ABSOLUTE CONTINUITY OF HEAT KERNEL MEASURE WITH PINNED WIENER MEASURE ON LOOP GROUPS

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Abstract. Let $t > 0$, K be a connected compact Lie group equipped with an Ad_K-invariant inner product on the Lie Algebra of K. Associated to this data are two measures μ_t^0 and ν_t^0 on $\mathcal{L}(K)$ – the space of continuous loops based at $e \in K$. The measure μ_t^0 is pinned Wiener measure with "variance t" while the measure ν_t^0 is a "heat kernel measure" on $\mathcal{L}(K)$. The measure μ_t^0 is constructed using a K – valued Brownian motion while the measure ν_t^0 is constructed using a $\mathcal{L}(K)$ – valued Brownian motion. In this paper we show that ν_t^0 is absolutely continuous with respect to μ_t^0 and the Radon-Nikodym derivative $d\nu_t^0/d\mu_t^0$ is bounded.

Contents

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1. Introduction

Let K be a connected compact Lie group, $\mathfrak{k} \equiv T_e K$ be the Lie algebra of K, and $\langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle_{\mathfrak{k}}$ be an Ad_K-invariant inner product on $\mathfrak{k}$. To simplify notation later we will assume that K is a matrix group. (Since K is compact, this is no restriction, see for example Theorem 4.1 on p. 136 in [7].)

Example 1.1. As an example, let $K = SO(3)$ be the group of 3×3 real orthogonal matrices with determinant 1. The Lie algebra of K is $\mathfrak{k} = so(3)$, the set of 3×3 real skew symmetric matrices, and the inner product $\langle A, B\rangle_{\mathfrak{k}} := -\mathrm{tr}(AB)$ is an example of an Ad_K – invariant inner product on $\mathfrak{k}$.

Elements $A \in \mathfrak{k}$ will be identified with the unique left invariant vector field on K agreeing with A at the identity in K, i.e. if $f \in C^\infty(K)$ then

$$Af(x) = \frac{d}{dt}|_0 f(xe^{tA}).$$

The path and loop groups on K are defined by

$$W(K) \equiv \{\sigma \in C([0,1] \to K)\,|\sigma(0) = e\} \tag{1.1}$$

and

$$\mathcal{L}(K) \equiv \{\sigma \in W(K)\,|\sigma(1) = e\} \tag{1.2}$$

respectively.

Notation 1.2. The constant path at e will be denote by $\mathbf{e}$, i.e. $\mathbf{e}(s) = e$ for $s \in [0,1]$.

Pinned Wiener measure (μ_t^0) on such a "loop group" (see [21], [23],[3], [17] and Definition 2.11 below) is the law of a K-valued Brownian motion starting at $e \in K$ and conditioned to end at $e \in K$. Heat kernel measure (ν_t^0) on $\mathcal{L}(K)$ (see [18], [13], and Carson [8, 9] and Definition 2.14 below) is the end point distribution of a "$\mathcal{L}(K)$ – valued Brownian motion." The main theorem (Theorem 2.16) in this paper asserts that ν_t^0 is absolutely continuous with respect to μ_t^0 and the Radon-Nikodym derivative $d\nu_t^0/d\mu_t^0$ is bounded. The proof of this theorem heavily relies on a theorem of Airault and Malliavin (Theorem 2.18 below) which shows that μ_t^0 solves a heat equation with a potential. A new proof of Theorem 2.18 will be given in Section 6.

One of our motivations for investigating Theorem 2.16 is L. Gross' logarithmic Sobolev inequality on $(\mathcal{L}(K), \mu_t^0)$. To state the inequality, let

$$\|\mathrm{grad} f\|^2 = \sum_{h \in S_0} (\partial_h f)^2,$$

where S_0 is an orthonormal basis for H_0 (H_0 is the $\mathfrak{k}$ – valued Cameron -Martin space in Definitions 3.1) and ∂_h is a left invariant vector field on $\mathcal{L}(K)$ defined in Definition 3.4. Also let us introduce the following notation. If μ is a measure on some measurable space Ω and $f : \Omega \to \mathbb{R}$ is a measurable function, let

$$\mu(f) = \int_\Omega f\, d\mu. \tag{1.3}$$

L. Gross proves in [20] that there is a constant $C < \infty$ such that

$$\int_{\mathcal{L}(K)} f^2 \log \frac{f^2}{\mu_t^0(f^2)} d\mu_t^0 \le C \int_{\mathcal{L}(K)} \left\{\|\mathrm{grad} f\|^2 + V f^2\right\} d\mu_t^0 \tag{1.4}$$

where V is essentially the same potential that appears in the Airault – Malliavin Theorem 2.18 below. It is still an open question as to whether the potential term Vf^2 in Eq. (1.4) is necessary or not.

On the other hand, it was shown in Driver and Lohrenz [18] that if μ_t^0 is replaced by ν_t^0, the potential term V is not needed, i.e. there is a constant $C < \infty$ such that

$$\int_{\mathcal{L}(K)} f^2 \log \frac{f^2}{\nu_t^0(f^2)} d\nu_t^0 \le C \int_{\mathcal{L}(K)} \|\mathrm{grad} f\|^2 \, d\nu_t^0. \tag{1.5}$$

Now Theorem 2.16 below shows that $Z_t := d\nu_t^0/d\mu_t^0$ is bounded. If one could show that Z_t^{-1} were also bounded, then the Holley – Stroock lemma (see [22] and Remark 1.20 in [10]) along with Eq. (1.5) would imply that Eq. (1.4) holds without the Vf^2 term. It is almost certainly seems too much to expect that Z_t is bounded from below in general. (It is not even known if $Z_t > 0$, μ_t^0 - a.s., when K is non-abelian.) So the authors do not expect this line of reasoning to work without modification. Nevertheless, better knowledge of the density Z_t may be useful in determining if potential is needed in Eq. (1.4).

1.1. **Conjecture on equivalence.**

Let us end this introduction with the following conjecture.

Conjecture. If K is simply connected (so that $\mathcal{L}(K)$ has only one connected component) then $Z_t > 0$, μ_t^0 - a.s. That is to say μ_t^0 is absolutely continuous relative to ν_t^0. If K is not simply connected, then we expect that μ_t^0 is absolutely continuous relative to a sum of left translates of ν_t^0 by finite energy loops from each homotopy class.

The explicit calculations in Section 7 shows that the conjecture is true for $K = \mathbb{R}^d$ and $K = S^1$, see Lemma 7.1 and Proposition 7.5. Moreover, the results in Srimurthy [31] also support the conjecture. Let $\mathcal{F}_\alpha$ be the σ – algebra consisting of the measurable sets in $W(K)$ depending only on the portion of the paths in $W(K)$ over the interval $[0, \alpha]$, see Definition 2.5 below. Srimurthy proves that μ_t^0 and ν_t^0 are equivalent on $\mathcal{F}_\alpha$ for any $\alpha < 1$. Of course these σ – algebras are not able to detect the homotopy classes in $\mathcal{L}(K)$ and it is certainly **not** true that μ_t^0 is absolutely continuous with respect to ν_t^0 if K is not simply connected. This is because pinned Wiener measure μ_t^0 charges all of the homotopy classes of K while the heat kernel measure ν_t^0 only charges the trivial homotopy class.

2. Notation and Statements of Results

2.1. **Brownian Sheets.**

Definition 2.1 ($\mathfrak{K}$ – valued Brownian Sheet)**.** Let $\{\beta(t,s)\}_{0\le s\le 1, 0\le t<\infty}$ be a $\mathfrak{k}$ – valued *Brownian sheet* and $\{\chi(t,s)\}_{0\le s\le 1, 0\le t<\infty}$ be a $\mathfrak{k}$ – valued *Brownian bridge sheet* defined on some probability space $(\Omega, \mathcal{G}, P)$. To be more precise, let $s \wedge \sigma \equiv \min(s,\sigma)$, $G_0(s,\sigma) = s\wedge\sigma - \sigma s$, $\beta^A(t,s) = \langle A, \beta(t,s)\rangle$ and $\chi^A(t,s) = \langle A, \chi(t,s)\rangle$. Then we are assuming the β and χ are centered Gaussian random fields with covariance functions

$$\mathbb{E}[\beta^A(t,s)\beta^B(\tau,\sigma)] = \langle A,B\rangle (t\wedge\tau)(s\wedge\sigma) \tag{2.1}$$

for all $s,\sigma,t,\tau \in [0,\infty)$ and $A, B \in \mathfrak{k}$ and

$$\mathbb{E}[\chi^A(t,s)\chi^B(\tau,\sigma)] = \langle A,B\rangle (t\wedge\tau) G_0(s,\sigma) \tag{2.2}$$

for all $s, \sigma \in [0,1]$, $t, \tau \in [0, \infty)$ and $A, B \in \mathfrak{k}$. (Here and in the sequel we will use $\mathbb{E}$ to denote the expectation relative to the measure P.)

It is well known that $\beta(t,s)$ and $\chi(t,s)$ may be chosen to have continuous sample paths, see for example the discussion after the proof of Corollary 1.3 in [33]. This fact may also be proved by abstract Wiener space considerations, see Remark 3.3 in [14]. So in the sequel we will assume that $(t,s) \to \beta(t,s)$ and $(t,s) \to \chi(t,s)$ are continuous processes.

Definition 2.2. A $\mathfrak{k}$ – valued process $\{B_s\}$ is said to be a *Brownian motion with variance t* if $\frac{1}{\sqrt{t}} B_s$ is a standard $\mathfrak{k}$ – valued Brownian motion. Alternatively, B may be described using Lévy's characterization (see for example Theorem 39 on p.80 in [27]) of Brownian motion, by requiring $\{B_s\}$ to be a mean zero martingale with quadratic co-variations given by $dB_s^C dB_s^D = t\langle C, D\rangle ds$ for all $C, D \in \mathfrak{k}$.

Remark 2.3. Notice that for fixed s; $t \to \beta(t,s)$ and $t \to \chi(t,s)$ are $\mathfrak{k}$ – valued Brownian motions with variance s and $G_0(s,s)$ respectively. This follows by the independent increments of these processes in the t variable, Lemma 8.1 of the Appendix, and Definition 2.2. Similarly for fixed t; $s \to \beta(s,t)$ is a $\mathfrak{k}$ – valued Brownian motion with variance t. The process $s \to \chi(t,s)$ is a Brownian Bridge for $0 \le s \le 1$ with quadratic co-variation given by $\chi^A(t,ds)\chi^B(t,ds) = t\langle A, B\rangle ds$, see Remark 2.12 below.

Definition 2.4 (Cylinder Functions)**.** For $0 \le s \le 1$, let $\pi_s : W(K) \to K$ be the projection map $\pi_s(\sigma) = \sigma(s)$. More generally if

$$\mathbb{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n < 1\} \tag{2.3}$$

is a partition of $[0,1]$, let $s_{n+1} = 1$ by convention and let $\pi_{\mathbb{P}} : W(K) \to K^n$ be given by

$$\pi_{\mathbb{P}}(\sigma) = (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n)). \tag{2.4}$$

A cylinder function f on $W(K)$ or $\mathcal{L}(K)$ is a function of the form $f = F \circ \pi_{\mathbb{P}}$ for some partition $\mathbb{P}$ and some measurable function $F : K^n \to \mathbb{R}$. The function f is said to be bounded (smooth) provided that F is bounded (smooth).

Definition 2.5. For $s \in [0,1]$, let $\mathcal{F}_s$ denote the σ – algebra on $W(K)$ generated by the smooth cylinder functions of the form $f = F \circ \pi_{\mathbb{P}}$ where $\mathbb{P}$ runs through partitions as in Eq. (2.3) with $s_n \le s$. We will write $\mathcal{F}$ for $\mathcal{F}_1$.

The σ – algebra, $\mathcal{F}$, is the same as the Borel σ – algebra on $W(K)$, where $W(K)$ is equipped with topology of uniform convergence relative to a metric on K derived from a Riemannian metric on TK.

Remark 2.6. i) For notational simplicity when working on $\mathcal{L}(K)$, we have defined $\pi_{\mathbb{P}}$ as in Eq. (2.4) rather than by $\pi_{\mathbb{P}}(\sigma) = (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n), \sigma(s_{n+1}))$ which would be more natural on $W(K)$. This results in a slightly smaller class of cylinder functions, but this is of no significance for our purposes.

The next result is well known, but we include it for the reader's convenience.

