denote the rectangle $[0,T]\times[0,s]$ in $\mathbb{R}^2$. Now

$$\int_0^s \frac{(1-s)}{(1-\sigma)^2}d\sigma = (1-s)\int d\left(\frac{1}{1-\sigma}\right) = s \leq 1,$$

and therefore

$$\|\Delta_i X(s)\|_{\mathfrak{K}}^2 \leq 4\sum_A \sup_{(\tau,\sigma)\in R_{1,1}} \left|b^A_{\tau\sigma}\right|^2. \tag{6.21}$$

By Theorem 6.4 we have

$$E\sup_{R_{1,1}} \left|b^A_{\tau\sigma}\right|^2 \leq 2^4 \sup_{R_{1,1}} E\left|b^A_{\tau\sigma}\right|^2 \leq 2^4 < \infty. \tag{6.22}$$

Therefore by Eq. [6.21]

$$\exp\left[C\left|\alpha\right|\|\Delta_i X(s)\|_{\mathfrak{K}}\right] \leq \exp\left[C\sum_A \sup_{R_{1,1}}\left|b^A_{\tau\sigma}\right|^2\right] =: \widetilde{C}.$$

By Eq. [6.22],

$$\widetilde{C} < \infty,\ P\text{-a.s.}$$

Thus returning to Eq. [6.20], we have P-a.s..,

$$\begin{aligned}\sum_{i=1}^{n_{\widetilde{\mathbb{P}}_r}} \|F(\alpha\Delta_i X(s)) - 1\|_{HS}^2 &\leq \|\alpha\Delta_i X(s)\|_{HS}^2 \exp C\left|\alpha\right|\|\Delta_i X(s)\|_{\mathfrak{K}} \\ &\leq \widetilde{C}\sum_{i=1}^{n_{\widetilde{\mathbb{P}}_r}} \|\alpha\Delta_i X(s)\|_{HS}^2 \\ &\leq \widetilde{C}\sum_A\sum_{i=1}^{n_{\widetilde{\mathbb{P}}_r}} \left|\Delta_i X^A(s)\right|^2. \end{aligned} \tag{6.23}$$

$\widetilde{C}$ is independent of the partition sequence $\left\{\widetilde{\mathbb{P}}_r\right\}$ as well as the partition points $\{t_i\}$. Thus it will suffice to show that we can find a subsequence of partitions $\{\mathbb{P}^z\}$ so that

$$\sup_r \sup_{[0,1-\varepsilon]} \sum_A \sum_{i=1}^{n_{\mathbb{P}^z_r}} \left|\Delta_i X^A(s)\right|^2 < \infty. \tag{6.24}$$

Let $\{\mathbb{P}_r^1\}$ be the subsequence of $\left\{\widetilde{\mathbb{P}}_r\right\}$ from Lemma 6.30. By Ito's Lemma,

$$\sum_{i=1}^{n_{\mathbb{P}_r^1}} \left|\Delta_i X^A(s)\right|^2 = sT + \sum_{i=1}^{n_{\mathbb{P}_r^1}} \int_0^s 2\Delta_i X^A(\sigma)\Delta_i X^A(d\sigma). \tag{6.25}$$

By Theorem 3.19

$$\Delta_i X^A(\sigma) = \Delta_i b^A(\sigma) - \int_0^\sigma \Delta_i b^A(u) \frac{(1-\sigma)}{(1-u)^2} du$$

and so

$$\Delta_i X^A(d\sigma) = \Delta_i b^A(d\sigma) - \frac{\Delta_i b^A(\sigma)}{(1-\sigma)^2} d\sigma + \int_0^\sigma \frac{\Delta_i b^A(u)\, du}{(1-u)^2} d\sigma.$$

Therefore

$$\begin{aligned}
&\int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^1}} \Delta_i X^A(\sigma)\Delta_i X^A(d\sigma) \\
&\quad\le \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^1}} \Delta_i X^A(\sigma)\Delta_i b^A(d\sigma) - \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^1}} \frac{\Delta_i X^A(\sigma)\Delta_i b^A(\sigma)}{(1-\sigma)^2} d\sigma \\
&\qquad + \int_0^s \int_0^\sigma \sum_{i=1}^{n_{\mathbb{P}_r^1}} \frac{\Delta_i X^A(\sigma)\Delta_i b^A(u)\, du}{(1-u)^2} d\sigma \\
&\quad= I_1 - I_2 + I_3. \qquad (6.26)
\end{aligned}$$

$$I_1 = \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^1}} \Delta_i X^A(\sigma)\Delta_i b^A(d\sigma) \text{ is an } \mathfrak{F}_{1s}\text{-martingale.}$$

By Doob's L^p-inequality, we have

$$\begin{aligned}
E\left[\sup_{s\in[0,1]} I_1\right]^2 &\le 2E\left[\int_0^1 \sum_{i=1}^{n_{\mathbb{P}_r^1}} \Delta_i X^A(\sigma)\Delta_i b^A(d\sigma)\right]^2 \\
&= 2\sum_{i=1}^{n_{\mathbb{P}_r^1}} (\Delta_i t)^2 \int_0^1 G_0(\sigma,\sigma)\, d\sigma \\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

Thus we can find a subsequence $\{\mathbb{P}_r^2\}$ of $\{\mathbb{P}_r^1\}$ so that $\sup_{s\in[0,1]} |I_1| \to 0$ P-a.s. as $r\to\infty$ and so that

$$\sup_r \sup_{s\in[0,1]} |I_1| < \infty \ P\text{-a.s.}$$

Applying Theorem 3.19 to I_2 yields

$$\begin{aligned}
&\sup_r \sup_{s\in[0,1-\varepsilon]} |I_2| \\
&\quad = \sup_r \sup_{s\in[0,1-\varepsilon]} \left| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^2}} \frac{\Delta_i X^A(\sigma)\Delta_i b^A(\sigma)}{(1-\sigma)^2} d\sigma \right| \\
&\quad \leq \sup_r \sup_{s\in[0,1]} \left| \sum_{i=1}^{n_{\mathbb{P}_r^2}} \Delta_i b^A(\sigma)\, \Delta_i b^A(\sigma) \right| \int_0^{1-\varepsilon} \frac{d\sigma}{(1-\sigma)^2} \\
&\qquad + \sup_r \sup_{s\in[0,1]} \left| \sum_{i=1}^{n_{\mathbb{P}_r^2}} \Delta_i b^A(\sigma)\, \Delta_i b^A(u) \right| \int_0^{1-\varepsilon} \int_0^{\sigma} \frac{d\sigma du}{(1-\sigma)(1-u)^2}.
\end{aligned}$$

This expression is finite by Lemma 6.30. A last invocation of Theorem 3.19 and Lemma 6.30 gives

$$\begin{aligned}
&\sup_r \sup_{s\in[0,1-\varepsilon]} |I_3| \\
&\quad = \sup_r \sup_{s\in[0,1-\varepsilon]} \left| \int_0^s \int_0^{\sigma} \sum_{i=1}^{n_{\mathbb{P}_r^2}} \frac{\Delta_i X^A(\sigma)\Delta_i b^A(u)\, du}{(1-u)^2} d\sigma \right| \\
&\quad \leq \sup_r \sup_{s\in[0,1]} \left| \sum_{i=1}^{n_{\mathbb{P}_r^2}} \Delta_i b^A(\sigma)\, \Delta_i b^A(u) \right| \int_0^{1-\varepsilon} \int_0^{\sigma} \frac{du d\sigma}{(1-u)^2} \\
&\qquad + \sup_r \sup_{s\in[0,1-\varepsilon]} \left| \sum_{i=1}^{n_{\mathbb{P}_r^2}} \Delta_i b^A(u)\, \Delta_i b^A(\nu) \right| \\
&\qquad\quad \times \int_0^{1-\varepsilon} \int_0^{\sigma} \int_0^{\sigma} \frac{(1-\sigma) d\nu du d\sigma}{(1-\nu)^2(1-u)^2} \\
&\quad < \infty.
\end{aligned}$$

Therefore returning to 6.26 we see that

$$\sup_r \sup_{s\in[0,1-\varepsilon]} \left| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^2}} \Delta_i X^A(\sigma)\Delta_i X^A(d\sigma) \right| < \infty \ P\text{-a.s.}$$

Thus by Eq. [6.25] we see that

$$\sup_r \sup_{s\in[0,1-\varepsilon]} \sum_{i=1}^{n_{\mathbb{P}_r^2}} \left|\Delta_i X^A(s)\right|^2 < \infty \ P\text{-a.s.}$$

Taking $\mathbb{P}_r^z = \mathbb{P}_r^2$ we see that Eq. [6.24] is satisfied and so we are done. ∎

Proof of Lemma 6.30. By Ito's Lemma,

$$\sum_{i=1}^{n_{\mathbb{P}_r}} \left|\Delta_i b^A(s)\right|^2 = sT + \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r}} 2\Delta_i b^A(\sigma)\Delta_i b^A(d\sigma). \tag{6.27}$$

Therefore,

$$\sup_{[0,1]} \sum_{i=1}^{n_{\mathbb{P}_r}} \left|\Delta_i b^A(s)\right|^2 = 1 + \sup_{[0,1]} \left| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r}} 2\Delta_i b^A(\sigma)\Delta_i b^A(d\sigma) \right|.$$

Since the process

$$s \to \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r}} 2\Delta_i b^A(\sigma)\Delta_i b^A(d\sigma)$$

is an $\mathfrak{F}_{1s}$-martingale, by Doob's L^p-inequality we have

$$\begin{aligned} E \sup_{s\in[0,1]} &\left| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r}} \Delta_i b^A(\sigma)\Delta_i b^A(d\sigma) \right|^2 \\ &\leq 4E \left| \int_0^1 \sum_{i=1}^{n_{\mathbb{P}_r}} \Delta_i b^A(\sigma)\Delta_i b^A(d\sigma) \right|^2 \\ &\leq \sum_{i=1}^{n_{\mathbb{P}_r}} \int_0^1 E\left(\Delta_i b^A\right)^2(\sigma)\Delta_i t d\sigma \\ &= \sum_{i=1}^{n_{\mathbb{P}_r}} \int_0^1 \left(\Delta_i t\right)^2 \sigma d\sigma \leq \frac{1}{2}\left|\mathbb{P}_r\right| \to 0. \end{aligned}$$

Therefore there exists a subsequence of partitions $\left\{\mathbb{P}_r^1\right\}$ so that the expression

$$\sup_{s\in[0,1]} \left| \int_0^s \sum_{i=1}^{n_{\mathbb{P}_r^1}} \Delta_i b^A(\sigma)\Delta_i b^A(d\sigma) \right| \to 0,\ P\text{-a.s. as } r \to \infty.$$

Returning to Eq.[6.27]we see that

$$\sup_r \sup_{s\in[0,1]} \sum_{i=1}^{n_{\mathbb{P}_r^1}} \left|\Delta_i b^A(s)\right|^2 < \infty,\ P\text{-a.s.} \tag{6.28}$$

If $s \geq \sigma$ then

$$s \to \sum_{i=1}^{n_{\mathbb{P}_r^1}} \left[\Delta_i b^A(s) - \Delta_i b^A(\sigma)\right] \Delta_i b^A(\sigma) \text{ is an } \mathfrak{F}_{1s}\text{-martingale},$$

and so by Doob's inequality, we have

$$\begin{aligned} E \sup_{s\in[0,1]} &\left| \sum_{i=1}^{n_{\mathbb{P}_r^1}} \left[\Delta_i b^A(s) - \Delta_i b^A(\sigma)\right] \Delta_i b^A(\sigma) \right|^2 \\ &\leq 2E \left| \sum_{i=1}^{n_{\mathbb{P}_r^1}} \left[\Delta_i b^A(1) - \Delta_i b^A(\sigma)\right] \Delta_i b^A(\sigma) \right|^2. \end{aligned}$$

By incremental independence of the Brownian sheet, this previous expression is just

$$\begin{aligned} &= E\sum_{i=1}^{n_{\mathbb{P}_r^1}} E\left[\Delta_i b^A(1)-\Delta_i b^A(\sigma)\right]^2 E\left(\Delta_i b^A\right)^2(\sigma) \\ &= E\sum_{i=1}^{n_{\mathbb{P}_r^1}} (\Delta_i t)^2 (1-\sigma)\sigma \\ &\to 0 \text{ as } r\to\infty. \end{aligned}$$

