UNIVERSITY OF CALIFORNIA, SAN DIEGO

Logarithmic Sobolev Inequalities for the Free Loop Group

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by

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ABSTRACT OF THE DISSERTATION

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For a compact Lie group, G, let $\mathcal{L}(G)$ be the free loop group consisting of the space of continuous paths $g : [0,1] \to G$ with $g(0) = g(1)$. For a given Ad_G-invariant inner product on the Lie algebra of G and a suitable related inner product on the "Lie algebra" of $\mathcal{L}(G)$ we derive the corresponding Riemannian structure on $\mathcal{L}(G)$. For this structure we have constructed finite dimensional approximations which are used in proving a logarithmic Sobolev inequality and integration by parts formulas on $\mathcal{L}(G)$. The underlying probability measure on $\mathcal{L}(G)$ is a heat kernel probability measure which can be derived by an $\mathcal{L}(G)$-valued Brownian motion.

Chapter 1

Introduction

1.1 Overview

The purpose of this paper is two-fold. As the title indicates, one major purpose is to prove a certain logarithmic Sobolev inequality which is of the form

$$\int_{\mathcal{L}} f^2 \log f^2 d\nu_t \leq \frac{2}{C}(e^{Ct} - 1)\mathcal{E}(f) + \int_{\mathcal{L}} f^2 d\nu_t \log(\int_{\mathcal{L}} f^2 d\nu_t), \;\; t > 0, \tag{1.1}$$

where $\mathcal{L}$ is a particular infinite dimensional Lie group called the free loop group, ν_t is a probability measure on $\mathcal{L}$ depending on t and a fixed element of $\mathcal{L}$, C is a constant, and $\mathcal{E}$ is a particular quadratic form. The other purpose of the work that follows is to develop the structure of the free loop group itself.

We start by developing a Riemannian geometric structure on $\mathcal{L}$, which includes a Levi-Civita covariant derivative and associated curvature and Ricci tensors. We then devote a considerable amount of effort to showing how elements of the geometry of $\mathcal{L}$, in particular, the Laplacian and Ricci tensor, can be approximated by corresponding elements of finite dimensional subspaces. These approximations will be important tools in the proof of the logarithmic Sobolev inequality.

Next, we construct a particular probability measure, ν_t, which is the "heat kernel" measure on $\mathcal{L}$ used for integration in (1.1). The construction of ν_t requires methods from stochastic analysis on manifolds. In particular, we show how to derive a Brownian motion stochastic process, Σ, on $\mathcal{L}$ by using the notion of parallel translation and the "rolling map" from a flat vector space to the manifold, $\mathcal{L}$. The measure ν_t is simply defined to be the distribution of Σ at time t.

We then show that the quadratic form, $\mathcal{E}$, appearing in (1.1) is closable. By extending $\mathcal{E}$ to its closure, we are able to prove that the inequality holds on a class of functions larger than that of the original domain of $\mathcal{E}$. The key to proving $\mathcal{E}$ is closable is the use of an integration by parts formula on $\mathcal{L}$, which in turn relies on the aforementioned finite dimensional geometrical approximations.

Finally, in proving the inequality itself, we choose a class of "cylinder" functions which depend only on finite dimensional subspaces of $\mathcal{L}$ (these comprise the domain of $\mathcal{E}$). For such a function, f, the inequality can be realized on a finite dimensional manifold for which the logarithmic Sobolev inequality is already know to hold when the corresponding Ricci tensor is bounded below (this bound is the constant, C , appearing in (1.1)). In this sense, it is appropriate to say that we are really extending a know inequality to an infinite dimensional space.

The idea of such an extension to an infinite dimensional space was first pioneered by B. K. Driver and T. Lohrenz [5] in their work on the pinned loop group. I owe much gratitude to Dr. Driver for his guidance and suggestions during the work of this paper, and for his altruistic sharing of time, patience, and knowledge.

1.2 Definitions and Results

In this section we give a more detailed summary of the mathematical structures and mappings that serve as a context for this paper, as well as the key results that will arise.

For the entirety of the paper, we assume that G is a compact, connected Lie group. In particular, G is a C^∞ manifold endowed with a group structure such that the group operations of multiplication and inverses are smooth. Furthermore, we assume that G has a Riemannian structure consisting of an inner product, $\langle \cdot, \cdot \rangle$, on the Lie algebra, $\mathfrak{g}$, of G (by definition, $\mathfrak{g} \equiv T_e G$ is the tangent space of G at the identity e). This inner product is assumed to be Ad_G-invariant, which implies that the adjoint operator, ad_A, is skew-symmetric for each $A \in \mathfrak{g}$. Note that $Ad : G \to Aut(\mathfrak{g})$ is the *adjoint representation* on G where, for each $\sigma \in G$, Ad_σ is the restriction to $\mathfrak{g}$ of the differential of the inner automorphism $\tau \mapsto \sigma\tau\sigma^{-1}$. The *adjoint operator* $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the restriction to $\mathfrak{g}$ of the differential of Ad. It turns out that for each $A, B \in \mathfrak{g}$, $ad_A B = [A, B]$ where $[A, B] \equiv AB - BA$ is the Lie bracket on $\mathfrak{g}$.

As an example, one may take $G \equiv SO(n)$, the set of $n \times n$ real orthogonal matrices with determinant 1. In this case, the Lie algebra $\mathfrak{g} = T_e G$ turns out to be $\mathfrak{g} = so(n)$, the set of $n \times n$ real matrices which are skew symmetric. A natural choice for an inner product would be $\langle A, B \rangle \equiv tr(A^T B)$ for all $A, B \in so(n)$.

The *free loop group* on G is defined to be the space of continuous loops:

$$\mathcal{L}(G) \equiv \{g : [0,1] \to G : \, g \text{ is continuous and } g(0) = g(1)\}.$$

Note that $\mathcal{L}(G)$ is itself an infinite dimensional topological group under the pointwise multiplication operation:

$$(g_1 g_2)(s) \equiv g_1(s) g_2(s) \;\; \forall g_1, g_2 \in \mathcal{L}(G), \, s \in [0,1].$$

We will take the topology on $\mathcal{L}(G)$ to be that of uniform convergence.

Embedded in $\mathcal{L}(G)$ is the space of finite energy free loops, $H(G)$. We will define the Lie algebra of $H(G)$ to be the Cameron-Martin Hilbert space, $H(\mathfrak{g})$, consisting of all absolutely continuous functions $h : [0,1] \to \mathfrak{g}$ such that $h(0) = h(1)$ and $(h, h) < \infty$, where $(\cdot, \cdot)$ is the inner product on H defined by

$$(h, k) \equiv \int_0^1 \{\langle h'(s), k'(s) \rangle + \langle h(s), k(s) \rangle\} ds$$

for $h, k \in H$.

The first step we take in the development of the geometry of $\mathcal{L}(G)$ is to find the Levi-Civita covariant derivative on $H(\mathfrak{g})$. By definition, this is the unique covariant derivative $\nabla : H(\mathfrak{g}) \otimes H(\mathfrak{g}) \to H(\mathfrak{g})$ that is metric compatible and torsion free, i.e.,

$$\begin{aligned} (\nabla_h k, l) + (k, \nabla_h l) &= 0 \\ \text{and } \nabla_h k - \nabla_k h &= [h, k] \end{aligned}$$

for all $h, k, l \equiv H(\mathfrak{g})$. The explicit formula for ∇ turns out be

$$\nabla_h l(s) = -\int_0^1 G_s(s,t)[h, l'](t) dt + \frac{1}{2} \int_0^1 G(s,t)[h, l](t) dt,$$

where $G(s,t)$ is the Green's function associated with the differential equation

$$-u'' + u = f$$

with certain periodic boundary conditions (see the proof of Lemma 2.1.7). It is important to note that this Green's function is what characterizes the majority of the development of the geometry of the free loop group and epitomizes the distinction between the free loop group and the pinned loop group. Its properties play a crucial role in many of our calculations.

Based on ∇, we derive explicit formulas for the curvature tensor $R : H(\mathfrak{g}) \otimes H(\mathfrak{g}) \otimes H(\mathfrak{g}) \to H(\mathfrak{g})$, defined by

$$R\langle h,k\rangle l \equiv \nabla_h \nabla_k l - \nabla_k \nabla_h l - \nabla_{[h,k]} l,$$

and the Ricci tensor $Ric : H(\mathfrak{g}) \otimes H(\mathfrak{g}) \to \mathbf{R}$, defined by

$$Ric\langle h,l\rangle \equiv \sum_{k\in S} (R\langle h,k\rangle k, l),$$

where S is an orthonormal basis of $H(\mathfrak{g})$. The curvature tensor is given in Theorem 2.2.1 and the Ricci tensor is given in (2.21). An important property of the Ricci tensor we derive is that it is bounded below (see Theorem 2.3.11) in the sense that there exists a constant $C > 0$ such that

$$Ric\langle h,h\rangle \geq -C\langle h,h\rangle \;\; \forall\, h \in H(\mathfrak{g}).$$

We next show the existence of a heat kernel probability measure on $\mathcal{L}(G)$. By definition, this is a family of measures $\{\nu_t(g_0,\cdot)\}_{t>0, g_0\in\mathcal{L}(G)}$ such that

$$u(t,g_0) \equiv \int_{\mathcal{L}(G)} f(g)\nu_t(g_0,dg)$$

solves the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \quad \lim_{t\downarrow 0} u(t,g_0) = f(g_0)$$

for all $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$, where $\mathcal{F}C^\infty(\mathcal{L}(G))$ are the *cylinder functions* on $\mathcal{L}(G)$ of the form $f = F \circ \pi_{\mathcal{P}}$ such that $\mathcal{P}$ is a finite partition of $[0,1]$ and $\pi_{\mathcal{P}} : \mathcal{L}(G) \to G^{\mathcal{P}}$ is the projection restricting to partition points.

To obtain the heat kernel measure, we first construct an $\mathcal{L}(G)$ -valued Brownian motion, $\{\Sigma_t\}_{t\geq 0}$. In Euclidean space, a Brownian motion is usually defined to be a continuous random process with independent, mean-zero Gaussian increments. In a more abstract space, such as $\mathcal{L}(G)$, we use the martingale characterization of Brownian motion found in Theorem 4.0.2 which requires $\{\Sigma_t\}_{t\geq 0}$ to be a diffusion process with generator $\frac{1}{2}\Delta$. We start, however, with a different Brownian motion, $\{\beta_t\}_{t\geq 0}$, which is valued in a vector space (an abstract Wiener space) containing our Cameron-Martin Hilbert space $H(\mathfrak{g})$, and is well-known to exist. $\{\Sigma_t\}_{t\geq 0}$ is then the image of $\{\beta_t\}_{t\geq 0}$ under the rolling map characterized by the property that the "velocity vector" of Σ at time t is the image of the "velocity vector" of β at time t under parallel translation along Σ. Since Brownian motions are almost nowhere differentiable however, we do not have the luxury of velocity vectors. We rely instead on the use of stochastic differential equations as given in Theorem 4.0.1. By taking $\nu_t(g_0, \cdot)$ to be the law of $g_0\Sigma$ at time t, we obtain our desired heat kernel measure (see Theorem 4.0.3).

For each $t > 0$ and $g_0 \in \mathcal{L}(G)$, the logarithmic Sobolev inequality in (1.1) can then be verified for all cylinder functions $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$ using the measure $\nu_t(g_0, \cdot)$. The quadratic form $\mathcal{E}$ depends on t and g_0 and is defined by

$$\mathcal{E}(f) = \mathcal{E}_{g_0,t}(f) \equiv \int_{\mathcal{L}(G)} \left\| \vec{\nabla} f(g) \right\|^2_{H(\mathfrak{g})} \nu_t(g_0, dg). \tag{1.2}$$

The key to proving the inequality is to note that for a given $f = F \circ \pi_{\mathcal{P}}$ and partition $\mathcal{P}$ of $[0,1]$, the measure $\nu_t(g_0, \cdot)$ is restricted to a corresponding measure on the finite dimensional compact manifold $G^{\mathcal{P}}$ on which it becomes absolutely continuous with respect to Riemannian volume measure and has an associated heat kernel density function. In this finite dimensional setting of $G^{\mathcal{P}}$, the inequality is known to hold provided the corresponding Ricci tensor, $Ric_{\mathcal{P}}$, is bounded below by some constant $-C_{\mathcal{P}}$. By proving the finite dimensional approximations to the Ricci tensor on $H(\mathfrak{g})$ found in Theorem 3.2.1, we are able to show that the finite dimensional Ricci tensors, $Ric_{\mathcal{P}}$, can be bounded below by a constant, $-C$, which is independent of the partitions $\mathcal{P}$.

Finally, we prove that the quadratic forms $\mathcal{E}_{g_0,t}$ are closable, which means that each $\mathcal{E}_{g_0,t}$ can be extended to a closed form, $\overline{\mathcal{E}}_{g_0,t}$, on a Hilbert space containing $\mathcal{F}C^\infty(\mathcal{L}(G))$. Thus, the inequality can be extended to this larger class of functions making it a truly infinite dimensional theorem. As we have mentioned before, the closability of $\mathcal{E}_{g_0,t}$ depends on an integration by parts formula (Theorem 5.20). Because of the dependence of $\mathcal{E}_{g_0,t}$ on the gradient $\vec{\nabla}$ in (1.2), $\mathcal{E}_{g_0,t}$ is closable if and only if $\vec{\nabla}$ is a closable operator. This holds precisely when $\vec{\nabla}$ has a densely defined adjoint, which we demonstrate to be the case in Theorem 5.6.1.

Chapter 2

The Geometry of the Free Loop Algebra

2.1 The Covariant Derivative

In this section we wish to derive the "Levi-Civita covariant derivative" on $H(\mathfrak{g})$. Since $H(\mathfrak{g})$ is equivalent to the left-invariant vector fields on $\mathcal{L}(G)$ (as illustrated by the next definition), we start by deriving the Levi-Civita covariant derivative in the context of the left-invariant vector fields.

Definition 2.1.1 *For $h \in T_e\mathcal{L}$, define $\tilde{h}$ to be the left-invariant vector field on $\mathcal{L}$ corresponding to h:*

$$\tilde{h}(g) \equiv L_{g*}h \;\; \forall g \in \mathcal{L},$$
$$i.e., \; (\tilde{h}(g))(s) = L_{g(s)*}h(s) \; \forall s \in [0,1].$$

Definition 2.1.2 *Define an inner product, also called $(\cdot,\cdot)$, on left-invariant vector fields by*

$$\begin{aligned}(\tilde{h}(g), \tilde{k}(g)) &= (L_{g*}h, L_{g*}k)\\ &\equiv (h,k) \; \forall g \in \mathcal{L}\end{aligned}$$

where the last inner product is the one on H. Note that $(\cdot,\cdot)$ is left- invariant.

Now let ∇ denote the Levi-Civita covariant derivative, with respect to $(\cdot,\cdot)$, which is uniquely determined on left-invariant vector fields. To find an explicit formula for ∇, we thus need to compute $\nabla_{\tilde{h}}\tilde{k}$ for all $h, k \in H$.

Using the requirements that ∇ be metric compatible and torsion free, by (a) on p.55 of Do Carmo [7], we must have, for all $h, k, l \in H$,

$$\begin{aligned}
(\tilde{k}, \nabla_{\tilde{h}}\tilde{l}) &= \frac{1}{2}\{\tilde{l}(\tilde{h},\tilde{k}) + \tilde{h}(\tilde{k},\tilde{l}) - \tilde{k}(\tilde{l},\tilde{h}) - (\tilde{k},[\tilde{l},\tilde{h}]) - (\tilde{l},[\tilde{h},\tilde{k}]) - (\tilde{h},[\tilde{l},\tilde{k}])\} \\
&= -\frac{1}{2}\{(\tilde{k},[\tilde{l},\tilde{h}]) + (\tilde{l},[\tilde{h},\tilde{k}]) + (\tilde{h},[\tilde{l},\tilde{k}])\} \\
&= -\frac{1}{2}\{(\tilde{k},[l,h]\tilde{\ }) + (\tilde{l},(ad_h k)\tilde{\ }) + (\tilde{h},(ad_l k)\tilde{\ })\} \\
&= -\frac{1}{2}\{(k,[l,h]) + (l, ad_h k) + (h, ad_l k)\} \\
&= -\frac{1}{2}\{(k,[l,h]) + (ad_h^* l, k) + (ad_l^* h, k)\} \\
&= -\frac{1}{2}\{(\tilde{k},[l,h]\tilde{\ }) + ((ad_h^* l)\tilde{\ },\tilde{k}) + ((ad_l^* h)\tilde{\ },\tilde{k}),
\end{aligned}$$

where ad_h^* is the adjoint of $ad_h : H(\mathfrak{g}) \to H(\mathfrak{g})^*$ with respect to $(\cdot,\cdot)$. In the above calculations, we are using the well-know fact from Lie group theory that if $h(s)$ and $k(s)$ are in $\mathfrak{g}$, then $[\widetilde{h(s)},\widetilde{k(s)}] = [h(s),k(s)]\tilde{\ }$. Note that $\tilde{l}(\tilde{h},\tilde{k}) = 0$ since $(\tilde{h},\tilde{k})$ is constant over $\mathcal{L}$. Since $\tilde{k}$ runs over all left-invariant vector fields, then

$$\nabla_{\tilde{h}}\tilde{l} \equiv -\frac{1}{2}\{[l,h] + ad_h^* l + ad_l^* h\}\tilde{\ }. \tag{2.1}$$

The existence of ad_h^* for each $h \in H(\mathfrak{g})$ follows by Riesz's representation theorem on Hilbert spaces. This theorem depends on ad_h being a bounded operator which is the case as follows. For all $h, k \in H(\mathfrak{g})$,

$$\begin{aligned}
\|ad_h k\|_{H(\mathfrak{g})}^2 &= \int_0^1 (\left\|[h,k]'(s)\right\|_{\mathfrak{g}}^2 + \|[h,k](s)\|_{\mathfrak{g}}^2)ds \\
&= \int_0^1 (\left\|[h',k](s) + [h,k'](s)\right\|_{\mathfrak{g}}^2 + \|[h,k](s)\|_{\mathfrak{g}}^2)ds \\
&\le \int_0^1 c(2\left\|h'(s)\right\|_{\mathfrak{g}}^2 \|k(s)\|_{\mathfrak{g}}^2 + 2\|h(s)\|_{\mathfrak{g}}^2 \left\|k'(s)\right\|_{\mathfrak{g}}^2 + \|h(s)\|_{\mathfrak{g}}^2 \|k(s)\|_{\mathfrak{g}}^2)ds \\
&\le c(2M_1 + 3M_2)\|k\|_{H(\mathfrak{g})}^2,
\end{aligned}$$

where c is a constant depending on $[\cdot,\cdot]$, $M_1 \equiv \sup_{s\in[0,1]} \|h'(s)\|_{\mathfrak{g}}^2$, $M_2 \equiv \sup_{s\in[0,1]} \|h(s)\|_{\mathfrak{g}}^2$.

Let us now determine ad_h^* for $h \in H(\mathfrak{g})$. For convenience in calculations, we will assume for the time being that $[h,l']$ and $[l,h']$ are in $H(\mathfrak{g})$ so that the derivatives

$[h, l']'$ and $[l, h']'$ exist almost everywhere and $[h, l'](0) = [h, l'](1)$ and $[l, h'](0) = [l, h'](1)$. However, we will show that our final expression for $\nabla_h l$ will not require $[h, l']$ and $[l, h']$ to be differentiable almost everywhere or periodic.

Proposition 2.1.3 *Let $h, l \in H(\mathfrak{g})$ such that $[h, l'] \in H(\mathfrak{g})$. Then*

$$(ad_h^* l)(s) = \int_0^1 a \cosh(r(s,t))([l, h] + [l', h'] + [h, l']')(t)dt$$

where $a = \frac{1}{2\sinh(\frac{1}{2})}$ and

$$r(s,t) = \begin{cases} s - t - \frac{1}{2} & 0 \le t < s \\ s - t + \frac{1}{2} & s \le t \le 1. \end{cases}$$

Proof. First note that by Riesz's Theorem, since $(l, [h, \cdot])$ is a bounded linear operator (being continuous), $ad_h^* l$ exists and is unique for all $l \in H(\mathfrak{g})$.

Let $k \in H(\mathfrak{g})$. Then ad_h^* satisfies

$$\begin{aligned} (ad_h^* l, k) &= (l, [h, k]) \\ &= \int_0^1 \{\langle l', [h, k]' \rangle + \langle l, [h, k] \rangle\} ds \\ &= \int_0^1 \{\langle l', [h', k] \rangle + \langle l', [h, k'] \rangle + \langle l, [h, k] \rangle\} ds \\ &= \int_0^1 \{-\langle [h', l'], k \rangle - \langle [h, l'], k' \rangle - \langle [h, l], k \rangle\} ds \end{aligned} \tag{2.2}$$

Since $(ad_h^* l, k) = \int_0^1 \{\langle (ad_h^* l)', k' \rangle + \langle ad_h^* l, k \rangle\} ds$, we would like to write (2.2) in the form of

$$(u, k) = \int_0^1 \{\langle u', k' \rangle + \langle u, k \rangle\} ds$$

for some $u \in H(\mathfrak{g})$. By the uniqueness of $ad_h^* l$, then we would have $ad_h^* l = u$.