Lemma 2.7. *Suppose that Q is a finite measure on $(W(K), \mathcal{F})$ and $1 \le p < \infty$. Then the smooth cylinder functions are dense in $L^p(W(K), \mathcal{F}, Q)$.*

Proof. Let $\mathcal{M}$ denote the smooth cylinder functions and $\mathcal{H}$ denote those functions in the $L^p(W(K),\mathcal{F},Q)$ – closure of $\mathcal{M}$ which are also bounded. Then $\mathcal{H}$ is a vector space containing the constant functions and which clearly satisfies the property; if $\{f_n\}_{n=1}^\infty$ is a sequence of functions in $\mathcal{H}$ such that $0\le f_1\le f_2\le f_3\le\dots$, and $f:=\lim_{n\to\infty}f_n$ is bounded, then $f\in\mathcal{H}$. Since $\mathcal{M}$ is closed under multiplication, we may apply the monotone class theorem (see Theorem 8 on p. 7 in [27]) to conclude $\mathcal{H}$ contains all bounded $\mathcal{F}=\sigma(\mathcal{M})$ – measurable functions. Since (by the dominated convergence theorem) $\mathcal{H}$ is dense in $L^p(W(K),\mathcal{F},Q)$, we are done. ∎

2.2. $K-$ valued Brownian motion and Wiener measures.

Definition 2.8 (Wiener Measure on $W(K)$)**.** Fix $t>0$, let $\{g_s\}_{s\in[0,1]}$ denote the solution to the stochastic differential equation

$$dg_s=g_s\beta(t,\delta s)\text{ with }g_0=e\in K, \tag{2.5}$$

where $\beta(t,\delta s)$ denotes the Stratonovich differential of the Brownian motion $s\to\beta(t,s)$. The *Wiener measure with variance* t on $\mathcal{F}$ is $\mu_t:=Law(g_\cdot)$.

Let $\mathfrak{k}_0\subset\mathfrak{k}$ be an orthonormal basis for $\mathfrak{k}$, and Δ_K be the second order elliptic operator,

$$\Delta_K=\sum_{A\in\mathfrak{k}_0}A^2. \tag{2.6}$$

Since K is compact and hence uni-modular, Δ_K is the Laplace Beltrami operator for the left invariant Riemannian metric on K determined by $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}=T_eK$, see for example Remark 2.2 in [15]. Using Itô's lemma, one easily shows that $\{g_s\}_{s\in[0,1]}$ is a diffusion process on K with generator $\frac{1}{2}t\Delta_K$. Such a K – valued process will be called a *Brownian motion on K with variance t.*

Definition 2.9 (Heat Kernel on K)**.** Let p_t^K denote the smooth function of K such that $Law(g_1)=p_t^K(x)dx$, where dx denotes normalized Haar measure on K.

The function p_t^K is the convolution kernel for the heat operator $e^{t\Delta_K/2}$. In particular, $(t,x)\to p_t^K(x)$ is a smooth positive function such that for any $f\in C(K)$, the function u defined by

$$u(t,x)\equiv\int_K f(y)\,p_t^K\left(x^{-1}y\right)dy\text{ for }(t,x)\in(0,\infty)\times K$$

satisfies the heat equation

$$\partial_t u=\frac{1}{2}\triangle_K u\text{ with }\lim_{t\to 0}u(t,x)=f(x)$$

where $\partial_t=\partial/\partial t$.

Remark 2.10. It is well known that $p_t^K(x)=p_t^K\left(x^{-1}\right)$ for all $x\in K$, see for example Item 2 of Proposition 3.1 in [15]. It also well known that p_t^K is a class function, i.e.

$$p_t^K(xy)=p_t^K(yx)\text{ for all }x,y\in K. \tag{2.7}$$

This is a consequence of the fact that Δ_K is a bi-invariant differential operator because of the Ad_K – invariance of $\langle\cdot,\cdot\rangle$. Thus for all bounded measurable functions

f on K,

$$\begin{aligned}\int_K f(y)p_t^K\left(x^{-1}y\right)dy &= \int_K f(xy)p_t^K(y)\,dy \\ &= \left(e^{t\Delta_K/2}f\circ L_x\right)(e) = \left(e^{t\Delta_K/2}f\right)(x) = \left(e^{t\Delta_K/2}f\circ R_x\right)(e) \\ &= \int_K f(yx)p_t^K(y)\,dy = \int_K f(y)p_t^K\left(yx^{-1}\right)dy,\end{aligned}$$

where L_x and R_x denote left and right multiplication by $x \in K$ respectively. The last displayed equation implies Eq. (2.7).

By the Markov property of $g.$ and the previous comments, if f is a bounded cylinder function of the form $f(\sigma) = F\circ\pi_{\mathbb{P}}$ where $\mathbb{P}$ is as in Eq. (2.3), then

$$\mu_t(f) \equiv \int_{K^n} F(x_1,\cdots,x_n)\prod_{i=1}^n p^K_{t\Delta_i s}\left(x_{i-1}^{-1}x_i\right)dx_1dx_2\dots dx_n, \tag{2.8}$$

where $x_0 := e$ and $\Delta_i s = s_i - s_{i-1}$.

Definition 2.11 (Doob's Construction of Pinned Wiener Measure)**.** Pinned Wiener measure, μ_t^0, on $W(K)$ with variance t, is the unique measure on $\mathcal{F}$ such that if f is a bounded $\mathcal{F}_\alpha$ measurable function for some $\alpha \in (0,1)$, then

$$\mu_t^0(f) \equiv \frac{1}{p_t^K(e)}\mu_t(fp^K_{t(1-\alpha)}(\pi_\alpha)).$$

In particular if f is a bounded cylinder function f of the form $f(\sigma) = F\circ\pi_{\mathbb{P}}$ where $\mathbb{P}$ is as in Eq. (2.3), then

$$\mu_t^0(f) \equiv \int_{K^n} F(x)\rho^{\mathbb{P}}(t,x)dx, \tag{2.9}$$

where $x = (x_1,\cdots,x_n)$, $dx = dx_1dx_2\dots dx_n$ is normalized Haar measure on K^n and

$$\rho^{\mathbb{P}}(t,x) := \frac{1}{p_t^K(e)}\prod_{i=1}^{n+1} p^K_{t(s_i-s_{i-1})}\left(x_{i-1}^{-1}x_i\right) \tag{2.10}$$

where by convention $x_0 = x_{n+1} = e$.

The existence of the probability measure μ_t^0 and the fact that $\mu_t^0(\mathcal{L}(K)) = 1$ is well known. A proof may be found, for example, in Theorem 2.3 in [11]. To apply this theorem, the reader should take the covariant derivative ∇ appearing in Theorem 2.3 in [11] to be the unique one for which left invariant vector fields on K are covariantly constant.

Remark 2.12. In Remark 2.3 it was asserted that the process $s \to \chi(t,s)$ is a Brownian bridge with quadratic co-variation given by $\chi^A(t,ds)\chi^C(t,ds) = t\langle A,C\rangle ds$, that is to say $Law(\chi(t,\cdot))$ is pinned Wiener measure on $\mathcal{L}(\mathfrak{k})$ with variance t. To check this let $p_t(x) = (2\pi t)^{-\dim\mathfrak{k}/2}e^{-|x|_{\mathfrak{k}}^2/2t}$ be the Euclidean heat kernel on $\mathfrak{k}$. Then for a cylinder function f on $\mathcal{L}(\mathfrak{k})$ based on a partition $\mathbb{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n < 1\}$, we must show that

$$\mathbb{E}f(\chi(t,\cdot)) = \mathbb{E}\left[f(B)\frac{p_{t(1-\alpha)}(B_\alpha)}{p_t(0)}\right], \tag{2.11}$$

where $B_s = \beta(t,s)$ – a $\mathfrak{k}$ – valued Brownian motion with variance t.

Proof. To prove Eq. (2.11), let $Z_s = p_t(0)^{-1} p_{t(1-s)}(\beta(t,s))$ for $0 \le s < 1$, then by Itô's lemma and the fact that

$$\frac{\partial}{\partial s} p_{t(1-s)}(x) = -\frac{1}{2} t \Delta_{\mathfrak{k}} p_{t(1-s)}(x) \text{ and } \nabla \log p_{t(1-s)}(x) = -\frac{1}{t(1-s)} x$$

we have

$$dZ_s = -Z_s \frac{1}{t(1-s)} \langle B_s, dB_s \rangle \text{ with } Z_0 = 1.$$

By Girsanov's theorem (see for example Theorem 20 on p.109 in [27])

$$M_s := B_s - \int_0^s \frac{1}{Z_r} dZ_r dB_r = B_s + \int_0^s \frac{1}{(1-r)} B_r dr \tag{2.12}$$

is a martingale on $[0, \alpha]$ relative to the measure $Z_\alpha P$. Since M has the same quadratic variation as $\beta(t, \cdot)$, by Lévy's criteria, M is a $\mathfrak{k}$ – valued Brownian motion with variance t under the measure $Z_\alpha P$. Interpreting (2.12) as stochastic differential equation for B,

$$dB = dM - \frac{1}{(1-s)} B_s ds \text{ with } B_0 = 0,$$

we find by variation of parameters that

$$B_s = \int_0^s e^{-\int_r^s \frac{1}{(1-\sigma)} d\sigma} dM_r = \int_0^s \frac{1-s}{1-r} dM_r.$$

This shows that, under $Z_\alpha P$, $\{B_s\}_{0 \le s \le \alpha}$ is still a Gaussian process. Moreover, for $0 \le \sigma \le s \le \alpha$,

$$\begin{aligned}
\mathbb{E}[B_s^C B_r^D Z_\alpha] &= \mathbb{E}[\left(\int_0^s \frac{1-s}{1-r} dM_r^C \right) \left(\int_0^\sigma \frac{1-\sigma}{1-r} dM_r^D \right) Z_\alpha] \\
&= t(1-s)(1-\sigma) \langle C, D \rangle \int_0^\sigma \frac{1}{(1-r)^2} dr \\
&= t(1-s)(1-\sigma) \left(1 - \frac{1}{1-\sigma} \right) \langle C, D \rangle \\
&= t\sigma(1-s) = tG_0(\sigma, s) \langle C, D \rangle
\end{aligned}$$

which is the same covariance function as $\chi(t, \cdot)$. Therefore $\{B_s\}_{0 \le s \le \alpha}$ under the measure $Z_\alpha P$ has the same law as $\{\chi(t,s)\}_{0 \le s \le \alpha}$ under the measure P. This is the assertion in Eq. (2.11). ∎

2.3. **Heat kernel measure on $W(K)$ and $\mathcal{L}(K)$.** In this section we are going to define heat kernel measures on $W(K)$ and $\mathcal{L}(K)$ by formally replacing K from the previous section by $W(K)$ and $\mathcal{L}(K)$ respectively. Following Malliavin [24], we have the following theorem.

Theorem 2.13 (Brownian Motion on $W(K)$ and $L(K)$)**.** *There are jointly continuous solutions $\Sigma(t,s)$ and $\Sigma^0(t,s)$ to the stochastic differential equations:*

$$\Sigma(\delta t, s) = \Sigma(t,s) \beta(\delta t, s) \text{ with } \Sigma(0,s) = e \quad \forall s \in [0, \infty), \tag{2.13}$$

and

$$\Sigma^0(\delta t, s) = \Sigma^0(t,s) \chi(\delta t, s) \text{ with } \Sigma(0,s) = e \quad \forall s \in [0,1]. \tag{2.14}$$

As before $\beta(\delta t, s)$ denotes the Stratonovich differentials of the processes $t \to \beta(t,s)$ and similarly for $\Sigma(\delta t, s)$, $\Sigma^0(\delta t, s)$, and $\chi(\delta t, s)$.

Proof. Such results may be found in Baxendale, [4], Malliavin [24], or in Theorem 3.8 in Driver [13]. The last two references cover the $\mathcal{L}(K)$ case, however the proof of the $W(K)$ case is the same, just replace $G_0(s,\sigma)$ by $s \wedge \sigma$ throughout. ∎

Definition 2.14 (Heat Kernel Measures on $W(K)$ and $L(K)$)**.** The measures $\nu_t = Law(\Sigma(t,\cdot)$ and $\nu_t^0 = Law(\Sigma^0(t,\cdot))$ are called heat kernel measures on $W(K)$ and $\mathcal{L}(K)$ respectively. So ν_t and ν_t^0 are determined by

$$\nu_t(f) = \mathbb{E}f(\Sigma(t,\cdot)) \text{ and } \nu_t^0(f) = \mathbb{E}f(\Sigma^0(t,\cdot)) \tag{2.15}$$

for all bounded $\mathcal{F}$ – measurable f. Notice that $\nu_t^0(\mathcal{L}(K)) = 1$ because $\Sigma^0(t,0) = \Sigma^0(t,1) = e$, P – almost surely.

Corollary 3.10 below justifies calling ν_t and ν_t^0 heat kernel measures.

2.4. **Statement of Results.** The following theorem is Lemma 1 in Airault and Malliavin [1].

Theorem 2.15. *Let $t > 0$, then $\nu_t = \mu_t$ on $W(K)$, i.e. heat kernel measure at time t and Wiener measure with variance t are the same on $W(K)$.*

This theorem is also proved in Lemma 3.3 of Srimurthy [31]. Since this theorem is crucial to the rest of the paper, we will give a proof in Section 4 below. The following theorem is the main result of the paper.