Thus there is a subsequence $\left\{\mathbb{P}_r^2\right\}$ so that

$$\sup_r \sup_{s>\sigma} \sum_{i=1}^{n_{\mathbb{P}_r^2}} \left[\Delta_i b^A(s)-\Delta_i b^A(\sigma)\right]\Delta_i b^A(\sigma) < \infty \ P\text{-a.s.} \tag{6.29}$$

If $1\ge s\ge\sigma$, we can always replace a given sequence $\{\mathbb{P}\}$of partitions by a subsequence $\left\{\mathbb{P}_r^2\right\}$ so that

$$\begin{aligned} &\sup_r \sup_{1\ge s\ge\sigma\ge 0}\left|\sum_{i=1}^{n_{\mathbb{P}_r^2}}\Delta_i b^A(s)\Delta_i b^A(\sigma)\right| \\ &\quad\le \sup_r \sup_{1\ge s\ge\sigma}\left|\sum_{i=1}^{n_{\mathbb{P}_r^2}}\left[\Delta_i b^A(s)-\Delta_i b^A(\sigma)\right]\Delta_i b^A(\sigma)\right| + \sup_r \sup_{\sigma\in[0,1]}\sum_{i=1}^{n_{\mathbb{P}_r^2}}\left(\Delta_i b^A\right)^2(\sigma) \\ &\quad< \infty \ P\text{-a.s. by Eqs. [6.29] and [6.28].} \end{aligned}$$

Letting $\{\mathbb{P}'\}$ denote this subsequence $\left\{\mathbb{P}_r^2\right\}$, we are done. ∎

7. $HKM\downarrow_{\mathfrak{F}_s}\backsim PWM\downarrow_{\mathfrak{F}_s}$

Theorem 7.1. *$\nu_1(e,\cdot)$[Heat Kernel measure on $L(K)$] is equivalent to μ_0[Pinned Wiener Measure on $L(K)$] as measures on $(L(K),\mathfrak{G}_z)$ where $\mathfrak{G}_z\equiv\sigma\langle x_t : t\in[0,z]\rangle$, for any $z<1$.*

We supply the proof of this result at the end of this section.

Definition 7.2. Let B_{ts} be defined to solve the Fisk-Stratonowicz equation $B_{t\delta s} = b_{t\delta s}B_{ts}$ with $B_{t0}=e$ where b is the Brownian sheet from Theorem 3.19. By the following Remark, we see that $t\to B_{ts}$ is a Brownian motion on K with parameter s.

Remark 7.3 ($t\to B_{ts}$ is a Brownian motion on K). Let $\overline{h}_{ts}$ solve $\overline{h}_{t\delta s}=b_{t\delta s}\overline{h}_{ts}$ with $\overline{h}_{t0}=e$. Let $\widetilde{h}_{ts}\equiv\overline{h}_{st}$ and $\widetilde{b}_{ts}\equiv b_{st}$. Then $s\to\overline{h}_{ts}$ is the same process as $s\to\widetilde{h}_{st}$ and so $\overline{h}_{t\delta s}=\widetilde{h}_{\delta st}$. Similarly, $b_{t\delta s}=\widetilde{b}_{\delta st}$. Thus $\widetilde{h}_{st}$ solves $\widetilde{h}_{\delta st}=\widetilde{b}_{\delta st}\widetilde{h}_{st}$ with $\widetilde{h}_{0t}=e$. To put it another way, $\widetilde{h}_{ts}$ solves $\widetilde{h}_{\delta ts}=\widetilde{b}_{\delta ts}\widetilde{h}_{ts}$ with $\widetilde{h}_{0s}=e$. Then $h\equiv\widetilde{h}^{-1}$ solves $h_{\delta ts}=-h_{ts}\widetilde{b}_{\delta ts}$ with $h_{0s}=e$. By Lemma 3.9 if β is a $\mathfrak{K}$-valued Brownian sheet and h_{ts} solves $h_{\delta ts}=h_{ts}\beta_{\delta ts}$ with $h_{0s}=e$ then the process $s\mapsto h_{ts}$ is a K-valued Brownian motion with parameter t. Taking $\beta=-\widetilde{b}$, we see that $s\mapsto\widetilde{h}_{ts}^{-1}$ is a K-valued Brownian motion with parameter t. Thus $s\mapsto\widetilde{h}_{ts}$ is also a K-valued Brownian motion with parameter t and so $s\mapsto\overline{h}_{st}$ is a Brownian motion on K with parameter t. Switching t and s yields $t\mapsto\overline{h}_{ts}$ is a Brownian motion on K with parameter s.

Remark 7.4. Let $\pi_s : C([0,1] \to L) \to C([0,s] \to L); \pi_s(x.)(r) = x(r)$ for any $r \leq s$. We make no distinction between a measure ν_1 on $(C([0,s] \to L), \sigma\langle x_r : r \leq s\rangle)$ and a measure ν_2 on $(C([0,1] \to L), \sigma\langle x_r : r \leq s\rangle)$ so long as $\nu_1(F \circ \pi_s) = \nu_2(F)$ for any $F : C([0,s] \to L) \to \mathbb{R}$. where L stands for either K or $\mathfrak{K}$.

Lemma 7.5. *If $\kappa_1 \backsim \kappa_2$ then $\kappa_1 \otimes \nu \backsim \kappa_2 \otimes \nu$, where κ_1, κ, ν are probability measures.*

Proof. Will suffice to show that if $\kappa_1 \ll \kappa_2$ then $\kappa_1 \otimes \nu \ll \kappa_2 \otimes \nu$. For rectangles, it is clear that $(\kappa_1 \otimes \nu)(1_A(x) 1_B(y)) = (\kappa_2 \otimes \nu)(1_A(x) f(x) 1_B(y))$. This extends to linear combinations of rectangles by linearity and all bounded measurable functions by dominated convergence. Thus $d(\kappa_1 \otimes \nu)/d(\kappa_2 \otimes \nu)(x,y) = d\kappa_1/d(\kappa_2)(x)$. Thus $\kappa_1 \otimes \nu \ll \kappa_2 \otimes \nu$. ∎

Theorem 7.6. *Let $t \mapsto g_{t\cdot}$ be our $L(K)$-valued Brownian motion from Definition 2.22 and let $s \to B_{ts}$ be the K-valued Brownian motion of Definition 7.2. Then $g_{T\cdot}$ and $B_{T\cdot}$ have equivalent laws as measures on $C([0,s] \to K)$ for any $s < \frac{\sqrt{1+4T}-1}{2T}$.*

We prove this result after the proof of Theorem 7.7.

Theorem 7.7. *$Law\ Y_{T\cdot} \backsim Law\ b_{T\cdot}$ as measures on $C([0,s] \to \mathfrak{K})$ for any $s < \frac{\sqrt{1+4T}-1}{2T}$. Here the random variable*

$$Y_{T\cdot} \equiv \int_{R_{T\cdot}} Ad_{g_{t\sigma}} b_{dtd\sigma} - \int_0^{\cdot} \frac{d\sigma}{1-\sigma} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma}$$

is as in Theorem 6.11.

The proof of Theorem 7.7 is given after that of Lemma 7.11.

Remark 7.8. Since for $s < 1$, $Law\ X_{T\cdot} \backsim Law\ b_{T\cdot}$ (as measures on $C([0,s] \to \mathfrak{K})$), one might suspect $Law\ X \backsim Law\ b$ (as measures on $C([0,1] \times [0,s] \to \mathfrak{K})$) which should then indicate that

$$Law\ Y_{T\cdot} = Law \int_{R_{T\cdot}} Ad_{g_{t\sigma}} X_{dtd\sigma} \backsim Law \int_{R_{T\cdot}} Ad_{g_{t\sigma}} b_{dtd\sigma} = Law\ (b_{T\cdot}).$$

Unfortunately in the t-variable, $X_{\cdot s}$ and $b_{\cdot s}$ are Brownian motions with parameters $s - s^2$ and s respectively.

Thus $Law X \perp Law b$ since

$$P_X\left(\sum_i |\Delta_i \omega(s)|^2 \to s - s^2\right) = 1,$$

while

$$P_b\left(\sum_i |\Delta_i \omega(s)|^2 \to s\right) = 1.$$

Hence these two measures live on different sets.

The proof of Theorem 7.7 relies heavily on Girsanov's Theorem which we state here for convenience.

Theorem 7.9 (Girsanov, see [22]). *Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\cdot\}, P)$ be a filtered probability space. Let $\beta_\cdot$ be a d-dimensional Brownian motion and let $Z_\cdot$ be an $\mathbb{R}^d$-valued adapted process so that $E \exp \frac{1}{2} \int_0^S |Z_s|^2 \, ds$ is finite and $\int_0^S \left(Z_s^i\right)^2 ds < \infty$ almost surely for any $i \in \{1, \cdots, d\}$. Define*

$$Z_\cdot \equiv \exp\left[\int_0^\cdot Z_s \cdot d\beta_s - \frac{1}{2}\int_0^\cdot |Z_s|^2 \, ds\right].$$

Define a new measure $\widetilde{P}_S$ on $\mathfrak{F}_S$ by setting $\widetilde{P}(A) = E 1_A Z_S$. Then $\widetilde{P}_S$ is a probability equivalent to P and the process $\{Y_t, \mathfrak{F}_t; 0 \le s \le S\}$ is a d-dimensional Brownian motion on $\left(\Omega, \mathfrak{F}_S, \widetilde{P}\right)$ where $Y_\cdot \equiv \beta_\cdot + \int_0^\cdot Z_s ds$.

Lemma 7.10. *The expression $\widetilde{X}_{t\sigma} \equiv \int_0^t Ad_{g_{u\sigma}} X_{du\sigma}$ has the same law as $X_{t\sigma}$.*

Proof. $\widetilde{X}_{t\sigma}$ is a $\mathfrak{F}_{t\sigma}$ martingale. To show $X_{\cdot s}$ and $\widetilde{X}_{\cdot s}$ have the same law it will suffice to show $\widetilde{X}_{\cdot\sigma}$ is a $\mathfrak{K}$-valued Brownian motion with parameter $\sigma - \sigma^2$. To this end, let $\{A\}$ run through an orthonormal basis of $\mathfrak{K}$. Then

$$\begin{aligned}
\widetilde{X}_{dt\sigma} \otimes \widetilde{X}_{dt\sigma} &= Ad_{g_{t\sigma}} X_{dt\sigma} \otimes Ad_{g_{t\sigma}} X_{dt\sigma} \\
&= \left(\sigma - \sigma^2\right) dt \sum_{A,B} \delta_{AB} Ad_{g_{t\sigma}} A \otimes Ad_{g_{t\sigma}} B \\
&= \left(\sigma - \sigma^2\right) dt \sum_{A} \left(Ad_{g_{t\sigma}} A\right)^{\otimes 2} \\
&= \left(\sigma - \sigma^2\right) dt \sum_{A} A^{\otimes 2}.
\end{aligned}$$

Thus we are done. ∎

Lemma 7.11. *Let X be the $\mathfrak{K}$-valued Brownian bridge sheet of Theorem 2.25. Then*

$$E \exp\left[\frac{1}{2}\int_0^s d\sigma \left|\int_0^T Ad_{g_{t\sigma}} \frac{X_{dt\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right] < \infty, \textit{if } s < \frac{\sqrt{1+4T}-1}{2T}.$$

Proof.