Let us first note that using integration by parts on the second term in (2.2) gives

$$\begin{aligned} \int_0^1 -\langle [h, l'], k' \rangle ds &= -\langle [h, l'], k \rangle |_0^1 - \int_0^1 -\langle [h, l']', k \rangle ds \\ &= \int_0^1 \langle [h, l']', k \rangle ds. \end{aligned}$$

So we may rewrite (2.2) as

$$\int_0^1 \langle [l, h] + [l', h'] + [h, l']', k \rangle ds. \tag{2.3}$$

At this point we need to prove some facts which we will state as lemmas since they will be useful to us for future calculations. For what follows, let

$$\begin{aligned} G(s,t) &\equiv a\cosh(r(s,t)) \qquad (2.4)\\ \text{and } F(s,t) &\equiv G_s(s,t) = -G_t(s,t) \\ &= a\sinh(r(s,t))\ \forall\,(s,t)\in[0,1]^2, \end{aligned}$$

where a and $r(s,t)$ are defined as in Proposition 2.1.3.

Lemma 2.1.4 *For $f \in L^1([0,1],\mathfrak{g})$ or $f \in L^1([0,1],\mathbf{R})$, $u(s) \equiv \int_0^1 G(s,t)f(t)dt$ is absolutely continuous and has the following derivative:*

$$u'(s) = \int_0^1 G(s,t)f(t)dt$$

for almost all $s \in [0,1]$.

Proof: Note that

$$\begin{aligned} &\int_0^1 G(s,t)f(t)dt \qquad (2.5)\\ =\ &\int_0^s a\cosh(s-t-\frac{1}{2})f(t)dt + \int_s^1 a\cosh(s-t+\frac{1}{2})f(t)dt \\ =\ &a\cosh(s)\int_0^s \cosh(-t-\frac{1}{2})f(t)dt + a\sinh(s)\int_0^s \sinh(-t-\frac{1}{2})f(t)dt \\ &+a\cosh(s)\int_s^1 \cosh(-t+\frac{1}{2})f(t)dt + a\sinh(s)\int_s^1 \sinh(-t+\frac{1}{2})f(t)dt. \qquad (2.6) \end{aligned}$$

Since cosh is bounded on $[0,1]$ and $f \in L^1([0,1])$, then $\int_0^s \cosh(-t-\frac{1}{2})f(t)dt$ is an indefinite integral of an L^1 function and is thus absolutely continuous by the fundamental theorem of calculus for Lebesgue integrals (see Folland [8], P.102, or Royden [19], P.110). Since cosh is absolutely continuous, then $a\cosh(s)\int_0^s \cosh(-t-\frac{1}{2})f(t)dt$ is also absolutely continuous being the product of two absolutely continuous functions (see Royden [19], p. 111).

To take the derivative in (2.6) we use the product rule and the fundamental theorem of calculus as follows:

$$\begin{aligned} &u'(s) \\ =\ &a\sinh(s)\int_0^s \cosh(-t-\frac{1}{2})f(t)dt + a\cosh(s)\cosh(-s-\frac{1}{2})f(s) \\ &+a\cosh(s)\int_0^s \sinh(-t-\frac{1}{2})f(t)dt + a\sinh(s)\sinh(-s-\frac{1}{2})f(s) \end{aligned}$$

$$
\begin{aligned}
&+a\sinh(s)\int_s^1 \cosh(-t+\frac{1}{2})f(t)dt - a\cosh(s)\cosh(-s+\frac{1}{2})f(s)\\
&+a\cosh(s)\int_s^1 \sinh(-t+\frac{1}{2})f(t)dt - a\sinh(s)\sinh(-s+\frac{1}{2})f(s)\\
= &\int_0^s a\sinh(s-t-\frac{1}{2})f(t)dt + \int_s^1 a\sinh(s-t+\frac{1}{2})f(t)dt\\
&+a\cosh(-\frac{1}{2}) - a\cosh(\frac{1}{2})\\
= &\int_0^1 G(s,t)f(t)dt
\end{aligned}
$$

for almost all $s \in [0,1]$. □

Lemma 2.1.5 *For $f \in L^1([0,1],\mathfrak{g})$ or $f \in L^1([0,1],\mathbf{R})$,*

$$\frac{d}{ds}(\int_0^1 F(s,t)f(t)dt) = -f(s) + \int_0^1 G(s,t)f(t)dt$$

for almost all $s\,[0,1]$. (If $u(s) \equiv \int_0^1 G(s,t)f(t)dt$, then $u''(s) = -f(s) + \int_0^1 G(s,t)f(t)dt$ for almost all $s \in [0,1]$.)

Proof: We first note that

$$
\begin{aligned}
&\int_0^1 F(s,t)f(t)dt\\
= \; & a\sinh(s)\int_0^s \cosh(-t-\frac{1}{2})f(t)dt + a\cosh(s)\int_0^s \sinh(-t-\frac{1}{2})f(t)dt\\
&+a\sinh(s)\int_s^1 \cosh(-t+\frac{1}{2})f(t)dt + a\cosh(s)\int_s^1 \sinh(-t+\frac{1}{2})f(t)dt.
\end{aligned}
$$

By the same reasoning used in Lemma 2.1.4, it is clear that $\int_0^1 F(s,t)f(t)dt$ is absolutely continuous. Recalling that $a = \frac{1}{2\sinh(\frac{1}{2})}$, we then have

$$
\begin{aligned}
&\frac{d}{ds}(\int_0^1 F(s,t)f(t)dt)\\
= \; & a\cosh(s)\int_0^s \cosh(-t-\frac{1}{2})f(t)dt + a\sinh(s)\cosh(-s-\frac{1}{2})f(s)\\
&+a\sinh(s)\int_0^s \sinh(-t-\frac{1}{2})f(t)dt + a\cosh(s)\sinh(-s-\frac{1}{2})f(s)
\end{aligned}
$$

$$
\begin{aligned}
& +a\cosh(s)\int_s^1 \cosh(-t+\frac{1}{2})f(t)dt - a\sinh(s)\cosh(-s+\frac{1}{2})f(s) \\
& +a\sinh(s)\int_s^1 \sinh(-t+\frac{1}{2})f(t)dt - a\cosh(s)\sinh(-s+\frac{1}{2})f(s) \\
= & \int_0^s a\cosh(s-t-\frac{1}{2})f(t)dt + \int_0^s a\cosh(s-t+\frac{1}{2})f(t)dt \\
& +a\sinh(-\frac{1}{2})f(s) - a\sinh(\frac{1}{2})f(s) \\
= & -f(s) + \int_0^1 G(s,t)f(t)dt
\end{aligned}
$$

for almost all $s \in [0,1]$. □

Definition 2.1.6 *Because* $\frac{d}{ds}(\int_0^1 G_s(s,t)f(t))dt = \int_0^1 G(s,t)f(t)dt - f(s)$ *as we have seen in the lemma above, let us reserve the notation* $G_{ss}(s,t)$ *to mean*

$$G_{ss}(s,t) \equiv G(s,t) - \delta(t-s)$$

where δ *is the Dirac delta generalized function, i.e.,*

$$\int_{-\infty}^{+\infty} \delta(t-s)g(t)dt \equiv g(s)$$

in the distributional sense.

Thus,

$$
\begin{aligned}
\frac{d}{ds}(\int_0^1 G_s(s,t)f(t)dt) &= \int_0^1 G_{ss}(s,t)f(t)dt \\
&\equiv \int_0^1 G(s,t)f(t)dt + f(s).
\end{aligned}
$$

Lemma 2.1.7 *For* $k, f \in H(\mathfrak{g})$*, the unique solution,* $u \in H$ *, to the following equation,*

$$(u,k) = \int_0^1 \{\langle u', k'\rangle + \langle u, k\rangle\}ds = \int_0^1 \langle f, k\rangle ds \tag{2.7}$$

is

$$u(s) = \int_0^1 G(s,t)f(t)\ dt.$$

Proof: Using integration by parts for the first term, we rewrite (2.7) as

$$\int_0^1 \langle -u'' + u, k\rangle ds = \int_0^1 \langle f, k\rangle ds.$$

So we seek a solution to the second order linear differential equation

$$-u'' + u = f$$

$$u(0) = u(1) \qquad f(0) = f(1)$$
$$u'(0) = u'(1) \qquad f'(0) = f'(1).$$

Checking that $u = \int_0^1 G(s,t)f(t)\,dt$ is indeed our desired solution, we note that by Lemmas 2.1.4 and 2.1.5

$$\frac{d}{ds}(u(s)) = \int_0^1 F(s,t)f(t)dt$$

and

$$\begin{aligned}\frac{d^2}{ds^2}(u(s)) &= \frac{d}{ds}(\int_0^1 F(s,t)f(t)dt)\\ &= \int_0^1 G_{ss}(s,t)f(t)dt\\ &= -f(s) + \int_0^1 G(s,t)f(t)dt\end{aligned}$$

It then follows that

$$-\frac{d^2}{ds^2}(u(s)) + u(s) = \int_0^1 \{-G_{ss}(s,t) + G(s,t)\}f(t)dt = f(s).$$

Finally, note that the boundary conditions are indeed satisfied by u:

$$u(0) = \int_0^1 a\cosh(-t+\frac{1}{2})f(t)dt = u(1)$$
$$u'(0) = \int_0^1 a\sinh(-t+\frac{1}{2})f(t)dt = u'(1). \ \square$$

(*Proof of Proposition 2.1.3 continued*)

Using the above lemma, we can solve

$$\begin{aligned}(ad_h^* l, k) &= \int_0^1 \{\langle (ad_h^* l)', k'\rangle + \langle ad_h^* l, k\rangle\}ds\\ &= \int_0^1 \langle [l,h] + [l',h'] + [h,l']', k\rangle ds\end{aligned}$$

to get

$$ad_h^* l(s) = \int_0^1 G(s,t)([l,h] + [l',h'] + [h,l']')(t)dt. \ \square$$

Returning to equation (2.1), we can now determine $\nabla_{\tilde{h}}\tilde{l}$:

$$\begin{aligned}\nabla_{\tilde{h}}\tilde{l} &= -\frac{1}{2}\{[l,h] + ad_h^* l + ad_l^* h\}^{\sim} \\ &= -\frac{1}{2}\{[l,h] + \int_0^1 G(s,t)([h,l']' + [l,h']')(t)dt\}^{\sim}.\end{aligned}$$

Let us define a derivative, also called ∇, on H in the obvious way:

$$\nabla_h l \equiv \nabla_{\tilde{h}}\tilde{l}|_e$$

where e is the identity of $\mathcal{L}(G)$. So

$$\nabla_h l(s) = -\frac{1}{2}\{[l,h](s) + \int_0^1 G(s,t)([h,l']' + [l,h']')(t)dt\}. \tag{2.8}$$

We would like to express (2.8) in a form that has no second derivatives. Noting that $G(s,t)$ is continuous with respect to t for all $s \in [0,1]$, and that

$$G(s,0) = a\cosh(s - \frac{1}{2}) = G(s,1)$$

we may use integration by parts in (2.8):

$$\begin{aligned}\nabla_h l(s) &= -\frac{1}{2}\{[l,h](s) + \int_0^1 G(s,t)([h,l'] + [l,h'])'(t)dt\} \\ &= -\frac{1}{2}\{[l,h](s) + \int_0^1 G_s(s,t)([h,l'] + [l,h'])(t)dt\} \\ &= -\frac{1}{2}\{[l,h](s) + \int_0^1 F(s,t)([h,l'] + [l,h'])(t)dt\} \\ &= -\frac{1}{2}\{[l,h](s) + \int_0^1 F(s,t)[h,l'](t)dt + \int_0^1 F(s,t)[l,h'](t)dt\}.\end{aligned} \tag{2.9}$$

Recall that we are assuming $[h,l']$ and $[l,h']$ are both in $H(\mathfrak{g})$.

By the continuity of G, the above integration by parts is straightforward. Before determining a final expression for $\nabla_h l$, we need to verify a number of other integration-by-parts formulas involving the Green's function, G.

Proposition 2.1.8 *Using the notation $G_{ss}(s,t)$ from Definition 2.1.6, for $h,k \in H(\mathfrak{g})$,*

$$\int_0^1 \langle \int_0^1 G_s(s,t)h(t)dt, k'(s)\rangle ds = \int_0^1 \langle \int_0^1 G_{ss}(s,t)h(t)dt, k(s)\rangle ds.$$

Proof: Applying integration by parts and Lemma 2.1.5 gives

$$\begin{aligned}\int_0^1 \langle \int_0^1 G_s(s,t)h(t)dt, k'(s)\rangle ds &= \int_0^1 \langle \frac{d}{ds}(\int_0^1 G_s(s,t)h(t)dt), k(s)\rangle ds \\ &= \int_0^1 \langle \int_0^1 G_{ss}(s,t)h(t)dt, k(s)\rangle ds. \ \square\end{aligned}$$

Proposition 2.1.9 *For $f \in H(R)$ or $f \in H(\mathfrak{g})$,*

$$\int_0^1 G_s(s,t)f'(s)ds = \int_0^1 -G_{ss}(s,t)f(s)ds.$$

Proof: Using integration by parts and the definition of G_s,

$$\begin{aligned}&\int_0^1 G_s(s,t)f'(s)ds \\ =\ & \int_0^t a\sinh(s-t+\frac{1}{2})f'(s)ds + \int_t^1 a\sinh(s-t-\frac{1}{2})f'(s)ds \\ =\ & a\sinh(t-t+\frac{1}{2})f(t) - a\sinh(0-t+\frac{1}{2})f(0) + \int_0^t a\cosh(s-t+\frac{1}{2})f(s)ds \\ & +a\sinh(1-t-\frac{1}{2})f(1) - a\sinh(t-t+\frac{1}{2})f(t) + \int_t^1 a\cosh(s-t-\frac{1}{2})f(s)ds \\ =\ & 2a\sinh(\frac{1}{2})f(t) + a\sinh(-t+\frac{1}{2})(f(1)-f(0)) + \int_0^1 a\cosh(r(s,t))f(s)ds \\ =\ & f(t) + \int_0^1 G(s,t)f(s)ds = \int_0^1 \{\delta(t-s)+G(s,t)\}f(s)ds = \int_0^1 -G_{ss}(s,t)f(s)ds. \square\end{aligned}$$

Proposition 2.1.10 *For $f \in H(R)$ or $f \in H(\mathfrak{g})$,*

$$\int_0^1 G_t(s,t)f'(s)ds = \int_0^1 -G_{ss}(s,t)f(s)ds.$$

Proof:

$$\begin{aligned}&\int_0^1 G_t(s,t)f'(s)ds \\ =\ & -\int_0^s a\sinh(s-t-\frac{1}{2})f'(t)dt + \int_s^1 a\sinh(s-t+\frac{1}{2})f'(t)dt \\ =\ & -a\sinh(s-s-\frac{1}{2})f(s) + a\sinh(s-0-\frac{1}{2})f(0) - \int_0^s a\cosh(s-t-\frac{1}{2})f(t)dt \\ & -a\sinh(s-1+\frac{1}{2})f(1) + a\sinh(s-s+\frac{1}{2})f(s) - \int_s^1 a\cosh(s-t+\frac{1}{2})f(t)dt \\ =\ & 2a\sinh(\frac{1}{2})f(s) - \int_0^1 G(s,t)f(t)dt = \int_0^1 \{-G(s,t)+\delta(t-s)\}f(t)dt \\ =\ & \int_0^1 -G_{ss}(s,t)f(t)dt.\square\end{aligned}$$

Proposition 2.1.11

$$\int_0^1 G_s(s,t_1)G_s(s,t_2)ds = \int_0^1 -G_{ss}(s,t_1)G(s,t_2)ds.$$

Proof: Assume first that $t_1 \le t_2$. Then, recalling that

$$G_s(s,t) = \begin{cases} a\sinh(s-t-\frac{1}{2}) \text{ if } 0 \le t < s \\ a\sinh(s-t+\frac{1}{2}) \text{ if } s \le t \le 1 \end{cases}$$

where $a = \frac{1}{2\sinh(\frac{1}{2})}$, we have the following:

$$\begin{aligned}
& \int_0^1 G_s(s,t_1)G_s(s,t_2)ds \\
= & \int_0^{t_1} a\sinh(s-t_1+\frac{1}{2})a\sinh(s-t_2+\frac{1}{2})ds \\
& + \int_{t_1}^{t_2} a\sinh(s-t_1-\frac{1}{2})a\sinh(s-t_2+\frac{1}{2})ds \\
& \int_{t_2}^{1} a\sinh(s-t_1-\frac{1}{2})a\sinh(s-t_2-\frac{1}{2})ds \\
= & a\sinh(t_1-t_1+\frac{1}{2})a\cosh(t_1-t_2+\frac{1}{2}) - a\sinh(-t_1+\frac{1}{2})a\cosh(-t_2+\frac{1}{2}) \\
& - \int_0^{t_1} a\cosh(s-t_1+\frac{1}{2})a\cosh(s-t_2+\frac{1}{2})ds \\
& + a\sinh(t_2-t_1-\frac{1}{2})a\cosh(t_2-t_2+\frac{1}{2}) - a\sinh(t_1-t_1+\frac{1}{2})a\cosh(t_1-t_2+\frac{1}{2}) \\
& - \int_{t_1}^{t_1} a\cosh(s-t_1-\frac{1}{2})a\cosh(s-t_2+\frac{1}{2})ds \\
& + a\sinh(1-t_1-\frac{1}{2})a\cosh(1-t_2-\frac{1}{2}) - a\sinh(t_2-t_1-\frac{1}{2})a\cosh(t_2-t_2-\frac{1}{2}) \\
& - \int_{t_1}^{1} a\cosh(s-t_1-\frac{1}{2})a\cosh(s-t_2-\frac{1}{2})ds.
\end{aligned}$$

After cancelling like terms, we get

$$\begin{aligned}
\int_0^1 G_s(s,t_1)G_s(s,t_2)ds &= 2a\sinh(\frac{1}{2})a\cosh(t_1-t_2+\frac{1}{2}) - \int_0^1 G(s,t_1)G(s,t_2)ds \\
&= a\cosh(t_1-t_2+\frac{1}{2}) - \int_0^1 G(s,t_1)G(s,t_2)ds.
\end{aligned}$$

Similarly, if $t_2 < t_1$,

$$\begin{aligned}
\int_0^1 G_s(s,t_1)G_s(s,t_2)ds &= 2a\sinh(\frac{1}{2})a\cosh(t_2-t_1+\frac{1}{2}) - \int_0^1 G(s,t_2)G(s,t_1)ds \\
&= a\cosh(t_1-t_2-\frac{1}{2}) - \int_0^1 G(s,t_1)G(s,t_2)ds.
\end{aligned}$$

Thus, for all $(t_1, t_2) \in [0,1]^2$,

$$\begin{aligned}\int_0^1 G_s(s,t_1)G_s(s,t_2)ds &= G(t_1,t_2) - \int_0^1 G(s,t_1)G(s,t_2)ds \\ &= \int_0^1 -G_{ss}(s,t_1)G(s,t_2)ds.\Box\end{aligned}$$

Proposition 2.1.12

$$\int_0^1 G(s,t_1)G_s(s,t_2)ds = -\int_0^1 G_s(s,t_1)G(s,t_2)ds.$$

Proof: This follows from normal integration by parts.□

Returning to our expression for $\nabla_h l$ in (2.9), we may use Proposition 2.1.9 with the antisymmetry of F to get

$$\begin{aligned}\int_0^1 F(s,t)[l,h']dt &= \int_0^1 F(s,t)([l,h]' - [l',h])(t)dt \\ &= -[l,h](s) - \int_0^1 G(s,t)[h,l](t)dt + \int_0^1 F(s,t)[h,l'](t)dt \qquad (2.10)\end{aligned}$$

Thus, (2.9) becomes

$$\begin{aligned}\nabla_h l(s) &= -\frac{1}{2}\{[l,h](s) + \int_0^1 F(s,t)[h,l'](t)dt - [l,h](s) \\ &\quad - \int_0^1 G(s,t)[h,l](t)dt + \int_0^1 F(s,t)[h,l'](t)dt\} \\ &= -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt.\end{aligned}$$

Since this expression does not involve $[h,l']'$ or $[l,h']'$, we may take this to be our definition of $\nabla_h l$ for general h and l:

Definition 2.1.13 *For $h, l \in H(\mathfrak{g}\ )$, let*

$$\nabla_h l(s) = -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt.$$

Before proceeding, it is important to verify that $\nabla_h l$ is actually an element of the Cameron-Martin Hilbert space $H(\mathfrak{g})$.

Proposition 2.1.14 *If $h, l \in H(\mathfrak{g})$, then $\nabla_h l \in H(\mathfrak{g})$. Forthermore, for almost all $s \in [0,1]$,*

$$(\nabla_h l)'(s) = [h,l'](s) - \int_0^1 G(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 F(s,t)[h,l](t)dt. \qquad (2.11)$$

Proof: The fact that $\nabla_h l$ is absolutely continuous follows directly from definition 2.1.13 and Lemmas 2.1.4 and 2.1.5. We may also obtain (2.11) from Lemmas 2.1.4 and 2.1.5. It is easy to check that $(\nabla_h l, \nabla_h l)_{H(\mathfrak{g})} < \infty$ by use of Cauchy-Schwartz inequality and the fact that $G(s,t)$ and $F(s,t)$ are bounded for $(s,t) \in [0,1]^2$. □

Proposition 2.1.15 *∇, as defined in Definition 2.1.13, is the "Levi-Civita covariant derivative" on $H(\mathfrak{g})$.*

Proof: We need to check directly that ∇ is torsion free and metric compatible.. Using Equation (2.10), we have

$$\begin{aligned}
\nabla_h l - \nabla_l h &= -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt \\
&\quad + \int_0^1 F(s,t)[l,h'](t)dt - \frac{1}{2}\int_0^1 G(s,t)[l,h](t)dt \\
&= -\int_0^1 F(s,t)[h,l'](t)dt + \int_0^1 G(s,t)[h,l](t)dt \\
&\quad -[l,h](s) - \int_0^1 G(s,t)[h,l](t)dt + \int_0^1 F(s,t)[h,l'](t)dt \\
&= -[l,h](s) = [h,l](s)
\end{aligned}$$

proving ∇ is torsion free.