Theorem 2.16. *Let $t > 0$, then $\nu_t^0 \ll \mu_t^0$, i.e. heat kernel measure at time t is absolutely continuous relative to pinned Wiener measure with variance t. Moreover, the Radon-Nikodym derivative, $d\nu_t^0/d\mu_t^0$, satisfies the bound*

$$\frac{d\nu_t^0}{d\mu_t^0} \le e^{C_t}$$

where

$$C_t \equiv \log\left[(2\pi t)^{\frac{1}{2}\dim \mathfrak{k}} p_t^K(e)\right]. \tag{2.16}$$

(Standard heat kernel asymptotics shows that $\lim_{t\to 0} C_t = 0$, see Lemma 6.1 below.)

The proof of this theorem (given in Section 6) will be a combination of the maximum principle along with a theorem of Airault and Malliavin [2]. In order to state the Airault-Malliavin theorem, let us recall that the coordinate process $\pi_s : \mathcal{L}(K) \to K$ (see Definition 2.4) is a semi-martingale relative to pinned Wiener measure, μ_t^0, see for example Bismut [6] or Theorem 2.3 in [12]. Hence we may define the $\mathfrak{k}$ – valued semi-martingale$\{b_s\}_{0\le s\le 1}$ by

$$b_s := \int_0^s \pi_r^{-1}\delta\pi_r. \tag{2.17}$$

Remarks 2.17. i) Technically speaking the stochastic integral in Eq. (2.17) depends on the measure μ_t^0 and in particular on $t > 0$. So a more appropriate notation would be to display this t dependence and write b_s^t for the μ_t^0 – a.e. defined stochastic integral $\int_0^s \pi_r^{-1}\delta\pi_r$. Since we will only need the process b_s for one fixed value of t, we will stick with the notation in Eq. (2.17).

ii) Gross shows (see Lemma 4.8 and Remark 4.9 in [20]) that $b_1 \in L^p(\mathcal{L}(K),\mu_t^0)$ and that $b_s \to b_1$ in $L^p(\mathcal{L}(K),\mu_t^0)$ as $s \to 1$ for all $1 \le p < \infty$.

Theorem 2.18 (Airault & Malliavin). *Let $V_t : \mathcal{L}(K) \to \mathbb{R}$ be the "potential,"*

$$V_t = \frac{1}{2t^2} |b_1|^2_{\mathfrak{k}} - c_t \tag{2.18}$$

where b_1 is defined in Eq. (2.17) and

$$c_t \equiv \frac{dC_t}{dt} = \frac{\dim \mathfrak{k}}{2t} + \partial_t \log p^K_t(e). \tag{2.19}$$

Then for any smooth cylindrical function $f : \mathcal{L}(K) \to \mathbb{R}$ (see Definition 2.4)

$$\partial_t \mu^0_t(f) = \mu^0_t \left[\left(\frac{1}{2} \triangle_{\mathcal{L}(K)} + V_t \right) f \right], \tag{2.20}$$

where $\triangle_{\mathcal{L}(K)}$ is the generator of the process $\Sigma^0(t, \cdot)$, see Definition 3.6 and Proposition 3.9 below.

We will give a simplified (in our view) proof of this theorem in Section 5. The proof relies on Theorem 2.15 and integration by parts on $(W(K), \mu_t)$.

3. Generators of $\Sigma(t, \cdot)$ and $\Sigma^0(t, \cdot)$

Much of the material in this section may be found in [18] and [13]. Nevertheless, in order to introduce the notation and for the readers convenience we will summarize some of the results in these papers.

3.1. **Cameron-Martin spaces.**

Definition 3.1. Given a continuous function $h : [0,1] \to \mathfrak{k}$ define

$$(h,h)_H = \begin{cases} \int_0^1 |h'(s)|^2 ds & \text{if} \quad h \text{ is absolutely continuous} \\ \infty & \text{otherwise.} \end{cases}$$

The *Cameron–Martin space* of $\mathfrak{k}$ is

$$H \equiv \{h \in C([0,1] \to \mathfrak{k}) | h(0) = 0 \text{ and } (h,h) < \infty\}$$

which we equip with the inner product

$$(h,k) = \int_0^1 \langle h'(s), k'(s) \rangle ds.$$

The *pinned Cameron–Martin space* is

$$H_0 \equiv \{h \in H(\mathfrak{k}) | h(1) = 0\}$$

which is a closed subspace of H. (The Hilbert spaces H and H_0 are to be thought of as the "Lie algebras" to the groups $W(K)$ and $\mathcal{L}(K)$.)

Notation 3.2. Let $S \subset H$ and $S_0 \subset H_0$ be orthonormal bases for H and H_0 respectively.

Lemma 3.3. *Let $\mathfrak{k}_0 \subset \mathfrak{k}$ be an orthonormal basis for $\mathfrak{k}$, $G(s,t) = s \wedge t$ and $G_0(s,t) \equiv s \wedge t - st$ for all $s, t \in [0,1]$. Then*

$$\sum_{h \in S} h(s) \otimes h(t) = G(s,t) \sum_{A \in \mathfrak{k}_0} A \otimes A \in \mathfrak{k} \otimes \mathfrak{k}. \tag{3.1}$$

$$\sum_{h \in S_0} h(s) \otimes h(t) = G_0(s,t) \sum_{A \in \mathfrak{k}_0} A \otimes A \in \mathfrak{k} \otimes \mathfrak{k}. \tag{3.2}$$

Proof. Let $A, B \in \mathfrak{k}$. Since $G(t,\cdot)B$ and $G(s,\cdot)A$ are in H,

$$(G(t,\cdot)B, G(s,\cdot)A) = \sum_{h\in S}(G(t,\cdot)B,h)(h,G(s,\cdot)A) \tag{3.3}$$

where the sum is absolutely convergent. By the fundamental theorem of calculus, G satisfies the reproducing property,

$$\int_0^1 \partial_s G(t,s)h'(s)ds = h(t) \text{ for all } h \in H.$$

Combined this equation with Eq. (3.3) shows that

$$G(s,t)\langle B,A\rangle = \sum_{h\in S}\langle B,h(t)\rangle\langle h(s),A\rangle$$

which implies Eq. (3.1) since A and B are arbitrary. Equation (3.2) is proved similarly, see Lemma 3.8 in [18] for more details. ∎

3.2. **Derivatives and Laplacians on $\mathcal{L}(K)$ and $W(K)$.**

Definition 3.4 (Left invariant derivatives)**.** Given $h \in H$ (or H_0) and $f : W(K) \to \mathbb{R}$ (or $f : \mathcal{L}(K) \to \mathbb{R}$) a smooth cylinder function, define

$$(\partial_h f)(\sigma) := \frac{d}{dt}|_0 f(\sigma e^{th}) \text{ for all } \sigma \in W(K) \ (\sigma \in \mathcal{L}(K))$$

where $\sigma e^{th} \in W(K)$ $(\sigma e^{th} \in \mathcal{L}(K))$ is defined by $\left(\sigma e^{th}\right)(s) := \sigma(s)e^{th(s)}$ for $s \in [0,1]$.

Remark 3.5. Suppose that $f = F\circ\pi_{\mathbb{P}}$ where $\mathbb{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n < 1\}$ is a partition of $[0,1]$ and $F : K^n \to \mathbb{R}$ is a smooth function. For $A \in \mathfrak{k}$ and $i \in \{1,2,\ldots,n\}$, let

$$A^{(i)}F(x_1,x_2,\ldots,x_n) = \frac{d}{dt}|_0 F(x_1,x_2,\ldots,x_{i-1},x_i e^{tA},x_{i+1},\ldots,x_n),$$

so that $A^{(i)}$ is the action of A on the ith variable of F. Then for $h \in H$ (or $h \in H_0$),

$$\partial_h f = \sum_{i=1}^n \left(h(s_i)^{(i)}F\right)\circ\pi_{\mathbb{P}}. \tag{3.4}$$

In particular $\partial_h f$ is still a smooth cylinder function. Therefore the operator $\partial_h^2 f$ is well defined and is given by

$$\partial_h^2 f = \sum_{i,j=1}^n \left(h(s_j)^{(j)}h(s_i)^{(i)}F\right)\circ\pi_{\mathbb{P}}. \tag{3.5}$$

Definition 3.6. Again suppose that $f = F\circ\pi_{\mathbb{P}}$ is a smooth cylinder function as in Definition 2.4. Define the *Laplacians* on $W(K)$ and $\mathcal{L}(K))$ by

$$\begin{aligned}\triangle_{W(K)}f &\equiv \sum_{h\in S}\partial_h^2 f \text{ and}\\ \triangle_{\mathcal{L}(K)}f &\equiv \sum_{h\in S_0}\partial_h^2 f\end{aligned}$$

respectively.

Remark 3.7. Combining Eqs. (3.1), (3.2) and (3.5) we find

$$\triangle_{W(K)} f = \sum_{h\in S}\sum_{i,j=1}^{n}\left(h(s_j)^{(j)}h(s_i)^{(i)}F\right)\circ\pi_{\mathbb{P}}$$

$$= \sum_{A\in\mathfrak{k}_0}\sum_{i,j=1}^{n} G(s_i,s_j)\left(A^{(j)}A^{(i)}F\right)\circ\pi_{\mathbb{P}} \tag{3.6}$$

and

$$\triangle_{\mathcal{L}(K)} f = \sum_{h\in S_0}\sum_{i,j=1}^{n}\left(h(s_j)^{(j)}h(s_i)^{(i)}F\right)\circ\pi_{\mathbb{P}}$$

$$= \sum_{A\in\mathfrak{k}_0}\sum_{i,j=1}^{n} G_0(s_i,s_j)\left(A^{(j)}A^{(i)}F\right)\circ\pi_{\mathbb{P}}. \tag{3.7}$$

Notation 3.8. Given, $\mathbb{P}=\{0=s_0<s_1<s_2<\cdots<s_n<1\}$, a partition of $[0,1]$ and $F\in C^{\infty}(K^n)$, let

$$L_{\mathbb{P}}F=\sum_{A\in\mathfrak{k}_0}\sum_{i,j=1}^{n} G(s_i,s_j)A^{(j)}A^{(i)}F \tag{3.8}$$

and

$$L^0_{\mathbb{P}}F=\sum_{A\in\mathfrak{k}_0}\sum_{i,j=1}^{n} G_0(s_i,s_j)A^{(j)}A^{(i)}F. \tag{3.9}$$

With this notation we may write Eqs. (3.6) and (3.7) as

$$\triangle_{W(K)}(F\circ\pi_{\mathbb{P}})=(L_{\mathbb{P}}F)\circ\pi_{\mathbb{P}} \text{ and } \triangle_{\mathcal{L}(K)}(F\circ\pi_{\mathbb{P}})=\left(L^0_{\mathbb{P}}F\right)\circ\pi_{\mathbb{P}}. \tag{3.10}$$

3.3. **Heat equations.**

Proposition 3.9. *The processes $\Sigma(t,\cdot)$ and $\Sigma^0(t,\cdot)$ are diffusion processes with $\triangle_{W(K)}$ and $\triangle_{\mathcal{L}(K)}$ as generators. More precisely, if $f=F\circ\pi_{\mathbb{P}}$ is a cylinder function as above, then*

$$M_t^f=f(\Sigma(t,\cdot))-f(\mathbf{e})-\frac{1}{2}\int_0^t\left(\Delta_{W(K)}f\right)(\Sigma(\tau,\cdot))d\tau \tag{3.11}$$

and

$$M_t^f=f(\Sigma^0(t,\cdot))-f(\mathbf{e})-\frac{1}{2}\int_0^t\left(\Delta_{\mathcal{L}(K)}f\right)(\Sigma^0(\tau,\cdot))d\tau \tag{3.12}$$

are martingales.