$$
\begin{aligned}
&E\exp\left[\frac{1}{2}\int_0^s d\sigma\left|\int_0^T Ad_{g_{t\sigma}}\frac{X_{dt\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right] \\
&\quad= E\sum_{P\geq 0}\left[\frac{1}{2}\int_0^s d\sigma\left|\int_0^T Ad_{g_{t\sigma}}\frac{X_{dt\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^2\right]^p \\
&\quad\leq \sum_{P\geq 0}\frac{s^{p-1}}{p!2^p}E\int_0^s d\sigma\left|\int_0^T Ad_{g_{t\sigma}}\frac{X_{dt\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^{2p} \text{ by Hölder's Inequality} \\
&\quad= \frac{1}{s}\sum_{P\geq 0}\frac{s^p}{p!2^p}\int_0^s d\sigma E\left|\int_0^T Ad_{g_{t\sigma}}\frac{X_{dt\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^{2p} \\
&\quad= \frac{1}{s}\sum_{P\geq 0}\frac{s^p}{p!2^p}\int_0^s d\sigma E\left|\int_0^T \frac{X_{dt\sigma}}{1-\sigma}\right|_{\mathfrak{K}}^{2p} \\
&\quad= \frac{1}{s}\int_0^s d\sigma\sum_{P\geq 0}\frac{1}{p!}E\left(\frac{s\left|X_{T\sigma}\right|_{\mathfrak{K}}^2}{2\left(1-\sigma\right)^2}\right)^p \\
&\quad= \frac{1}{s}\int_0^s d\sigma E\exp\left(\frac{s\left|X_{T\sigma}\right|_{\mathfrak{K}}^2}{2\left(1-\sigma\right)^2}\right) \\
&\quad= \frac{1}{s}\int_0^s d\sigma\int_{\mathbb{R}^{\dim\mathfrak{K}}}\exp\left(\frac{s\left|x\right|^2}{2\left(1-\sigma\right)^2}\right)\exp\left(\frac{-\left|x\right|^2}{2T\sigma\left(1-\sigma\right)}\right)\frac{dx}{\left[2\pi T\sigma\left(1-\sigma\right)\right]^{\frac{\dim\mathfrak{K}}{2}}} \\
&\quad= \frac{1}{s}\int_0^s d\sigma\int_{\mathbb{R}^{\dim\mathfrak{K}}}\exp\left[-\frac{\left|x\right|^2}{2T\sigma\left(1-\sigma\right)}\left(1-\frac{sT\sigma}{\left(1-\sigma\right)}\right)\right]\frac{dx}{\left[2\pi T\sigma\left(1-\sigma\right)\right]^{\frac{\dim\mathfrak{K}}{2}}} \\
&\quad= \frac{1}{s}\int_0^s d\sigma\left[2\pi T\sigma\left(1-\sigma\right)\right]^{-\frac{\dim\mathfrak{K}}{2}} \\
&\quad< \infty \iff 1-\frac{sT\sigma}{\left(1-\sigma\right)}>0, \forall\sigma\in\left[0,s\right].
\end{aligned}
$$

$$
\begin{aligned}
1-&\frac{sT\sigma}{\left(1-\sigma\right)}>0 \text{ for } \sigma\in\left[0,s\right] \\
&\iff \frac{\sigma}{1-\sigma}<\frac{1}{sT} \text{ for } \sigma\in\left[0,s\right] \\
&\iff \frac{s}{1-s}<\frac{1}{sT} \\
&\iff Ts^2+s-1<0 \\
&\iff s\in\left[0,\frac{\sqrt{1+4T}-1}{2T}\right).
\end{aligned}
$$

∎

We are now able to prove Theorem 7.7.

Proof. Define

$$\begin{aligned}\beta_{Ts} &= \int_{R_{Ts}} Ad_{g_{t\sigma}} b_{dtd\sigma};\\ Z_T(\sigma) &\equiv \frac{-1}{(1-\sigma)} \int_0^T Ad_{g_{t\sigma}} X_{dt\sigma}.\end{aligned}$$

By definition of $Y_{T\cdot}$ in Theorem 6.11

$$Y_{T\cdot} \equiv \beta_{T\cdot} + \int_0^{\cdot} d\sigma Z_T(\sigma).$$

By Lemma 7.11,

$$E\exp \int_0^S |Z_T(\sigma)|^2_{\mathfrak{K}} d\sigma < \infty \text{ whenever } S < \frac{\sqrt{1+4T}-1}{2T}.$$

Thus the measure

$$d\widetilde{P}_S \equiv \exp\left[\int_0^S Z_T(s)\cdot d\beta_{Ts} - \frac{1}{2}\int_0^S |Z_T(s)|^2 ds\right] dP$$

is a probability on $\mathfrak{F}_{TS}$ and the process $\{Y_{Ts}, \mathfrak{F}_{Ts}; 0 \le s \le S\}$ is a $\widetilde{P}_S$-Brownian motion on $\mathfrak{K}$. Thus for any set $\mathcal{A} \subset (C[0,S] \to \mathfrak{K})$

$$E1_{\mathcal{A}} \circ \beta_{T\cdot} = 0 \iff \widetilde{E}1_{\mathcal{A}} \circ Y_{T\cdot} = 0 \iff E1_{\mathcal{A}} \circ Y_{T\cdot} = 0,$$

since the measures $\widetilde{P}_S$ and P are equivalent on $\mathfrak{F}_{TS}$. [Note:- it is essential that $\mathcal{A}$ only depend on the path to time S or else $1_{\mathcal{A}} \circ Y_{T\cdot}$ will cease to be $\mathfrak{F}_{TS}$-measurable.] ∎

We now return to the proof of Theorem 7.6.

Proof. Fix s. Pick T so that $s < \frac{\sqrt{1+4T}-1}{2T}$.

Define a map , from $C([0,z] \to \mathfrak{K})$ to $C([0,z] \to K)$ so that , $(x_\cdot) = y_\cdot$, where $y_{\delta s} = x_{\delta s} y_s$ with $y_0 = e$, the integration being done with respect to the Wiener Measure on $C([0,z] \to \mathfrak{K})$ with parameter T.

If we can show that , $: b_{T\cdot} \mapsto B_{T\cdot}$, and , $: Y_{T\cdot} \mapsto g_{T\cdot}$ we shall be done. This is so because by Theorem 7.7 *Law* $Y_{T\cdot}$ is equivalent to *Law* $b_{T\cdot}$. Thus

$$\begin{aligned} & E1_A \circ g_{T\cdot} = 0 \\ \iff & E1_{\Gamma_1^{-1}(A)} \circ Y_{T\cdot} = 0 \\ \iff & E1_{\Gamma_1^{-1}(A)} \circ b_{T\cdot} = 0 \\ \iff & E1_A \circ B_{T\cdot} = 0. \end{aligned}$$

Hence by the Radon-Nikodym Theorem *Law* $g_{T\cdot}$ is equivalent to *Law* $B_{T\cdot}$.

To show , $: b_{T\cdot} \mapsto B_{T\cdot}$, and , $: Y_{T\cdot} \mapsto g_{T\cdot}$, we shall invoke Lemma 8.1. Let Ω_0 be $C([0,1] \to \mathfrak{K})$ where K is identified with $\mathbb{R}^{\dim \mathfrak{K}}$. Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_{T\cdot}\}, P)$ be our standard probability space as in Definition 2.22. The stochastic differential equation we will use in Lemma 8.1 will be

$$y_{ds} = x_{\delta s} y_s = x_{ds} y_s + \frac{T ds}{2}\left(\sum_A A^2\right) y_s \text{ with } y_0 = 1.$$

Here the $\{A\}$ run through an orthonormal basis of the Lie algebra $\mathfrak{K}$. Clearly the boundedness conditions of the Lemma are satisfied. Also both *Law* $b_{T\cdot}$ and *Law* $Y_{T\cdot}$

are absolutely continuous with respect to $\mu' = Law\ b_{T\cdot}$. Thus either $b_{T\cdot}$ or $Y_{T\cdot}$ can be taken to be the $X.$ in the Lemma. Thus we see that , $(b_{T\cdot})(\delta s) = b_{T\delta s},\ (b_{T\cdot})(\delta s)$ and , $(Y_{T\cdot})(\delta s) = Y_{T\delta s},\ (Y_{T\cdot})(\delta s)$. By Definition 7.2 $B_{t\delta s} \equiv b_{t\delta s} B_{ts}$ with $B_{t0} = 1$. Hence , $(b_{T\cdot}) = B_{T\cdot}$. By Theorem 6.11, $g_{t\delta s} \equiv Y_{t\delta s} g_{ts}$ with $g_{t0} = 1$. Hence , $(Y_{T\cdot}) = g_{T\cdot}$ and so we are done. ∎

We now return to the proof of Theorem 7.1.

Proof. Let μ_0 be Pinned Wiener measure on $L(K)$ and let μ be Wiener measure on $C([0,1] \to K)$ as in Definitions 2.7 and 2.9. Then $\mu = Law[B_{1\cdot}]$ since $B_{1\cdot}$ is a standard K-valued Brownian motion by Definition 7.2. A key fact that we shall exploit in this proof is μ_0 is equivalent to μ on $\mathfrak{G}_z$ for any $z < 1$.

Fix $z < 1$.Now $\lim_{T\to 0} \frac{\sqrt{1+4T}-1}{2T} = 1$ so there exists an $N \in \mathbb{N}$ large so that $z < \frac{\sqrt{1+4/N}-1}{2/N}$. Let $T \equiv 1/N$. Let $F : C([0,1] \to K) \to \mathbb{R}$ so that $F \in \mathfrak{G}_z$. Then

$$\begin{aligned} \nu_1(e, A) &= P\{g_{1\cdot} \in A\} \\ &= P\left\{g_{(1/N)\cdot}\left(g^{-1}_{(1/N)\cdot} g_{(2/N)\cdot}\right)\cdots\left(g^{-1}_{(N-1/N)\cdot} g_{1\cdot}\right) \in A\right\} \\ &= \left(\otimes_{i=1}^N Law_{g_{T\cdot}}\right)(k_1 \cdots k_N \in A), \end{aligned}$$

where $A' \equiv \{(k_1, \cdots, k_n) : k_1 \cdots k_N \in A\}$. Now by Theorem 7.6, since the condition $z < \frac{\sqrt{1+4T}-1}{2T}$ obtains, $g_{T\cdot}$ has a law equivalent to that of $B_{T\cdot}$, on the restricted σ-algebra $\mathfrak{G}_z$. Invoking Lemma 7.5 repeatedly, we see that $\otimes_{i=1}^N Law_{g_{T\cdot}} \backsim \otimes_{i=1}^N Law_{B_{T\cdot}}$, on the restricted σ-algebra $\mathfrak{G}_z$. Thus if A is $\mathfrak{G}_z$-measurable, $\nu_1(e, A) = 0$ holds if and only if $\left(\otimes_{i=1}^N Law_{B_{T\cdot}}\right)(k_1 \cdots k_N \in A) = 0$ holds. Since $t \mapsto B_{ts}$ is a K-valued Brownian motion (see Remark 7.3), it exhibits incremental independence. Thus $(Law_{B_{1\cdot}})(A) = 0$ if and only if $\left(\otimes_{i=1}^N Law_{B_{T\cdot}}\right)(k_1 \cdots k_N \in A) = 0$. Thus we have $\nu_1(e, \cdot) \downarrow \mathfrak{G}_z \backsim \mu \downarrow \mathfrak{G}_z \backsim \mu_0 \downarrow \mathfrak{G}_z$ and so we are done. ∎

8. Appendix

8.1. General Technical results.

Lemma 8.1 (General Technical Lemma). *Let X be an $(\Omega, \mathfrak{F}_t, P)$ continuous semimartingale taking values in $\mathbb{R}^d$. $\Omega_0 \equiv C\left([0,1] \to \mathbb{R}^d\right)$ is to be thought of as the measure space $(\Omega_0, \mathfrak{H}_t, \mu')$ where $Law\ X. \ll \mu'$, and $\mathfrak{H}_t \equiv \sigma\langle X_r : r \le t\rangle$. Let x, ω denote members of the probability spaces Ω_0 and Ω respectively. Let a be an $\mathbb{R}^d$-valued Ω_0-random variable that solves the following stochastic differential equation*

$$a(dt, x) = \sum_j c(t, a(t), x)\, r(dt, x) + c^0(t, a(t), x)\, dt \text{ with } a_i(0, x) = K_i, \tag{8.1}$$

where $c \in C_b\left(\mathbb{R} \times \mathbb{R}^d \times \Omega_0 \to \mathbb{R}^d \otimes \mathbb{R}^{d*}\right)$, $c^0 \in C_b\left(\mathbb{R} \times \mathbb{R}^d \times \Omega_0 \to \mathbb{R}^d\right)$. Then $A(t, \omega) \equiv a(t, X.(\omega))$ solves the Stochastic differential equation.

$$A(dt, \omega) = \sum_j C(t, A(t, \omega), \omega)\, R(dt, \omega) + C^0(t, A(t, \omega), \omega)\, dt \tag{8.2}$$

with $A_i(0, \omega) = K_i$; such that $C(t, \xi, \omega) \equiv c(t, \xi, X.(\omega))$, $R(t, \omega) \equiv r(t, X.(\omega))$, $C^0(t, \xi, \omega) \equiv c^0(t, \xi, X.(\omega))$, and $\xi \in \mathbb{R}^d$.