For metric compatibility we want to show that

$$(\nabla_h l, k) + (l, \nabla_h k) = 0 \ \ \forall h, l, k \in H.$$

Note the following:

$$\begin{aligned}
&(\nabla_h l, k) + (l, \nabla_h k) \\
= &\int_0^1 \{\langle (\nabla_h l)', k' \rangle + \langle l', (\nabla_h k)' \rangle + \langle \nabla_h l, k \rangle + \langle l, \nabla_h k \rangle\} ds \\
= &\int_0^1 \{\langle [h,l'](s) - \int_0^1 G(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 F(s,t)[h,l](t)dt, k' \rangle \\
&+\langle l'(s), [h,k'](s) - \int_0^1 G(s,t)[h,k'](t)dt + \frac{1}{2}\int_0^1 F(s,t)[h,k](t)dt \rangle \\
&+\langle -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt, k \rangle \\
&+\langle l(s), -\int_0^1 F(s,t)[h,k'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,k](t)dt \rangle\} ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \{\langle [h,l'],k'\rangle + \langle l',[h,k']\rangle\} ds \\
&\quad - \int_0^1 \langle -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}(-[h,l](s) + \int_0^1 G(s,t)[h,l](t)dt), k\rangle ds \\
&\quad - \int_0^1 \langle l(s), -\int_0^1 F(s,t)[h,k'](t)dt + \frac{1}{2}(-[h,k](s) + \int_0^1 G(s,t)[h,k](t)dt)\rangle ds \\
&\quad + \int_0^1 \langle -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt, k\rangle ds \\
&\quad + \int_0^1 \langle l(s), -\int_0^1 F(s,t)[h,k'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,k](t)dt\rangle ds \\
&= \int_0^1 \{\langle [h,l'],k'\rangle - \langle [h,l'],k'\rangle\} ds \\
&\quad - \int_0^1 \frac{1}{2}\{\langle -[h,l],k\rangle + \langle l,-[h,k]\rangle\} ds \\
&= -\int_0^1 \frac{1}{2}\{-\langle [h,l],k\rangle + \langle [h,l],k\rangle\} ds \\
&= 0. \ \square
\end{aligned}
$$

Note that since $G_s = -G_t$, we may write

$$\nabla_h l(s) = \int_0^1 G_t(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt.$$

By comparison, the covariant derivative, D, for the Lie algebra of the pinned loop group, as found in Driver [5], is

$$D_h l(s) = \int_0^s [h,l']dt - s\int_0^1 [h,l']dt = \int_0^1 \{\theta(s-t) - s\}[h,l'](t)dt.$$

The Green's function for that case is $G_0(s,t) \equiv s \wedge t - st$. So

$$
\begin{aligned}
G_{0t}(s,t) &= \begin{cases} 1-s & t \le s \\ -s & s < t. \end{cases} \\
&= (1-s)1_{t\le s} - s1_{t>s} \\
&= 1_{t\le s} - s \\
&= \theta(s-t) - s.
\end{aligned}
$$

Thus,

$$D_h l(s) = \int_0^1 G_{0t}(s,t)[h,l'](t)dt.$$

2.2 Curvature

Recall that the covariant derivative turned out to be

$$\nabla_h l(s) = -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt.$$

We wish to calculate the curvature tensor

$$R\langle h,k\rangle l = [\nabla_h, \nabla_k]l - \nabla_{[h,k]}l.$$

for $h,k,l \in H$. First note:

$$\begin{aligned}
&\nabla_h \nabla_k l(x) \\
= &\int_0^1 -F(x,s)[h,(\nabla_k l)'](s)ds + \frac{1}{2}\int_0^1 G(x,s)[h,\nabla_k l](s)ds \\
= &\int_0^1 -F(x,s)[h(s),[k,l'](s) - \int_0^1 G(s,t)[k,l'](t)dt + \frac{1}{2}\int_0^1 F(s,t)[k,l](t)dt]ds \\
&+\frac{1}{2}\int_0^1 G(x,s)[h(s), -\int_0^1 F(s,t)[k,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[k,l](t)dt]ds.
\end{aligned}$$

So

$$\begin{aligned}
&[\nabla_h, \nabla_k]l(x) \\
= &\int_0^1 -F(x,s)[h,[k,l']]ds - \int_0^1 -F(x,s)[k,[h,l']]ds \\
&+\int_0^1 -F(x,s)[h(s), -\int_0^1 G(s,t)[k,l'](t)dt + \frac{1}{2}\int_0^1 F(s,t)[k,l](t)dt]ds \\
&-\int_0^1 -F(x,s)[k(s), -\int_0^1 G(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 F(s,t)[h,l](t)dt]ds \\
&+\frac{1}{2}\int_0^1 G(x,s)[h(s), -\int_0^1 F(s,t)[k,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[k,l](t)dt]ds \\
&-\frac{1}{2}\int_0^1 G(x,s)[k(s), -\int_0^1 F(s,t)[h,l'](t)dt + \frac{1}{2}\int_0^1 G(s,t)[h,l](t)dt]ds \\
= &\int_0^1 F(x,s)[l',[h,k]](s)ds \\
&+\int_0^1\int_0^1 \{F(x,s)G(s,t) - \frac{1}{2}G(x,s)F(s,t)\}[h(s),[k,l'](t)]dtds \\
&+\int_0^1\int_0^1 \{\frac{1}{2}F(x,s)F(s,t) - \frac{1}{4}G(x,s)G(s,t)\}[h(s),[l,k](t)]dtds
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^1 \int_0^1 \{F(x,s)G(s,t) - \frac{1}{2}G(x,s)F(s,t)\}[k(s),[h,l'](t)]dtds \\
&- \int_0^1 \int_0^1 \{\frac{1}{2}F(x,s)F(s,t) - \frac{1}{4}G(x,s)G(s,t)\}[k(s),[l,h](t)]dtds \\
= \; & \int_0^1 F(x,s)[l',[h,k]](s)ds \\
&+ \int_0^1 \int_0^1 M(x,s,t)([h(s),[k,l'](t)] - [k(s),[h,l'](t)])dtds \\
&+ \frac{1}{2}\int_0^1 \int_0^1 N(x,s,t)([h(s),[l,k](t)] - [k(s),[l,h](t)])dtds
\end{aligned}
$$

where we have used the Jacobi identity for the first term in the second equation above, and we define M and N by

$$M(x,s,t) \equiv F(x,s)G(s,t) - \frac{1}{2}G(x,s)F(s,t) \tag{2.12}$$

$$N(x,s,t) \equiv F(x,s)F(s,t) - \frac{1}{2}G(x,s)G(s,t). \tag{2.13}$$

Finally, noting that

$$\nabla_{[h,k]}l(x) = -\int_0^1 F(x,s)[[h,k],l'](s)ds + \frac{1}{2}\int_0^1 G(x,s)[[h,k],l](s)ds$$

we get the following theorem.

Theorem 2.2.1 *For $h,k,l \in H(\mathfrak{g})$, the curvature tensor can be written as*

$$
\begin{aligned}
(R\langle h,k\rangle l)(x) = & \int_0^1 \int_0^1 M(x,s,t)([h(s),[k,l'](t)] - [k(s),[h,l'](t)])dtds \\
&+ \frac{1}{2}\int_0^1 \int_0^1 N(x,s,t)([h(s),[l,k](t)] - [k(s),[l,h](t)])dtds \\
&+ \frac{1}{2}\int_0^1 G(x,s)[l,[h,k]](s)ds,
\end{aligned}
$$

where M and N are defined in (2.12) and (2.13).

2.3 Ricci Tensor

If $h, k \in H$ and k has the property that $[k', k] = 0$, then

$$\begin{aligned}
(R\langle h,k\rangle k)(x) &= \int_0^1 \int_0^1 M(x,s,t)([h(s),[k,k'](t)] - [k(s),[h,k'](t)])dtds \\
&+\frac{1}{2}\int_0^1 \int_0^1 N(x,s,t)([h(s),[k,k](t)] - [k(s),[k,h](t)])dtds \\
&+\frac{1}{2}\int_0^1 G(x,s)[k,[h,k]](s)ds \\
&= \int_0^1 \int_0^1 M(x,s,t)[k(s),[k',h](t)]dtds \\
&-\frac{1}{2}\int_0^1 \int_0^1 N(x,s,t)[k(s),[k,h](t)]dtds \\
&+\frac{1}{2}\int_0^1 G(x,s)[k,[h,k]](s)ds. \qquad (2.14)
\end{aligned}$$

In anticipation of the Ricci tensor, we wish now to compute $(R\langle h,k\rangle k, h)$. For what follows, the x, s and t in $M(x,s,t)$ and $N(x,s,t)$ will be suppressed. Also, let us keep in mind that our Green's function, $G(x,s)$, has the following properties, as discussed in Definition 2.1.6:

$$-G_{xx}(x,s) + G(x,s) = \delta(s-x)$$

$$F(x,s) = G_x(x,s)$$

$$G_s(x,s) = -G_x(x,s).$$

First consider

$$\begin{aligned}
&(\int_0^1 G(\cdot,s)[k,[h,k]](s)ds, h(\cdot)) \\
&= \int_0^1 \{\langle \int_0^1 G_x(x,s)[k,[h,k]](s)ds, h'(x)\rangle + \langle \int_0^1 G(x,s)[k,[h,k]](s)ds, h(x)\rangle\}dx \\
&= \int_0^1 \{-\langle \int_0^1 G_{xx}(x,s)[k,[h,k]](s)ds, h(x)\rangle + \langle \int_0^1 G(x,s)[k,[h,k]](s)ds, h(x)\rangle\}dx \\
&= \int_0^1 \langle \int_0^1 (-G_{xx}(x,s) + G(x,s))[k,[h,k]](s)ds, h(x)\rangle dx \\
&= \int_0^1 \langle \int_0^1 \delta(s-x)[k,[h,k]](s)ds, h(x)\rangle dx \\
&= \int_0^1 \langle [k,[h,k]](x), h(x)\rangle dx.
\end{aligned}$$

Thus,

$$(\int_0^1 G(\cdot,s)[k,[h,k]](s)ds, h(\cdot)) = \int_0^1 \langle [k,h](x),[k,h](x)\rangle dx. \tag{2.15}$$

Next, consider

$$\begin{aligned}
&(\int_0^1 \int_0^1 M(\cdot,s,t)[k(s),[k',h](t)]dtds, h(\cdot)) \\
= &\int_0^1 \{\langle \int_0^1 \int_0^1 M_x[k(s),[k',h](t)]dtds, h'(x)\rangle \\
&+\langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle\}dx \\
= &\int_0^1 \{\langle \int_0^1 \int_0^1 \{G_{xx}(x,s)G(s,t) - \frac{1}{2}G_x(x,s)F(s,t)\}[k(s),[k',h](t)]dtds, h'(x)\rangle \\
&+\langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle\}dx \\
= &\int_0^1 \{\langle \int_0^1 \int_0^1 \{(G(x,s) - \delta(s-x))G(s,t) \\
&-\frac{1}{2}G_x(x,s)F(s,t)\}[k(s),[k',h](t)]dtds, h'(x)\rangle \\
&+\langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle\}dx.
\end{aligned} \tag{2.16}$$

Let us calculate different parts of (2.16) separately:

$$\begin{aligned}
\text{a)} &\int_0^1 \langle \int_0^1 \int_0^1 -\delta(s-x)G(s,t)[k(s),[k',h](t)]dtds, h'(x)\rangle dx \\
= &\int_0^1 \langle \int_0^1 [\int_0^1 -\delta(s-x)G(s,t)k(s)ds,[k',h](t)]dt, h'(x)\rangle dx \\
= &\int_0^1 \langle \int_0^1 -[G(x,t)k(x),[k',h](t)]dt, h'(x)\rangle dx \\
= &\int_0^1 \langle \int_0^1 [G_x(x,t)k(x) + G(x,t)k'(x),[k',h](t)]dt, h(x)\rangle dx \\
= &\int_0^1 \int_0^1 F(x,t)\langle [k(x),[k',h](t)], h(x)\rangle dtdx \\
&+\int_0^1 \int_0^1 G(x,t)\langle [k'(x),[k',h](t)], h(x)\rangle dtdx
\end{aligned}$$

$$\text{b)} \int_0^1 \langle \int_0^1 \int_0^1 \{G(x,s)G(s,t)$$
$$-\frac{1}{2}G_x(x,s)F(s,t)\}[k(s),[k',h](t)]dtds, h\prime(x)\rangle dx$$
$$= -\int_0^1 \langle \int_0^1 \int_0^1 \{G_x(x,s)G(s,t) - \frac{1}{2}G_{xx}(x,s)F(s,t)\}[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$= -\int_0^1 \langle \int_0^1 \int_0^1 \{G_x(x,s)G(s,t) - \frac{1}{2}(G(x,s)-\delta(s-x))F(s,t)\}$$
$$\cdot [k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$= -\int_0^1 \langle \int_0^1 \int_0^1 \{F(x,s)G(s,t) - \frac{1}{2}G(x,s)F(s,t)\}[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$-\int_0^1 \langle \int_0^1 \int_0^1 \frac{1}{2}\delta(s-x)F(s,t)[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$= -\int_0^1 \langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$-\int_0^1 \langle \int_0^1 \frac{1}{2}F(x,t)[k(x),[k',h](t)]dt, h(x)\rangle dx$$
$$= -\int_0^1 \langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$-\frac{1}{2}\int_0^1 \int_0^1 F(x,t)\langle [k(x),[k',h](t)], h(x)\rangle dtdx$$

Using these calculations in (2.16) we get

$$(\int_0^1 \int_0^1 M(\cdot,s,t)[k(s),[k',h](t)]dtds, h(\cdot))$$
$$= \int_0^1 \int_0^1 F(x,t)\langle [k(x),[k',h](t)], h(x)\rangle dtdx$$
$$+\int_0^1 \int_0^1 G(x,t)\langle [k'(x),[k',h](t)], h(x)\rangle dtdx$$
$$-\int_0^1 \langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$-\frac{1}{2}\int_0^1 \int_0^1 F(x,t)\langle [k(x),[k',h](t)], h(x)\rangle dtdx$$
$$+\int_0^1 \langle \int_0^1 \int_0^1 M[k(s),[k',h](t)]dtds, h(x)\rangle dx$$
$$= \int_0^1 \int_0^1 G(x,t)\langle [k'(x),[k',h](t)], h(x)\rangle dtdx$$
$$+\frac{1}{2}\int_0^1 \int_0^1 F(x,t)\langle [k(x),[k',h](t)], h(x)\rangle dtdx,$$

and thus

$$\begin{aligned}
&(\int_0^1 \int_0^1 M(\cdot,s,t)[k(s),[k\prime,h](t)]dtds, h(\cdot)) \\
= \quad & -\int_0^1 \int_0^1 G(x,t)\langle [k',h](t),[k',h](x)\rangle dtdx \\
& -\frac{1}{2}\int_0^1 \int_0^1 F(x,t)\langle [k',h](t),[k,h](x)\rangle dtdx. \qquad (2.17)
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
&(\int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)) \\
= \quad & \int_0^1 \{\langle \int_0^1 \int_0^1 N_x[k(s),[k,h](t)]dtds, h'(x)\rangle \\
& +\langle \int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)\rangle\}dx \\
= \quad & \int_0^1 \{\langle \int_0^1 \int_0^1 \{G_{xx}(x,s)F(s,t) - \frac{1}{2}G_x(x,s)G(s,t)\}[k(s),[k,h](t)]dtds, h'(x)\rangle \\
& +\langle \int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)\rangle\}dx \\
= \quad & \int_0^1 \{\langle \int_0^1 \int_0^1 \{(G(x,s)-\delta(s-x))F(s,t) \\
& -\frac{1}{2}G_x(x,s)G(s,t)\}[k(s),[k,h](t)]dtds, h'(x)\rangle \\
& +\langle \int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)\rangle\}dx \qquad (2.18)
\end{aligned}$$

Let us calculate different parts of (2.18) separately:

$$\begin{aligned}
\text{a)} &\int_0^1 \langle \int_0^1 \int_0^1 -\delta(s-x)F(s,t)[k(s),[k,h](t)]dtds, h\prime(x)\rangle dx \\
= \quad & \int_0^1 \langle \int_0^1 [\int_0^1 -\delta(s-x)F(s,t)k(s)ds, [k,h](t)]dt, h'(x)\rangle dx \\
= \quad & \int_0^1 -\langle \int_0^1 [F(x,t)k(x),[k,h](t)]dt, h'(x)\rangle dx \\
= \quad & \int_0^1 -\langle [k(x), \int_0^1 F(x,t)[k,h](t)dt], h'(x)\rangle dx \\
= \quad & \int_0^1 \langle [k'(x), \int_0^1 F(x,t)[k,h](t)dt] + [k(x), \int_0^1 G_{xx}(x,t)[k,h](t)dt], h(x)\rangle dx \\
= \quad & \int_0^1 \langle [k'(x), \int_0^1 F(x,t)[k,h](t)dt] + [k(x), \int_0^1 \{G(x,t)-\delta(t-x)\}[k,h](t)dt], h(x)\rangle dx
\end{aligned}$$

$$
\begin{aligned}
= & \int_0^1 \int_0^1 G(x,t)\langle [k(x),[k,h](t)],h(x)\rangle dtdx \\
& + \int_0^1 \int_0^1 F(x,t)\langle [k'(x),[k,h](t)],h(x)\rangle dtdx \\
& + \int_0^1 \langle [k,h](t),[k,h](t)\rangle dt
\end{aligned}
$$

b) $\int_0^1 \langle \int_0^1 \int_0^1 \{(G(x,s)F(s,t) - \frac{1}{2}G_x(x,s)G(s,t)\}[k(s),[k,h](t)]dtds, h\prime(x)\rangle dx$

$$
\begin{aligned}
= & -\int_0^1 \langle \int_0^1 \int_0^1 \{(G_x(x,s)F(s,t) - \frac{1}{2}G_{xx}(x,s)G(s,t)\}[k(s),[k,h](t)]dtds, h(x)\rangle dx \\
= & -\int_0^1 \langle \int_0^1 \int_0^1 \{(G_x(x,s)F(s,t) \\
& -\frac{1}{2}(G(x,s)-\delta(s-x))G(s,t)\}[k(s),[k,h](t)]dtds, h(x)\rangle dx \\
= & -\int_0^1 \langle \int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)\rangle dx \\
& -\int_0^1 \langle \int_0^1 \int_0^1 \frac{1}{2}\delta(s-x)G(s,t)[k(s),[k,h](t)]dtds, h(x)\rangle dx \\
= & -\int_0^1 \langle \int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)\rangle dx \\
& -\frac{1}{2}\int_0^1 \int_0^1 G(x,t)\langle [k(x),[k,h](t)],h(x)\rangle dtdx
\end{aligned}
$$

Using these calculations in (2.18) we get

$$
\begin{aligned}
& (\int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)) \\
= & \int_0^1 \int_0^1 F(x,t)\langle [k'(x),[k,h](t)],h(x)\rangle dtdx \\
& +\frac{1}{2}\int_0^1 \int_0^1 G(x,t)\langle [k(x),[k,h](t)],h(x)\rangle dtdx \\
& + \int_0^1 \langle [k,h](t),[k,h](t)\rangle dt,
\end{aligned}
$$

and so

$$
\begin{aligned}
(\int_0^1 \int_0^1 N[k(s),[k,h](t)]dtds, h(x)) = & -\int_0^1 \int_0^1 F(x,t)\langle [k,h](t),[k',h](x)\rangle dtdx \\
& -\frac{1}{2}\int_0^1 \int_0^1 G(x,t)\langle [k,h](t),[k,h](x)\rangle dtdx \\
& + \int_0^1 \langle [k,h](t),[k,h](t)\rangle dt. \qquad (2.19)
\end{aligned}
$$

Now we are ready to determine $(R\langle h,k\rangle k,h)$. Putting Equations (2.15), (2.17), and (2.19) together gives

$$
\begin{aligned}
(R\langle h,k\rangle k,h) &= (\int_0^1\int_0^1 M[k(s),[k',h](t)]dtds,h) \\
&\quad -\frac{1}{2}(\int_0^1\int_0^1 N(x,s,t)[k(s),[k,h](t)]dtds,h) \\
&\quad +\frac{1}{2}(\int_0^1 G(x,s)[k,[h,k]]ds,h) \\
&= -\int_0^1\int_0^1 G(x,t)\langle[k',h](t),[k',h](x)\rangle dtdx \\
&\quad -\frac{1}{2}\int_0^1\int_0^1 F(x,t)\langle[k',h](t),[k,h](x)\rangle dtdx \\
&\quad +\frac{1}{2}\int_0^1\int_0^1 F(x,t)\langle[k,h](t),[k',h](x)\rangle dtdx \\
&\quad +\frac{1}{4}\int_0^1\int_0^1 G(x,t)\langle[k,h](t),[k,h](x)\rangle dtdx \\
&\quad -\frac{1}{2}\int_0^1 \langle[k,h](t),[k,h](t)\rangle dt \\
&\quad +\frac{1}{2}\int_0^1 \langle[k,h](x),[k,h](x)\rangle dx \\
&= -\int_0^1\int_0^1 G(x,t)\langle[k',h](t),[k',h](x)\rangle dtdx \\
&\quad +\frac{1}{4}\int_0^1\int_0^1 G(x,t)\langle[k,h](t),[k,h](x)\rangle dtdx \\
&\quad -\frac{1}{2}\int_0^1\int_0^1 F(x,t)\langle[k',h](t),[k,h](x)\rangle dtdx \\
&\quad +\frac{1}{2}\int_0^1\int_0^1 F(t,x)\langle[k,h](x),[k',h](t)\rangle dtdx.
\end{aligned}
$$

We have thus proved the following theorem.