Proof. We will only prove Eq. (3.11) since the proof of Eq. (3.12) is completely analogous. Let $\Sigma_{\mathbb{P}}(t):=\pi_{\mathbb{P}}(\Sigma(t,\cdot))\in K^n$ and $B_{\mathbb{P}}(t)=(\beta(t,s_1),\ldots,\beta(t,s_n))$, then

$f(\Sigma(t,\cdot)) = F(\Sigma_{\mathbb{P}}(t))$ and by Itô's Lemma we have that

$$\begin{aligned}
df(\Sigma(t,\cdot)) &= dF(\Sigma_{\mathbb{P}}(t)) \\
&= \sum_{i=1}^{n}\sum_{A\in\mathfrak{k}_0} A^{(i)}F(\Sigma_{\mathbb{P}}(t))\beta^A(\delta t, s_i) \\
&= \sum_{i=1}^{n}\sum_{A\in\mathfrak{k}_0} A^{(i)}F(\Sigma_{\mathbb{P}}(t))\beta^A(dt, s_i) \\
&\quad +\frac{1}{2}\sum_{i,j=1}^{n}\sum_{A,B\in\mathfrak{k}_0} B^{(j)}A^{(i)}F(\Sigma_{\mathbb{P}}(t))\beta^A(dt, s_i)\beta^B(dt, s_j) \\
&= \sum_{i=1}^{n}\sum_{A\in\mathfrak{k}_0} A^{(i)}F(\Sigma_{\mathbb{P}}(t))\beta^A(dt, s_i) \\
&\quad +\frac{1}{2}\sum_{i,j=1}^{n}\sum_{A\in\mathfrak{k}_0} G(s_i,s_j)A^{(j)}A^{(i)}F(\Sigma_{\mathbb{P}}(t))dt \\
&= \sum_{i=1}^{n}\sum_{A\in\mathfrak{k}_0} A^{(i)}F(\Sigma_{\mathbb{P}}(t))\beta^A(dt, s_i) + \frac{1}{2}\left(\Delta_{W(K)}f\right)(\Sigma(t,\cdot))dt.
\end{aligned}$$

This shows that M_t^f is the martingale;

$$M_t^f = \sum_{i=1}^{n}\sum_{A\in\mathfrak{k}_0}\int_0^t A^{(i)}F(\Sigma_{\mathbb{P}}(\tau))\beta^A(d\tau, s_i).$$

∎

Corollary 3.10. *The measures ν_t and ν_t^0 satisfy the heat equations on $W(K)$ and $\mathcal{L}(K)$ in the following weak sense. If $f: W(K)\to\mathbb{R}$ is a smooth cylinder function then*

$$\partial_t\nu_t(f) = \frac{1}{2}\nu_t(\Delta_{W(K)}f) \tag{3.13}$$

and

$$\partial_t\nu_t^0(f) = \frac{1}{2}\nu_t^0(\Delta_{\mathcal{L}(K)}f). \tag{3.14}$$

Proof. Taking expectations of Eq. (3.11) shows that

$$\begin{aligned}
0 &= \mathbb{E}M_t^f = \mathbb{E}f(\Sigma(t,\cdot)) - f(\mathbf{e}) - \frac{1}{2}\int_0^t \mathbb{E}\left(\Delta_{W(K)}f\right)(\Sigma(\tau,\cdot))d\tau \\
&= \nu_t(f) - f(\mathbf{e}) - \frac{1}{2}\int_0^t \nu_\tau\left(\Delta_{W(K)}f\right)d\tau.
\end{aligned}$$

Differentiating this equation in t proves Eq. (3.13). Eq. (3.14) is proved analogously. ∎

Corollary 3.11 (Heat solution)**.** *Suppose that $u: \mathcal{L}(K)\to\mathbb{R}$ is a smooth cylinder function and let*

$$H(t,\sigma) = \int_{\mathcal{L}(K)} u(\sigma\gamma^{-1})d\nu_t^0(\gamma), \tag{3.15}$$

then

$$\partial_t H(t,\sigma) = \frac{1}{2}\triangle_{\mathcal{L}(K)} H(t,\sigma) \text{ and } \lim_{t\to 0} H(t,\sigma) = u(\sigma) \tag{3.16}$$

Proof. For $\sigma \in \mathcal{L}(K)$, let $u_\sigma : \mathcal{L}(K) \to \mathbb{R}$ be the cylinder function defined by $u_\sigma(\gamma) = u(\sigma\gamma^{-1})$. Notice that for $h \in H$,

$$(\partial_h u_\sigma)(\gamma) = \frac{d}{d\epsilon}|_0 u_\sigma(\gamma e^{\epsilon h}) = \frac{d}{d\epsilon}|_0 u(\sigma e^{-\epsilon h}\gamma^{-1}) = -\partial_h(\sigma \to u_\sigma(\gamma))$$

and therefore

$$\left(\Delta_{\mathcal{L}(K)} u_\sigma\right)(\gamma) = \Delta_{\mathcal{L}(K)}(\sigma \to u_\sigma(\gamma)).$$

Thus by Corollary 3.10,

$$\begin{aligned}
\partial_t H(t,\sigma) &= \frac{1}{2}\int_{\mathcal{L}(K)} \left(\triangle_{\mathcal{L}(K)} u_\sigma\right)(\gamma) d\nu_t^0(\gamma) \\
&= \frac{1}{2}\int_{\mathcal{L}(K)} \Delta_{\mathcal{L}(K)}(\sigma \to u_\sigma(\gamma)) d\nu_t^0(\gamma) \\
&= \frac{1}{2}\Delta_{\mathcal{L}(K)}\left(\sigma \to \int_{\mathcal{L}(K)} u_\sigma(\gamma) d\nu_t^0(\gamma)\right) \\
&= \frac{1}{2}\triangle_{\mathcal{L}(K)} H(t,\sigma).
\end{aligned}$$

Working with the explicitly representation of u as a cylinder function and using Eq. (3.10), it is easy to justify the interchange of $\Delta_{\mathcal{L}(K)}$ with the integral in the third equality. This proves the first assertion in Eq. (3.16). The second follows from the dominated convergence theorem and the identity,

$$H(t,\sigma) = \mathbb{E}\left[u(\sigma\Sigma^0(t,\cdot)^{-1})\right],$$

where $\Sigma^0(t,s)$ is the process defined in Eq. (2.14) of Theorem 2.13. ∎

4. The path group case

In the next subsection we will give a proof of Theorem 2.15. However, before doing this let us record the following trivial Corollary of Theorem 2.15 and Corollary 3.10 above. This corollary will be key to our proof of the Airault Malliavin theorem in Section 5.

Corollary 4.1. *The Wiener measure μ_t with variance t satisfies (weakly) the heat equation on $W(K)$, i.e. if $f : W(K) \to \mathbb{R}$ is a smooth cylinder function then*

$$\partial_t \mu_t(f) = \frac{1}{2}\mu_t(\Delta_{W(K)} f). \tag{4.1}$$

4.1. Proof of Theorem 2.15.

Proof. As mentioned in Section 2, the reader may find this theorem in Lemma 1 of Airault and Malliavin [1] or Lemma 3.3 of Srimurthy [31]. It would also be possible to give a proof using two parameter stochastic calculus as developed in Norris [26]. Rather than introduce this machinery, we will give a more pedestrian but perhaps less illuminating proof. Our proof is similar to that in [31].

Let Σ denote the process defined in Theorem 2.13 and $\mathbb{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n < 1\}$ be a partition of $[0,1]$. Let

$$U_i(t) := \Sigma(t, s_i)\Sigma(t, s_{i-1})^{-1} \tag{4.2}$$

and

$$B_i(t) := \int_0^t Ad_{\Sigma(\tau, s_{i-1})}\left(\beta(\delta\tau, s_i) - \beta(\delta\tau, s_{i-1})\right) \tag{4.3}$$

for $i = 1, 2, \ldots, n$. By Eq. (2.13) and Itô's Lemma,

$$\delta_t \Sigma(t,s)^{-1} = -\beta(\delta t, s)\Sigma(t,s)^{-1}$$

and therefore

$$\begin{aligned} \delta U_i(t) &= \Sigma(t, s_i)\left(\beta(\delta t, s_i) - \beta(\delta t, s_{i-1})\right)\Sigma(t, s_{i-1})^{-1} \\ &= U_i(t) Ad_{\Sigma(t, s_{i-1})}\left(\beta(\delta t, s_i) - \beta(\delta t, s_{i-1})\right) \\ &= U_i(t)\delta B_i(t). \end{aligned} \tag{4.4}$$

Because, $t \to \beta(t, s_{i-1})$ and $t \to \beta(t, s_i) - \beta(t, s_{i-1})$ are independent Brownian motions on $\mathfrak{k}$,

$$\begin{aligned} &Ad_{\Sigma(t,s_{i-1})}\left(\beta(\delta t, s_i) - \beta(\delta t, s_{i-1})\right) \\ &\quad = Ad_{\Sigma(t,s_{i-1})}\left(\beta(dt, s_i) - \beta(dt, s_{i-1})\right) \\ &\quad + \frac{1}{2} Ad_{\Sigma(t,s_{i-1})}[\beta(dt, s_{i-1}), (\beta(dt, s_i) - \beta(dt, s_{i-1}))]_{\mathfrak{k}} \\ &\quad = Ad_{\Sigma(t,s_{i-1})}\left(\beta(dt, s_i) - \beta(dt, s_{i-1})\right). \end{aligned}$$

Therefore the Stratonovich differentials in Eq. (4.3) may be replaced by Itô differentials to learn that $B_i(t)$ is the martingale

$$B_i(t) := \int_0^t Ad_{\Sigma(\tau, s_{i-1})}\left(\beta(d\tau, s_i) - \beta(d\tau, s_{i-1})\right).$$

Claim. The processes $B_1, B_2, \ldots B_n$ are independent $\mathfrak{k}$ – valued Brownian motions with variances $\Delta_i s := s_i - s_{i-1}$ for $i = 1, 2, \ldots, n$.

To prove this claim, let $C, D \in \mathfrak{k}$, and let $B_i^C(t) = \langle B_i(t), C\rangle$, $B_j^D(t) = \langle B_j(t), D\rangle$ and $\Delta_i\beta(t) := \beta(t, s_i) - \beta(t, s_{i-1})$. Then because $\langle\cdot,\cdot\rangle$ is Ad_K – invariant,

$$dB_i^C(t) = \langle Ad_{\Sigma(t,s_{i-1})} d\Delta_i\beta(t), C\rangle = \langle d\Delta_i\beta(t), Ad^{-1}_{\Sigma(t,s_{i-1})} C\rangle.$$

Thus the differential of the quadratic co-variation of B_i^C and B_j^D is given by,

$$\begin{aligned} dB_i^C(t) dB_j^D(t) &= \langle d\Delta_i\beta(t), Ad^{-1}_{\Sigma(t,s_{i-1})} C\rangle \langle d\Delta_j\beta(t), Ad^{-1}_{\Sigma(t,s_{j-1})} D\rangle \\ &= \sum_{A \in \mathfrak{k}_0} \langle A, Ad^{-1}_{\Sigma(t,s_{i-1})} C\rangle \langle A, Ad^{-1}_{\Sigma(t,s_{j-1})} D\rangle \, d\Delta_i\beta^A(t)\, d\Delta_j\beta^A(t) \\ &= \delta_{ij} \sum_{A \in \mathfrak{k}_0} \langle A, Ad^{-1}_{\Sigma(t,s_{i-1})} C\rangle \langle A, Ad^{-1}_{\Sigma(t,s_{i-1})} D\rangle \Delta_i s dt \\ &= \delta_{ij} \langle Ad^{-1}_{\Sigma(t,s_{i-1})} C, Ad^{-1}_{\Sigma(t,s_{i-1})} D\rangle \Delta_i s dt \\ &= \delta_{ij}\langle C, D\rangle \Delta_i s dt, \end{aligned} \tag{4.5}$$

wherein the third equality we have used: i) $\Delta_i\beta^A(\cdot) = \beta^A(\cdot, s_i) - \beta^A(\cdot, s_{i-1})$ and $\Delta_j\beta^A(\cdot) = \beta^A(\cdot, s_j) - \beta^A(\cdot, s_{j-1})$ are independent if $i \neq j$ and $\Delta_i\beta^A(\cdot)$ is a $\mathfrak{k}$ – valued Brownian motion with variance $\Delta_i s$. In the last equality we again have used the Ad_K – invariance of $\langle\cdot,\cdot\rangle$. Eq. (4.5) along with Lévy's criteria proves the claim.

Since the U_i's in Eq. (4.2) satisfy Eq. (4.4), the claim implies that $U_1(t), U_2(t), \ldots, U_n(t)$ are independent K – valued Brownian motion with variance $\Delta_1 s, \Delta_2 s, \ldots, \Delta_n s$ respectively. Suppose that $f = F \circ \pi_{\mathbb{P}}$ is a bounded cylinder function on $W(K)$. Define $\tilde{F} : K^n \to \mathbb{R}$ so that

$$F(x_1, x_2, x_3, \ldots, x_n) = \tilde{F}(x_1, x_2 x_1^{-1}, x_3 x_2^{-1} \ldots, x_n x_{n-1}^{-1})$$

for all $x_i \in K$. Then

$$f(\Sigma(t, \cdot)) = \tilde{F}(U_1(t), U_2(t), \ldots, U_n(t))$$

and therefore

$$\begin{aligned} \nu_t(f) &= \mathbb{E} f(\Sigma(t, \cdot)) = \mathbb{E}\tilde{F}(U_1(t), U_2(t), \ldots, U_n(t)) \\ &= \int_{K^n} \tilde{F}(x_1, \cdots, x_n) \prod_{i=n}^{n} p^K_{t\Delta_i s}(x_i) dx_i. \end{aligned} \tag{4.6}$$

Let $x_0 := e$. Using the invariance of Haar measure, make the translations

$$\begin{aligned} x_2 &\longrightarrow x_2 x_1^{-1} \text{ then} \\ x_3 &\longrightarrow x_3 x_2^{-1} \text{ then} \\ &\vdots \\ x_n &\longrightarrow x_n x_{n-1}^{-1} \end{aligned}$$

in the last integral of Eq. (4.6) to find

$$\begin{aligned} \nu_t(f) &= \int_{K^n} \tilde{F}(x_1, x_2 x_1^{-1} \cdots, x_n x_{n-1}^{-1}) \prod_{i=n}^{n} p^K_{\Delta_i st}(x_i x_{i-1}^{-1}) dx_i \\ &= \int_{K^n} F(x_1, x_2, x_3, \ldots, x_n) \prod_{i=n}^{n} p^K_{\Delta_i st}(x_i x_{i-1}^{-1}) dx_i \\ &= \int_{K^n} F(x_1, x_2, x_3, \ldots, x_n) \prod_{i=n}^{n} p^K_{\Delta_i st}(x_{i-1}^{-1} x_i) dx_i \end{aligned} \tag{4.7}$$

wherein the last equality we have use the fact that $p_t^K(\cdot)$ is a class function, see Remark 2.10.