Proof. For convenience, let $\mu \equiv \textit{Law } X.$, For explicitness we will not suppress dependence on the probability space as is done traditionally.

$$a_i(T,x) = K_i + \int_0^T \sum_j c_{ij}(t,x,a(t))\, r_i(dt,x) + c_i^0(t,x,a(t))\,dt.$$

This means that the expression

$$\sum_k \sum_j c_{ij}\left(t_{k-1}^{\mathbb{P}}, x, a\left(t_{k-1}^{\mathbb{P}}\right)\right)\left(r_i\left(t_k^{\mathbb{P}}, x\right) - r_i\left(t_{k-1}^{\mathbb{P}}, x\right)\right)$$

converges in $L^2(\mu')$ as $|\mathbb{P}| \to 0$ to the expression

$$a_i(T,x) - K_i - \int_0^T c_i^0(t,x,a(t))\,dt.$$

Since L^2-convergence implies convergence in measure, we see that the expression

$$\sum_k \sum_j c_{ij}\left(t_{k-1}^{\mathbb{P}}, x, a\left(t_{k-1}^{\mathbb{P}}\right)\right)\left(r_i\left(t_k^{\mathbb{P}}, x\right) - r_i\left(t_{k-1}^{\mathbb{P}}, x\right)\right)$$

converges in the measure μ' as $|\mathbb{P}| \to 0$ to

$$a_i(T,x) - K_i - \int_0^T c_i^0(t,x,a(t))\,dt.$$

Since $\mu \ll \mu'$, the following statement holds. For any $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ so that if $\mu'(A) < \delta(\varepsilon)$ then $\mu(A) < \varepsilon$.(if this were not so can have $\mu'(A) = 0$ and $\mu(A) > \varepsilon$ for some ε). Hence

$$\lim_{|\mathbb{P}|\to 0} \mu\left(\left|\begin{array}{c} \sum_k \sum_j c_{ij}\left(t_{k-1}^{\mathbb{P}}, x, a\left(t_{k-1}^{\mathbb{P}}\right)\right)\left(r_i\left(t_k^{\mathbb{P}}, x\right) - r_i\left(t_{k-1}^{\mathbb{P}}, x\right)\right) \\ -\left(a_i(T,x) - K_i - \int_0^T c_i^0(t,x,a(t))\,dt\right) \end{array}\right| < \varepsilon\right) = 0.$$

This in turn implies that as $|\mathbb{P}| \to 0$,

$$P\left(\left|\begin{array}{c} \sum_{kj} C_{ij}\left(t_{k-1}^{\mathbb{P}}, \omega, a\left(t_{k-1}^{\mathbb{P}}, X(\cdot,\omega)\right)\right)\left(R_i\left(t_k^{\mathbb{P}}, \omega\right) - R_i\left(t_{k-1}^{\mathbb{P}}, \omega\right)\right) \\ +K_i + \int_0^T c_i^0(t, X(\cdot,\omega), a(t, X(\cdot,\omega)))\,dt - a_i(T, X(\cdot,\omega)) \end{array}\right| < \varepsilon\right) \to 0.$$

However since, essentially by assumption, the expression

$$\begin{aligned} &K_i + \int_0^T C_i^0(t, X(\cdot,\omega), a(t, X(\cdot,\omega)))\,dt \\ &+ \sum_{kj} C_{ij}\left(t_{k-1}^{\mathbb{P}}, \omega, a\left(t_{k-1}^{\mathbb{P}}, X(\cdot,\omega)\right)\right)\left(R_i\left(t_k^{\mathbb{P}}, \omega\right) - R_i\left(t_{k-1}^{\mathbb{P}}, \omega\right)\right) \end{aligned}$$

has an L^2-limit, it must be equal to $a_i(T, X(\cdot,\omega))$. Hence $A_i(T,\omega) \equiv a_i(T, X(\cdot,\omega))$ satisfies Eq. (8.2) as desired. ∎

Theorem 8.2 (Kolmogorov's extension Theorem)**.** *Let $(\mathbb{E}, d)$ be a complete metric space and let U_x be an $\mathbb{E}$-valued process for all dyadic rationals x in $\mathbb{R}^n$. Suppose that for all x, y we $d(U_x, U_y)$ is a random variable and that there exist strictly positive constants ε, c, β so that*

$$P\left[d(U_x, U_y)^{\varepsilon}\right] \le C\,\|x-y\|^{n+\beta}.$$

Then P-a.s. the function $x \to U_x$ can be extended uniquely to a continuous function from $\mathbb{R}^n$ to $\mathbb{E}$.

Proof. See Theorem 53 of Chapter IV of Protter [30]. ∎

8.2. **Brownian Sheets and bridges.**

Lemma 8.3 (Quadratic Variations)**.** *We compute some quadratic variations we shall later find useful. Let A, B be two perpendicular unit vectors. As in Definitions 3.5 and 2.12, let $G(s,\sigma)$ denote $s \wedge \sigma$, and let $G_0(s,\sigma)$ denote $s\wedge\sigma - s\sigma$. Then letting b^A_{ts} denote $\langle b_{ts}, A\rangle_{\mathfrak{K}}$ and X^A_{ts} denote $\langle X_{ts}, A\rangle_{\mathfrak{K}}$ we have:-*

1. $$\begin{aligned} b^A_{dts} b^B_{dt\sigma} &= \langle A,B\rangle_{\mathfrak{K}}\, G(s,\sigma)\, dt \\ b^A_{tds} b^B_{\tau ds} &= \langle A,B\rangle_{\mathfrak{K}}\, G(t,\tau)\, ds \end{aligned}$$
2. $$X^A_{dts} X^B_{dt\sigma} = \langle A,B\rangle_{\mathfrak{K}}\, G_0(s,\sigma)\, dt.$$
3. $$EX^A_{t\sigma} b^B_{ts} = \langle A,B\rangle_{\mathfrak{K}} (1-\sigma)\log\left(\frac{1}{1-s\wedge\sigma}\right) dt.$$

Proof.

$$\text{Let } \mathfrak{G}_t \equiv \sigma\left\{b_{rs} | s\in[0,1],\, r\in[0,t]\right\}.$$

Then Remark 6.2 implies that

$$\mathfrak{G}_t = \sigma\left\{X_{rs} | s\in[0,1],\, r\in[0,t]\right\}.$$

Thus by computing correlations, one sees that the increments $X_{ts} - X_{\tau s}$, $b_{ts} - b_{\tau s}$ are independent of $\mathfrak{G}_\tau$ if $t > \tau$. We want to show if two mean-zero $\mathfrak{G}_\tau$-martingales M_τ, N_τ have independent increments then

$$M_t N_t - EM_t N_t \text{ is a martingale.}$$

Then we shall be able to conclude that

$$\begin{aligned} b^A_{dts} b^B_{dt\sigma} &= d_t E b^A_{ts} b^B_{t\sigma} = \langle A,B\rangle_{\mathfrak{K}}\, G(s,\sigma)\, dt \\ b^A_{tds} b^B_{\tau ds} &= d_s E b^A_{ts} b^B_{\tau s} = \langle A,B\rangle_{\mathfrak{K}}\, G(t,\tau)\, ds \\ X^A_{dts} X^B_{dt\sigma} &= d_t E X^A_{ts} X^B_{t\sigma} = \langle A,B\rangle_{\mathfrak{K}}\, G_0(s,\sigma)\, dt \\ X^A_{dt\sigma} b^B_{dts} &= d_t E X^A_{ts} b^B_{t\sigma}. \end{aligned}$$

Let $t > s$. Then

$$\begin{aligned} E\left(M_t N_t - EM_t N_t | \mathfrak{G}_s\right) &= E\left(M_t N_t | \mathfrak{G}_s\right) - EM_t N_t \\ &= E\left(M_t - M_s\right)\left(N_t - N_s\right) + N_s E\left(M_t - M_s\right) \\ &\quad + M_s N_s + M_s E\left(N_t - N_s\right) - EM_t N_t \\ &= M_s N_s + E\left(M_t - M_s\right)\left(N_t - N_s\right) - EM_t N_t \\ &= M_s N_s - EM_s N_s. \end{aligned}$$

Thus the joint quadratic variation

$$M_{dt} N_{dt} = d_t E M_t N_t.$$

It remains only to find $X^A_{dt\sigma}b^B_{dts}$ by computing $EX^A_{ts}b^B_{t\sigma}$. But this is just

$$EX^A_{t\sigma}b^B_{ts} = EX^A_{t\sigma}X^B_{ts} + \int_0^s \frac{EX^A_{t\sigma}X^B_{tu}}{1-u}du = \langle A,B\rangle_{\mathfrak{K}}\, t\left[G_0(s,\sigma) + \int_0^s \frac{G_0(u,\sigma)}{1-u}du\right]. \tag{8.3}$$

$$\begin{aligned}
\int_0^s \frac{G_0(u,\sigma)}{1-u}du &= \int_0^s \frac{u\wedge\sigma}{1-u}du - \int_0^s \frac{u\sigma}{1-u}du \\
&= \int_0^{s\wedge\sigma} \frac{u}{1-u}du + \int_{s\wedge\sigma}^s \frac{\sigma}{1-u}du - \int_0^s \frac{u\sigma}{1-u}du \\
&= (1-\sigma)\int_0^{s\wedge\sigma} \frac{u}{1-u}du + \int_{s\wedge\sigma}^s \frac{\sigma-u\sigma}{1-u}du \\
&= (1-\sigma)\int_0^{s\wedge\sigma}\left(\frac{1}{1-u}-1\right)du + \sigma(s-s\wedge\sigma) \\
&= (1-\sigma)\log\frac{1}{1-s\wedge\sigma} - (1-\sigma)(s\wedge\sigma) + \sigma(s-s\wedge\sigma) \\
&= (1-\sigma)\log\frac{1}{1-s\wedge\sigma} - G_0(s,\sigma).
\end{aligned}$$

Returning to Eq.[8.3] yields

$$EX^A_{t\sigma}b^B_{ts} = \langle A,B\rangle_{\mathfrak{K}}\, t(1-\sigma)\log\frac{1}{1-s\wedge\sigma}.$$

∎

Lemma 8.4. *b_{ts} is a $\mathfrak{K}$-valued Brownian Sheet starting at 0 such that $Eb^i_{ts}b^j_{\tau\sigma} = \delta_{ij}(t\wedge\tau)(s\wedge\sigma)$ where $b^i_{ts} \equiv \langle b_{ts}, A_i\rangle_{\mathfrak{K}}$ where A_i runs through an orthonormal basis of $\mathfrak{K}$.*

Proof.

$$\begin{aligned}
Eb^i_{ts}b^j_{\tau\sigma} &= E[X^i_{ts} + \int_0^s \frac{X^i_{tu}}{(1-u)}du][X^j_{\tau\sigma} + \int_0^\sigma \frac{X^j_{\tau u}}{(1-u)}du] \\
&= EX^i_{ts}X^j_{\tau\sigma} + EX^i_{ts}[\int_0^\sigma \frac{X^j_{\tau u}}{(1-u)}du] + EX^j_{\tau\sigma}[\int_0^s \frac{X^i_{tu}}{(1-u)}du] \\
&\quad + E[\int_0^s \frac{X^i_{tu}}{(1-u)}du][\int_0^\sigma \frac{X^j_{\tau u}}{(1-u)}du] \\
&= I + J + K + L.
\end{aligned}$$

$$I = \delta_{ij}(t\wedge\tau)(s\wedge\sigma - s\sigma).$$

Now by Tonelli we have ,

$$\begin{aligned}
E\int_0^\sigma \left|X^i_{ts}\frac{X^j_{\tau u}}{(1-u)}\right|du &= \int_0^\sigma E\left|X^i_{ts}X^j_{\tau u}\right|\frac{du}{(1-u)} \\
&< \int_0^\sigma [E\left|X^i_{ts}\right|^2 E\left|X^j_{\tau u}\right|^2]^{\frac12}\frac{du}{(1-u)} \\
&= \int_0^\sigma [t\tau(s-s^2)(u-u^2)]^{\frac12}\frac{du}{(1-u)} < \infty.
\end{aligned}$$