Theorem 2.3.1

$$
\begin{aligned}
(R\langle h,k\rangle k,h) = &-\int_0^1\int_0^1 G(x,t)\langle[k',h](x),[k',h](t)\rangle dtdx \\
&+\frac{1}{4}\int_0^1\int_0^1 G(x,t)\langle[k,h](x),[k,h](t)\rangle dtdx \\
&-\int_0^1\int_0^1 F(x,t)\langle[k,h](x),[k',h](t)\rangle dtdx. \qquad (2.20)
\end{aligned}
$$

Recalling that $F(x,t)=G_x(x,t)$, we may use integration by parts to see that

$$
\begin{aligned}
&\int_0^1\int_0^1 F(x,t)\langle[k,h](x),[k',h](t)\rangle dtdx \\
= &\int_0^1\int_0^1 -G(x,t)\langle[k',h](x)+[k,h'](x),[k',h](t)\rangle dtdx.
\end{aligned}
$$

Thus, (2.20) can be written in the alternative form

$$(R\langle h,k\rangle k,h) = \int_0^1\int_0^1 G(x,t)\{\langle [k,h'](x),[k',h](t)\rangle + \frac{1}{4}\langle [k,h](x),[k,h](t)\rangle\}dtdx.$$

This verifies the following symmetry property of the curvature tensor.

Corollary 2.3.2 *For $h, k \in H(\mathfrak{g})$,*

$$(R\langle h,k\rangle k,h) = (R\langle k,h\rangle h,k).$$

Before determining a suitable expression for the Ricci tensor, it will be useful to first point out some properties of our recurring Green's function, G.

Definition 2.3.3 *Let $H(R)$ be the space of absolutely continuous functions $a : [0,1] \to R$ with $a(0) = a(1)$.*

Definition 2.3.4 *Let S be the orthonormal basis of $H(g)$ (the Lie algebra of our free loop group $\mathcal{L}(G)$), with the form $S = \{aA : a \in h_0, A \in g_0\}$, where g_0 is an orthonormal basis of $\mathfrak{g}$, and h_0 is an orthonormal basis of $H(R)$.*

Throughout this paper we will take "orthonormal basis" to mean a Hilbert basis (i.e., total and orthonormal in the given Hilbert space).

Lemma 2.3.5 *For all $k \in S$, $[k', k] = 0$.*

Proof: Since k is of the form $k(s) = a(s)A$, $a \in H(R)$, $A \in g_0$, then

$$[k',k](s) = [a'(s)A, a(s)A] = a'(s)a(s)[A,A] = 0.$$

Q.E.D.

We will call such an S (i.e., with the property that $[k,k'] = 0$ for all $k \in S$) a "good" basis of $H(\mathfrak{g})$.

Lemma 2.3.6 *The Green's function $G(s,t)$ is the reproducing Kernel for $H(R)$:*

$$(G(s,\cdot),a) = a(s) \;\; \forall a \in H(R).$$

Proof: The inner product, $(\cdot,\cdot)$, that we are assuming gives

$$\begin{aligned}(G(s,\cdot),a) &= \int_0^1 \{G_t(s,t)a'(t)+G(s,t)a(t)\}dt \\ &= G_t(s,1)a(1)-G_t(s,0)a(0)+\int_0^1(-G_{ss}(s,t)+G(s,t))a(t)dt \\ &= \int_0^1 \delta(t-s)a(t)dt \\ &= a(s).\end{aligned}$$

Q.E.D.

The next lemma provides a convenient way of writing $G(s,t)$ as a series.

Lemma 2.3.7 *For any orthonormal basis h_0 of $H(R)$, $\sum_{a\in h_0} a(s)a(t)=G(s,t)\ \ \forall s,t\in[0,1]$*

Proof: Since $G(s,t)$ is the reproducing kernel for $H(R)$, and h_0 is an orthonormal basis, then

$$\begin{aligned}G(s,t) &= (G(s,\cdot),G(\cdot,t)) \\ &= (\sum_{a\in h_0}(G(s,\cdot),a)a, \sum_{b\in h_0}(G(\cdot,t),b)b) \\ &= \sum_{a\in h_0}(G(s,\cdot),a)(G(\cdot,t),a) \\ &= \sum_{a\in h_0} a(s)a(t).\end{aligned}$$

Note that convergence of the series $\sum_{a\in h_0} a(s)a(t)$ is guaranteed by the fact that h_0 is a Hilbert basis. □

Lemma 2.3.8 *For any orthonormal basis h_0 of $H(R)$,*

$$\sum_{a\in h_0}|a(s)a(t)|\leq\frac{e+1}{2(e-1)}\quad\forall s,t\in[0,1].$$

Thus, $\sum_{a\in h_0} a(s)a(t)$ is absolutely convergent.

Proof: From the previous lemma, for $s \in [0,1]$

$$\begin{aligned}
\sum_{a \in h_0} a^2(s) &= G(s,s) \\
&= a\cosh(\frac{1}{2}) \\
&= \frac{1}{2\sinh(\frac{1}{2})}\cosh(\frac{1}{2}) \\
&= \frac{1}{2}\cdot\frac{e^{\frac{1}{2}}+e^{-\frac{1}{2}}}{e^{\frac{1}{2}}-e^{-\frac{1}{2}}} \\
&= \frac{e+1}{2(e-1)}
\end{aligned}$$

Thus, using the Cauchy Schwartz inequality,

$$\begin{aligned}
\sum_{a \in h_0} |a(s)a(t)| &\leq \sqrt{\sum_{a \in h_0} a^2(s)}\sqrt{\sum_{a \in h_0} a^2(t)} \\
&= \frac{e+1}{2(e-1)}. \square
\end{aligned}$$

Definition 2.3.9 *Let $K\langle\cdot,\cdot\rangle$ denote the following form on $\mathfrak{g}$:*

$$\begin{aligned}
K\langle A,B\rangle &\equiv -tr(ad_A ad_B) \\
&= -\sum_{C \in g_0} \langle ad_A ad_B C, C\rangle \\
&= \sum_{C \in g_0} \langle ad_B C, ad_A C\rangle.
\end{aligned}$$

One should note that $-K\langle\cdot,\cdot\rangle$ is traditionally known as the Killing form *of $\mathfrak{g}$ (see p. 131 in Helgason [11]).*

Using the above lemmas and definitions, the Ricci tensor can now be computed as follows.

$$\mathrm{Ric}\langle h,h\rangle = \sum_{k \in S} (R\langle h,k\rangle k, h)$$

$$
\begin{aligned}
&= \sum_{k\in S}\int_0^1\int_0^1 \{-G(x,t)\langle [k',h](x),[k',h](t)\rangle \\
&\quad +\frac{1}{4}G(x,t)\langle [k,h](x),[k,h](t)\rangle \\
&\quad -F(x,t)\langle [k,h](x),[k',h](t)\rangle\} dtdx \\
&= \sum_{a\in h_0, A\in g_0}\int_0^1\int_0^1 \{-G(x,t)\langle [a'(x)A,h(x)],[a'(t)A,h(t)]\rangle \\
&\quad +\frac{1}{4}G(x,t)\langle [a(x)A,h(x)],[a(t)A,h(t)]\rangle \\
&\quad -F(x,t)\langle [a(x)A,h(x)],[a'(t)A,h(t)]\rangle\} dtdx \\
&= \sum_{a\in h_0, A\in g_0}\int_0^1\int_0^1 \{G_x(x,t)\langle [a(x)A,h(x)],[a'(t)A,h(t)]\rangle \\
&\quad +G(x,t)\langle [a(x)A,h'(x)],[a'(t)A,h(t)]\rangle \\
&\quad +\frac{1}{4}G(x,t)\langle [a(x)A,h(x)],[a(t)A,h(t)]\rangle \\
&\quad -G_x(x,t)\langle [a(x)A,h(x)],[a'(t)A,h(t)]\rangle\} dtdx \\
&= \sum_{a\in h_0, A\in g_0}\int_0^1\int_0^1 \{G_x(x,t)\langle [a(x)A,h(x)],[a(t)A,h(t)]\rangle \\
&\quad -G(x,t)\langle [a(x)A,h(x)],[a(t)A,h'(t)]\rangle \\
&\quad +\frac{1}{4}G(x,t)\langle [a(x)A,h(x)],[a(t)A,h(t)]\rangle\} dtdx \\
&= \int_0^1\int_0^1 \{G_x(x,t)G(x,t)K\langle h'(x),h(t)\rangle \\
&\quad -G^2(x,t)K\langle h'(x),h'(t)\rangle + \frac{1}{4}G^2(x,t)K\langle h(x),h(t)\rangle\} dtdx \\
&= \int_0^1\int_0^1 \{(-G_x(x,t)G(x,t) - G_x^2(x,t))K\langle h(x),h(t)\rangle \\
&\quad +2G_x(x,t)G(x,t)K\langle h(x),h'(t)\rangle + \frac{1}{4}G^2(x,t)K\langle h(x),h(t)\rangle\} dtdx \\
&= \int_0^1 G(x,x)K\langle h(x),h(x)\rangle dx \\
&\quad +\int_0^1\int_0^1 \{-G^2(x,t) - G_x^2(x,t) + 2G_{xx}(x,t)G(x,t) \\
&\quad +2G_x(x,t)^2 + \frac{1}{4}G^2(x,t)\}K\langle h(x),h(t)\rangle dtdx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 G(x,x)K\langle h(x),h(x)\rangle dx - 2\int_0^1 G(x,x)K\langle h(x),h(x)\rangle dx \\
&\quad +\int_0^1\int_0^1 \{-G^2(x,t) - G_x^2(x,t) + 2G^2(x,t) \\
&\quad +2G_x(x,t)^2 + \frac{1}{4}G^2(x,t)\}K\langle h(x),h(t)\rangle dtdx.
\end{aligned}
$$

We have thus proved the following theorem.

Theorem 2.3.10 *For all $h \in H(\mathfrak{g})$,*

$$\mathrm{Ric}\langle h,h\rangle = \int_0^1\int_0^1 \{\frac{5}{4}G^2(x,t)+F^2(x,t)\}K\langle h(x),h(t)\rangle dtdx - \int_0^1 G(x,x)K\langle h(x),h(x)\rangle dx, \tag{2.21}$$

where, again, G and F are defined as in (2.4).

An important key to proving our logarithmic Sobolev inequality in Theorem 6.0.2 will be based on the fact the Ricci tensor is bounded below, which is what we next verify.

Theorem 2.3.11 *There exists a constant $C > 0$ such that*

$$Ric\langle h,h\rangle \geq -C(h,h) \tag{2.22}$$

for all $h \in H(\mathfrak{g})$.

Proof: By the definition of G and F, it is clear that there exists bounds, C_1 and C_2, on such that $|G(x,t)| \leq C_1$ and $|F(x,t)| \leq C_2$ for all $(x,t) \in [0,1]^2$. It then follows that for all $h \in H(\mathfrak{g})$,

$$\begin{aligned}
|Ric\langle h,h\rangle| &\leq \int_0^1\int_0^1 \{\frac{5}{4}C_1^2+C_2^2\}\,|K\langle h(x),h(t)\rangle|\,dtdx + \int_0^1 C_1\,|K\langle h(x),h(x)\rangle|\,dx \\
&\leq \int_0^1\int_0^1 \{\frac{5}{4}C_1^2+C_2^2\}c\,|\langle h(x),h(t)\rangle_{\mathfrak{g}}|\,dtdx + \int_0^1 C_1c\langle h(x),h(x)\rangle_{\mathfrak{g}}dx \\
&\leq \{\frac{5}{4}C_1^2+C_2^2\}c\int_0^1 \|h(t)\|_{\mathfrak{g}}\,dt\int_0^1 \|h(x)\|_{\mathfrak{g}}\,dx + C_1c\int_0^1 \|h(x)\|_{\mathfrak{g}}^2\,dx \\
&\leq \{\frac{5}{4}C_1^2+C_2^2\}c(\int_0^1 \|h(x)\|_{\mathfrak{g}}\,dx)^2 + C_1c\int_0^1 \|h(x)\|_{\mathfrak{g}}^2\,dx \\
&\leq \{\frac{5}{4}C_1^2+C_2^2\}c\int_0^1 \|h(x)\|_{\mathfrak{g}}^2\,dx + C_1c\int_0^1 \|h(x)\|_{\mathfrak{g}}^2\,dx \\
&\leq \{\frac{5}{4}C_1^2+C_2^2+C_1\}c\int_0^1 \{\|h'(x)\|_{\mathfrak{g}}^2+\|h(x)\|_{\mathfrak{g}}^2\}dx \\
&= \{\frac{5}{4}C_1^2+C_2^2+C_1\}c(h,h),
\end{aligned}$$

where we have used Hölder's and Cauchy Schwartz inequalities for the third inequality above and c is a constant depending on K. Letting $C \equiv \{\frac{5}{4}C_1^2+C_2^2+C_1\}c$ we see that (2.22) holds. $\Box$

Chapter 3

Finite Dimensional Approximations

For the finite dimensional approximations to the space $\mathcal{L}(G)$, we consider, for each partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$, the finite dimensional product Lie group $G^{\mathcal{P}} \equiv G^n = G \times \cdots \times G$. Recall the Green's function $G(s,t) = a\cosh(r(s,t))$ where $a = \frac{1}{2\sinh(\frac{1}{2})}$ and

$$r(s,t) = \begin{cases} s - t - \frac{1}{2} & 0 \le t < s \\ s - t + \frac{1}{2} & s \le t \le 1. \end{cases}.$$

We define a metric on the Lie algebra $\mathfrak{g}^{\mathcal{P}}$ by

$$(\vec{A}, \vec{B})_{\mathcal{P}} \equiv \sum_{i,j=1}^{n} Q_{ij} \langle A_i, B_j \rangle, \tag{3.1}$$

where $\vec{A} = (A_1, \ldots, A_n)$, $\vec{B} = (B_1, \ldots, B_n) \in \mathfrak{g}^{\mathcal{P}}$, $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{g}$, and Q is the inverse of the matrix $\{G(s_i, s_j)\}_{i,j=1}^n$. We can then extend $(\cdot,\cdot)_{\mathcal{P}}$ to a unique left-invariant Riemannian metric, which we will also denote by $(\cdot,\cdot)_{\mathcal{P}}$, on $G^{\mathcal{P}}$.

Before continuing, we should verify that the matrix $\{G(s_i, s_j)\}_{i,j=1}^n$is actually invertible. Let $\tilde{\pi}_{\mathcal{P}} : H(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ be the projection defined by

$$\tilde{\pi}_{\mathcal{P}}(h) \equiv (h(s_0), \ldots, h(s_{n-1}))$$

for $h \in H(\mathfrak{g})$..It is sufficient to show that it is positive definite. So assume that $\mathbf{x} =$

$(x_0, x_1, \ldots, x_{n-1}) \in \mathbf{R}^n \backslash \{0\}$. Then, by Lemma 2.3.7,

$$\begin{aligned} \sum_{i,j=0}^{n-1} G(s_i, s_j) x_i x_j &= \sum_{i,j=0}^{n-1} \sum_{a \in h_0} a(s_i) a(s_j) x_i x_j \\ &= \sum_{a \in h_0} (\sum_{i=0}^{n-1} a(s_i) x_i)(\sum_{j=0}^{n-1} a(s_j) x_j) \\ &= \sum_{a \in h_0} (\mathbf{x} \cdot (\tilde{\pi}_{\mathcal{P}}(a)))^2, \end{aligned}$$

where h_0 is an orthonormal basis of $H(\mathfrak{g})$. The summation in the last equation will be greater than zero if we just choose the basis h_0 to include an element a such that $\mathbf{x} \cdot (\tilde{\pi}_{\mathcal{P}}(a)) \neq 0$ (one may start with such an a and use the Gram-Schmidt process to construct the rest of h_o).

Next, define the subspace $H_{\mathcal{P}}(\mathfrak{g}) \subseteq H(\mathfrak{g})$ by

$$H_{\mathcal{P}}(\mathfrak{g}) \equiv (Ker(\tilde{\pi}_{\mathcal{P}}))^{\perp}.$$

Lemma 3.0.1 *If* $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$ *is a given partition of* $[0,1]$, *then the following are true:*

a) If $k, l \in H_{\mathcal{P}}(\mathfrak{g})$ *are such that* $k(s_i) = l(s_i)\ \forall\, s_i \in \mathcal{P}$, *then* $k \equiv l$.

b) $H_{\mathcal{P}}(\mathfrak{g}) = \{k \in H(\mathfrak{g}) : k(s) = \sum_{i=0}^{n-1} G(s, s_i) A_i,\ A_i \in \mathfrak{g}\}$.

c) $H_{\mathcal{P}}(\mathfrak{g}) = \{k \in H(\mathfrak{g}) :\ -k''(s) + k(s) = 0$ *for all* $s \in [0,1] \backslash \mathcal{P}\}$.

Proof: a)Suppose $k, l \in H_{\mathcal{P}}(\mathfrak{g})$ are such that $k(s_i) = l(s_i)\ \forall\, s_i \in \mathcal{P}$. Then $h \equiv k - l$ has the property that $h(s_i) = 0\ \forall\, s_i \in \mathcal{P}$, and thus $h \in Ker(\tilde{\pi}_{\mathcal{P}}) \cap H_{\mathcal{P}}(\mathfrak{g}) = Ker(\tilde{\pi}_{\mathcal{P}}) \cap (Ker(\tilde{\pi}_{\mathcal{P}}))^{\perp} = \{0\}$. Hence, $h \equiv 0$ and so $k \equiv l$.

b) Suppose $k(s) = \sum_{i=0}^{n-1} G(s, s_i) A_i$ where $A_i \in \mathfrak{g}$ for all $i \in \{0, 1, \ldots, n-1\}$. Then for an arbitrary $h \in Ker(\tilde{\pi}_{\mathcal{P}})$, the reproducing property of G gives

$$\begin{aligned} (h, k) &= (h, \sum_{i=0}^{n-1} G(s, s_i) A_i) \\ &= \sum_{i=0}^{n-1} \langle h(s_i), A_i \rangle = 0, \end{aligned}$$

where, for the last equation, we have used the fact that $h(s_i) = 0$ for all i. Since $h \in Ker(\tilde{\pi}_{\mathcal{P}})$ was arbitrary, we have shown that $k \in H_{\mathcal{P}}(\mathfrak{g}) = (Ker(\tilde{\pi}_{\mathcal{P}}))^{\perp}$.

Now suppose that $k \in H_{\mathcal{P}}(\mathfrak{g})$. We wish to show that it must be of the form $k(s) = \sum_{i=0}^{n-1} G(s, s_i)A_i$ where $A_i \in \mathfrak{g}$ for all i. Let $A_i \equiv \sum_{j=0}^{n-1} Q_{ij}k(s_j)$ for each i (Q is the matrix defined in (3.1)). Thus, $k(s_j) = \sum_{i=0}^{n-1} G(s_j, s_i)A_i$ for $j \in \{0, 1, \dots, n-1\}$. Let $l(s) \equiv \sum_{i=0}^{n-1} G(s, s_i)A_i$ for $s \in [0, 1]$. By the first part of the proof for part (b) we know that $l \in H_{\mathcal{P}}(\mathfrak{g})$. Since, for each $s_j \in \mathcal{P}$, $l(s_j) = \sum_{i=0}^{n-1} G(s_j, s_i)A_i = k(s_j)$, it follows by part (a) that $k - l \equiv 0$.

c) Let $k \in H_{\mathcal{P}}(\mathfrak{g})$. Then by part (b), k is of the form $k(s) = \sum_{i=0}^{n-1} G(s, s_i)A_i$ where $A_i \in \mathfrak{g}$ for all i. From the definition of G in (2.4) it is clear that $G_{ss}(s, s_i) = G(s, s_i)$ for each $s_i \in \mathcal{P}$ and $s \notin \mathcal{P}$. Thus $k''(s) = k(s)$ for $s \in [0, 1]\backslash\mathcal{P}$.

Conversely, suppose that $k \in H(\mathfrak{g})$ is such that $-k''(s) + k(s) = 0$ for all $s \in [0, 1]\backslash\mathcal{P}$. For each $h \in Ker(\tilde{\pi}_{\mathcal{P}})$, integration by parts gives

$$
\begin{aligned}
(h, k) &= \int_0^1 \{\langle h'(s), k'(s)\rangle + \langle h(s), k(s)\rangle\} ds \\
&= \sum_{i=0}^{n-1} \{\langle h(s_i), k'(s_i)\rangle - \int_{s_i}^{s_{i+1}} \langle h(s), k''(s)\rangle ds\} + \int_0^1 \langle h(s), k(s)\rangle ds \\
&= \sum_{i=0}^{n-1} \langle h(s_i), k'(s_i)\rangle + \int_0^1 \langle h(s), -k''(s) + k(s)\rangle ds = 0.
\end{aligned}
$$

Thus, $k \in (Ker(\tilde{\pi}_{\mathcal{P}}))^{\perp} = H_{\mathcal{P}}(\mathfrak{g})$. $\square$

Our next goal is to show that $\tilde{\pi}_{\mathcal{P}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ is an isometric isomorphism. This will give us a more convenient finite dimensional space, namely $H_{\mathcal{P}}(\mathfrak{g})$, in which to perform calculations. But first we need to give $H_{\mathcal{P}}(\mathfrak{g})$ a Lie algebra structure. Note that $H_{\mathcal{P}}(\mathfrak{g})$ does not inherit the Lie algebra structure from $H(\mathfrak{g})$ since if $h, k \in H_{\mathcal{P}}(\mathfrak{g})$, then $[h, k]$ is not necessarily back in $H_{\mathcal{P}}(\mathfrak{g})$. We start with the following definitions.