Comparing Eq. (4.7) with Eq. (2.8), shows that $\nu_t(f) = \mu_t(f)$ for all bounded cylinder functions f on $W(K)$ which implies that $\nu_t = \mu_t$ by Lemma 2.7. ∎

5. Proof of the Airault-Malliavin Theorem 2.18

This subsection is devoted to the proof of Theorem 2.18. We will need some, mostly well known, preliminary results regarding integration by parts on $(W(K), \mu_t)$. These results will be gathered in the next subsection.

5.1. **Integration by parts and strong differentiability.**
The key result here for the remainder of the paper is Corollary 5.6. The reader may skip this subsection if she/he is willing to accept Corollary 5.6 below.

Definition 5.1. Let $L^{\infty-}(W(K), \mu_t) = \cap_{1 \le p < \infty} L^p(W(K), \mu_t)$ and $h \in H$. A function $f \in L^{\infty-}(W(K), \mu_t)$ is said to be *strongly h differentiable* provided there is a function $g \in L^{\infty-}(W(K), \mu_t)$ such that

$$g = L^p(\mu_t)\text{-}\lim_{\epsilon\to 0} \frac{f(\sigma e^{\epsilon h}) - f(\sigma)}{\epsilon}$$

for all $1 \le p < \infty$. We will denote the function g, if it exists, by $\partial_h f$.

Cylinder functions are strongly h – differentiable for all $h \in H$ and $\partial_h f$ is given by Eq. (3.4). Another example is given in Lemma 5.5 below.

Definition 5.2. An element $k \in W(K)$ is a finite energy path if

$$k'(s) \text{ exists } ds\text{-a.s. and } \int_0^1 \left|k^{-1}(s)k'(s)\right|_{\mathfrak{k}}^2 ds < \infty.$$

Letting $k \in W(K)$ be a finite energy path and b_s being as in Eq. (2.17), then for μ_t – a.e. $\sigma \in W(K)$,

$$\begin{aligned} b_s(\sigma k) &= \int_0^s (\sigma(r)k(r))^{-1}\,\delta(\sigma k)(r) \\ &= \int_0^s k^{-1}(r)\sigma^{-1}(r)[\delta\sigma(r)k(r) + \sigma(r)k'(r)\,dr] \\ &= \int_0^s Ad_{k^{-1}(r)} db_r(\sigma) + \int_0^s k^{-1}(r)k'(r)\,dr. \end{aligned} \tag{5.1}$$

Since $Ad_{k^{-1}(r)}$ is orthogonal on $\mathfrak{k}$, Lévy's characterization of Brownian motion shows that $B_s := \int_0^s Ad_{k^{-1}(r)} db_r$ on $(W(K), \mu_t)$ is still a Brownian motion with variance t. This observation and the Cameron-Martin theorem is essentially the proof of the following quasi invariance theorem of Albeverio and Hoegh-Krohn, see [3], [29], and [28].

Theorem 5.3 (Albeverio & Hoegh-Krohn)**.** *Let k be a finite-energy path on K and $f : W(K) \to \mathbb{R}$ be a bounded measurable function. Then*

$$\int_{W(K)} f(\sigma)d\mu_t(\sigma) = \int_{W(K)} f(\sigma k)J_k(\sigma)d\mu_t(\sigma), \tag{5.2}$$

where

$$J_k := \exp\left(-\frac{1}{t}\int_0^1 \langle k'(s)k^{-1}(s), db_s\rangle - \frac{1}{2t}\int_0^1 \left|k'(s)k^{-1}(s)\right|^2 ds\right). \tag{5.3}$$

Proof. Let $h(s) := \int_0^s k^{-1}(r)k'(r)\,dr$, $B_s := \int_0^s Ad_{k^{-1}(r)} db_r$, and $\tilde f$ be a measurable function on $C([0,1] \to \mathfrak{k})$ such that $\tilde f(b_\cdot(\sigma)) = f(\sigma)$ for μ_t – a.e. σ. Using the Ad_K – invariance of the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}$, we have

$$\int_0^1 \left|k'(r)k^{-1}(r)\right|^2 dr = \int_0^1 \left|k^{-1}(r)k'(r)\right|^2 dr$$

and

$$\begin{aligned} \int_0^1 \langle k'(r)k^{-1}(r), db_r\rangle &= \int_0^1 \langle k'(r)k^{-1}(r), Ad_{k(r)}dB_r\rangle \\ &= \int_0^1 \langle Ad_{k^{-1}(r)}k'(r)k^{-1}(r), dB_r\rangle \\ &= \int_0^1 \langle k^{-1}(r)k'(r), dB_r\rangle. \end{aligned}$$

Combining these equations show that J_k may be written as

$$J_k := \exp\left(-\frac{1}{t}\int_0^1 \langle h', dB_r\rangle - \frac{1}{2t}\int_0^1 |h'(r)|^2\, dr\right). \tag{5.4}$$

By Eq. (5.1),

$$\begin{aligned}\int_{W(K)} f(\sigma k)J_k(\sigma)d\mu_t(\sigma) &= \int_{W(K)} \tilde{f}(b_\cdot(\sigma k))J_k(\sigma)d\mu_t(\sigma)\\ &= \int_{W(K)} \tilde{f}(B_\cdot(\sigma)+h)J_k(\sigma)d\mu_t(\sigma)\\ &= \int_{W(K)} \tilde{f}(B_\cdot(\sigma))d\mu_t(\sigma) \qquad (5.5)\end{aligned}$$

wherein the last equality we have used the Cameron-Martin (or Girsanov's) theorem. Since B and b have the same laws, being $\mathfrak{k}$ – valued Brownian motions with variance t,

$$\int_{W(K)} \tilde{f}(B_\cdot(\sigma))d\mu_t(\sigma) = \int_{W(K)} \tilde{f}(b_\cdot(\sigma))d\mu_t(\sigma) = \int_{W(K)} f(\sigma)d\mu_t(\sigma). \tag{5.6}$$

Combining Eqs. (5.4), (5.5) and (5.6) proves the theorem. ∎

Corollary 5.4. *Let $h \in H(\mathfrak{k})$ and suppose that f and g are strongly h – differentiable, then*

$$\mu_t(g\partial_h f) = \mu_t((-\partial_h g + j_h g)f) \tag{5.7}$$

where

$$j_h := \frac{1}{t}\int_0^1 \langle h'(s), db_s\rangle. \tag{5.8}$$

This corollary has been proved in the more general context of Wiener measure on a Riemannian manifold in Driver [11].

Proof. Let $k = e^{\epsilon h}$ and replace f by fg in Eq. (5.2) of Theorem 5.3 to find

$$\mu_t(fg) = \int_{W(K)} f(\sigma e^{\epsilon h})g(\sigma e^{\epsilon h})J_{e^{\epsilon h}}(\sigma)d\mu_t(\sigma).$$

Differentiate this equation in ϵ implies

$$0 = \mu_t\left(\partial_h f\cdot g + f\partial_h g + fg\frac{d}{d\epsilon}|_0 J_{e^{\epsilon h}}\right)$$

which proves the corollary provided that

$$\frac{d}{d\epsilon}|_0 J_{e^{\epsilon h}} = -j_h \text{ in } L^p(\mu_t) \text{ for all } p \in [1,\infty).$$

We will not carry out the convergence details here which are fairly routine. The interested reader may refer to Gross [21] or Section 9 in [11]. However, let us check "algebraically" that the formula in Eq. (5.8) is correct. Computing $\frac{d}{d\epsilon}|_0 J_{e^{\epsilon h}}$ gives

$$\begin{aligned}\frac{d}{d\epsilon}|_0 J_{e^{\epsilon h}} &= \frac{d}{d\epsilon}|_0 \exp\left(-\frac{1}{t}\int_0^1 \langle (\frac{d}{dr}e^{\epsilon h(r)})e^{-\epsilon h(r)}, db_r\rangle - \frac{1}{2t}\int_0^1 \left|(\frac{d}{dr}e^{\epsilon h(r)})e^{\epsilon h(r)}\right|^2 dr\right) \\ &= -\frac{1}{t}\frac{d}{d\epsilon}|_0\left[\int_0^1 \langle (\frac{d}{dr}e^{\epsilon h(r)})e^{-\epsilon h(r)}, db_r\rangle - \frac{1}{2t}\int_0^1 \left|(\frac{d}{dr}e^{\epsilon h(r)})e^{\epsilon h(r)}\right|^2 dr\right] \\ &= -\frac{1}{t}\int_0^1 \langle h'(r), db_r\rangle.\end{aligned}$$

because

$$\frac{d}{d\epsilon}|_0 (\frac{d}{dr}e^{\epsilon h(r)})e^{-\epsilon h(r)} = h'(r)$$

and

$$\begin{aligned}\frac{d}{d\epsilon}|_0 \left|(\frac{d}{dr}e^{\epsilon h(r)})e^{\epsilon h(r)}\right|^2 &= 2\left\langle (\frac{d}{dr}e^{\epsilon h(r)})e^{\epsilon h(r)}|_{\epsilon=0}, \frac{d}{d\epsilon}|_0(\frac{d}{dr}e^{\epsilon h(r)})e^{\epsilon h(r)}\right\rangle \\ &= 2\left\langle 0, \frac{d}{d\epsilon}|_0(\frac{d}{dr}e^{\epsilon h(r)})e^{\epsilon h(r)}\right\rangle = 0.\end{aligned}$$

∎

Lemma 5.5. *For each $h \in H$, the function j_h is strongly h differentiable and*

$$\partial_h j_h = \frac{1}{t}\int_0^1 \left\langle ad_{h(r)}h'(r), db_r(\sigma)\right\rangle + \frac{1}{t}\int_0^1 |h'(r)|^2 dr. \tag{5.9}$$

Proof. According to Eq. (5.1),

$$\begin{aligned}j_h(\sigma e^{\epsilon h}) &= \frac{1}{t}\int_0^1 \left\langle h'(r), Ad_{e^{-\epsilon h(r)}} db_r(\sigma) + e^{-\epsilon h(r)}\frac{d}{dr}e^{\epsilon h(r)}\, dr\right\rangle \\ &= \frac{1}{t}\int_0^1 \langle Ad_{e^{\epsilon h(r)}} h'(r), db_r(\sigma)\rangle + \frac{1}{t}\int_0^1 \left\langle h'(r), e^{-\epsilon h(r)}\frac{d}{dr}e^{\epsilon h(r)}\, dr\right\rangle.\end{aligned}$$

Therefore, again ignoring convergence questions,

$$\partial_h j_h(\sigma) = \frac{d}{d\epsilon}|_0 j_h(\sigma e^{\epsilon h}) = \frac{1}{t}\int_0^1 \left\langle ad_{h(r)}h'(r), db_r(\sigma)\right\rangle + \frac{1}{t}\int_0^1 |h'(r)|^2 dr.$$

Here the convergence questions are even easier since we only have jointly Gaussian random variables to contend with and L^2 – convergence of Gaussian random variables implies L^p convergence for $p < \infty$. The reader may find more details in Section 4 in Gross [20]. ∎

The following Corollary is a key ingredient in our proof of the Airault – Malliavin Theorem 2.18.

Corollary 5.6. *Let f be a smooth cylinder function (see Definition 2.4) and $h \in H(\mathfrak{k})$ such that the Lie bracket $[h(s), h'(s)] = 0$ for a.e. s. Then*

$$\mu_t(\partial_h^2 f) = \mu_t\left(\left[j_h^2 - \frac{1}{t}\int_0^1 |h'(r)|^2 dr\right] f\right),$$

where j_h, is as in Eq. (5.8).