Therefore by Fubini,

$$
\begin{aligned}
J &= E\int_0^\sigma X^i_{ts}\frac{X^j_{\tau u}}{(1-u)}du \\
&= \int_0^\sigma \delta_{ij}(t\wedge\tau)(s\wedge u - su)\frac{du}{(1-u)} \\
&= \delta_{ij}(t\wedge\tau)[\int_0^{s\wedge\sigma}(1-s)\frac{udu}{(1-u)} + \int_{s\wedge\sigma}^\sigma sdu] \\
&= \delta_{ij}(t\wedge\tau)[(1-s)\int_0^{s\wedge\sigma}(\frac{1}{(1-u)}-1)du + s(\sigma - s\wedge\sigma)] \\
&= \delta_{ij}(t\wedge\tau)[(1-s)\log(\frac{1}{1-s\wedge\sigma}) - (1-s)s\wedge\sigma + s(\sigma - s\wedge\sigma)] \\
&= -\delta_{ij}(t\wedge\tau)[(1-s)\log(1-s\wedge\sigma) + s\wedge\sigma - s\sigma].
\end{aligned}
$$

Similarly,

$$K = -\delta_{ij}(t\wedge\tau)[(1-\sigma)\log(1-s\wedge\sigma) + s\wedge\sigma - s\sigma]$$

By Tonelli,

$$
\begin{aligned}
E &\quad [\int_0^s\int_0^\sigma \left|\frac{X^i_{tu}}{(1-u)}\frac{X^j_{\tau\nu}}{(1-\nu)}\right| dud\nu] \\
&= \int_0^s\int_0^\sigma E\left|\frac{X^i_{tu}}{(1-u)}\frac{X^j_{\tau\nu}}{(1-\nu)}\right| dud\nu \\
&< \int_0^s\int_0^\sigma \frac{dud\nu}{(1-u)(1-\nu)}\left(E\left(X^i_{tu}\right)^2 E\left(X^j_{\tau\nu}\right)^2\right)^{\frac{1}{2}} \\
&= t\tau\int_0^s\int_0^\sigma (u\nu)dud\nu < \infty.
\end{aligned}
$$

Therefore by Fubini,

$$\begin{aligned}
L &= E\Big[\int_0^s\int_0^\sigma \frac{X^i_{tu}}{(1-u)}\frac{X^j_{\tau\nu}}{(1-\nu)}dud\nu\Big] \\
&= \int_0^s\int_0^\sigma E\frac{X^i_{tu}}{(1-u)}\frac{X^j_{\tau\nu}}{(1-\nu)}dud\nu \\
&= \int_0^s\int_0^\sigma \delta_{ij}(t\wedge\tau)(u\wedge\nu-u\nu)\frac{dud\nu}{(1-u)(1-\nu)} \\
&= \delta_{ij}(t\wedge\tau)\left(\int_{\{s>u>\nu>0\}\cup\{\sigma>\nu\}} dud\nu\frac{\nu}{1-\nu}+\int_{\{\sigma>\nu>u>0\}\cup\{s>u\}} dud\nu\frac{u}{1-u}\right) \\
&= \delta_{ij}(t\wedge\tau)\int_0^{s\wedge\sigma} d\nu\left(\frac{\nu(s-\nu)}{1-\nu}+\frac{\nu(\sigma-\nu)}{1-\nu}\right) \\
&= \delta_{ij}(t\wedge\tau)\int_0^{s\wedge\sigma} d\nu\Big(\frac{1}{1-\nu}-1\Big)(s+\sigma-2\nu) \\
&= \delta_{ij}(t\wedge\tau)\left[\int_0^{s\wedge\sigma}\frac{s+\sigma-2\nu}{1-\nu}d\nu-\big((s+\sigma)(s\wedge\sigma)-(s\wedge\sigma)^2\big)\right] \\
&= \delta_{ij}(t\wedge\tau)\left[\int_0^{s\wedge\sigma}\left(\frac{s+\sigma-2}{1-\nu}+2\right)d\nu-\big((s+\sigma)(s\wedge\sigma)-(s\wedge\sigma)^2\big)\right] \\
&= \delta_{ij}(t\wedge\tau)\left[\int_0^{s\wedge\sigma}\frac{s+\sigma-2}{1-\nu}d\nu+\big((s\wedge\sigma)^2+2(s\wedge\sigma)-(s+\sigma)(s\wedge\sigma)\big)\right] \\
&= \delta_{ij}(t\wedge\tau)\left[(s\wedge\sigma)^2+2(s\wedge\sigma)-(s+\sigma)(s\wedge\sigma)-(s+\sigma-2)\log(1-s\wedge\sigma)\right].
\end{aligned}$$

So putting it all together, we have

$$\begin{aligned}
&I+J+K+L \\
&\quad=\delta_{ij}(t\wedge\tau)\left[(s\wedge\sigma)^2+s\sigma+(s\wedge\sigma)-(s+\sigma)(s\wedge\sigma)\right] \\
&\quad=\delta_{ij}(t\wedge\tau)\,(\sigma\wedge s).
\end{aligned}$$

Therefore,

$$Eb^i_{ts}b^j_{\tau\sigma}=\delta_{ij}(t\wedge\tau)(s\wedge\sigma).$$

b a linear transformation of a Gaussian process and is hence Gaussian. Hence the above assertion is enough to show b is a $\mathfrak{K}$-valued Brownian Sheet. ∎

Theorem 8.5. *There exists a Brownian sheet b_{ts} such that X_{ts} can be expressed as:*

$$X_{ts}=b_{ts}-\int_0^s b_{t\sigma}\frac{(1-s)}{(1-\sigma)^2}d\sigma, \tag{8.4}$$

where b is a $\mathfrak{K}$-valued 2-parameter Brownian Sheet.

Proof. Define $b_{ts}\equiv S(X_{t\cdot})(s)$. Then by Lemma 8.4, b_{ts} is a $\mathfrak{K}$-valued Brownian Sheet. Now by Lemma 8.6, Eq. (8.4) holds. ∎

Lemma 8.6. $T: b_{t\cdot}\to X_{t\cdot}$

Proof.

$$
\begin{aligned}
T(b_{t.})(s) &= b_{ts} - \int_0^s b_{t\sigma}\frac{(1-s)}{(1-\sigma)^2}d\sigma \\
&= [X_{ts} + \int_0^t \frac{X_{t\sigma}}{(1-\sigma)}d\sigma] - \int_0^s [X_{t\sigma} + \int_0^\sigma \frac{X_{tu}}{(1-u)}du]\frac{(1-s)}{(1-\sigma)^2}d\sigma \\
&= X_{ts} + \int_0^s X_{ts}\frac{(s-\sigma)}{(1-\sigma)^2}d\sigma - (1-s)\int_0^s\int_0^\sigma \frac{X_{tu}}{(1-u)}dud(\frac{1}{1-\sigma}) \\
&= X_{ts} + \int_0^s X_{t\sigma}\frac{(s-\sigma)}{(1-\sigma)^2}d\sigma - \int_0^s \frac{X_{tu}}{(1-u)}du + (1-s)\int_0^s(\frac{1}{1-\sigma})\frac{X_{t\sigma}}{(1-\sigma)}d\sigma \\
&= X_{ts} + \int_0^s X_{t\sigma}[\frac{(s-\sigma)}{(1-\sigma)^2} - \frac{1}{(1-\sigma)} + \frac{1-s}{(1-\sigma)^2}]d\sigma \\
&= X_{ts}.
\end{aligned}
$$

∎

Lemma 8.7 (Evaluation of L^2-norms)**.** *Let $\mathcal{M}_m(\mathbb{R})$ be as in Remark 2.18. Let $\{f_{i,A}(\cdot)\}$ be a collection of continuous adapted $\mathcal{M}_m(\mathbb{R})$-valued processes. (i.e. $f_{i,A}(\sigma) \in \mathfrak{F}_{t_{i-1}\sigma}$). Let $\Delta_i b^A(\sigma) \equiv b^A(t_i,\sigma) - b^A(t_{i-1},\sigma)$ and recall that $b^A = \langle b, A\rangle_{\mathfrak{K}}$ where b is the Brownian sheet from Theorem 3.19. Then*

$$E\left\|\sum_{i,A}\int_0^s f_{i,A}(\sigma)d\Delta_i b^A(\sigma)\right\|_{HS}^2 = E\sum_{i,A}\Delta_i t\int_0^s \|f_{i,A}(\sigma)\|_{HS}^2 d\sigma.$$

Proof. Let $(A)_{pq}$ denote the p,q entry of the matrix $A \in \mathcal{M}_m(\mathbb{R})$. Let

$$J \equiv E\left\|\sum_{i,A}\int_0^s f_{i,A}(\sigma)d\Delta_i b^A(\sigma)\right\|_{HS}^2.$$

Then

$$
\begin{aligned}
J &= \sum_{p,q} E\left(\sum_{i,A}\int_0^s (f_{i,A}(\sigma))_{pq}\, d\Delta_i b^A(\sigma)\right)^2 \\
&= \sum_{p,q} E\sum_{i,A,i',A'}\int_0^s (f_{i,A}(\sigma))_{pq}(f_{i',A'}(\sigma))_{pq}\, d\Delta_i b^A(\sigma)d\Delta_{i'}b^{A'}(\sigma) \\
&= \sum_{p,q} E\sum_{i,A}\int_0^s \Delta_i t\,(f_{i,A}(\sigma)_{pq})^2 d\sigma \\
&= E\sum_{i,A}\Delta_i t\int_0^s \|f_{i,A}(\sigma)\|_{HS}^2 d\sigma.
\end{aligned}
$$

∎

8.3. **Proof of Lemma 8.8.**

Lemma 8.8. *Recall that $K \subset GL_m(\mathbb{R})$ as in Remark 2.18. Let F and G be the exponential and inverse map respectively on matrices as in Definition 6.9. Then the following relations hold where $A \in U$ and $B, C \in \mathfrak{K}$:-*

1.

$$F'(A)B = \int_0^1 F\left[(1-\tau)A\right] BF\left[\tau A\right] d\tau. \tag{8.5}$$

2.

$$\begin{aligned} F''(A)B\otimes C &= \int_0^1 d\tau \int_0^1 (1-u) F\left[(1-\tau)(1-u)A\right] \\ &\qquad \times CF\left[\tau(1-u)A\right] BF\left[uA\right] du \\ &\quad + \int_0^1 d\tau \int_0^1 uF\left[(1-u)A\right] B \\ &\qquad \times F\left[(1-\tau)uA\right] CF\left[\tau uA\right] du. \end{aligned} \tag{8.6}$$

3.

$$G'(A)B = -A^{-1}BA^{-1}. \tag{8.7}$$

4.

$$G''(A)B\otimes C = A^{-1}BA^{-1}CA^{-1} + A^{-1}CA^{-1}BA^{-1}. \tag{8.8}$$

5.

$$\sup_{A\in\mathfrak{K}} \left\|F'(A)B\right\|_{HS} \le Const \left\|B\right\|_{HS}. \tag{8.9}$$

6.

$$\sup_{A\in\mathfrak{K}} \left\|F^{(n)}(A)B_1\otimes\cdots\otimes B_n\right\|_{HS} \le Const \left\|B_1\right\|_{HS}\cdots\left\|B_n\right\|_{HS}. \tag{8.10}$$

Proof.