Definition 3.0.2 *For each partition $\mathcal{P}$ and each $h \in H(\mathfrak{g})$ (or $h \in H(R)$), define the projection $P_{\mathcal{P}} : H(\mathfrak{g}) \to H_{\mathcal{P}}(\mathfrak{g})$ (or $P_{\mathcal{P}} : H(R) \to H_{\mathcal{P}}(R)$) by*

$$(P_{\mathcal{P}}h)(s) \equiv \sum_{i,j=0}^{n-1} G(s, s_j)Q_{ij}h(s_j).$$

Note that for each $s_l \in \mathcal{P}$, $(P_{\mathcal{P}}h)(s_l) = \sum_{i,j=0}^{n-1} G(s_l, s_i)Q_{ij}h(s_j) = h(s_l)$ (by definition of the matrix Q). In other words, $P_{\mathcal{P}}h$ and h agree on the partition points of $\mathcal{P}$.

Proposition 3.0.3 *For each partition $\mathcal{P}$ $P_{\mathcal{P}} : H(\mathfrak{g}) \to H_{\mathcal{P}}(\mathfrak{g})$ is the orthogonal projection onto $H_{\mathcal{P}}(\mathfrak{g})$. Furthermore, for each $h \in H(\mathfrak{g})$, $P_{\mathcal{P}}h$ is the piece-wise exponential approximation of h that is uniquely determined by its values on $\mathcal{P}$ as follows:*

$$(P_{\mathcal{P}}h)(t) = \frac{h(s_i) + h(s_{i+1})}{\varsigma_i} \cosh(t - \bar{s}_i) + \frac{\Delta_i h}{\eta_i} \sinh(t - \bar{s}_i) \ \forall t \in [s_i, s_{i+1}], \tag{3.2}$$

where $\bar{s}_i \equiv \frac{s_i + s_{i+1}}{2}$, $\Delta_i h \equiv h(s_{i+1}) - h(s_i)$, $\varsigma_i \equiv 2\cosh(\bar{s}_i)$ *and* $\eta_i \equiv 2\sinh(\frac{s_{i+1}-s_i}{2})$.

Proof: Note that for any $h \in H(\mathfrak{g})$, since $(h - P_{\mathcal{P}}h)(s_i) = h(s_i) - (P_{\mathcal{P}}h)(s_i) = h(s_i) - h(s_i) = 0$ for all $s_i \in \mathcal{P}$, then $h - P_{\mathcal{P}}h \in Ker(\tilde{\pi}_{\mathcal{P}}) = (H_{\mathcal{P}}(\mathfrak{g}))^{\perp}$. This, along with the fact that $\tilde{\pi}_{\mathcal{P}}$ is an isometry, shows that $H(\mathfrak{g}) = H_{\mathcal{P}}(\mathfrak{g}) \oplus Ker(\tilde{\pi}_{\mathcal{P}}) = H_{\mathcal{P}}(\mathfrak{g}) \oplus H_{\mathcal{P}}(\mathfrak{g})^{\perp}$ since any $h \in H(\mathfrak{g})$ can be written as $h = P_{\mathcal{P}}h + (h - P_{\mathcal{P}}h)$, and if $h \in H_{\mathcal{P}}(\mathfrak{g}) \cap Ker(\tilde{\pi}_{\mathcal{P}})$, then $h(s_i) = 0 \ \forall s_i \in \mathcal{P}$ and thus $\tilde{\pi}_{\mathcal{P}}(h) = 0 \Rightarrow |h| = |\tilde{\pi}_{\mathcal{P}}(h)| = 0 \Rightarrow h = 0$. Hence, $P_{\mathcal{P}}h$ is the orthogonal projection of h onto $H_{\mathcal{P}}(\mathfrak{g})$.

To see that (3.2) holds, note that $P_{\mathcal{P}}h$ is defined in terms of the Green's function, $G(s,t)$, which is a sum of exponential functions. Thus, on a subinterval, $[s_i, s_{i+1}]$, of the partition, $P_{\mathcal{P}}h$ must be exponential of the form $(P_{\mathcal{P}}h)(t) = A\cosh(t - \bar{s}_i) + B\sinh(t - \bar{s}_i)$. To determine A and B, we use the fact that $(P_{\mathcal{P}}h)(s_i) = h(s_i)$ and $(P_{\mathcal{P}}h)(s_{i+1}) = h(s_{i+1})$. □

Definition 3.0.4 *For $h, k \in H_{\mathcal{P}}(\mathfrak{g})$, define a Lie bracket, $[\cdot,\cdot]_{\mathcal{P}}$, on $H_{\mathcal{P}}(\mathfrak{g})$ by $[h,k]_{\mathcal{P}} \equiv P_{\mathcal{P}}[h,k]$.*

We claim that $H_{\mathcal{P}}(\mathfrak{g})$ is a Lie algebra under $[\cdot,\cdot]_{\mathcal{P}}$. For this we need to show the Jacobi identity holds. In other words, for any $h, k, l \in H_{\mathcal{P}}(\mathfrak{g})$, we want

$$J \equiv [h, [k,l]_{\mathcal{P}}]_{\mathcal{P}} + [l, [h,k]_{\mathcal{P}}]_{\mathcal{P}} + [k, [l,h]_{\mathcal{P}}]_{\mathcal{P}} = 0.$$

By our remarks in the definition of $P_{\mathcal{P}}$, for each $s_i \in \mathcal{P}$ we have

$$\begin{aligned} J(s_i) &= [h(s_i), [k,l](s_i)] + [l(s_i), [h,k](s_i)] + [k(s_i), [l,h](s_i)] \\ &= 0, \end{aligned}$$

where we have used the Jacobi identity for $\mathfrak{g}$ in the last equality. But since J is in $H_{\mathcal{P}}(\mathfrak{g})$, by part (a) of Lemma 3.0.1 it is uniquely determined by its values on $\mathcal{P}$ and hence $J \equiv 0$.

The next two propositions verify that $\tilde{\pi}_{\mathcal{P}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ is indeed an isometric isomorphism.

Proposition 3.0.5 *Given a partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$, suppose that $h, k \in H_{\mathcal{P}}(\mathfrak{g})$. Then*

$$(h, k) = \sum_{i,j=1}^{n-1} Q_{ij} \langle h(s_i), k(s_j) \rangle,$$

Thus, the map $\tilde{\pi}_{\mathcal{P}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ is an isometry, where the inner product on $\mathfrak{g}^{\mathcal{P}}$ is given in (3.1).

Proof: Suppose $k(s) = \sum_{i=0}^{n-1} G(s, s_i) A_i$ and $h(s) = \sum_{i=0}^{n-1} G(s, s_i) B_i$, where A_i and $B_i \in \mathfrak{g}$ for all i. Then

$$\begin{aligned} (h, k) &= \sum_{i=0}^{n-1} \langle h(s_i), A_i \rangle \\ &= \sum_{i=0}^{n-1} \langle h(s_i), \sum_{j=0}^{n-1} Q_{ij} k(s_j) \rangle \\ &= \sum_{i,j=0}^{n-1} Q_{ij} \langle h(s_i), k(s_j) \rangle, \end{aligned}$$

where we have used Lemma 2.3.6 for the first equation.□

Corollary 3.0.6 *For $k(s) = \sum_{i=0}^{n-1} G(s, s_i) A_i$ and $h(s) = \sum_{i=0}^{n-1} G(s, s_i) B_i$ as above,*

$$(h, k) = \sum_{i,j=0}^{n-1} G(s_i, s_j) \langle B_i, A_j \rangle.$$

Proof: As in the proof above,

$$\begin{aligned} (h, k) &= \sum_{j=0}^{n-1} \langle h(s_j), A_j \rangle \\ &= \sum_{j=0}^{n-1} \langle \sum_{i=0}^{n-1} G(s_i, s_j) B_i, A_j \rangle \\ &= \sum_{i,j=0}^{n-1} G(s_i, s_j) \langle B_i, A_j \rangle . \square \end{aligned}$$

Proposition 3.0.7 *The map $\tilde{\pi}_{\mathcal{P}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ is a Lie algebra isomorphism.*

Proof: From the proof of Proposition 3.0.3, it is clear that $\tilde{\pi}_{\mathcal{P}}$ is one-to-one. For any $g = (g_0, g_1, \ldots g_{n-1}) \in \mathfrak{g}^{\mathcal{P}}$, choose $h \in H_{\mathcal{P}}(\mathfrak{g})$ to be $h(s) \equiv \sum_{i,j=0}^{n-1} G(s, s_j) Q_{ij} g_j$. Then $\tilde{\pi}_{\mathcal{P}}(h) = g$, and thus $\tilde{\pi}_{\mathcal{P}}$ is onto. To see that $\tilde{\pi}_{\mathcal{P}}$ is a Lie algebra homomorphism, note that for $h, k \in H_{\mathcal{P}}(\mathfrak{g})$,

$$\begin{aligned}
\tilde{\pi}_{\mathcal{P}}([h,k]_{\mathcal{P}}) &= \tilde{\pi}_{\mathcal{P}}(P_{\mathcal{P}}[h,k]) \\
&= \tilde{\pi}_{\mathcal{P}}\Big(\sum_{i,j=0}^{n-1} G(\cdot, s_j) Q_{ij} [h(s_j), k(s_j)]\Big) \\
&= ([h(s_0), k(s_0)], [h(s_1), k(s_1)], \ldots, [h(s_{n-1}), k(s_{n-1})]) \\
&= [(h(s_0), \ldots, h(s_{n-1})), (k(s_0), \ldots, k(s_{n-1}))]_{\mathfrak{g}^{\mathcal{P}}} \\
&= [\tilde{\pi}_{\mathcal{P}}(h), \tilde{\pi}_{\mathcal{P}}(k)]_{\mathfrak{g}^{\mathcal{P}}}. \ \square
\end{aligned}$$

At this point it is useful to discuss the geometry of $H_{\mathcal{P}}(\mathfrak{g})$.

Definition 3.0.8 *Let ∇ be the "Levi-Civita covariant derivative" on $H(\mathfrak{g})$. Identifying $H_{\mathcal{P}}(\mathfrak{g})$ with the left invariant vector fields on $G^{\mathcal{P}}$, we will define on $H_{\mathcal{P}}(\mathfrak{g})$ a covariant derivative, $\nabla^{\mathcal{P}}$, as follows: for each $h, k \in H_{\mathcal{P}}(\mathfrak{g})$ let $\nabla_h^{\mathcal{P}} k \equiv P_{\mathcal{P}}(\nabla_h k)$, i.e., $\nabla_h^{\mathcal{P}} = P_{\mathcal{P}} \nabla_h$.*

Definition 3.0.9 *We define the Laplacian Δ on $H(\mathfrak{g})$ by*

$$\Delta \equiv \sum_{k \in S} \nabla_k \nabla_k,$$

and the Laplacian $\Delta_{\mathcal{P}}$ on $H_{\mathcal{P}}(\mathfrak{g})$ by

$$\Delta_{\mathcal{P}} \equiv \sum_{k \in S_{\mathcal{P}}} \nabla_k^{\mathcal{P}} \nabla_k^{\mathcal{P}},$$

where S is a good orthonormal basis for $H(\mathfrak{g})$ and $S_{\mathcal{P}}$ is any orthonormal basis for $H_{\mathcal{P}}(\mathfrak{g})$, respectively.

Note that for $h \in H(\mathfrak{g})$ and $l \in H_{\mathcal{P}}(\mathfrak{g})$,

$$\begin{aligned}
(P_{\mathcal{P}} h, l) &= (P_{\mathcal{P}} h + h - P_{\mathcal{P}} h, l) \\
&= (h, l).
\end{aligned}$$

We can use this to show that $\nabla^{\mathcal{P}}$ is metric compatible based on the metric compatibility of ∇: for all $h, k, l \in H_{\mathcal{P}}(\mathfrak{g})$

$$(\nabla_h^{\mathcal{P}} k, l) = (P_{\mathcal{P}} \nabla_h k, l) = (\nabla_h k, l) = (k, \nabla_h l) = (k, P_{\mathcal{P}} \nabla_h l) = (k, \nabla_h^{\mathcal{P}} l).$$

Also, since ∇ is torsion free, then so is $\nabla^{\mathcal{P}}$ since

$$\nabla^{\mathcal{P}}_h k - \nabla^{\mathcal{P}}_k h = P_{\mathcal{P}}(\nabla_h k - \nabla_k h) = P_{\mathcal{P}}[h,k] = [h,k]_{\mathcal{P}}.$$

Thus, $\nabla^{\mathcal{P}}$ deserves to be called the Levi-Civita covariant derivative on $H_{\mathcal{P}}(\mathfrak{g})$.

For the following lemmas, $(\cdot,\cdot)$ will denote the inner product on $H(R)$ defined by $(h,h) = \int_0^1 \{(h'(s))^2 + (h(s))^2\} ds$ and $S_{\mathcal{P}}(R)$ will be an arbitrary orthonormal basis of $H_{\mathcal{P}}(R)$.

Lemma 3.0.10 *For all* $(t_1,t_2) \in [0,1]^2$, $(P_{\mathcal{P}}G(\cdot,t_1), G(\cdot,t_2)) = G_{\mathcal{P}}(t_1,t_2)$, *where*

$$G_{\mathcal{P}}(t_1,t_2) \equiv \sum_{a \in S_{\mathcal{P}}(R)} a(t_1)a(t_2).$$

Proof: Let $S_{\mathcal{P}}(\perp)$ be an orthonormal basis of $(H_{\mathcal{P}}(R))^{\perp}$ and let S be the orthonormal basis of $H(R)$ defined by $S = S_{\mathcal{P}}(R) \cup S_{\mathcal{P}}(\perp)$. Then

$$\begin{aligned}
(P_{\mathcal{P}}G(\cdot,t_1), G(\cdot,t_2)) &= \sum_{a \in S} (P_{\mathcal{P}}G(\cdot,t_1), a)(G(\cdot,t_2), a) \\
&= \sum_{a \in S_{\mathcal{P}}(R)} (P_{\mathcal{P}}G(\cdot,t_1), a)(G(\cdot,t_2), a) \\
&\quad + \sum_{b \in S_{\mathcal{P}}(\perp)} (P_{\mathcal{P}}G(\cdot,t_1), b)(G(\cdot,t_2), b) \\
&= \sum_{a \in S_{\mathcal{P}}(R)} (G(\cdot,t_1), a)(G(\cdot,t_2), a) \\
&= \sum_{a \in S_{\mathcal{P}}(R)} a(t_1)a(t_2) = G_{\mathcal{P}}(t_1,t_2),
\end{aligned}$$

where we have used Lemma 2.3.6 for the last equation.

It is clear from the proof of Lemma 3.0.10 that $G_{\mathcal{P}}(t_1,t_2)$ is well-defined since the summation in the first equation of the proof is independent of choice of orthonormal basis S.

Lemma 3.0.11 *For all* $(t_1,t_2) \in [0,1]^2$, $G_{\mathcal{P}}(t_1,t_2) = (P_{\mathcal{P}}G(t_1,\cdot))(t_2)$.

Proof: Using Lemma 3.0.10 and the fact that G is a reproducing kernel,

$$G_{\mathcal{P}}(t_1,t_2) = (P_{\mathcal{P}}G(\cdot,t_1), G(\cdot,t_2)) = (P_{\mathcal{P}}G(\cdot,t_1))(t_2) = (P_{\mathcal{P}}G(t_1,\cdot))(t_2).\square$$

Lemma 3.0.12 *For $h \in L^1([0,1])$ and $t_1 \in [0,1]$,*

$$(P_{\mathcal{P}} \int_0^1 F(\cdot, t_1)h(t_1)dt_1)(t_2) = \int_0^1 -(G_{\mathcal{P}})_s(t_1, t_2)h(t_1)dt_1$$

.

Proof:

$$\begin{aligned}
(P_{\mathcal{P}} \int_0^1 F(\cdot, t_1)h(t_1)dt_1)(t_2) &= \sum_{i,j=0}^{n-1} G(t_2, s_i)Q_{ij} \int_0^1 G_s(s_j, t_1)h(t_1)dt_1 \\
&= \int_0^1 \sum_{i,j=0}^{n-1} G(t_2, s_i)Q_{ij}G_s(s_j, t_1)h(t_1)dt_1 \\
&= \int_0^1 \sum_{i,j=0}^{n-1} -G_s(t_1, s_j)Q_{ji}G(s_i, t_2)h(t_1)dt_1 \\
&= \int_0^1 -(G_{\mathcal{P}})_s(t_1, t_2)h(t_1)dt_1. \ \square
\end{aligned}$$

Lemma 3.0.13 *Let $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$ be a given partition as before. Then*

$$G(s,t) = G_{\mathcal{P}}(s,t) + \sum_{i=0}^{n-1} G_i(s,t),$$

where

$$G_i(s,t) = \begin{cases} 0 \text{ if } (s,t) \notin [s_i, s_{i+1}]^2 \\ \frac{\sinh(t-s_i)\sinh(s_{i+1}-s)}{\sinh(s_{i+1}-s_i)} \text{ if } s_i \le t < s < s_{i+1} \\ \frac{\sinh(s-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} \text{ if } s_i \le s \le t < s_{i+1} \end{cases}$$

Proof: Let $t \in [0,1]$ be fixed. Since $G(\cdot, t) \in H(R) = H_{\mathcal{P}}(R) \oplus Ker(\tilde{\pi}_{\mathcal{P}})$, and since, by Lemma 3.0.11, $(P_{\mathcal{P}}G(\cdot,t))(s) = G_{\mathcal{P}}(s,t)$, then we can write

$$G(s,t) = G_{\mathcal{P}}(s,t) + V(s,t) \quad \forall\, s \in [0,1]$$

where $V(\cdot,t) \in Ker(\tilde{\pi}_{\mathcal{P}})$ (i.e., $V(s_i,t) = 0 \ \forall\, s_i \in \mathcal{P}$). Assume $t \in [s_i, s_{i+1}]$ for some i. Define

$$G_i(s,t) \equiv G(s,t) - G_{\mathcal{P}}(s,t) \ \ \forall\, s \in [s_i, s_{i+1}].$$

Since $G_{\mathcal{P}}(s,t) = (P_{\mathcal{P}}G(\cdot,t))(s) = \sum_{j,l=0}^{n-1} G(s,s_j)Q_{jl}G(s_l,t)$, then $G_{\mathcal{P}}(s_i,t) = G(s_i,t)$ and $G_{\mathcal{P}}(s_{i+1},t) = G(s_{i+1},t)$, so $G_i(s_i,t) = G_i(s_{i+1},t) = 0$.

Let L be the differential operator defined by $L \equiv -\frac{d^2}{ds^2} + 1$. Since we know that $L(G(\cdot, s_j))(s) = 0$ for $s \neq s_j$, then $L(G_i(\cdot, t))(s) = 0$ for $s_i < s < t$ and $t < s < s_{i+1}$. Since $G_s(t^+, t) - G_s(t^-, t) = -1$ and $G_s(s, s_j)$ is continuous on (s_i, s_{i+1}) for all j, then $(G_i)_s(t^+, t) - (G_i)_s(t^-, t) = -1$. Also, since G is continuous, then G_i is continuous.

Putting all of the above facts together, we see that G_i must be the Green's function on $[s_i, s_{i+1}]$ satisfying the following conditions:

1. $L(G_i(\cdot, t))(s) = 0$ for $s \in (s_i, t) \cup (t, s_{i+1})$
2. $G_i(s_i, t) = G_i(s_{i+1}, t) = 0$
3. $(G_i)_s(t^+, t) - (G_i)_s(t^-, t) = -1$
4. $G_i(\cdot, t)$ is continuous on $[s_i, s_{i+1}]$.