Proof. Two applications of Corollary 5.4 gives

$$\mu_t(\partial_h^2 f) = \mu_t(j_h \partial_h f) = \mu_t\left(\left(-\partial_h j_h + j_h^2\right) f\right)$$

which combined with Eq. (5.9) of Lemma 5.5 proves the Corollary. ∎

5.2. **Proof of Theorem 2.18.**

Proof. Let $f = F \circ \pi_{\mathbb{P}}$ be a cylinder function on $\mathcal{L}(K)$ (see Definition 2.4) and let $\alpha \in (s_n, 1)$. (We will eventually let $\alpha \to 1$.) Recall the definition of pinned Wiener measure μ_t^0 (see Definition 2.11) says that $\mu_t^0(f) = \mu_t(f\eta_t)$ where $\eta_t := p_{t(1-\alpha)}^K(\pi_\alpha)/p_t^K(e)$. Therefore, by Corollary 4.1,

$$\begin{aligned}
\partial_t \mu_t^0(f) &= \partial_t \mu_t(f\eta_t) \\
&= \mu_t(f\partial_t\eta_t) + \frac{1}{2}\mu_t\left(\triangle_{W(K)}(f\eta_t)\right) \\
&= I_\alpha + J_\alpha.
\end{aligned} \tag{5.10}$$

Now

$$\begin{aligned}
I_\alpha &= \mu_t(f\partial_t\eta_t) \\
&= \frac{1}{2p_t^K(e)}(1-\alpha)\mu_t\left(f\Delta_K p_{t(1-\alpha)}^K(\pi_\alpha)\right) - \partial_t \ln p_t^K(e)\mu_t^0(f).
\end{aligned} \tag{5.11}$$

By Eq. (2.8),

$$\begin{aligned}
\mu_t(f\Delta_K p_{t(1-\alpha)}^K(\pi_\alpha)) &= \int_K G(\alpha, x)\Delta_K p_{t(1-\alpha)}^K(x)dx \\
&= \int_K \Delta_K G(\alpha, x) p_{t(1-\alpha)}^K(x)dx
\end{aligned}$$

where

$$G(\alpha, x) := \int_{K^n} F(x_1, \cdots, x_n)\, p_{t(\alpha - s_n)}^K(x_n^{-1}x) \prod_{i=1}^n p_{t\triangle_i s}^K\left(x_{i-1}^{-1}x_i\right) dx_i.$$

From this expression we see that $\Delta_K G(\alpha, x)$ remains bounded as $\alpha \to 1$, so that letting $\alpha \to 1$ in Eq. (5.11) gives

$$\lim_{\alpha \to 1} I_\alpha = -\mu_t^0(f\partial_t \log p_t^K(e)). \tag{5.12}$$

We proceed to work on the second term, J_α, in Eq. (5.10). Let $\mathbb{P}_\alpha$ be the partition of $[0,1]$,

$$\mathbb{P}_\alpha = \{0 = s_0 < s_1 < s_2 < \cdots < s_n < \alpha < 1\},$$

and set $s_{n+1} = \alpha$. Define $G_0^\alpha(s,t) = (s \wedge t - \alpha^{-1}st)$ so that

$$G(s,t) = s \wedge t = G_0^\alpha(s,t) + \alpha^{-1}st.$$

Let $\tilde{\eta}_t(x_1, x_2, \ldots, x_{n+1}) = p_{t(1-\alpha)}^K(x_{n+1})/p_t^K(e)$ and by abuse of notation use F again to denote the function $(x_1, x_2, \ldots, x_{n+1}) \in K^{n+1} \to F(x_1, x_2, \ldots, x_n)$. Then by Eqs. (3.10) and (3.8) applied to the partition $\mathbb{P}_\alpha$,

$$\begin{aligned}\Delta_{W(K)}(f\eta_t) &= L_{\mathbb{P}_\alpha}(F\tilde{\eta}_t)\circ\pi_{\mathbb{P}_\alpha}\\ &= \sum_{i,j=1}^{n+1}\sum_{A\in\mathfrak{k}_0} G_0^\alpha(s_i,s_j)\,A^{(i)}A^{(j)}(F\tilde{\eta}_t)\circ\pi_{\mathbb{P}_\alpha}\\ &\quad+\sum_{i,j=1}^{n+1}\sum_{A\in\mathfrak{k}_0}\alpha^{-1}s_is_jA^{(i)}A^{(j)}(F\tilde{\eta}_t)\circ\pi_{\mathbb{P}_\alpha}\\ &= S_\alpha+T_\alpha. \end{aligned}\tag{5.13}$$

Now for $A\in\mathfrak{k}$, let

$$h_A^\alpha(s):=\alpha^{-1/2}(s\wedge\alpha)A.$$

Then by Eq. (3.5),

$$T_\alpha=\sum_{A\in\mathfrak{k}_0}\partial^2_{h_A^\alpha}(f\eta_t).\tag{5.14}$$

For the S_α term in Eq. (5.13), notice that by construction $G_0^\alpha(s,t)=0$ if s or t is in $\{0,\alpha\}$. Therefore $G_0^\alpha(s_i,s_j)=0$ if i or $j=n+1$ (i.e. s_i or s_j is α) so that

$$\begin{aligned}S_\alpha &= \sum_{i,j=1}^{n}\sum_{A\in\mathfrak{k}_0} G_0^\alpha(s_i,s_j)\,A^{(i)}A^{(j)}(F\tilde{\eta}_t)\circ\pi_{\mathbb{P}_\alpha}\\ &= \tilde{\eta}_t\circ\pi_{\mathbb{P}_\alpha}\cdot\sum_{i,j=1}^{n}\sum_{A\in\mathfrak{k}_0} G_0^\alpha(s_i,s_j)\,(A^{(i)}A^{(j)}F)\circ\pi_{\mathbb{P}}.\end{aligned}\tag{5.15}$$

Taking the μ_t expectation of Eq. (5.13) and making use of Eq. (5.14) and (5.15) shows that J_α from Eq. (5.10) satisfies

$$\begin{aligned}J_\alpha &= \frac{1}{2}\mu_t^0\left(\sum_{i,j=1}^{n}\sum_{A\in\mathfrak{k}_0} G_0^\alpha(s_i,s_j)\,(A^{(i)}A^{(j)}F)\circ\pi_{\mathbb{P}}\right)\\ &\quad+\frac{1}{2}\mu_t\left(\sum_{A\in\mathfrak{k}_0}\partial^2_{h_A^\alpha}(f\eta_t)\right)\\ &= J_\alpha^{(1)}+J_\alpha^{(2)}.\end{aligned}$$

Since $G_0^\alpha\to G_0$ as $\alpha\to 1$,

$$\begin{aligned}\lim_{\alpha\to1}J_\alpha^{(1)} &= \frac{1}{2}\mu_t^0\left(\sum_{i,j=1}^{n}\sum_{A\in\mathfrak{k}_0} G_0(s_i,s_j)\,(A^{(i)}A^{(j)}F)\circ\pi_{\mathbb{P}}\right)\\ &= \frac{1}{2}\mu_t^0\left(\sum_{i,j=1}^{n+1}\sum_{A\in\mathfrak{k}_0} G_0(s_i,s_j)\,(A^{(i)}A^{(j)}F)\circ\pi_{\mathbb{P}}\right)\\ &= \frac{1}{2}\mu_t^0\left(\Delta_{\mathcal{L}(K)}f\right),\end{aligned}\tag{5.16}$$

where we have used Eq. (3.10) for the last equality.

By Corollary 5.6,

$$\begin{aligned} 2J_\alpha^{(2)} &= \sum_{A\in\mathfrak{k}_0} \mu_t\left(\left[j^2_{h_A^\alpha} - \frac{1}{t}\int_0^1 |\frac{d}{dr}h_A^\alpha(r)|^2 dr\right] f\eta_t\right) \\ &= \sum_{A\in\mathfrak{k}_0} \mu_t^0\left(\left[j^2_{h_A^\alpha} - \frac{1}{t}\int_0^1 |\frac{d}{dr}h_A^\alpha(r)|^2 dr\right] f\right), \end{aligned}$$

where (by Eq. (5.8))

$$j_{h_A^\alpha} = \frac{1}{\sqrt{\alpha t}}\int_0^\alpha \langle A, db_s\rangle = \frac{1}{\sqrt{\alpha t}}\int_0^\alpha \langle A, db_s\rangle = \frac{1}{\sqrt{\alpha t}}\langle A, b_\alpha\rangle.$$

Using these facts and

$$\int_0^1 |\frac{d}{dr}h_A^\alpha(r)|^2 dr = \frac{1}{\alpha}\int_0^\alpha |A|^2 dr = |A|^2.$$

we see that

$$J_\alpha^{(2)} = \frac{1}{2}\mu_t^0\left(\frac{1}{\alpha t^2}|b_\alpha|^2 - \frac{1}{t}\dim\mathfrak{k}\right)$$

and hence by Remark 2.17,

$$\lim_{\alpha\to 1} J_\alpha^{(2)} = \frac{1}{2}\mu_t^0\left(\frac{1}{t^2}|b_1|^2 - \frac{1}{t}\dim\mathfrak{k}\right). \tag{5.17}$$

Assembling Eqs. (5.16) and (5.17) shows that

$$\lim_{\alpha\to 1} J_\alpha = \frac{1}{2}\mu_t^0\left(\Delta_{\mathcal{L}(K)}f + \left[\frac{1}{t^2}|b_1|^2 - \frac{1}{t}\dim\mathfrak{k}\right] f\right). \tag{5.18}$$

Combining Eqs. (5.10), (5.12) and (5.18) proves the Theorem. ∎

Corollary 5.7. *Suppose that $u : \mathcal{L}(K) \to \mathbb{R}$ is a smooth cylinder function and let*

$$G(t,\sigma) = \int_{\mathcal{L}(K)} u(\sigma\gamma^{-1})d\mu_t^0(\gamma), \tag{5.19}$$

then

$$\partial_t G(t,\sigma) = \frac{1}{2}\triangle_{\mathcal{L}(K)}G(t,\sigma) + \int_{\mathcal{L}(K)} V_t(\gamma)u(\sigma\gamma^{-1})d\mu_t^0(\gamma). \tag{5.20}$$

Proof. As in the proof of Corollary 3.11, let $u_\sigma : \mathcal{L}(K) \to \mathbb{R}$ be the cylinder function defined by $u_\sigma(\gamma) = u(\sigma\gamma^{-1})$. By the Airault Malliavin Theorem 2.18,

$$\partial_t G(t,\sigma) = \int_{\mathcal{L}(K)} \left[\frac{1}{2}\left(\triangle_{\mathcal{L}(K)}u_\sigma\right)(\gamma) + V_t(\gamma)u_\sigma(\gamma)\right] d\mu_t^0(\gamma).$$

Using the same method of proof used for Corollary 3.11, we see that this equation is the same as Eq. (5.20). ∎

6. Absolute continuity of heat kernel with respect to pinned Wiener measure

In this section we will prove the main Theorem 2.16. We will first need a couple of preliminary results.

Lemma 6.1 (Asymptotic properties of heat Kernels on K). *The heat kernel, p_t^K, on K has the following properties:*

1. $\lim_{t\to 0}(2\pi t)^{\frac{1}{2}\dim\mathfrak{k}}p_t^K(e) = 1$.
2. *For every $T < \infty$, there is a constant $M_T < \infty$ such that*
$$p_t^K(x) \le M_T t^{-\dim\mathfrak{k}/2}e^{-\frac{1}{4t}d^2(e,x)} \text{ for all } x \in K \text{ and } 0 < t \le T$$
where $d(x,y)$ is the distance associated to the bi-invariant Riemannian metric on K which agrees with $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ at $e \in K$.

Proof. These are standard properties of heat kernels. For item 1., see Theorem 2.30 of [5]. See also [25]. For the second item see, for example, Theorem IX.1.2 in [32]. To apply this theorem, use the fact that K is compact so the modular function is constant. It is also necessary to note that the time parameter in [32] is twice our time parameter t. ∎

Lemma 6.2 ($\mu_t^0 \to \delta_e$ as $t \to 0$). *Let $f : \mathcal{L}(K) \to \mathbb{R}$ be a continuous cylinder function, then*

$$\lim_{t\to 0+}\mu_t^0(f) = f(\mathbf{e}), \tag{6.1}$$

where $\mathbf{e}$ denotes the identity loop in $\mathcal{L}(K)$, see Notation 1.2.

Proof. This result can be proved in a number of ways. For example one could use the Kolmogorov's continuity criteria to show that μ_t^0 concentrates near the identity loop as $t \to 0$. See the argument in the proof of Item 1 of Theorem 2.3 in [12]. Rather than carry this out in full detail, we will only prove what we need.