$$\frac{d}{dt}F\left(t\left(A+sB\right)\right) = \frac{d}{dt}e^{t(A+sB)} = \left(A+sB\right)e^{t(A+sB)}.$$

$$\frac{d}{ds}\frac{d}{dt}F\left(t\left(A+sB\right)\right)|_{s=0} = Be^{tA} + tA\left(F'\left(tA\right)tB\right).$$

$$\frac{d}{dt}\left[\frac{d}{ds}F\left(t\left(A+sB\right)\right)|_{s=0}\right] = Be^{tA} + A\left[\frac{d}{ds}F\left(t\left(A+sB\right)\right)|_{s=0}\right].$$

By Duhammel's Principle (or method of integrating factors) we get Eq. (8.5)

$$\frac{d}{dt}\left\{e^{-tA}\left[\frac{d}{ds}F\left(t\left(A+sB\right)\right)|_{s=0}\right]\right\} = e^{-tA}Be^{tA}.$$

$$\frac{d}{ds}F\left(A+sB\right)|_{s=0} = e^{A}\int_0^1 e^{-tA}Be^{tA}dt.$$

$$\frac{d}{ds}F\left(A+sB\right)|_{s=0} = e^{A}\int_0^1 e^{-tA}Be^{tA}dt.$$

$$\begin{aligned}
F''(A)\,B\otimes C &= \frac{d}{dt}F'(A+tC)B|_{t=0} \\
&= \int_0^1 \frac{d}{dt}F\left[(1-u)(A+tC)\right]BF\left[u(A+tC)\right]|_{t=0}du \\
&= \int_0^1 (1-u)\left(F'\left[(1-u)A\right]C\right)Be^{uA}du \\
&\qquad + \int_0^1 ue^{(1-u)A}BF'(uA)\,Cdu,
\end{aligned}$$

which is Eq. (8.6).

$$\begin{aligned}
(A+tB)\,G\,(A+tB) &= 1 \\
BG\,(A) + A\,(G'(A)B) &= 0 \\
G'(A)B &= -G(A)BG(A),
\end{aligned}$$

which is Eq. (8.7).

$$\begin{aligned}
&G''(A)B\otimes C \\
&\qquad = \frac{d}{dt}G'(A+tC)B|_{t=0} \\
&\qquad = -\frac{d}{dt}G(A+tC)BG(A+tC)|_{t=0} \\
&\qquad = G(A)CG(A)BG(A) + G(A)BG(A)CG(A),
\end{aligned}$$

which is Eq. (8.8).

It will not be necessary to prove Eq. (8.9) if we can show Eq. (8.10). Let $\widetilde{B}_i \equiv B_i / \|B_i\|_{HS}$

$$\begin{aligned}
&F^{(n)}(A)B_1\otimes\cdots\otimes B_n \\
&\qquad = \|B_1\|_{HS}\cdots\|B_n\|_{HS}\,F^{(n)}(A)\widetilde{B}_1\otimes\cdots\otimes\widetilde{B}_n.
\end{aligned}$$

Now

$$\sup_{A,\widetilde{B}_1\cdots\widetilde{B}_n}\left\|F^{(n)}(A)\widetilde{B}_1\otimes\cdots\otimes\widetilde{B}_n\right\|_{HS}$$

is the supremum of a continuous function over a compact set and is hence finite. Call this supremum C. Therefore,

$$\sup_{A\in\Re}\left\|F^{(n)}(A)B_1\otimes\cdots\otimes B_n\right\|_{HS} \le C\,\|B_1\|_{HS}\cdots\|B_n\|_{HS}\,,$$

which is Eq. (8.10). ∎

8.4. **Gaussian Measures.** For further information on this topic see Kuo [[23]].

Definition 8.9 (Gaussian Measure)**.** Ω is a separable Banach Space with $|\cdot|_\Omega$. μ a (mean-zero, non-degenerate) Gaussian measure on Ω iff

$$\widehat{\mu}(\phi) \equiv \int \mu(dx)\exp i\phi(x) = \exp\left[-\frac{1}{2}q(\phi,\phi)\right],$$

and $q:\Omega^*\times\Omega^*\to\mathbb{R}$ an inner product on Ω^*. Ω^* denotes the dual to Ω; i.e. the set of all bounded linear maps from Ω to $\mathbb{R}$. $\widehat{\mu}$ denotes the Fourier transform of the measure μ.

Remark 8.10. An alternate (and equivalent definition) of a Gaussian measure μ on a Banach space Ω is a measure so that any $\psi \in \Omega^*$ (an $\mathbb{R}$-valued random variable) is a mean-zero normal random variable. One can see that if μ satisfies Definition 8.9 then it satisfies the alternate Definition in Remark 8.10. This is because for any $\psi \in \Omega^*$,

$$\widehat{Law_\psi}(\lambda) \equiv Law_\psi \exp i\lambda x = \mu \exp i\lambda\psi(x) = \exp -\frac{\lambda^2}{2} q(\psi,\psi),$$

which means that ψ has a normal distribution with mean 0 and variance $q(\psi,\psi)$. (Although this second definition appears weaker, it is possible to prove Definition 8.9 from it.)

We shall repeatedly use the following well-known Theorem due to Fernique:

Theorem 8.11 (Fernique)**.** *Let μ be a Gaussian measure on a Banach Space Ω. Then there exists an $\varepsilon > 0$ so that*

$$\mu\left[\exp \varepsilon |x|_\Omega^2\right] < \infty.$$

(Note:- this also means that $\mu |x|_\Omega^2 < \infty$)

Lemma 8.12 (Bochner Integrals)**.** *Define*

$$\mathcal{S} \equiv \{f : \Omega \to \Omega |\ Ran\ f \text{ is a finite set}\}.$$

Let $L^1(\mu,\Omega)$ denote $\left\{f : \Omega \to \Omega | \int |f(x)|_\Omega \mu(dx) < \infty\right\}$. Then there exists a linear functional $I : L^1(\mu,\Omega) \to \Omega$ with the following properties:-

1.
$$I(f) = \sum_{x\in\Omega} x\mu\left(f^{-1}\{x\}\right) \text{ if } f \in \mathcal{S}.$$

2.
$$|I(f)|_\Omega \le \int |f|_\Omega \mu(dx) = \|f\|_{L^1(\mu,\Omega)}.$$

3.
$$\psi(I(f)) = \int \psi(f)\mu(dx) \text{ for any } \psi \in \Omega^*.$$

Henceforth we shall write $\int f(x)\,\mu(dx)$ in place of the less intuitive $I(f)$.

Proof. First we show that $\mathcal{S}$ is dense in $L^1(\mu,\Omega)$. We shall use the separability of Ω to do this. Let $\{x_n\}$ be a countable dense subset of Ω. Cover Ω with measurable sets B_i as follows $B_1 \equiv B(\varepsilon,x_1),\cdots,B_n \equiv B(\varepsilon,x_n) - \cup_{i=1}^n B(\varepsilon,x_i),\cdots$. Given $f \in L^1(\mu,\Omega)$, let $\phi_\varepsilon \equiv \sum_i x_i 1_{f^{-1}(B_i)}$. Then

$$\int |f-\phi_\varepsilon|_\Omega \mu(dx) \le \sum_i \int |f - x_i|_\Omega 1_{f^{-1}(B_i)}\mu(dy) \le \varepsilon\mu(\Omega).$$

Since the ϕ_ε were all in $\mathcal{S}$, $\mathcal{S}$ is dense in $L^1(\mu,\Omega)$.

Define $I(f) = \sum_{x\in\Omega} x\mu\left(f^{-1}\{x\}\right)$ on $\mathcal{S}$. Hence property 1. is already satisfied.

Notice that I is a linear functional on $\mathcal{S}$. Secondly, if $\psi \in \Omega^*$, on $\mathcal{S}$

$$\begin{aligned}
\psi(I(f)) &= \sum_{x\in\Omega} \psi(x)\,\mu\left(f^{-1}\{x\}\right) \\
&= \int \mu(dy) \sum_{x\in\Omega} \psi(x)\, 1_{f^{-1}\{x\}}(y) \\
&= \int \psi\left(\sum_{x\in\Omega} x 1_{f^{-1}\{x\}}\right) d\mu \\
&= \int \psi \circ f(y)\,\mu(dy).
\end{aligned}$$

Also

$$|I(f)|_\Omega \leq \sum_{x\in\Omega} |x|_\Omega\,\mu\left(f^{-1}\{x\}\right) = \|f\|_{L^1(\mu,\Omega)}.$$

Extend our definition of I to $L^1(\mu,\Omega)$ by defining $I(f) \equiv \lim_{n\to\infty} I(f_n)$ whenever $f_n \to f$ in $L^1(\mu,\Omega)$ with $\{f_n\} \in \mathcal{S}$.

Property 1 holds by definition. The linearity of I holds despite the extension.

Property 2 holds as well since

$$|I(f)|_\Omega = |\lim I(f_n)|_\Omega = \lim |I(f_n)|_\Omega < \lim \|f_n\|_{L^1(\mu,\Omega)} = \|f\|_{L^1(\mu,\Omega)}.$$

If $\psi \in \Omega^*$ we have $f_n \to f$ in $L^1(\mu,\Omega)$. This means $I(f_n) \to I(f)$ in Ω. Thus $\int \psi \circ f_n d\mu = \psi \circ I(f_n) \to I(f)$ in Ω.However,

$$\begin{aligned}
\left|\int \psi(f)\,d\mu - \int \psi(f_n)\,d\mu\right|_\Omega &\leq \int |\psi(f) - \psi(f_n)|_\Omega\,d\mu \\
&\leq |\psi|_{\Omega^*}\,\|f - f_n\|_{L^1(\mu,\Omega)} \\
&\to 0.
\end{aligned}$$

Thence we have $\int \psi \circ f d\mu = I(f)$ and Property 3 holds. ∎

Theorem 8.13 (Cameron-Martin space)**.** *We construct H, the Cameron-Martin space associated with the Gaussian measure μ:-*

1. $\Omega^* \subset L^p(\mu)$ for all p.
2. There is a linear map $J : \Omega^* \to \Omega$ so that $\phi(J\psi) = \langle \psi, \phi \rangle_{L^2(\mu)}$.
3. $q(\phi,\psi) = \int \phi(x)\,\psi(x)\,\mu(dx)$.
4. $H \equiv \left\{x \in \Omega : \sup_{\phi\in\Omega^*} |\phi(x)|^2 / q(\phi,\phi) < \infty\right\}$is a subspace of Ω.
5. $|h|_\Omega \leq \mathit{const}\,|h|_H$ for any $h \in H$.
6. Let K be the closure of Ω^* in $L^2(\mu)$. Then there exists an extension of the map J from $K \to H$ so that $J : f \to \int x f(x)\,\mu(dx)$. Furthermore, J is an isometry onto H and H is dense in Ω. In particular, H is a Hilbert space under the isometry from K.
7. Letting h run through an orthonormal basis of H,