Using the Green's function method of solving linear differential equations with linear boundary conditions (see p. 225 in Zwillinger [21]]), we use (1) above to determine that G_i must be of the form

$$G_i(s,t) = \begin{cases} Ae^s + Be^{-s} \text{ for } s_i \leq t < s \\ Ee^s + Fe^{-s} \text{ for } s \leq t \leq s_{i+1}. \end{cases}$$

From (1)-(4) above, we get the following system of equations:

$$\begin{aligned} Ee^{s_i} + Fe^{-s_i} &= 0 \\ Ae^{s_{i+1}} + Be^{-s_{i+1}} &= 0 \\ Ae^t - Be^{-t} - Ee^t + Fe^{-t} &= -1 \\ Ae^t + Be^{-t} &= Ee^t + Fe^{-t}. \end{aligned}$$

Solving this system of linear equations for A, B, E and F will give

$$\begin{aligned} A &= -\frac{1}{2}e^{-t} + \frac{e^{-s_i}\sinh(s_{i+1} - t)}{2\sinh(s_{i+1} - s_i)} \\ B &= \frac{1}{2}e^{t} - \frac{e^{s_i}\sinh(s_{i+1} - t)}{2\sinh(s_{i+1} - s_i)} \\ E &= \frac{e^{-s_i}\sinh(s_{i+1} - t)}{2\sinh(s_{i+1} - s_i)} \\ F &= -\frac{e^{-s_i}\sinh(s_{i+1} - t)}{2\sinh(s_{i+1} - s_i)}. \end{aligned}$$

Substituting these terms in our formula for $G_i(s,t)$ then gives

$$G_i(s,t) = \begin{cases} \left\{ \begin{array}{c} -\frac{1}{2}e^{s-t} + \frac{e^{s-s_i}\sinh(s_{i+1}-t)}{2\sinh(s_{i+1}-s_i)} \\ +\frac{1}{2}e^{-s+t} - \frac{e^{-s+s_i}\sinh(s_{i+1}-t)}{2\sinh(s_{i+1}-s_i)} \end{array} \right\} & \text{for } s_i \le t < s \le s_{i+1} \\ \frac{e^{s-s_i}\sinh(s_{i+1}-t)}{2\sinh(s_{i+1}-s_i)} - \frac{e^{-s+s_i}\sinh(s_{i+1}-t)}{2\sinh(s_{i+1}-s_i)} & \text{for } s_i \le s \le t \le s_{i+1} \end{cases}$$

$$= \begin{cases} \sinh(t-s) + \frac{\sinh(s-s_i)}{\sinh(s_{i+1}-s_i)}\sinh(s_{i+1}-t) & \text{for } s_i \le t < s \le s_{i+1} \\ \frac{\sinh(s-s_i)}{\sinh(s_{i+1}-s_i)}\sinh(s_{i+1}-t) & \text{for } s_i \le s \le t \le s_{i+1} \end{cases}.$$

For the part of $G_i(s,t)$ where $s_i \le t < s \le s_{i+1}$, we use the identity $\sinh(a)\sinh(b) = \frac{1}{2}(\cosh(a+b) - \cosh(a-b))$ twice as follows:

$$\begin{aligned} & \sinh(t-s) + \frac{\sinh(s-s_i)}{\sinh(s_{i+1}-s_i)}\sinh(s_{i+1}-t) \\ = & \frac{\sinh(t-s)\sinh(s_{i+1}-s_i) + \sinh(s-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} \\ = & \frac{\frac{1}{2}\left\{ \begin{array}{c} \cosh(t-s+s_{i+1}-s_i) - \cosh(t-s-s_{i+1}+s_i) \\ +\cosh(s-s_i+s_{i+1}-t) - \cosh(s-s_i-s_{i+1}+t) \end{array} \right\}}{\sinh(s_{i+1}-s_i)} \\ = & \frac{\frac{1}{2}\{\cosh(t-s_i+s_{i+1}-s) - \cosh(t-s_i-(s_{i+1}-s))\}}{\sinh(s_{i+1}-s_i)} \\ = & \frac{\sinh(t-s_i)\sinh(s_{i+1}-s)}{\sinh(s_{i+1}-s_i)}. \end{aligned}$$

Hence, $G_i(s,t)$ may now be written as

$$G_i(s,t) = \begin{cases} \frac{\sinh(t-s_i)\sinh(s_{i+1}-s)}{\sinh(s_{i+1}-s_i)} & \text{for } s_i \le t < s \le s_{i+1} \\ \frac{\sinh(s-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} & \text{for } s_i \le s \le t \le s_{i+1} \end{cases}.$$

The following verifies that conditions (1)-(4) are satisfied by G_i.

1. Since $L(\sinh(s_{i+1}-\cdot))(s) = -\sinh(s_{i+1}-s)+\sinh(s_{i+1}-s) = 0$ and $L(\sinh(\cdot-s_i))(s) = -\sinh(s-s_i)+\sinh(s-s_i) = 0$, then clearly $L(G_i(\cdot,t))(s) = 0$ for $s \in (s_i,t)\cup(t,s_{i+1})$.

2.

$$\begin{aligned} G_i(s_i,t) &= \frac{\sinh(s_i-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} = 0 \\ G_i(s_{i+1},t) &= \frac{\sinh(t-s_i)\sinh(s_{i+1}-s_{i+1})}{\sinh(s_{i+1}-s_i)} = 0 \end{aligned}$$

3.

$$\begin{aligned}
(G_i)_s(t^+,t) &= -\frac{\sinh(t-s_i)\cosh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} \\
&= -\frac{\frac{1}{2}\{\sinh(s_{i+1}-s_i)+\sinh(2t-(s_{i+1}+s_i))\}}{\sinh(s_{i+1}-s_i)} \\
(G_i)_s(t^-,t) &= \frac{\cosh(t-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} \\
&= \frac{\frac{1}{2}\{\sinh(s_{i+1}-s_i)+\sinh(-2t+(s_{i+1}+s_i))\}}{\sinh(s_{i+1}-s_i)}.
\end{aligned}$$

Thus, $(G_i)_s(t^+,t)-(G_i)_s(t^-,t) = -\frac{1}{2}-\frac{1}{2} = -1$.

4. Since sinh is continuous, we just need to check the continuity of $G_i(s,t)$ at $s = t$. Indeed,

$$G_i(t^-,t) = \frac{\sinh(t-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} = G_i(t^+,t).$$

We have thus shown that for $(s,t) \in [s_i, s_{i+1}]$, $V(s,t) = G(s,t) - G_{\mathcal{P}}(s,t) = G_i(s,t)$, where $G_i(s,t)$ is defined as above. If $t \in [s_i, s_{i+1}]$ for some i, then on each interval $[s_j, s_{j+1}]$ where $i \neq j$, it is easy to see that $V(\cdot,t)$ must be of the form $V(s,t) = Ae^s + Be^{-s}$ with $V(s_j,t) = V(s_{j+1},t) = 0$. Thus $A = B = 0$, and so $V(s,t) = 0$ for $s \in [s_j, s_{j+1}]$. We can then conclude that

$$V(s,t) = \begin{cases} 0 \text{ for } s \notin [s_i, s_{i+1}] \\ G_i(s,t) \text{ for } s \in [s_i, s_{i+1}] \end{cases}.$$

Extending $G_i(s,t)$ to be zero outside of $[s_i, s_{i+1}]$, we can then write $V(s,t) = \sum_{i=0}^{n-1} G_i(s,t)$ $\forall\, s,t \in [0,1]$ and thus $G(s,t) = G_{\mathcal{P}}(s,t) + \sum_{i=0}^{n-1} G_i(s,t)$. $\square$

One clearly sees by the formula for each $G_i(s,t)$ that G_i is symmetric since G and $G_{\mathcal{P}}$ are both symmetric. Using our convention that $(G_i)_s$ is the partial derivative of G_i with respect to its first component and $(G_i)_t$ is the partial derivative of G_i with respect to its second component we have the following for $(x,y) \in [s_i, s_{i+1}]^2$:

$$\begin{aligned}
(G_i)_s(x,y) &= \begin{cases} -\frac{\sinh(y-s_i)\cosh(s_{i+1}-x)}{\sinh(s_{i+1}-s_i)} \text{ for } s_i \le y < x \le s_{i+1} \\ \frac{\cosh(x-s_i)\sinh(s_{i+1}-y)}{\sinh(s_{i+1}-s_i)} \text{ for } s_i \le x \le y \le s_{i+1} \end{cases} \\
&= \begin{cases} \frac{\sinh(s_i-y)\cosh(s_{i+1}-x)}{\sinh(s_{i+1}-s_i)} \text{ for } s_i \le y < x \le s_{i+1} \\ \frac{\cosh(x-s_i)\sinh(s_{i+1}-y)}{\sinh(s_{i+1}-s_i)} \text{ for } s_i \le x \le y \le s_{i+1} \end{cases}.
\end{aligned}$$

Using the identity $\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$, we then get

$$(G_i)_s(x,y) = \begin{cases} \frac{\sinh(-(y+x)+s_{i+1}+s_i) - \cosh(s_i - y)\sinh(s_{i+1}-x)}{\sinh(s_{i+1}-s_i)} & \text{for } s_i \le y < x \le s_{i+1} \\ \frac{\sinh(-(y+x)+s_{i+1}+s_i) - \sinh(s_i - x)\cosh(s_{i+1}-y)}{\sinh(s_{i+1}-s_i)} & \text{for } s_i \le x \le y \le s_{i+1} \end{cases}.$$

Noting that

$$(G_i)_t(x,y) = \begin{cases} \frac{\cosh(y-s_i)\sinh(s_{i+1}-x)}{\sinh(s_{i+1}-s_i)} & \text{for } s_i \le y < x \le s_{i+1} \\ \frac{\sinh(s_i-x)\cosh(s_{i+1}-y)}{\sinh(s_{i+1}-s_i)} & \text{for } s_i \le x \le y \le s_{i+1} \end{cases}$$

we then have the following corollary.

Corollary 3.0.14

$$\begin{aligned}(G_i)_s(x,y) &= \begin{cases} -(G_i)_t(x,y) + \frac{\sinh(-(y+x)+s_{i+1}+s_i)}{\sinh(s_{i+1}-s_i)} \; \textit{for } (x,y) \in [s_i, s_{i+1}]^2 \\ 0 \; \textit{for } (x,y) \notin [s_i, s_{i+1}]^2 \end{cases} \\ &= \begin{cases} -(G_i)_s(y,x) + \frac{\sinh(-(y+x)+s_{i+1}+s_i)}{\sinh(s_{i+1}-s_i)} \; \textit{for } (x,y) \in [s_i, s_{i+1}]^2 \\ 0 \; \textit{for } (x,y) \notin [s_i, s_{i+1}]^2 \end{cases}.\end{aligned}$$

3.1 Convergence of Finite Dimensional Laplacian

Theorem 3.1.1

$$\|P_{\mathcal{P}}(\;\Delta - \Delta_{\mathcal{P}})P_{\mathcal{P}}\|_{op} \to 0 \;\; as \;\; |\mathcal{P}| \to 0.$$

Proof: Our goal is to find, for each partition $\mathcal{P}$, an $\epsilon(\mathcal{P})$ such that $\epsilon(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and, for each $h, J \in H_{\mathcal{P}}(\mathfrak{g})$,

$$|((\;\Delta - \Delta_{\mathcal{P}})h, J)| \le \epsilon(\mathcal{P}) \, \|h\| \, \|J\| \, . \tag{3.3}$$

Due to the length of the calculations that will unfold for the proof of this theorem, we will use several lemmas to focus on separate estimations which combined will give the inequality in (3.3). Since we are really comparing $(\Delta h, J)$ with $(\Delta_{\mathcal{P}} h, J)$, we start by expanding these two terms in the following two lemmas.

Lemma 3.1.2 *For $h, J \in H(\mathfrak{g})$,*

$$\begin{aligned}(\Delta h, J) = &- \int_0^1 a\cosh(\frac{1}{2})K\langle h'(t), J'(t)\rangle dt \\ &+ \frac{1}{2}\int_0^1\int_0^1 G(x,t)^2 K\langle h'(x), J'(t)\rangle dxdt \\ &- \frac{1}{4}\int_0^1\int_0^1 G(x,t)^2 K\langle h(x), J(t)\rangle dxdt.\end{aligned} \tag{3.4}$$

Proof: Letting S be an orthonormal basis of $H(\mathfrak{g})$ of the form $S = h_0\mathfrak{g}_0$ where h_0 is an orthonormal basis of $H(R)$ and $\mathfrak{g}_0$ is an orthonormal basis of $\mathfrak{g}$, we have:

$$
\begin{aligned}
& (\Delta h, J) \\
= & \sum_{k\in S} -(\nabla_k h, \nabla_k J) \\
= & -\sum_{k\in S}\int_0^1 \{\langle(\nabla_k h)'(x), (\nabla_k J)'(x)\rangle + \langle(\nabla_k h)(x), (\nabla_k J)(x)\rangle\}dx \\
= & -\sum_{k\in S}\int_0^1 \{\langle [k,h'](x) - \int_0^1 G(x,t)[k,h']dt + \frac{1}{2}\int_0^1 F(x,t)[k,h]dt, \\
& \qquad [k,J'](x) - \int_0^1 G(x,t)[k,J']dt + \frac{1}{2}\int_0^1 F(x,t)[k,J]dt\rangle \\
& +\langle -\int_0^1 F(x,t)[k,h']dt + \frac{1}{2}\int_0^1 G(x,t)[k,h]dt, \\
& -\int_0^1 F(x,t)[k,J']dt + \frac{1}{2}\int_0^1 G(x,t)[k,J]dt\rangle\}dx \\
= & -\sum_{a\in h_0}\sum_{B\in\mathfrak{g}_0}\{\int_0^1 \langle [a(x)B, h'(x)], [a(x)B, J'(x)]\rangle dx \\
& +\int_0^1 \langle -\int_0^1 G(x,t)[a(t)B, h'(t)]dt + \frac{1}{2}\int_0^1 F(x,t)[a(t)B, h(t)]dt, [a(x)B, J'(x)]\rangle dx \\
& +\int_0^1 \langle [a(x)B, h'(x)], -\int_0^1 G(x,t)[a(t)B, J'(t)]dt + \frac{1}{2}\int_0^1 F(x,t)[a(t)B, J(t)]dt\rangle dx \\
& +\int_0^1 \langle -\int_0^1 G(x,t)[a(t)B, h'(t)]dt + \frac{1}{2}\int_0^1 F(x,t)[a(t)B, h(t)]dt, \\
& -\int_0^1 G(x,t)[a(t)B, J'(t)]dt + \frac{1}{2}\int_0^1 F(x,t)[a(t)B, J(t)]dt\rangle dx \\
& +\int_0^1 \langle -\int_0^1 F(x,t)[a(t)B, h'(t)]dt + \frac{1}{2}\int_0^1 G(x,t)[a(t)B, h(t)]dt, \\
& -\int_0^1 F(x,t)[a(t)B, J'(t)]dt + \frac{1}{2}\int_0^1 G(x,t)[a(t)B, J(t)]dt\rangle dx\} \\
= & -\int_0^1 (\sum_{a\in h_0} a(x)(x)) \sum_{B\in\mathfrak{g}_0}\langle [B, h'(x)], [B, J'(x)]\rangle dx \\
& -\int_0^1\int_0^1 (\sum_{a\in h_0} a(t)(x))\{-G(x,t)\sum_{B\in\mathfrak{g}_0}\langle [B, h'(t)], [B, J'(x)]\rangle \\
& \qquad +\frac{1}{2}F(x,t)\sum_{B\in\mathfrak{g}_0}\langle [B, h(t)], [B, J'(x)]\rangle\}dtdx
\end{aligned}
$$

$$
\begin{aligned}
& -\int_0^1\int_0^1 (\sum_{a\in h_0} a(x)(t))\{-G(x,t)\sum_{B\in\mathfrak{g}_0}\langle [B,h'(t)],[B,J'(x)]\rangle \\
& \qquad +\frac{1}{2}F(x,t)\sum_{B\in\mathfrak{g}_0}\langle [B,h'(t)],[B,J(x)]\rangle\} dtdx \\
& -\int_0^1\int_0^1\int_0^1 (\sum_{a\in h_0} a(t_1)(t_2))\{G(x,t_1)G(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h'(t_1)],[B,J'(t_2)]\rangle \\
& -\frac{1}{2}G(x,t_1)F(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h'(t_1)],[B,J(t_2)]\rangle \\
& -\frac{1}{2}F(x,t_1)G(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h(t_1)],[B,J'(t_2)]\rangle \\
& +\frac{1}{4}F(x,t_1)F(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h(t_1)],[B,J(t_2)]\rangle \\
& +F(x,t_1)F(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h'(t_1)],[B,J'(t_2)]\rangle \\
& -\frac{1}{2}G(x,t_1)F(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h(t_1)],[B,J'(t_2)]\rangle \\
& -\frac{1}{2}F(x,t_1)G(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h'(t_1)],[B,J(t_2)]\rangle \\
& +\frac{1}{4}G(x,t_1)G(x,t_2)\sum_{B\in\mathfrak{g}_0}\langle [B,h(t_1)],[B,J(t_2)]\rangle\} dt_1dt_2dx \\
= \; & -\int_0^1 (G(x,x)K\langle h'(x),J'(x)\rangle dx \\
& +\int_0^1\int_0^1 G(t,x)\{G(x,t)K\langle h'(t),J'(x)\rangle - \frac{1}{2}F(x,t)K\langle h(t),J'(x)\rangle\} dtdx \\
& +\int_0^1\int_0^1 G(t,x)\{G(x,t)K\langle h'(t),J'(x)\rangle - \frac{1}{2}F(x,t)K\langle h'(t),J(x)\rangle\} dtdx \\
& -\int_0^1\int_0^1\int_0^1 G(t_1,t_2)\{G(x,t_1)G(x,t_2)K\langle h'(t_1),J'(t_2)\rangle \\
& -\frac{1}{2}G(x,t_1)F(x,t_2)K\langle h'(t_1),J(t_2)\rangle \\
& -\frac{1}{2}F(x,t_1)G(x,t_2)K\langle h(t_1),J'(t_2)\rangle + \frac{1}{4}F(x,t_1)F(x,t_2)K\langle h(t_1),J(t_2)\rangle \\
& +F(x,t_1)F(x,t_2)K\langle h'(t_1),J'(t_2)\rangle - \frac{1}{2}G(x,t_1)F(x,t_2)K\langle h'(t_1),J(t_2)\rangle \\
& -\frac{1}{2}F(x,t_1)G(x,t_2)K\langle h(t_1),J'(t_2)\rangle + \frac{1}{4}G(x,t_1)G(x,t_2)K\langle h(t_1),J(t_2)\rangle\} dt_1dt_2dx
\end{aligned}
$$

$$
\begin{aligned}
= \; & -\int_0^1 (G(x,x)K\langle h'(x), J'(x)\rangle dx + 2\int_0^1\int_0^1 G(x,t)^2 K\langle h'(x), J'(t)\rangle dtdx \\
& +\frac{1}{4}\int_0^1\int_0^1 \frac{d}{dt}(G(x,t))^2 K\langle h(x), J'(t)\rangle dtdx \\
& +\frac{1}{4}\int_0^1\int_0^1 \frac{d}{dt}(G(x,t))^2 K\langle h'(x), J(t)\rangle dtdx \\
& -\int_0^1\int_0^1 G(t_1,t_2)\{\int_0^1 G(x,t_1)G(x,t_2)dx \cdot K\langle h'(t_1), J'(t_2)\rangle \\
& -\frac{1}{4}\int_0^1 G_{xx}(x,t_1)G(x,t_2)dx \cdot K\langle h(t_1), J(t_2)\rangle \\
& -\int_0^1 G_{xx}(x,t_1)G(x,t_2)dx \cdot K\langle h'(t_1), J'(t_2)\rangle \\
& -\frac{1}{4}\int_0^1 G(x,t_1)G(x,t_2)dx \cdot K\langle h(t_1), J(t_2)\rangle\}dt_1dt_2 \\
= \; & -\int_0^1 (G(x,x)K\langle h'(x), J'(x)\rangle dx + 2\int_0^1\int_0^1 G(x,t)^2 K\langle h'(x), J'(t)\rangle dtdx \\
& -\frac{1}{2}\int_0^1\int_0^1 G(x,t)^2 K\langle h'(t), J'(x)\rangle dtdx \\
& -\int_0^1\int_0^1 G(t_1,t_2)\{G(t_1,t_2)K\langle h'(t_1), J'(t_2)\rangle + \frac{1}{4}G(t_1,t_2)K\langle h(t_1), J(t_2)\rangle\}dt_1dt_2.
\end{aligned}
$$

Renaming variables and noting that $G(x,x) = a\cosh(\frac{1}{2})$, we see that (3.4) holds.□

Lemma 3.1.3 *For a given partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$ and $h, J \in H_{\mathcal{P}}(\mathfrak{g})$,*

$$
\begin{aligned}
& (\Delta_{\mathcal{P}} h, J) \\
= \; & -\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i \cosh(s_i - t_1 + s_{i+1} - t_2) G_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1), J'(t_2)\rangle dt_1dt_2 \\
& +\frac{1}{2}\int_0^1\int_0^1 G_{\mathcal{P}}^2(t_1,t_2)K\langle h'(t_1), J'(t_2)\rangle dt_1dt_2 \\
& -\frac{1}{4}\int_0^1\int_0^1 G_{\mathcal{P}}^2(t_1,t_2)K\langle h(t_1), J(t_2)\rangle dt_1dt_2 \\
& -\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i \sinh(-(t_2+t_1) + s_{i+1} + s_i) \\
& \qquad .G_{\mathcal{P}}(t_1,t_2)(K\langle h'(t_1), J(t_2)\rangle + K\langle h(t_1), J'(t_2)\rangle)dt_1dt_2. \qquad (3.5)
\end{aligned}
$$

Proof: Let $S_{\mathcal{P}}$ be an orthonormal basis of $H_{\mathcal{P}}(\mathfrak{g})$ of the form $S = h_P\mathfrak{g}_0$ where

h_P is an orthonormal basis of $H_{\mathcal{P}}(R)$ and $\mathfrak{g}_0$ is an orthonormal basis of $\mathfrak{g}$, Then

$$\begin{aligned}
(\Delta_{\mathcal{P}} h, J) &= -\sum_{k\in S_{\mathcal{P}}} (\nabla_k^{\mathcal{P}} h, \nabla_k^{\mathcal{P}} J) \qquad (3.6)\\
&= -\sum_{k\in S_{\mathcal{P}}} (-P_{\mathcal{P}}(\int_0^1 F(\cdot,t_1)[k,h'](t_1)dt_1) + \frac{1}{2}P_{\mathcal{P}}(\int_0^1 G(\cdot,t_1)[k,h](t_1)dt_1),\\
&\quad -P_{\mathcal{P}}(\int_0^1 F(\cdot,t_2)[k,J'](t_2)dt_2) + \frac{1}{2}P_{\mathcal{P}}(\int_0^1 G(\cdot,t_2)[k,J](t_2)dt_2))\\
&= -\sum_{k\in S_{\mathcal{P}}} \{(\int_0^1 (G_{\mathcal{P}})_s(t_1,\cdot)[k,h'](t_1)dt_1, \int_0^1 (G_{\mathcal{P}})_s(t_2,\cdot)[k,J'](t_2)dt_2)\\
&\quad +\frac{1}{2}(\int_0^1 (G_{\mathcal{P}})_s(t_1,\cdot)[k,h'](t_1)dt_1, \int_0^1 G_{\mathcal{P}}(\cdot,t_2)[k,J](t_2)dt_2)\\
&\quad +\frac{1}{2}(\int_0^1 G_{\mathcal{P}}(\cdot,t_1)[k,h](t_1)dt_1, \int_0^1 (G_{\mathcal{P}})_s(t_2,\cdot)[k,J'](t_2)dt_2)\\
&\quad +\frac{1}{4}(\int_0^1 G_{\mathcal{P}}(\cdot,t_1)[k,h](t_1)dt_1, \int_0^1 G_{\mathcal{P}}(\cdot,t_2)[k,J](t_2)dt_2)\}, \qquad (3.7)
\end{aligned}$$

where we have used Lemmas (3.0.11) and (3.0.12) for the last equation. We will write the last summation in (3.7) as

$$-\sum_{k\in S_{\mathcal{P}}} \{A_k + \frac{1}{2}B_k + \frac{1}{2}C_k + \frac{1}{4}D_k\} \qquad (3.8)$$

and calculate each of the inner products, A_k, B_k, C_k, and D_k, individually in what follows.