Let $\mathbb{P}$ be a partition of $[0,1]$ as in Eq. (2.3), $f = F\circ\pi_{\mathbb{P}}$ and $\rho^{\mathbb{P}} : (0,\infty)\times K^n \to (0,\infty)$ be as in Eq. (2.10). By Lemma 6.1, there is a constant $M < \infty$ such that

$$\rho^{\mathbb{P}}(t,x) \le Mt^{\frac{1}{2}\dim\mathfrak{k}}\prod_{i=1}^{n+1}(t\Delta_i s)^{-\dim\mathfrak{k}/2}\exp\left(-\frac{1}{4t\Delta_i s}d^2(e,x_{i-1}^{-1}x_i)\right) \text{ for all } t \in (0,1],$$

where $x = (x_1,\cdots,x_n)$, $\Delta_i s = s_i - s_{i-1}$, and $x_0 = x_{n+1} = e \in K$. By the left invariance of the Riemannian metric on K, $d(x,y) = d(e,x^{-1}y)$, so the previous inequality may be written as

$$\rho^{\mathbb{P}}(t,x) \le M_{\mathbb{P}}t^{-\frac{n}{2}\dim\mathfrak{k}}\exp\left(-\frac{1}{4t}\sum_{i=1}^{n+1}\frac{d^2(x_{i-1},x_i)}{\Delta_i s}\right) \tag{6.2}$$

where $M_{\mathbb{P}} := M\prod_{i=1}^{n+1}(\Delta_i s)^{-\dim\mathfrak{k}/2}$. Now let $\delta > 0$ be given, and suppose that $d(e,x_i) \ge \delta$ for some $i \in \{1,2,\ldots,n\}$, then by the triangle inequality and the

Cauchy-Schwarz inequality,

$$\begin{aligned}\delta^2 &\le d^2(e,x_i) \le \left(\sum_{j=1}^{i} d(x_{j-1},x_j)\right)^2 \\ &\le \left(\sum_{i=1}^{n+1} \frac{d(x_{j-1},x_j)}{\sqrt{\Delta_j s}}\sqrt{\Delta_j s}\right)^2 \le \sum_{i=1}^{n+1}\Delta_i s \cdot \sum_{i=1}^{n+1}\frac{d^2(x_{i-1},x_i)}{\Delta_i s} \\ &= \sum_{i=1}^{n+1}\frac{d^2(x_{i-1},x_i)}{\Delta_i s}\end{aligned}$$

Combining this estimate with Eq. (6.2) implies

$$\rho^{\mathbb{P}}(t,x) \le M_{\mathbb{P}} t^{-\frac{n}{2}\dim\mathfrak{k}} \exp\left(-\frac{1}{4t}|x|^2\right) \tag{6.3}$$

where

$$|x| := \max\{d(e,x_i) : i = 1,2,\ldots,d\}.$$

Therefore $\rho^{\mathbb{P}}(t,\cdot)$ satisfies:

1. $\rho^{\mathbb{P}}(t,x) > 0$.
2. $\int_{K^n} \rho^{\mathbb{P}}(t,x)dx = 1$ where dx is Haar measure on K^n.
3. For any $\delta > 0$, $\rho^{\mathbb{P}}(t,x) \to 0$ uniformly in $x \in K^n$ with $|x| \ge \delta$.

It is now routine to show, using these three properties, that

$$\lim_{t\to 0}\int_{K^n} F(x)\rho^{\mathbb{P}}(t,x)dx = F(e,e,\ldots,e)$$

which is equivalent to Eq. (6.1). ∎

6.1. **Proof of Theorem 2.16.**

Proof. Let u be a smooth non-negative cylinder function on $\mathcal{L}(K)$ and let C_t be as in Eq. (2.16). Notice that $\frac{d}{dt}C_t = c_t$ (c_t is defined in Eq. (2.19) of the Airault – Malliavin Theorem 2.18) and because of Lemma 6.1, $\lim_{t\to 0} C_t = 0$. Define

$$\begin{aligned}H(t,\sigma) &= \int_{\mathcal{L}(K)} u(\sigma\gamma^{-1})d\nu_t^0(\gamma), \text{ and} \\ F(t,\sigma) &= e^{C_t}\int_{\mathcal{L}(K)} u\left(\sigma\gamma^{-1}\right)d\mu_t^0(\gamma),\end{aligned}$$

then by Corollary 3.11

$$\partial_t H(t,\sigma) = \frac{1}{2}\triangle_{\mathcal{L}(K)}H(t,\sigma) \text{ and } \lim_{t\to 0} H(t,\sigma) = u(\sigma) \tag{6.4}$$

and by Corollary 5.7

$$\begin{aligned}\partial_t F(t,\sigma) &= \frac{1}{2}\triangle_{\mathcal{L}(K)}F(t,\sigma) + e^{C_t}\int_{\mathcal{L}(K)} (V_t(\gamma)+c_t)\,u(\sigma\gamma^{-1})d\mu_t^0(\gamma) \\ &= \frac{1}{2}\triangle_{\mathcal{L}(K)}F(t,\sigma) + e^{C_t}\int_{\mathcal{L}(K)} \frac{1}{2t^2}|b_1(\gamma)|_{\mathfrak{k}}^2\,u(\sigma\gamma^{-1})d\mu_t^0(\gamma) \\ &\ge \frac{1}{2}\triangle_{\mathcal{L}(K)}F(t,\sigma).\end{aligned}$$

Combining this with Lemma 6.2, shows that

$$\partial_t F(t,\sigma) \geq \frac{1}{2}\triangle_{\mathcal{L}(K)} F(t,\sigma) \text{ and } \lim_{t\to 0} F(t,\sigma) = u(\sigma). \tag{6.5}$$

The idea now is to use Eqs. (6.4), (6.5) and the maximum principle to conclude that

$$F(t,\sigma) \geq H(t,\sigma) \text{ for all } 0 \leq t < \infty \text{ and } \sigma \in \mathcal{L}(K). \tag{6.6}$$

We will postpone the full justification of Eq. (6.6) to Lemma 6.3 below.

Writing out Eq. (6.6) when σ is the constant loop $\mathbf{e}$, shows that

$$\int_{\mathcal{L}(K)} u(\gamma^{-1}) d\nu_t^0(\gamma) \leq e^{C_t} \int_{\mathcal{L}(K)} u\left(\gamma^{-1}\right) d\mu_t^0\left(\gamma\right)$$

for all non-negative smooth cylinder functions u. Replacing u by the cylinder function $\tilde{u}(\gamma) = u(\gamma^{-1})$ then implies that

$$\int_{\mathcal{L}(K)} u(\gamma) d\nu_t^0(\gamma) \leq e^{C_t} \int_{\mathcal{L}(K)} u\left(\gamma\right) d\mu_t^0\left(\gamma\right) \tag{6.7}$$

for all non-negative smooth cylinder functions u.

Since, by Lemma 2.7, bounded smooth cylinder functions are dense in

$$L^2(\mathcal{L}(K), \mathcal{F}, \mu_t^0 + \nu_t^0),$$

by passing to the limit, we may conclude that Eq. (6.7) is valid for all bounded non-negative $\mathcal{F}$ – measurable functions u. By taking u to be characteristic functions and using the Radon-Nikodym theorem, Eq. (6.7) implies that ν_t^0 is absolutely continuous relative to μ_t^0. Letting $Z_t := d\nu_t^0/d\mu_t^0$ we may conclude from Eq. (6.7) that

$$\int_{\mathcal{L}(K)} u \cdot \left(Z_t - e^{C_t}\right) d\mu_t^0 \leq 0$$

for all bounded measurable functions u and hence that $Z_t - e^{C_t} \leq 0$. ∎

Lemma 6.3. *Keeping the same notation as above, Eq. (6.6) is valid.*

Proof. In order to justify the use of the maximum principle to prove Eq. (6.6), write $u = U \circ \pi_{\mathbb{P}}$, where $\mathbb{P}$ is a partition as in Eq. (2.3) and $U : K^n \to [0,\infty)$ is a smooth function. Then

$$\begin{aligned} H(t,\sigma) &= \int_{\mathcal{L}(K)} u(\sigma\gamma^{-1}) d\nu_t^0(\gamma) = \int_{\mathcal{L}(K)} U(\pi_{\mathbb{P}}(\sigma)\pi_{\mathbb{P}}(\gamma)^{-1}) d\nu_t^0(\gamma) \\ &= H_{\mathbb{P}}(t, \pi_{\mathbb{P}}(\sigma)), \end{aligned} \tag{6.8}$$

where for $x \in K^n$

$$\begin{aligned} H_{\mathbb{P}}(t,x) &= \int_{\mathcal{L}(K)} U(x\pi_{\mathbb{P}}(\gamma)^{-1}) d\nu_t^0(\gamma) \\ &= \int_{K^n} U(xy^{-1}) p_t^{\mathbb{P}}(y) dy, \end{aligned} \tag{6.9}$$

and $p_t^{\mathbb{P}}(y)dy = Law(\pi_{\mathbb{P}}(\Sigma(t,\cdot))$. By the proof of Proposition 3.9, $\pi_{\mathbb{P}}(\Sigma(t,\cdot)$ is a diffusion on K^n with elliptic generator $L_{\mathbb{P}}^0$ defined in Eq. (3.9). Thus $p_t^{\mathbb{P}}(y)$ is the smooth heat kernel for the operator $e^{tL_{\mathbb{P}}^0/2}$. This shows that $H_{\mathbb{P}}(t,x)$ is smooth on $(0,\infty)\times K^n$. Using this information, Eq. (6.4) may be recast as the finite dimensional statement

$$\partial_t H_{\mathbb{P}}(t,x)=\frac{1}{2}L^0_{\mathbb{P}}H_{\mathbb{P}}(t,x) \text{ and } \lim_{t\to 0}H_{\mathbb{P}}(t,x)=U(x). \tag{6.10}$$

Similarly

$$\begin{aligned} F(t,\sigma) &= e^{C_t}\int_{\mathcal{L}(K)} u\left(\sigma\gamma^{-1}\right)d\mu^0_t(\gamma)=e^{C_t}\int_{\mathcal{L}(K)} U(\pi_{\mathbb{P}}(\sigma)\pi_{\mathbb{P}}(\gamma)^{-1})d\mu^0_t(\gamma)\\ &= F_{\mathbb{P}}(t,\pi_{\mathbb{P}}(\sigma)),\end{aligned}$$

where for $x\in K^n$

$$\begin{aligned} F_{\mathbb{P}}(t,x) &= e^{C_t}\int_{\mathcal{L}(K)} U(x\pi_{\mathbb{P}}(\gamma)^{-1})d\mu^0_t(\gamma)\\ &= e^{C_t}\int_{K^n} U(xy^{-1})\rho^{\mathbb{P}}(t,y)dy\end{aligned}$$

where $\rho^{\mathbb{P}}:(0,\infty)\times K^n\to(0,\infty)$ is the smooth function defined in Eq. (2.10) of Definition 2.11. This shows that $F_{\mathbb{P}}(t,x)$ is smooth on $(0,\infty)\times K^n$. Using this information, Eq. (6.4) may be recast as the finite dimensional statement

$$\partial_t F_{\mathbb{P}}(t,x)\geq\frac{1}{2}L^0_{\mathbb{P}}F_{\mathbb{P}}(t,x) \text{ and } \lim_{t\to 0}F_{\mathbb{P}}(t,x)=U(x). \tag{6.11}$$

Now there is no problem in applying the maximum principle on K^n, using Eqs. (6.10) and (6.11), to conclude that

$$F_{\mathbb{P}}(t,x)\geq H_{\mathbb{P}}(t,x) \text{ for all } 0\leq t<\infty \text{ and } x\in K^n.$$

This finishes the proof since this last assertion is equivalent (6.6). ∎

7. The $K=\mathbb{R}^d$ and S^1 cases

In this section, we will work out the explicit relationship between μ^0_t and ν^0_t in the case that K is the abelian Lie group $\mathbb{R}^d$ or S^1.

7.1. The $K=\mathbb{R}^d$ case.

Let K be the Lie group $\mathbb{R}^d$ with group operation being addition. The Lie algebra of $\mathbb{R}^d$ is $\mathfrak{k}=\mathbb{R}^d$ with the trivial lie bracket, $[a,b]=0$ for all $a,b\in\mathbb{R}^d$. Although $\mathbb{R}^d$ is not compact and is not being represented as a matrix group, the theory above easily extends to this case. There is one notational point to take care of now. Namely, the matrix expression of the form $g^{-1}\delta g$ must now be interpreted as $L_{g^{-1}*}\delta g=\delta g$. We will assume that $\langle a,b\rangle=a\cdot b$ is the usual dot product, although any inner product would work.