$$\int \phi(x)\,\psi(x)\,\mu(dx) = \sum \psi(h)\,\phi(h).$$

Proof. Proceeding in order, we prove:-

1. If $\phi \in \Omega^*$ then
$$\int |\phi(x)|_\Omega^p \mu(dx) \le |\phi|_{\Omega^*}^p \int |x|_\Omega^p \mu(dx) < \infty$$
by Theorem 8.11 (Fernique).
2. Define $J(\phi) \equiv \int x\phi(x)\mu(dx)$ for any $\phi \in \Omega^*$. If $\psi \in \Omega^*$, then by property 3 of Lemma 8.12
$$\psi(J(\phi)) = \int \psi(x)\phi(x)\mu(dx) = \langle \psi, \phi \rangle_{L^2(\mu)}.$$
3. By Definition 8.9, we have
$$\widehat{\mu}(t\phi) = \mu(\exp it\phi(x)) = \exp -t^2\left[\frac{1}{2}q(\phi,\phi)\right].$$
If we can show $[\partial_t\partial_t\widehat{\mu}(t\phi)]\downarrow_{t=0} = -\mu\phi(x)^2$ we would have
$$\mu\phi(x)^2 = q(\phi,\phi)$$
by taking two derivatives on the right hand side. Then
$$\mu[\phi\psi] = \frac{1}{4}\mu(\phi+\psi)^2 - \frac{1}{4}\mu(\phi-\psi)^2 = q(\phi,\psi),$$
and we would be done. So the problem reduces to computing $\partial_t\partial_t\widehat{\mu}(t\phi)$.
We shall show
$$\partial_t\partial_t\widehat{\mu}(t\phi) = \mu\left[-\phi(x)^2\exp it\phi(x)\right]$$
by Dominated Convergence and Theorem 8.11 as follows:
$$\begin{aligned}\partial_t\widehat{\mu}(t\phi) &= \lim_{\varepsilon\downarrow 0}\mu\left(\frac{\exp i(t+\varepsilon)\phi(x) - \exp it\phi(x)}{\varepsilon}\right)\\ &= \lim_{\varepsilon\downarrow 0}\mu\left(\frac{\exp i\varepsilon\phi(x)-1}{\varepsilon}\right)\exp it\phi(x).\end{aligned}$$
Now $|\exp it\phi(x)| < 1$ so to apply dominated convergence, it will suffice to dominate $\frac{\exp i\varepsilon\phi(x)-1}{\varepsilon}$ by an $L^1(d\mu)$ function.
$$\begin{aligned}\left|\frac{\exp i\varepsilon\phi(x)-1}{\varepsilon}\right| &= \frac{1}{\varepsilon}\sum_{n>1}|i\varepsilon\phi(x)|^n/n! \le |\phi(x)|\sum_{n>0}|\varepsilon\phi(x)|^n/(n+1)!\\ &\le |\phi(x)|\exp\varepsilon|\phi|_{\Omega^*}|x|_\Omega.\end{aligned}$$
By Theorem 8.11 (Fernique), we have
$$\mu|\phi(x)|\exp\varepsilon|\phi|_{\Omega^*}|x|_\Omega \le \sqrt{\mu|\phi|^2\|x\|_\Omega^2}\sqrt{\mu\exp(\varepsilon|\phi|_{\Omega^*})^2|x|_\Omega^2} < \infty.$$
Thus by dominated convergence, the limit goes through and we have
$$\partial_t\widehat{\mu}(t\phi) = \mu(i\phi(x)\exp it\phi(x)).$$
Similarly to take the second derivative, we have to again verify that the limit can be passed through the integral in Eq.(8.11) below.
$$\partial_t\partial_t\widehat{\mu}(t\phi) = \lim_{\varepsilon\downarrow 0}\mu\left(i\phi(x)\exp it\phi(x)\left[\frac{\exp i\varepsilon\phi(x)-1}{\varepsilon}\right]\right). \tag{8.11}$$

Now using Fernique yet again, $i\phi(x)\left[\frac{\exp i\varepsilon\phi(x)-1}{\varepsilon}\right]$ is bounded by the $L^1(d\mu)$ function $|\phi(x)|^2\exp\varepsilon|\phi|_{\Omega^*}|x|_\Omega$. Thus by dominated convergence, we have

$$\partial_t\partial_t\widehat{\mu}(t\phi)=\mu\left[-\phi(x)^2\exp it\phi(x)\right].$$

Hence we have shown part 3.

4. If $h,k\in H$, then
$$\frac{|\phi(\alpha h+k)|}{\sqrt{q(\phi,\phi)}}\leq|\alpha|\frac{|\phi(h)|}{\sqrt{q(\phi,\phi)}}+\frac{|\phi(k)|}{\sqrt{q(\phi,\phi)}},$$
which implies that H is a subspace of Ω.

5. By Fernique's Theorem (Theorem 8.11)$\int\mu(dx)|x|_\Omega^2\leq\int\mu(dx)\exp\varepsilon|x|_\Omega^2<\infty$. As a consequence,
$$\begin{aligned}q(\phi,\phi)&=\int\mu(dx)\phi(x)^2\\&\leq|\phi|_{\Omega^*}^2\int\mu(dx)|x|_\Omega^2\\&=C^2|\phi|_{\Omega^*}^2.\end{aligned}$$
Therefore
$$\begin{aligned}|h|_\Omega&=\sup_{\phi\in\Omega^*}\frac{|\phi(x)|}{|\phi|_{\Omega^*}}\\&\leq C\sup_{\phi\in\Omega^*}\frac{|\phi(x)|}{q(\phi,\phi)}\\&\leq C|h|_H.\end{aligned}$$

6. Clearly, $\Omega^*\subset H$. Let $\{\psi_n\}$ be Cauchy in $L^2(d\mu)$. Then
$$\begin{aligned}|J\psi_n-J\psi_m|&\leq\mu|x|_\Omega|(\psi_n-\psi_m)(x)|\\&\leq\left(\mu|x|_\Omega^2\right)\mu(\psi_n-\psi_m)^2\to0\text{ as }n,m\text{ go to }\infty.\end{aligned}$$
Thus by completeness of Ω, $\{J\psi_n\}$ converges in Ω. Thus we extend the map J to the space $K\equiv\overline{\Omega^*}^{L^2(d\mu)}$ by continuity.

Let $x\in\operatorname{Im}J$. Then there is some sequence $\{J\psi_n\}$ so that $\psi_n\in\Omega^*$, ψ_n converges in $L^2(d\mu)$ to some $\psi\in K$. But then
$$\begin{aligned}\sup_{\phi\in\Omega^*}|\phi(x)|^2/q(\phi,\phi)&=\sup_{\phi\in\Omega^*}\lim_{n\to\infty}\left(\mu[\phi\psi_n]/\sqrt{\mu[\phi^2]}\right)^2\\&=\sup_{\phi\in\Omega^*}\left(\mu[\phi\psi]/\sqrt{\mu[\phi^2]}\right)^2\\&=\mu\left[\psi^2\right].\end{aligned}$$
Thus $\operatorname{Im}J\subset H$, and $J:K\to\operatorname{Im}H$ is an isometry.

$\operatorname{Im}J$ dense in Ω. If not there's a non-trivial $\psi\in\Omega^*$ so that $\psi(\operatorname{Im}J)=0$. This means that $\mu\left(\psi^2\right)=0$ which is a contradiction. Thus $\operatorname{Im}J$ and hence also H are dense in Ω.

Let $h\in H$. Let $\widehat{h}(\phi)\equiv\phi(h)$ for any $\phi\in\Omega^*$. Note that $\left|\widehat{h}(\phi)\right|=|\phi(h)|<|\phi|_{L^2(d\mu)}|h|_H$. Thus $h\in K^*$. Thus there exists an $f\in K$ so that

$\phi(h) = \mu[\phi f]$ for any $\phi \in \Omega^*$. But now $\phi(Jf) = \mu[\phi(x) f(x)] = \phi(h)$. Thus $h = Jf \Rightarrow H \subset \operatorname{Im} J$. Thus J is a unitary map from K to H and in particular, H is a Hilbert space that's dense in Ω.

7. Let h run through an orthonormal basis of H. Then

$$\begin{aligned}\int \phi(x)\psi(x)\mu(dx) &= \langle J_\phi, J_\psi\rangle_H \\ &= \sum \langle J_\phi, h\rangle_H \langle h, J_\psi\rangle_H \\ &= \sum \phi(h)\psi(h).\end{aligned}$$

∎

Example 8.14 (Computing Cameron-Martin Spaces)**.** Let Ω be the Banach space

$$L(\mathfrak{K}) \equiv \{x \in C([0,1] \to \mathfrak{K}) \,|x(0) = x(1) = 0\}$$

equipped with the uniform norm. $\mu = LawX_{t\cdot}.(X_{t\cdot}$ is a Brownian bridge from 0 to 0 with parameter t). Let $H_{0,t}$ denote the Cameron-Martin space associated to the measure μ. Then $H_{0,t}$ is the space $H_0(\mathfrak{K})$ equipped with the inner product

$$\langle k(\cdot), l(\cdot)\rangle_{H_{0,t}} = \frac{1}{t}\int_0^1 \langle k'(u), l'(u)\rangle_{\mathfrak{K}}\, du.$$

Recall from Definition 2.1 that

$$H_0(\mathfrak{K}) = \left\{x \in \Omega : x \text{ has one } L^2([0,1], d\lambda)\text{-derivative}\right\}.$$

Proof. $X_{t\cdot}$ is a Brownian bridge with parameter t. So $\mu = LawX_{t\cdot}$ is already a Gaussian measure. Furthermore,

$$\begin{aligned}&\int \mu(dx)\langle x(s), A\rangle_{\mathfrak{K}} \langle x(\sigma), B\rangle_{\mathfrak{K}} \\ &\quad= E\langle X_{ts}, A\rangle_{\mathfrak{K}} \langle X_{\tau\sigma}, B\rangle_{\mathfrak{K}} \\ &\quad= \langle A, B\rangle_{\mathfrak{K}}\, tG_0(s,\sigma).\end{aligned}$$

Define an element $\psi_{A,s}$ of Ω^* by setting $\psi_{A,s}(x) = \langle x(s), A\rangle_{\mathfrak{K}}$. Then

$$q\left(\psi_{A,s}, \psi_{B,\sigma}\right) = E\langle X_{ts}, A\rangle_{\mathfrak{K}} \langle X_{\tau\sigma}, B\rangle_{\mathfrak{K}} = \langle A, B\rangle_{\mathfrak{K}}\, tG_0(s,\sigma).$$

Let J be the standard inclusion of Ω^* into $H_{0,t}$ as in Theorem 8.13. Then from abstract nonsense (i.e. Theorem 8.13)

$$\begin{aligned}\left\langle \left(J\psi_{A,s}\right)(\sigma), B\right\rangle_{\mathfrak{K}} &= \psi_{B,\sigma}\left(J\psi_{A,s}\right) \\ &= q\left(\psi_{B,\sigma}, \psi_{A,s}\right) \\ &= \langle AtG_0(s,\sigma), B\rangle_{\mathfrak{K}}.\end{aligned}$$

Therefore $J\psi_{A,s} = AtG_0(s,\cdot)$. Let $\omega \in H_{0,t}$. $\omega \perp J\psi_{A,s} \iff \langle \omega(s), A\rangle_{\mathfrak{K}} = 0$. Therefore,

$$\Lambda \equiv span\left\langle J\psi_{A,s} : s \in (0,1),\, A \in \mathfrak{K}\right\rangle \text{ is dense in } H_{0,t}.$$

Thus it will suffice for us to specify the norm of $H_{0,t}$ on Λ.

$$\begin{aligned}
&\frac{1}{t}\int_0^1 \left\langle \left(J\psi_{B,\sigma}\right)'(u), \left(J\psi_{A,s}\right)'(u)\right\rangle_{\mathfrak{K}} du \\
&= \frac{1}{t}\int_0^1 \left\langle \frac{d}{du}tG_0(u,\sigma)B, \frac{d}{du}tG_0(u,s)A\right\rangle_{\mathfrak{K}} du \\
&= t\langle B,A\rangle_{\mathfrak{K}} \int_0^1 \left(1_{\{u\le\sigma\}}-\sigma\right)\left(1_{\{u\le s\}}-s\right) du \\
&= t\langle B,A\rangle_{\mathfrak{K}} [s\wedge\sigma - s\sigma - \sigma s + s\sigma] \\
&= t\langle B,A\rangle_{\mathfrak{K}} G_0(s,\sigma) \\
&= q\left(\psi_{A,s},\psi_{B,\sigma}\right).
\end{aligned}$$

Thus the inner product $\frac{1}{t}\int_0^1 \langle k'(u), l'(u)\rangle_{\mathfrak{K}} du$ works on Λ and $H_{0,t}$ is the closure of Λ under this norm.

Define

$$\widetilde{H} \equiv \left\{x\in\Omega : x \text{ has one } L^2([0,1],d\lambda)\text{-derivative}\right\}.$$

We want to show $\widetilde{H} = H_{0,t}$.