First, using Lemma 3.0.10, we have

$$\begin{aligned}
&\sum_{k\in S_{\mathcal{P}}} A_k\\
&= \sum_{k\in S_{\mathcal{P}}} (\int_0^1 (\sum_{l\in h_{\mathcal{P}}} l'(t_1)l(\cdot))[k,h'](t_1)dt_1, \int_0^1 (\sum_{\tilde{l}\in h_{\mathcal{P}}} \tilde{l}'(t_2)\tilde{l}(\cdot))[k,J'](t_2)dt_2 \qquad (3.9)\\
&= \sum_{k\in S_{\mathcal{P}}} \{\int_0^1 \langle \frac{d}{dx}\int_0^1 (\sum_{l\in h_{\mathcal{P}}} l'(t_1)l(x))[k,h'](t_1)dt_1, \frac{d}{dx}\int_0^1 (\sum_{\tilde{l}\in h_{\mathcal{P}}}{}' \tilde{l}'(t_2)\tilde{l}(x))[k,J'](t_2)dt_2\rangle dx\\
&\quad + \int_0^1 \langle \int_0^1 (\sum_{l\in h_{\mathcal{P}}} l'(t_1)l(x))[k,h'](t_1)dt_1, \int_0^1 (\sum_{\tilde{l}\in h_{\mathcal{P}}}{}' \tilde{l}'(t_2)\tilde{l}(x))[k,J'](t_2)dt_2\rangle dx\}\\
&= \sum_{k\in S_{\mathcal{P}}} \sum_{l,\tilde{l}\in h_{\mathcal{P}}} \{\int_0^1 \langle \int_0^1 l'(t_1)l'(x)[k,h'](t_1)dt_1, \int_0^1 \tilde{l}'(t_2)\tilde{l}(x)[k,J'](t_2)dt_2\rangle dx\\
&\quad + \int_0^1 \langle \int_0^1 l'(t_1)l(x)[k,h'](t_1)dt_1, \int_0^1 \tilde{l}'(t_2)\tilde{l}(x)[k,J'](t_2)dt_2\rangle dx\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in S_{\mathcal{P}}}\sum_{l,\tilde{l}\in h_{\mathcal{P}}}\int_0^1\{l'(x)\tilde{l}'(x)+l(x)\tilde{l}(x)\}dx\cdot\langle\int_0^1 l'(t_1)[k,h'](t_1)dt_1,\int_0^1\tilde{l}'(t_2)[k,J'[(t_2]dt_2\rangle \\
&= \sum_{k\in S_{\mathcal{P}}}\sum_{l,\tilde{l}\in h_{\mathcal{P}}}(l(\cdot),\tilde{l}(\cdot))\cdot\langle\int_0^1 l'(t_1)[k,h'](t_1)dt_1,\int_0^1\tilde{l}'(t_2)[k,J'[(t_2]dt_2\rangle \\
&= \int_0^1\int_0^1(\sum_{l\in h_{\mathcal{P}}}l'(t_1)l'(t_2))\sum_{a\in h_{\mathcal{P}}}\sum_{B\in\mathfrak{g}_0}\langle[a(t_1)B,h'(t_1)],[a(t_2)B,J'(t_2)]\rangle dt_1dt_2 \\
&= \int_0^1\int_0^1(G_{\mathcal{P}})_{st}(t_1,t_2)G_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \qquad (3.10)
\end{aligned}
$$

where we have used the fact that $h_{\mathcal{P}}$ is an orthonormal basis of $H_{\mathcal{P}}(R)$ to get the second to last equation above.

At this point we make note that $(G_{\mathcal{P}})_{st}(t_1,t_2)$ can be expressed as follows:

$$
\begin{aligned}
&(G_{\mathcal{P}})_{st}(t_1,t_2) \\
&= \begin{cases}
\left\{\begin{array}{c} -a\cosh(t_1-t_2+\frac{1}{2}) \\ +\sum_{i=0}^{n-1}a_i\cosh(s_i-t_1)\cosh(s_{i+1}-t_2)1_{[s_i,s_{i+1}]^2}(t_1,t_2)\end{array}\right\} & \text{if } t_1\le t_2 \\
\left\{\begin{array}{c} -a\cosh(t_1-t_2-\frac{1}{2}) \\ +\sum_{i=0}^{n-1}a_i\cosh(s_i-t_2)\cosh(s_{i+1}-t_1)1_{[s_i,s_{i+1}]^2}(t_1,t_2)\end{array}\right\} & \text{if } t_2< t_1
\end{cases} \\
&= \begin{cases}
\left\{\begin{array}{c} -a\cosh(t_1-t_2+\frac{1}{2})+\sum_{i=0}^{n-1}\{-a_i\sinh(s_i-t_1)\sinh(s_{i+1}-t_2) \\ +a_i\cosh(s_i-t_1+s_{i+1}-t_2)\}1_{[s_i,s_{i+1}]^2}(t_1,t_2)\end{array}\right\} & \text{if } t_1\le t_2 \\
\left\{\begin{array}{c} -a\cosh(t_1-t_2-\frac{1}{2})+\sum_{i=0}^{n-1}\{-a_i\sinh(s_i-t_2)\sinh(s_{i+1}-t_1) \\ +a_i\cosh(s_i-t_2+s_{i+1}-t_1)\}1_{[s_i,s_{i+1}]^2}(t_1,t_2)\end{array}\right\} & \text{if } t_2< t_1
\end{cases} \\
&= -G_{\mathcal{P}}(t_1,t_2)+\sum_{i=0}^{n-1}a_i\cosh(s_I-t_1+s_{i+1}-t_2)1_{[s_i,s_{i+1}]^2}(t_1,t_2)
\end{aligned}
$$

We can then conclude that

$$
\begin{aligned}
&\sum_{k\in S_{\mathcal{P}}}A_k \\
&= \int_0^1\int_0^1 -G_{\mathcal{P}}^2(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \\
&\quad+\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}}a_i\cosh(s_I-t_1+s_{i+1}-t_2)G_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2.
\end{aligned}
$$

For the second term in (3.8), the same reasoning used in (3.10) gives

$$\sum_{k\in S_{\mathcal{P}}} B_k = \int_0^1 \int_0^1 (G_{\mathcal{P}})_s(t_1,t_2) G_{\mathcal{P}}(t_1,t_2) K\langle h'(t_1), J(t_2)\rangle dt_1 dt_2.$$

But by Corollary 3.0.14,

$$\begin{aligned}
(G_{\mathcal{P}})_s(t_1,t_2) &= G_s(t_1,t_2) - \sum_{i=0}^{n-1}(G_i)_s(t_1,t_2) \\
&= -G_t(t_1,t_2) \\
&\quad + \sum_{i=0}^{n-1}\{(G_i)_t(t_1,t_2) - a_i \sinh(-(t_1+t_2)+s_{i+1}+s_i)\}1_{[s_i,s_{i+1}]^2}(t_1,t_2).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{k\in S_{\mathcal{P}}} B_k \\
= &\int_0^1 \int_0^1 -(G_{\mathcal{P}})_t(t_1,t_2) G_{\mathcal{P}}(t_1,t_2) K\langle h'(t_1), J(t_2)\rangle dt_1 dt_2 \\
&- \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \sinh(-(t_1+t_2)+s_{i+1}+s_i) G_{\mathcal{P}}(t_1,t_2) K\langle h'(t_1), J(t_2)\rangle dt_1 dt_2 \\
= &-\frac{1}{2}\int_0^1 \int_0^1 \frac{d}{dt_2}(G_{\mathcal{P}}(t_1,t_2))^2 K\langle h'(t_1), J(t_2)\rangle dt_1 dt_2 \\
&- \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \sinh(-(t_1+t_2)+s_{i+1}+s_i) G_{\mathcal{P}}(t_1,t_2) K\langle h'(t_1), J(t_2)\rangle dt_1 dt_2 \\
= &\frac{1}{2}\int_0^1 \int_0^1 G_{\mathcal{P}}^2(t_1,t_2) K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 \\
&- \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \sinh(-(t_1+t_2)+s_{i+1}+s_i) G_{\mathcal{P}}(t_1,t_2) K\langle h'(t_1), J(t_2)\rangle dt_1 dt_2,
\end{aligned}$$

where we have used integration by parts for the last equation.

For the third term in (3.8), the same reasoning used above for the second term gives

$$\begin{aligned}
&\sum_{k\in S_{\mathcal{P}}} C_k \\
= &\frac{1}{2}\int_0^1 \int_0^1 G_{\mathcal{P}}^2(t_1,t_2) K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 \\
&- \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \sinh(-(t_1+t_2)+s_{i+1}+s_i) G_{\mathcal{P}}(t_1,t_2) K\langle h(t_1), J'(t_2)\rangle dt_1 dt_2
\end{aligned}$$

Finally, the forth term in (3.8) can be calculated as follows:

$$\begin{aligned}
&\sum_{k\in S_{\mathcal{P}}} D_k \\
= &\sum_{k\in S_{\mathcal{P}}}\{\int_0^1\langle\int_0^1 (G_{\ \mathcal{P}})_s(x,t_1)[k,h](t_1)dt_1, \int_0^1 (G_{\mathcal{P}})_s(x,t_2)[k,J](t_2)dt_2\rangle dx \\
&+\int_0^1\langle\int_0^1 G_{\mathcal{P}}(x,t_1)[k,h](t_1)dt_1, \int_0^1 G_{\mathcal{P}}(x,t_2)[k,J](t_2)dt_2\rangle dx \\
= &\int_0^1\{\int_0^1 (G_{\mathcal{P}})_s(x,t_1)(G_{\mathcal{P}})_s(x,t_2)dx \\
&+\int_0^1 G_{\mathcal{P}}(x,t_1)G_{\mathcal{P}}(x,t_2)dx\}G_{\mathcal{P}}(t_1,t_2)K\langle h(t_1),J(t_2)\rangle dt_1dt_2 \\
= &\int_0^1\int_0^1 (G_{\mathcal{P}}(\cdot,t_1),G_{\mathcal{P}}(\cdot,t_2))G_{\mathcal{P}}(t_1,t_2)K\langle h(t_1),J(t_2)\rangle dt_1dt_2 \\
= &\int_0^1\int_0^1 G_{\mathcal{P}}(t_1,t_2)^2K\langle h(t_1),J(t_2)\rangle dt_1dt_2,
\end{aligned}$$

where we have used Lemmas (3.0.10) and (3.0.11) for the last equation.

Having calculated the four terms in (3.8), we can now conclude that

$$\begin{aligned}
&(\Delta_{\mathcal{P}}h,J) \\
= &-\sum_{k\in S_{\mathcal{P}}}\{A_k+\frac{1}{2}B_k+\frac{1}{2}C_k+\frac{1}{4}D_k\} \\
= &\int_0^1\int_0^1 G^2_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \\
&-\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i\cosh(s_I-t_1+s_{i+1}-t_2)G_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \\
&-\frac{1}{4}\int_0^1\int_0^1 G^2_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \\
&-\frac{1}{2}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i\sinh(-(t_1+t_2)+s_{i+1}+s_i)G_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J(t_2)\rangle dt_1dt_2 \\
&-\frac{1}{4}\int_0^1\int_0^1 G^2_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \\
&-\frac{1}{2}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i\sinh(-(t_1+t_2)+s_{i+1}+s_i)G_{\mathcal{P}}(t_1,t_2)K\langle h'(t_1),J(t_2)\rangle dt_1dt_2 \\
&-\frac{1}{4}\int_0^1\int_0^1 G_{\mathcal{P}}(t_1,t_2)^2K\langle h(t_1),J(t_2)\rangle dt_1dt_2,
\end{aligned}$$

which gives (3.5).□

Lemma 3.1.4 *$G_{\mathcal{P}}$ converges to G uniformly on $[0,1]^2$ as $|\mathcal{P}|\to 0$.*

Proof: Recall that $G(s,t) = G_{\mathcal{P}}(s,t) + \sum_{i=0}^{n-1} G_i(s,t)$ for $(s,t) \in [0,1]^2$, where

$$G_i(s,t) = \begin{cases} 0 & \text{if } (s,t) \notin [s_i, s_{i+1}]^2 \\ \frac{\sinh(t-s_i)\sinh(s_{i+1}-s)}{\sinh(s_{i+1}-s_i)} & \text{if } s_i \le t < s \le s_{i+1} \\ \frac{\sinh(s-s_i)\sinh(s_{i+1}-t)}{\sinh(s_{i+1}-s_i)} & \text{if } s_i \le s \le t \le s_{i+1} \end{cases}.$$

If $(s,t) \notin [s_i, s_{i+1}]^2$ $\forall i$, then clearly $G(s,t) = G_{\mathcal{P}}(s,t)$. If $(s,t) \in [s_i, s_{i+1}]^2$ for some i, then for $|\mathcal{P}|$ sufficiently small,

$$\begin{aligned} |G(s,t) - G_{\mathcal{P}}(s,t)| &= |G_i(s,t)| \\ &\le \sinh(s_{i+1} - s_i) \le \sinh(|\mathcal{P}|), \end{aligned}$$

since sinh is an increasing function. In any case, $|G(s,t) - G_{\mathcal{P}}(s,t)| \le \sinh(|\mathcal{P}|) \to 0$ as $|\mathcal{P}| \to 0$. $\Box$

The next three lemmas demonstrate the appropriate approximation of the terms in $(\Delta_{\mathcal{P}} h, J)$ to those in $(\Delta h, J)$.

Lemma 3.1.5 *For a given partition $\mathcal{P}$ and $h, J \in H_{\mathcal{P}}(\mathfrak{g})$, let*

$$\begin{aligned} E(h,J,\mathcal{P}) \equiv \Bigg| \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} & a_i \sinh(-(t_1+t_2) + s_{i+1} + s_i) \\ & \cdot G_{\mathcal{P}}(t_1,t_2)(K\langle h'(t_1), J(t_2)\rangle + K\langle h(t_1), J'(t_2)\rangle) dt_1 dt_2 \Bigg|. \end{aligned}$$

Then

$$\lim_{|\mathcal{P}|\to 0} \sup_{\|h\|,\|J\| \le 1} E(h,J,\mathcal{P}) = 0.$$

Proof: First note that

$$\begin{aligned} & E(h,J,\mathcal{P}) \\ \le & \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} \left| \frac{\sinh(s_{i+1}-t_2)}{\sinh(s_{i+1}-s_i)} \cosh(t_1 - s_i) - \frac{\sinh(t_1 - s_i)}{\sinh(s_{i+1}-s_i)} \cosh(s_{I+1} - t_2) \right| \\ & \cdot M \cdot \left| K\langle h'(t_1), J(t_2)\rangle + K\langle h(t_1), J'(t_2)\rangle \right| dt_1 dt_2, \end{aligned}$$

where M is a bound for $G_{\mathcal{P}}(t_1,t_2)$ on $[0,1]^2$. For small enough $|\mathcal{P}|$, we have $\left|\frac{\sinh(s_{i+1}-t_2)}{\sinh(s_{i+1}-s_i)}\right|$ and $\left|\frac{\sinh(t_1-s_i)}{\sinh(s_{i+1}-s_i)}\right|$ less than 1 , and $|\cosh(t_1 - s_i)|$ and $|\cosh(s_{I+1} - t_2)|$ less than 2. Thus, by use of the Cauchy-Schwartz inequality,

$$\begin{aligned}
& E(h,J,\mathcal{P}) \\
\leq\ & 4M\int_0^1\int_0^1 (\sum_{i=0}^{n-1} 1_{[s_i,s_{i+1}]^2}) \left|K\langle h'(t_1),J(t_2)\rangle + K\langle h(t_1),J'(t_2)\rangle\right| dt_1dt_2 \\
\leq\ & 4M(\int_0^1\int_0^1 (\sum_{i=0}^{n-1} 1_{[s_i,s_{i+1}]^2})^2 dt_1dt_2)^{\frac{1}{2}} \\
& \cdot(\int_0^1\int_0^1 \left|K\langle h'(t_1),J(t_2)\rangle + K\langle h(t_1),J'(t_2)\rangle\right|^2 dt_1dt_2)^{\frac{1}{2}} \\
\leq\ & 4M(\sum_{i=0}^{n-1}(s_{i+1}-s_i)^2)^{\frac{1}{2}}(\int_0^1\int_0^1 2(\left|K\langle h'(t_1),J(t_2)\rangle\right|^2 + \left|K\langle h(t_1),J'(t_2)\rangle\right|^2)dt_1dt_2)^{\frac{1}{2}} \\
\leq\ & 4\sqrt{2}M(\sum_{i=0}^{n-1}(s_{i+1}-s_i)^2)^{\frac{1}{2}} \\
& \cdot(\int_0^1\int_0^1 c^2\left|h'(t_1)\right|^2\left|J(t_2)\right|^2 dt_1dt_2 + \int_0^1\int_0^1 c^2\left|h(t_1)\right|^2\left|J'(t_2)\right|^2 dt_1dt_2)^{\frac{1}{2}} \\
\leq\ & 4\sqrt{2}Mc(\sum_{i=0}^{n-1}(s_{i+1}-s_i)^2)^{\frac{1}{2}}(2\left\|h\right\|^2\left\|J\right\|^2)^{\frac{1}{2}} = 8Mc(\sum_{i=0}^{n-1}(s_{i+1}-s_i)^2)^{\frac{1}{2}}\left\|h\right\|\left\|J\right\|,
\end{aligned}$$

where c is a constant depending on $K\langle\cdot,\cdot\rangle$. Therefore,

$$\lim_{|\mathcal{P}|\to 0}\sup_{\|h\|,\|J\|\leq 1} E(h,J,\mathcal{P}) \leq \lim_{|\mathcal{P}|\to 0} 8Mc(\sum_{i=0}^{n-1}(s_{i+1}-s_i)^2)^{\frac{1}{2}} = 0,$$

which proves the lemma.□

Lemma 3.1.6 *For each partition $\mathcal{P}$ of $[0,1]$ there exists an $\epsilon(\mathcal{P})\geq 0$ such that $\epsilon(\mathcal{P})\to 0$ as $|\mathcal{P}|\to 0$ and for $h,J\in H_{\mathcal{P}}(\mathfrak{g})$,*

$$\begin{aligned}
& \left| \begin{array}{c} \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i[\cosh(s_I-t_1+s_{i+1}-t_2)G_{\mathcal{P}}(t_1,t_2) - a\cosh(\frac{1}{2})] \\ \cdot K\langle h'(t_1),J'(t_2)\rangle dt_1dt_2 \end{array} \right| \\
\leq\ & \epsilon(\mathcal{P})\left\|h\right\|\left\|J\right\|
\end{aligned}$$

Proof: Note first that cosh is uniformly continuous on $[0,1]$, $G_{\mathcal{P}}$ converges uniformly to G on $[0,1]$ as $|\mathcal{P}|\to 0$ (by Lemma 3.1.4), and G is uniformly continuous on $[0,1]^2$. For a given $\mathcal{P}$, let

$$\epsilon(\mathcal{P}) \equiv \max_{i\in\{1,\cdots,n-1\}}\left\{\sup_{(t_1,t_2)\in[s_i,s_{i+1}]^2}\left\{\left|\cosh(s_I-t_1+s_{i+1}-t_2)G_{\mathcal{P}}(t_1,t_2) - a\cosh(\frac{1}{2})\right|\right\}\right\}.$$

Letting $A(t_1,t_2,s_i,s_{i+1}) \equiv \left|\cosh(s_I - t_1 + s_{i+1} - t_2)G_{\mathcal{P}}(t_1,t_2) - a\cosh(\frac{1}{2})\right|$, then we see that

$$\begin{aligned}
& A(t_1,t_2,s_i,s_{i+1}) \\
= \; & |\cosh(s_I - t_1 + s_{i+1} - t_2)G_{\mathcal{P}}(t_1,t_2) - \cosh(s_I - t_1 + s_{i+1} - t_2)G(t_1,t_2) \\
& + \cosh(s_I - t_1 + s_{i+1} - t_2)G(t_1,t_2) - G(t_1,t_2) \\
& + G(t_1,t_2) - G(t_2,t_2)| \\
\leq \; & |\cosh(s_I - t_1 + s_{i+1} - t_2)| \, |G_{\mathcal{P}}(t_1,t_2) - G(t_1,t_2)| \\
& + |\cosh(s_I - t_1 + s_{i+1} - t_2) - 1| \, |G(t_1,t_2)| + |G(t_1,t_2) - G(t_1,t_2)|
\end{aligned}$$

where we have used the fact that $a\cosh(\frac{1}{2}) = aG(t_2,t_2)$. It is then clear that $\epsilon(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ by our remarks at the beginning of the proof.