Lemma 7.1. *On the loop space of $\mathbb{R}^d$, $\mathcal{L}(\mathbb{R}^d)$, the heat kernel measures ν^0_t and the pinned Wiener measures, μ^0_t, are the same.*

Proof. The process $\Sigma^0(t,s)$ in Theorem 2.13 and the process g in Eq. (2.5) of Definition 2.8 are explicitly given by $\Sigma^0(t,s)=\chi(t,s)$ and $g_s=\beta(t,s)$ respectively. Since $g_s=\beta(t,s)$ is a standard Brownian motion with variance t, the pinned Wiener measure $\mu^0_t=Law(g_\cdot|g_1=0)$ is the law of an $\mathbb{R}^d$ – valued Brownian bridge with variance t. But $s\to\chi(t,s)$ is a Brownian bridge with variance t (see Remark 2.12), so that $\mu^0_t=Law(\chi(t,\cdot))=Law(\Sigma^0(t,\cdot))=\nu^0_t$. ∎

7.2. **The $K = S^1$ case.** Let $K = S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The Lie algebra of K is $\mathfrak{k} = i\mathbb{R}$ with the trivial Lie bracket. We will identify with $\mathfrak{k} = i\mathbb{R}$ with $\mathbb{R}$, putting in the i explicitly when needed. Let $\langle a, b\rangle = ab$ for $a, b \in \mathbb{R} \cong i\mathbb{R} = \mathfrak{k}$.

Remark 7.2. Let $p_t(x) = p_t^{\mathbb{R}}(x) = (2\pi t)^{-1/2} \exp\left(-\frac{1}{2t}x^2\right)$ be the heat kernel on $\mathbb{R}$, and $q_t(z) = \frac{1}{2\pi} p_t^{S^1}(z)$ denote the heat kernel on S^1 relative to the un-normalized Haar measure "$d\theta$," i.e. for $f : S^1 \to \mathbb{R}$,

$$\int_{S^1} f d\theta := \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

The well know relationship between q_t and p_t is

$$q_t(e^{i\theta}) = \sum_{n=-\infty}^{\infty} p_t(\theta - 2\pi n) \text{ for } \theta \in \mathbb{R}. \tag{7.1}$$

To check this, suppose that $f_0 : S^1 \to \mathbb{R}$ is a continuous function. Then

$$f(t, z) = \int_0^{2\pi} f_0(ze^{-i\alpha}) q_t(e^{i\alpha}) d\alpha$$

solves the heat equation on S^1 which is equivalent to saying that $F(t, \theta) := f(t, e^{i\theta})$ solves the heat equation on $\mathbb{R}$. Since F is a bounded solution to the heat equation it is given by

$$\begin{aligned} \int_0^{2\pi} f_0(ze^{-i\alpha}) q_t(e^{i\alpha}) d\alpha &= F(t, \theta) = \int_{-\infty}^{\infty} F(0, \theta - \alpha) p_t(\alpha) d\alpha \\ &= \int_{-\infty}^{\infty} f_0(ze^{-i\alpha}) p_t(\alpha) d\alpha \\ &= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f_0(ze^{-i(\alpha - 2\pi n)}) p_t(\alpha - 2\pi n) d\alpha \\ &= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f_0(ze^{-i\alpha}) p_t(\alpha - 2\pi n) d\alpha \end{aligned}$$

where $z = e^{i\theta}$. This equation, holding for all continuous $f_0 : S^1 \to \mathbb{R}$, proves Eq. (7.1).

Definition 7.3. For $n \in \mathbb{Z}$, let $h_n(s) := 2\pi n s$,

$$z_n(s) = e^{i2\pi n s} = e^{i h_n(s)}$$

and let ν_t^n be the left translation of ν_t^0 by z_n, i.e. ν_t^n is the probability measure on $\mathcal{L}(S^1)$ such that

$$\int_{\mathcal{L}(S^1)} f(\sigma) d\nu_t^n(\sigma) = \int_{\mathcal{L}(S^1)} f(z_n \sigma) d\nu_t^0(\sigma).$$

Also let $\mathcal{L}_n(S^1)$ denote those $\sigma \in \mathcal{L}(S^1)$ which are homotopic to z_n.

Remark 7.4. The loops $\{z_n\}_{n=-\infty}^{\infty}$ are representatives from each of the homotopy classes of $\mathcal{L}(S^1)$, i.e. $\mathcal{L}(S^1)$ is the disjoint union of $\left\{\mathcal{L}_n(S^1)\right\}_{n=-\infty}^{\infty}$. By the construction of ν_t^0 in Theorem 2.13, the measure ν_t^0 is concentrated on $\mathcal{L}_0(S_1)$ and therefore ν_t^n is concentrated on $\mathcal{L}_n(S^1)$, i.e. $\nu_t^n(\mathcal{L}_m(S_1)) = \delta_{mn}$.

Proposition 7.5. *The relationship between pinned Wiener measure μ_t^0 and heat kernel measure ν_t^0 on $\mathcal{L}(S^1)$ is*

$$\mu_t^0 = \frac{1}{q_t(1)} \sum_{n=-\infty}^{\infty} p_t(2\pi n)\nu_t^n.$$

In particular

$$\mu_t^0|_{\mathcal{L}_0(S^1)} = \frac{1}{q_t(1)} p_t(0)\nu_t^0 = \left(\sum_{n=-\infty}^{\infty} e^{-\frac{1}{2t}(2\pi n)^2}\right)^{-1} \nu_t^0. \tag{7.2}$$

Proof. To simplify notation, let $B_s = \beta(t,s)$. Using Itô's formula, one easily shows that the process $\Sigma^0(t,s)$ in Theorem 2.13 and the process g in Eq. (2.5) of Definition 2.8 are given by $\Sigma^0(t,s) = e^{i\chi(t,s)}$ and $g_s = e^{i\beta(t,s)} = e^{iB_s}$ respectively. Suppose that $f : \mathcal{L}(S^1) \to \mathbb{R}$ is a cylinder function as in Definition 2.4. then for $\alpha \in (s_n, 1)$,

$$\begin{aligned}
\mu_t^0(f) &= \mathbb{E}\left[f(g)\frac{q_{t(1-\alpha)}(g_\alpha)}{q_t(1)}\right] \\
&= \mathbb{E}\left[f(e^{iB_\cdot})\frac{q_{t(1-\alpha)}(e^{iB_\alpha})}{q_t(1)}\right] = \frac{1}{q_t(1)}\mathbb{E}\left[f(e^{iB_\cdot})\sum_{n=-\infty}^{\infty} p_{t(1-\alpha)}(B_\alpha - 2\pi n)\right] \\
&= \frac{1}{q_t(1)} \sum_{n=-\infty}^{\infty} \mathbb{E}\left[f(e^{iB_\cdot})p_{t(1-\alpha)}(B_\alpha - 2\pi n)\right]. \qquad (7.3)
\end{aligned}$$

Let $h_n(s) := 2\pi n s, h_n^\alpha(s) = 2\pi n\,(s \wedge \alpha)$, and $F(B) = f(e^{iB_\cdot})$, so that F is a bounded cylinder function $W(\mathbb{R})$. By the Cameron-Martin theorem (making the translation $B_\cdot \to B_\cdot + h_n^\alpha$),

$$\begin{aligned}
&\mathbb{E}[F(B)p_{t(1-\alpha)}(B_\alpha - 2\pi n)] \\
&= \mathbb{E}\left[F(B + h_n^\alpha)p_{t(1-\alpha)}(B_\alpha - 2\pi n(1-\alpha))\exp\left(-\frac{1}{t}\int_0^\alpha 2\pi n\, dB_s - \frac{1}{2t}\int_0^\alpha (2\pi n)^2 ds\right)\right] \\
&= \mathbb{E}\left[F(B + h_n)p_{t(1-\alpha)}(B_\alpha - 2\pi n(1-\alpha))\exp\left(-\frac{2\pi n}{t}B_\alpha - \frac{1}{2t}\alpha(2\pi n)^2\right)\right]. \qquad (7.4)
\end{aligned}$$

By direct computation,

$$p_{t(1-\alpha)}(x - y(1-\alpha)) = p_{t(1-\alpha)}(x) \cdot \exp\left(\frac{1}{t}xy - \frac{1}{2t}(1-\alpha)y^2\right)$$

and thus taking $x = B_\alpha$ and $y = 2\pi n$,

$$\begin{aligned}
p_{t(1-\alpha)}(B_\alpha - 2\pi n(1-\alpha)) \cdot \exp\left(-\frac{2\pi n}{t}B_\alpha - \frac{1}{2t}\alpha(2\pi n)^2\right) \\
= p_{t(1-\alpha)}(B_\alpha) \cdot \exp\left(-\frac{1}{2t}(2\pi n)^2\right). \qquad (7.5)
\end{aligned}$$

Combining Eqs. (7.4) and (7.5) shows that

$$\begin{aligned}
&\mathbb{E}[F(B)p_{t(1-\alpha)}(B_\alpha - 2\pi n)] \\
&\quad = (2\pi t)^{-1/2}\exp\left(-\frac{1}{2t}(2\pi n)^2\right)\mathbb{E}\left[F(B + h_n)\frac{p_{t(1-\alpha)}(B_\alpha)}{p_t(0)}\right] \\
&\quad = p_t(2\pi n)\mathbb{E}\left[F(\chi(t,\cdot) + h_n)\right]
\end{aligned}$$

wherein the second equality we have used Eq. (2.11) of Remark 2.12. Using this equation, with $F(B) = f(e^{iB})$, in Eq. (7.3) gives

$$\begin{aligned}\mu_t^0(f) &= \frac{1}{q_t(1)} \sum_{n=-\infty}^{\infty} p_t(2\pi n)\mathbb{E}\left[f(e^{i(\chi(t,\cdot)+h_n)})\right] \\ &= \frac{1}{q_t(1)} \sum_{n=-\infty}^{\infty} p_t(2\pi n)\mathbb{E}\left[f(z_n e^{i\chi(t,\cdot)})\right] \\ &= \frac{1}{q_t(1)} \sum_{n=-\infty}^{\infty} p_t(2\pi n)\nu_t^n(f).\end{aligned}$$

∎

8. Appendix (Quadratic variations)

Lemma 8.1. *As above, for $A \in \mathfrak{k}$ let $\beta^A(t,s) = \langle \beta(t,s), A\rangle_{\mathfrak{k}}$ and $\chi^A(t,s) = \langle \chi(t,s), A\rangle_{\mathfrak{k}}$. Let $A, B \in \mathfrak{k}$ and $s, \sigma \in [0,1]$, then*

$$\begin{aligned}\beta^A(dt,s)\beta^B(dt,\sigma) &= \langle A,B\rangle_{\mathfrak{k}}\, G(s,\sigma)\, dt, \\ \chi^A(dt,s)\chi^B(dt,\sigma) &= \langle A,B\rangle_{\mathfrak{k}}\, G_0(s,\sigma)\, dt\end{aligned}$$

and for $t, \tau \in [0,\infty)$,

$$\beta^A(t,ds)\beta^B(\tau,ds) = \langle A,B\rangle_{\mathfrak{k}}\, G(t,\tau)ds.$$

Proof. Let $\{\mathfrak{G}_t\}$ be an abstract filtration (satisfying the "usual hypothesis") and suppose that M_t and N_t are two continuous $\{\mathfrak{G}_t\}$ adapted processes such that $(M_t - M_s, N_t - N_s)$ is independent of $\mathfrak{G}_s$ for all $t > s$ and $\mathbb{E}M_t = \mathbb{E}N_t = 0$ for all $t \geq 0$. Then clearly M and N are $\{\mathfrak{G}_t\}$ – martingales. We now also assert that

$$M_t N_t - \mathbb{E}\left[M_t N_t\right] \text{ is a martingale} \tag{8.1}$$

Assuming Eq. (8.1) for the moment, we may conclude the differential $M_{dt}N_{dt}$ of the quadratic co-variation of M and N is given by

$$M_{dt}N_{dt} = d_t\mathbb{E}[M_t N_t]. \tag{8.2}$$

The lemma then follows from repeated application of Eq. (8.2). For example, taking $M_t = \beta^A(t,s)$ and $N_t = \beta^B(t,\sigma)$, we learn that

$$\beta^A(dt,s)\beta^B(dt,\sigma) = d_t\mathbb{E}\left[\beta^A(t,s)\beta^B(t,\sigma)\right] = \langle A,B\rangle_{\mathfrak{k}}\, G(s,\sigma)\, dt.$$

To prove Eq. (8.1), let $t > s$, $\Delta M = M_t - M_s$, $\Delta N = N_t - N_s$ and $\mathbb{E}_s = \mathbb{E}(\cdot|\mathfrak{G}_s)$. Then using the martingale properties of M and N and the independent increment assumption we find

$$\begin{aligned}\mathbb{E}_s\left[M_t N_t - M_s N_s\right] &= \mathbb{E}_s\left[(M_s + \Delta M)(N_s + \Delta N) - M_s N_s\right] = \mathbb{E}_s\left[\Delta M \Delta N\right] \\ &= \mathbb{E}[(M_t - M_s)(N_t - N_s)] = \mathbb{E}[(M_t - M_s)(N_t + N_s)] \\ &= \mathbb{E}[M_t N_t] - \mathbb{E}[M_s N_s].\end{aligned}$$

Rearranging the terms of the result of this computation shows that

$$\mathbb{E}_s\left[M_t N_t - \mathbb{E}[M_t N_t]\right] = M_s N_s - \mathbb{E}[M_s N_s]$$

as desired. ∎

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