Let $y\in C^\infty[0,1]$ so that $y(0)=y(1)=0$. Define $\psi_y\in\Omega^*$ as follows:

$$\psi_y : x \mapsto -\frac{1}{t}\int_0^1 \langle y''(u), x(u)\rangle_{\mathfrak{K}} du.$$

Also

$$\begin{aligned}
\left\langle J\psi_y(s), A\right\rangle_{\mathfrak{K}} &= \psi_y\left(J\psi_{A,s}\right) = -\frac{1}{t}\int_0^1 \langle y''(u), t(s\wedge u - su)A\rangle_{\mathfrak{K}} du \\
&= -\int_0^1 \langle y, A\rangle''_{\mathfrak{K}}(u)(s\wedge u - su)\,du \\
&= -\int_0^1 (s\wedge u - su)\, d_u \langle y, A\rangle'_{\mathfrak{K}} \\
&= \int_0^1 \left(1_{\{u\le s\}} - s\right)\langle y, A\rangle'_{\mathfrak{K}}(u)\,du \\
&= \langle y(s), A\rangle_{\mathfrak{K}}.
\end{aligned}$$

Thus $J\psi_y = y$ which implies that smooth loops are in $\operatorname{Im} J$ and hence in $H_{0,t}$.

Let $x\in\widetilde{H}$. Then let y_n be smooth so that $y_n\to x'$ in L^2. Then

$$\widetilde{y}_n \equiv y_n - \int_0^1 y_n(u)\,du$$

also converges to x' in L^2, since

$$\int_0^1 y_n(u)\,du \to \int_0^1 x'(u)\,du = 0.$$

But then $\int_0^{\cdot}\widetilde{y}_n$ converge to x in the norm $\frac{1}{t}\int_0^1 \langle k'(u), l'(u)\rangle_{\mathfrak{K}} du$. Thus $\widetilde{H}\subset H_{0,t}$. Since $\widetilde{H}$ is complete, with respect to the inner product $\frac{1}{t}\int_0^1 \langle k'(u), l'(u)\rangle_{\mathfrak{K}} du$, we have $\widetilde{H} = H_{0,t}$ and we are done. ∎

9. Conjectures on Gaussian measure equivalence

Let g be the $C([0,1] \to L(K))$-valued random variable so that $g(t)$ is the loop g_t of Definition 2.22. Let P_σ be the law of σg a measure on $C([0,1] \to L(K))$. Then for any probability μ on $L(K)$ we define a measure P_μ on $C([0,1] \to L(K))$ by

$$P_\mu[f] \equiv \int \mu(d\sigma) P_\sigma[f].$$

Conjecture 1. Let the energy, $\widetilde{E}$, of an absolutely continuous loop be given by

$$\widetilde{E}\left(\sigma_{[g_0]}\right) \equiv \int_0^1 \left|\sigma_{[g_0]}(ds)\, \sigma_{[g_0]}(s)^{-1}\right|_{\mathfrak{K}}^2.$$

Let $\sigma_{[g_0]}$ be an Energy-minimizing loop in the homotopy class $[g_0]$. Recall the $\mu_{0,t}$-a.s. function V_t of Theorem 4.1;

$$V_t(x) = \frac{1}{2t^2}\left|\int_0^1 x(ds)\, x(s)^{-1}\right|_{\mathfrak{K}}^2 - \left(\frac{\dim \mathfrak{K}}{2t} + \partial_t \log P_t^K(e)\right).$$

Then if $f : L(K) \to \mathbb{R}$ is "nice" it is reasonable to expect

$$\mu_{0,T}[f] = \sum_{[g_0]} \int f(x_T) \exp\left[\lim_{\varepsilon \to 0} \int_\varepsilon^T V_t(x_t)\, dt - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right] \nu_{\sigma_{[g_0]}}(dx).$$

where the $[g_0]$ run through all the homotopy classes of K.

Proof. Let $\mu_{0,t} = \int \mathcal{M}_t \mathcal{D}x$ where $\mathcal{D}x$ is "Lebesgue measure" on $L(K)$. Let $g.$ be an $L(K)$-valued Brownian motion. Then

$$\begin{aligned}
&\int f(\partial_t \mathcal{M}_t)\, \mathcal{D}x \\
&\quad = \partial_t \int f \mathcal{M}_t \mathcal{D}x \\
&\quad = \int \mathcal{M}_t \left(\frac{1}{2}\triangle_{L(K)} f + V_t f\right) \mathcal{D}x \\
&\quad = \int f \left(\frac{1}{2}\triangle_{L(K)} + V_t\right) \mathcal{M}_t \mathcal{D}x.
\end{aligned}$$

Thus $\mathcal{M}_t$ "satisfies"

$$\partial_t \mathcal{M}_t = \left(\frac{1}{2}\triangle_{L(K)} + V_t\right) \mathcal{M}_t.$$

Working in this vein we have

$$\begin{aligned}
&\partial_t \left[\mathcal{M}_{T-t}\left(g_t\right) \exp \int_0^t V_{T-\tau}\left(g_\tau\right) d\tau\right] \\
&= -\left(\left(\frac{1}{2}\triangle_{L(K)} + V_{T-t}\right)\mathcal{M}_{T-t}\right)\left(g_t\right) \exp\left(\int_0^t V_{T-\tau}\left(g_\tau\right) d\tau\right) dt \\
&\quad + dt V_{T-t}\left(g_t\right)\mathcal{M}_{T-t}\left(g_t\right) \exp\left(\int_0^t V_{T-\tau}\left(g_\tau\right) d\tau\right) dt \\
&\quad\quad + \frac{1}{2}\triangle_{L(K)}\mathcal{M}_{T-t}\left(g_t\right) \exp\left(\int_0^t V_{T-\tau}\left(g_\tau\right) d\tau\right) dt \\
&\quad\quad\quad + dmartingale \\
&= dmartingale.
\end{aligned}$$

Therefore, Let $\sigma_{[g_0]}$ denote the energy-minimizing path in the homotopy class of $[g_0]$. Let $\widetilde{E}(x)$ denote the energy $\int_0^1 \left|x(ds)\, x(s)^{-1}\right|_{\mathfrak{K}}^2$ of a path in $L(K)$ (defined only for absolutely continuous paths with one L^2-derivative). Then we have

$$\begin{aligned}
&E\mathcal{M}_T\left(g_0\right) \\
&= E \lim_{\varepsilon\to 0} \mathcal{M}_\varepsilon\left(g_T\right) \exp \int_\varepsilon^T V_{T-\tau}\left(g_\tau\right) d\tau \\
&= \int \exp \frac{-\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon} \delta_{\sigma_{[g_0]}}\left(x_T\right) \exp\left[\int_0^T V_{T-\tau}\left(x_\tau\right) d\tau\right] \nu_{g_0}(dx) \\
&= \int \delta_{\sigma_{[g_0]}}\left(x_T\right) \exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_{T-\tau}\left(x_\tau\right) d\tau - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right] \nu_{g_0}(dx) \\
&= \int \delta_{\sigma_{[g_0]}}\left(x_T\right) \delta_{g_0}\left(x_0\right) \exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_{T-\tau}\left(x_\tau\right) d\tau - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right] \nu(dx) \\
&= \int \delta_{\sigma_{[g_0]}}\left(x_0\right) \delta_{g_0}\left(x_T\right) \exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_{T-\tau}\left(x_{T-\tau}\right) d\tau - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right] \nu(dx) \\
&= \nu_{\sigma_{[g_0]}}\left[\delta_{g_0}\left(x_T\right) \exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_t\left(x_t\right) dt - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right]\right].
\end{aligned}$$

Here we have used the fact that backwards Brownian motion has the same law as a forwards Brownian motion. $\sigma_{[g_0]}$ has to be homotopic to g_0 because $g.$ explicitly describes such a homotopy.

Now letting $[g_0]$ run through the homotopy classes, we have

$$\begin{aligned}
&\mu_{0,T} f \\
&\quad = \int f(g_0)\, \mathcal{M}_T(g_0)\, \mathcal{D}g_0 \\
&\quad = \sum_{[g_0]} \int f(g_0)\, \delta_{g_0}(x_T) \exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_t(x_t)\,dt - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right] \nu_{\sigma_{[g_0]}}(dx)\, \mathcal{D}g_0 \\
&\quad = \sum_{[g_0]} \int f(x_T) \exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_t(x_t)\,dt - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}\right] \nu_{\sigma_{[g_0]}}(dx),
\end{aligned}$$

which is the desired result. ∎

Example 9.1 (The S^1 case)**.** Define a measure $\widetilde{\nu}_T$ by setting

$$\widetilde{\nu}_T \equiv \sum_{\alpha\in\mathbb{Z}} C_{\alpha,T}\, \nu_T^{S^1}(\sigma_\alpha, \cdot)$$

where

$$C_{\alpha,T} \equiv \frac{P_T^{\mathbb{R}}(0)}{P_T^{S^1}(e)} \exp\left(-\frac{1}{2T}\alpha^2\right).$$

Proof. Let $\sigma_\alpha \equiv (\cos 2\pi\alpha s, \sin 2\pi\alpha s)$ be the energy-minimizing geodesic in the α^{th} homotopy class for any $\alpha \in \mathbb{Z}$. Then for any loop x homotopic to σ_α, we have

$$\widetilde{E}(\sigma_\alpha) = \int_0^1 |\alpha|^2\, ds = \alpha^2.$$

and

$$\begin{aligned}
V_t(x) &= \frac{1}{2t^2}\left|\int_0^1 x(ds)\, x(s)^{-1}\right|_{\mathfrak{K}}^2 - \left(\frac{1}{2t}\dim\mathfrak{K} + \partial_t \log p_t^K(e)\right) \\
&= \frac{\alpha^2}{2t^2} - \left(\frac{1}{2t} + \partial_t \log p_t^K(e)\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\varepsilon^T V_t(x_t)\,d\tau - \frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon} \\
&\qquad = \lim_{\varepsilon\to 0}\int_\varepsilon^T \frac{\alpha^2}{2t^2} - \left(\frac{1}{2t} + \partial_t \log p_t^K(e)\right) dt - \frac{\alpha^2}{2\varepsilon} \\
&\qquad = \lim_{\varepsilon\to 0} \frac{\alpha^2}{2\varepsilon} - \frac{\alpha^2}{2T} - \int_\varepsilon^T \left(\frac{1}{2t} + \partial_t \log P_t^{S^1}(e)\right) dt - \frac{\alpha^2}{2\varepsilon} \\
&\qquad = \lim_{\varepsilon\to 0} \frac{-\alpha^2}{2T} - \int_\varepsilon^T \left(\frac{1}{2t} + \partial_t \log P_t^{S^1}(e)\right) dt \\
&\qquad = \lim_{\varepsilon\to 0} \frac{-\alpha^2}{2T} - \int_\varepsilon^T \left(\partial_t \log \sqrt{t} P_t^{S^1}(e)\right) dt \\
&\qquad = \frac{-\alpha^2}{2T} - \log\sqrt{T} P_T^{S^1}(e) + \log\sqrt{\varepsilon} P_\varepsilon^{S^1}(e).
\end{aligned}$$

By Lemma 5.3,

$$\lim_{\varepsilon\to 0}\varepsilon^{d/2}P_\varepsilon^K(e)=(2\pi)^{-d/2},$$

and so

$$\lim_{\varepsilon\to 0}\int_\varepsilon^T V_t(x_t)\,d\tau-\frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2\varepsilon}=\frac{-\alpha^2}{2T}-\log\sqrt{T}P_T^{S^1}(e)-\log\sqrt{2\pi}.$$

Thus we are done since

$$\begin{aligned}\mu_{0,T}f &= \sum_{[g_0]}\int f(x_T)\exp\left[\lim_{\varepsilon\to 0}\int_\varepsilon^T V_t(x_t)\,dt-\frac{\widetilde{E}\left(\sigma_{[g_0]}\right)}{2T\varepsilon}\right]\nu_{\sigma_{[g_0]}}(dx)\\ &= \sum_\alpha\int f(x_T)\exp\left[-\frac{\alpha^2}{2T}-\log\sqrt{T}P_T^{S^1}(e)-\log\sqrt{2\pi}\right]\nu_{\sigma_\alpha}(dx)\\ &= \sum_\alpha\frac{P_T^{\mathbb{R}}(0)}{P_T^{S^1}(e)}\int f(x_T)\exp\left[-\frac{\alpha^2}{2T}\right]\nu_{\sigma_\alpha}(dx).\end{aligned}$$

∎

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