We now have that

$$\begin{aligned}
& \left|\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i A(t_1,t_2,s_i,s_{i+1}) K\langle h'(t_1), J'(t_2)\rangle dt_1 dtw\right| \\
\leq \; & \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \left|A(t_1,t_2,s_i,s_{i+1})\right| \left|K\langle h'(t_1), J'(t_2)\rangle\right| dt_1 dt_2 \\
\leq \; & \epsilon(\mathcal{P}) \int_{[0,1]^2} (\sum_{i=0}^{n-1} \frac{1}{\sinh(s_{i+1} - s_i)} 1_{[s_i,s_{i+1}]^2}(t_1,t_2)) \left|K\langle h'(t_1), J'(t_2)\rangle\right| dt_1 dt_2 \\
\leq \; & \epsilon(\mathcal{P}) \int_{[0,1]^2} (\sum_{i=0}^{n-1} \frac{1}{\Delta_i} 1_{[s_i,s_{i+1}]^2}(t_1,t_2)) \left|K\langle h'(t_1), J'(t_2)\rangle\right| dt_1 dt_2,
\end{aligned}$$

where we have used the fact that $\Delta_i \equiv s_{i+1} - s_i < \sinh(s_{i+1} - s_i)$ for all i (note that $\sinh'(x) = \cosh(x) > 1 \Rightarrow \sinh(x) > x$ for $x > 0$). Letting

$$\delta_{\mathcal{P}}(t_1,t_2) \equiv \sum_{i=0}^{n-1} \frac{1}{\Delta_i} 1_{[s_i,s_{i+1}]^2}(t_1,t_2)$$

for $(t_1,t_2) \in [0,1]$, the Cauchy-Schwartz inequality gives

$$\begin{aligned}
& \left|\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i A(t_1,t_2,s_i,s_{i+1}) K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2\right| \\
\leq \; & \epsilon(\mathcal{P}) \int_{[0,1]^2} \delta_{\mathcal{P}}(t_1,t_2) \left|h'(t_1)\right| \left|J'(t_2)\right| dt_1 dt_2 \\
= \; & \epsilon(\mathcal{P}) \int_0^1 \left|h'(t_1)\right| (T_{\mathcal{P}} \left|J'(\cdot)\right|)(t_1) dt_1 \\
\leq \; & \epsilon(\mathcal{P}) \left\|h'\right\|_{L^2(H_{\mathcal{P}}(\mathfrak{g}))} \left\|T_{\mathcal{P}} \left|J'(\cdot)\right|\right\|_{L^2([0,1]).}
\end{aligned}$$

where $(T_{\mathcal{P}}f)(s) \equiv \int_0^1 \delta_{\mathcal{P}}(t,s)f(s)ds$ for all $f \in .L^2([0,1])$.

We claim that $T_{\mathcal{P}}$ is a bounded operator on $L^2([0,1])$. Indeed, for $f \in .L^2([0,1])$,

$$\begin{aligned}
\int_0^1 |(T_{\mathcal{P}}f)(t)|^2\, dt &= \int_0^1 \left| \int_0^1 \delta_{\mathcal{P}}(t,s)f(s)ds \right|^2 dt \\
&= \int_0^1 \left| \int_0^1 (\sum_{i=0}^{n-1} \frac{1}{\Delta_i} 1_{[s_i,s_{i+1}]^2}(t,s))f(s)ds \right|^2 dt \\
&= \int_0^1 \left| \sum_{i=0}^{n-1} 1_{[s_i,s_{i+1}]}(t) \int_{s_i}^{s_{i+1}} f(s)\frac{1}{\Delta_i}ds \right|^2 dt \\
&= \int_0^1 \sum_{i=0}^{n-1} 1_{[s_i,s_{i+1}]}(t)(\int_{s_i}^{s_{i+1}} f(s)\frac{1}{\Delta_i}ds)^2 dt \\
&= \sum_{i=0}^{n-1} \Delta_i(\int_{s_i}^{s_{i+1}} f(s)\frac{1}{\Delta_i}ds)^2 \\
&\leq \sum_{i=0}^{n-1} \Delta_i \int_{s_i}^{s_{i+1}} |f(s)|^2 \frac{1}{\Delta_i}ds \\
&= \int_0^1 |f(s)|^2\, ds = \|f\|_{L^2([0,1])},
\end{aligned}$$

where we have used Jensen's inequality in the last inequality.

Hence, $\|T_{\mathcal{P}}\|_{L^2 \to L^2} \leq 1$, and we then have

$$\begin{aligned}
&\left| \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i A(t_1,t_2,s_i,s_{i+1}) K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 \right| \\
\leq\ & \epsilon(\mathcal{P}) \left\|h'\right\|_{L^2(H_{\mathcal{P}}(\mathfrak{g}))} \|T_{\mathcal{P}}\|_{L^2\to L^2} \left\|J'\right\|_{L^2(H_{\mathcal{P}}(\mathfrak{g}))} \leq \epsilon(\mathcal{P}) \|h\|_{H(\mathfrak{g})} \|J\|_{H(\mathfrak{g})}
\end{aligned}$$

which proves the lemma.□

Lemma 3.1.7 *For each partition $\mathcal{P}$ of $[0,1]$ there exists an $\epsilon(\mathcal{P}) \geq 0$ such that $\epsilon(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and for $h, J \in H_{\mathcal{P}}(\mathfrak{g})$,*

$$\left| \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 - \int_0^1 K\langle h'(t), J'(t)\rangle dt \right| \leq \epsilon(\mathcal{P}) \|h\| \|J\|. \tag{3.11}$$

Proof: There are certain simplifications we can make in (3.11) to make calculations easier. First, it is sufficient to prove the lemma with $a_i = \frac{1}{\sinh(s_{i+1}-s_i)}$ replaced by $\frac{1}{\Delta_i} = \frac{1}{s_{i+1}-s_i}$. Indeed, by Taylor's theorem, for $|\mathcal{P}|$ sufficiently small,

$$0 \leq \sinh(s_{i+1}-s_i) = s_{i+1} - s_i + \frac{\cosh(\xi_i)(s_{i+1}-s_i)^3}{3!} \quad \forall i,$$

where $0 < \xi_i < s_{i+1} - s_i$ and $1 < \cosh(\xi_i) < 2$. Thus, for all i,

$$\begin{aligned}\left|\frac{1}{\sinh(s_{i+1}-s_i)} - \frac{1}{s_{i+1}-s_i}\right| &= \frac{\frac{\cosh(\xi_i)}{3!}(s_{i+1}-s_i)^3}{(s_{i+1}-s_i+\frac{\cosh(\xi_i)}{3!}(s_{i+1}-s_i)^3)(s_{i+1}-s_i)} \\ &\leq s_{i+1}-s_i \leq |\mathcal{P}|\,.\end{aligned}$$

Second, it is sufficient to replace J by h. If we define the bilinear form B by

$$E\langle h, J\rangle \equiv \sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 - \int_0^1 K\langle h'(t), J'(t)\rangle dt,$$

then by the symmetry of K, E is symmetric. Since $H_{\mathcal{P}}(\mathfrak{g})$ is finite dimensional, then for all $h \in H_{\mathcal{P}}(\mathfrak{g})$ the linear functional $E\langle h, \cdot\rangle : H_{\mathcal{P}}(\mathfrak{g}) \to R$ is bounded. Thus, by the Riesz representation theorem, there exists a linear operator $A : H_{\mathcal{P}}(\mathfrak{g}) \to H_{\mathcal{P}}(\mathfrak{g})$ such that $E\langle h, J\rangle = \langle Ah, J\rangle$ for all $h, J \in H_{\mathcal{P}}(\mathfrak{g})$. Since E is symmetric, then so is A. If we assume that $|E\langle h, h\rangle| \leq \epsilon(\mathcal{P})\,\|h\|^2$ for all $h \in H_{\mathcal{P}}(\mathfrak{g})$, i.e., $|\langle Ah, h\rangle| \leq \epsilon(\mathcal{P})\,\|h\|^2$ for all $h \in H_{\mathcal{P}}(\mathfrak{g})$, then by Lemma 2.1 in Lang [15] and by the symmetry of A,

$$\begin{aligned}|E\langle h, J\rangle| &= |\langle Ah, J\rangle| \\ &\leq \epsilon(\mathcal{P})\,\|h\|^2\,\|J\|^2\end{aligned}$$

for all $h, J \in H_{\mathcal{P}}(\mathfrak{g})$.

Third, it is sufficient to prove (3.11) assuming h is real and with $K\langle\cdot,\cdot\rangle$ replaced by scalar multiplication. To see why, recall that $H_{\mathcal{P}}(\mathfrak{g}) = \{G(\cdot, s_i)A :\ A \in \mathfrak{g},\ s_i \in \mathcal{P}\} = H_{\mathcal{P}}(R)\mathfrak{g}$. We can choose a basis, $\{e_1, \ldots, e_d\}$ (where $d = \dim(\mathfrak{g})$), which is orthonormal with respect to $K\langle\cdot,\cdot\rangle$, i. e., $K\langle e_i, e_j\rangle = \delta_{ij}$ for $i, j \in \{1, \ldots, d\}$. We may then write $h = \sum_{j=1}^d h_j e_j$, where $h_j \in H_{\mathcal{P}}(R)$ for each $j \in \{1, \ldots, d\}$. Assuming (3.11) is true for the scalar case, then

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i K\langle h'(t_1), h'(t_2)\rangle dt_1 dt_2 - \int_0^1 K\langle h'(t), h'(t)\rangle dt\right| \\ =& \left|\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{s_i}^{s_{i+1}} a_i \{\sum_{j=1}^d h_j'(t_1)h_j'(t_2)\} dt_1 dt_2 - \int_0^1 \{\sum_{j=1}^d h_j(t)h_j(t)\} dt\right| \\ \leq& \sum_{j=1}^d \epsilon(\mathcal{P})\,\|h_j\|^2_{H_{\mathcal{P}}(R)}\end{aligned}$$

$$
\begin{aligned}
&= \epsilon(\mathcal{P}) \int_0^1 \sum_{j=1}^{d} \{ \left| h_j'(t) \right|^2 + |h_j(t)|^2 \} dt \\
&= \epsilon(\mathcal{P}) \int_0^1 \{ K\langle h'(t), h'(t) \rangle + K\langle h(t), h(t) \rangle \} dt \\
&\leq \epsilon(\mathcal{P}) \int_0^1 \{ c \left| h'(t) \right|^2 + c \left| h(t) \right|^2 \} dt \\
&= c\epsilon(\mathcal{P}) \left\| h \right\|^2_{H_{\mathcal{P}}(\mathfrak{g})}
\end{aligned}
$$

where c is a constant depending on K.

By the fundamental theorem of calculus,

$$
\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} \frac{1}{\Delta_i} h'(t_1) \cdot h'(t_2) dt_1 dt_2 = \sum_{i=0}^{n-1} \frac{\Delta_i h}{\Delta_i} \cdot \frac{\Delta_i h}{\Delta_i} \Delta_i,
$$

where $\Delta_i h \equiv h(s_{i+1}) - h(s_i)$. We thus need to prove that for each partition $\mathcal{P}$, there exists an $\epsilon(\mathcal{P})$ such that

$$
\left| \sum_{i=0}^{n-1} \frac{\Delta_i h}{\Delta_i} \cdot \frac{\Delta_i h}{\Delta_i} \Delta_i - \int_0^1 h'(t) \cdot h'(t) \rangle dt \right| \leq \epsilon(\mathcal{P}) \left\| h \right\|^2,
$$

where $\epsilon(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$. But note that

$$
\begin{aligned}
\sum_{i=0}^{n-1} \frac{\Delta_i h}{\Delta_i} \cdot \frac{\Delta_i h}{\Delta_i} \Delta_i &= \sum_{i=0}^{n-1} \tilde{h}_i'(t) \cdot \tilde{h}_i'(t) \Delta_i \\
&= \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \tilde{h}_i'(t) \cdot \tilde{h}_i'(t) dt,
\end{aligned}
$$

where for each i, $\tilde{h}_i$ is the line segment with $\tilde{h}_i(s_i) = h(s_i)$ and $\tilde{h}_i(s_{i+1}) = h(s_{i+1})$.

For each i, consider the integral $\int_{s_i}^{s_{i+1}} \{ \left| \tilde{h}_i'(t) \right|^2 - |h'(t)|^2 \} dt$. Recall that $h \in H_{\mathcal{P}}(R)$ is of the form $h(t) = \sum_{j=0}^{n-1} c_j G(t, s_j)$ and can thus be expressed as $h(t) = ae^t + be^{-t}$ when restricted to $[s_i, s_{i+1}]$. In light of this, h is then uniquely determined by its values at the endpoints: $\alpha \equiv h(s_i)$ and $\beta \equiv h(s_{i+1})$. Since integration is translation invariant, then, for convenience in calculations, we may write

$$
\int_{s_i}^{s_{i+1}} \{ \left| \tilde{h}_i'(t) \right|^2 - \left| h'(t) \right|^2 \} dt = \int_{-\delta}^{\delta} \{ \left| \tilde{k}'(t) \right|^2 - \left| k'(t) \right|^2 \} dt
$$

where k is of the form $k = ce^t + de^{-t}$ with $k(-\delta) = \alpha = h(s_i)$ and $k(\delta) = \beta = h(s_{i+1})$, and $\tilde{k}$ is a line with $\tilde{k}(-\delta)=\alpha$ and $\tilde{k}(\delta)=\beta$. One can easily verify that

$$\begin{aligned}
k(t) &= \frac{1}{\sinh(2\delta)}(\beta\sinh(\delta+t)+\alpha\sinh(\delta-t)),\\
\tilde{k}(t) &= \alpha+\frac{\beta-\alpha}{2\delta}(t+\delta)=\frac{1}{2\delta}(\beta(\delta+t)+\alpha(\delta-t)),\\
k'(t) &= \frac{1}{\sinh(2\delta)}(\beta\cosh(\delta+t)-\alpha\cosh(\delta-t)),\\
\tilde{k}'(t) &= \frac{\beta-\alpha}{2\delta}.
\end{aligned}$$

Letting $b\equiv\sinh(2\delta)$, we then have the following:

$$\begin{aligned}
&\int_{-\delta}^{\delta}\left|k'(t)\right|^2 dt\\
=\ & \frac{1}{b^2}\int_{-\delta}^{\delta}(\beta\cosh(\delta+t)-\alpha\cosh(\delta-t))^2 dt\\
=\ & \frac{1}{b^2}\int_{-\delta}^{\delta}(\beta(\cosh(\delta)\cosh(t)+\sinh(\delta)\sinh(t))\\
& -\alpha(\cosh(\delta)\cosh(t)-\sinh(\delta)\sinh(t)))^2 dt\\
=\ & \frac{1}{b^2}\int_{-\delta}^{\delta}((\beta-\alpha)\cosh(\delta)\cosh(t)+(\beta+\alpha)\sinh(\delta)\sinh(t))^2 dt\\
=\ & \frac{1}{b^2}\int_{-\delta}^{\delta}\{(\beta-\alpha)^2\cosh^2(\delta)\cosh^2(t)+(\beta+\alpha)^2\sinh^2(\delta)\sinh^2(t)\\
& +2(\beta-\alpha)(\beta+\alpha)\cosh(\delta)\cosh(t)\sinh(\delta)\sinh(t)\}dt\\
=\ & \frac{1}{b^2}\int_{-\delta}^{\delta}\{(\beta-\alpha)^2\cosh^2(\delta)\cosh^2(t)+(\beta+\alpha)^2\sinh^2(\delta)\sinh^2(t)\}dt\\
=\ & \frac{1}{b^2}\int_{-\delta}^{\delta}\{(\beta-\alpha)^2\cosh^2(\delta)\frac{1}{2}(\cosh(2t)+1)+(\beta+\alpha)^2\sinh^2(\delta)\frac{1}{2}(\cosh(2t)-1)\}dt\\
=\ & \frac{1}{b^2}\{(\beta-\alpha)^2\cosh^2(\delta)\frac{1}{2}(\sinh(2t)+2\delta)+(\beta+\alpha)^2\sinh^2(\delta)\frac{1}{2}(\sinh(2t)-2\delta)\}.
\end{aligned}$$

Noting that $\int_{-\delta}^{\delta}\left|\tilde{k}'(t)\right|^2 dt=\int_{-\delta}^{\delta}\frac{(\beta-\alpha)^2}{4\delta^2}dt=\frac{(\beta-\alpha)^2}{2\delta}$, we then have

$$\begin{aligned}
&\left|\int_{-\delta}^{\delta}\{\left|\tilde{k}'(t)\right|^2-\left|k'(t)\right|^2\}dt\right|\\
=\ & \Big|(\beta-\alpha)^2\{\frac{1}{2\delta}-\frac{1}{\sinh^2(2\delta)}\cosh^2(\delta)\frac{1}{2}(\sinh(2\delta)+2\delta)\}\\
& -(\beta+\alpha)^2\frac{1}{\sinh^2(2\delta)}\sinh^2(\delta)\frac{1}{2}(\sinh(2\delta)-2\delta)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq (\beta-\alpha)^2 \frac{1}{\sinh(2\delta)} \left| \frac{\sinh(2\delta)}{2\delta} - \cosh^2(\delta)\frac{1}{2}(1+\frac{2\delta}{\sinh(2\delta)}) \right| \\
&\quad +(\beta+\alpha)^2 \frac{1}{\sinh(2\delta)} \sinh^2(\delta)\frac{1}{2}\left|1-\frac{2\delta}{\sinh(2\delta)}\right| \\
&\leq \frac{1}{b}\{(\beta-\alpha)^2+\sinh^2(\delta)(\beta+\alpha)^2\frac{1}{2}\}\epsilon(\delta) \\
&\leq \frac{1}{b}\{(\beta-\alpha)^2+(\cosh^2(\delta)-1)(\beta^2+\alpha^2)\}\epsilon(\delta) \\
&\leq \frac{1}{b}\{(\beta-\alpha)^2+(\cosh(2\delta)-1)(\beta^2+\alpha^2)\}\epsilon(\delta), \qquad (3.12)
\end{aligned}
$$

where $\epsilon(\delta) \equiv \max\{\left|\frac{\sinh(2\delta)}{2\delta} - \cosh^2(\delta)\frac{1}{2}(1+\frac{2\delta}{\sinh(2\delta)})\right|, \left|1-\frac{2\delta}{\sinh(2\delta)}\right|\}$, and, for the last inequality, we used the fact that $\cosh^2(\delta) \leq \cosh(2\delta)$ since $\cosh(2\delta) = \cosh^2(\delta)+\sinh^2(\delta)$. It is easy to check that $\epsilon(\delta) \to 0$ as $\delta \to 0$.

We now compare the expression in (3.12) to $(k,k)_\delta \equiv \int_{-\delta}^{\delta}\{|k'(t)|^2+|k(t)|^2\}dt$. Since, in the interval $(-\delta,\delta)$, k has the property that $k''=k$, we can use integration by parts to get

$$
\begin{aligned}
(k,k)_\delta &= k'k|_{-\delta}^{\delta} - \int_{-\delta}^{\delta}|k(t)|^2\,dt + \int_{-\delta}^{\delta}|k(t)|^2\,dt \\
&= \frac{1}{b^2}\,(\beta\cosh(\delta+t)-\alpha\cosh(\delta-t))(\beta\sinh(\delta+t)+\alpha\sinh(\delta-t))|_{-\delta}^{\delta} \\
&= \frac{1}{b^2}\{(\beta\cosh(2\delta)-\alpha)\beta\sinh(2\delta)-(\beta-\alpha\cosh(2\delta))\alpha\sinh(2\delta)\} \\
&= \frac{1}{b}\{\cosh(2\delta)(\alpha^2+\beta^2)-2\alpha\beta\} \\
&= \frac{1}{b}\{(\beta-\alpha)^2+(\cosh(2\delta)-1)(\alpha^2+\beta^2)\}.
\end{aligned}
$$

Therefore, $\left|\int_{-\delta}^{\delta}\{\left|\tilde{k}'(t)\right|^2-|k'(t)|^2\}dt\right| \leq \epsilon(\delta)(k,k)_\delta$.

Returning to our original notation and using translation invariance of integration, we have

$$
\left|\int_{s_i}^{s_{i+1}}\{\left|\tilde{h}'(t)\right|^2-|h(t)|^2\}dt\right| \leq \epsilon(s_{i+1}-s_i)(h,h)_i,
$$

where $(h,h)_i = \int_{s_i}^{s_{i+1}}\{|h'(t)|^2+|h(t)|^2\}dt$. Thus,

$$
\begin{aligned}
\left|\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}}\{\left|\tilde{h}'(t)\right|^2-|h(t)|^2\}dt\right| &\leq \sum_{i=0}^{n-1}\epsilon(s_{i+1}-s_i)(h,h)_i \\
&\leq \epsilon(|\mathcal{P}|)(h,h) = \epsilon(|\mathcal{P}|)\,\|h\|^2\,.
\end{aligned}
$$

This completes the proof of the lemma. □

We are now ready to prove (3.3). By Lemma 3.4 and Lemma 3.5 we get

$$
\begin{aligned}
&|(\Delta h, J) - (\Delta_{\mathcal{P}} h, J)| \\
\leq\ & \left|\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \cosh(s_i - t_1 + s_{i+1} - t_2) G_{\mathcal{P}}(t_1,t_2) K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 \right. \\
& \left. - \int_0^1 a\cosh(\frac{1}{2}) K\langle h'(t), J'(t)\rangle dt \right| \\
& + \left|\frac{1}{2}\int_0^1\int_0^1 (G^2_{\mathcal{P}}(t_1,t_2) - G^2(t_1,t_2)) K\,\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2\right| \\
& + \left|\frac{1}{4}\int_0^1\int_0^1 (G^2_{\mathcal{P}}(t_1,t_2) - G^2(t_1,t_2)) K\langle h(t_1), J(t_2)\rangle dt_1 dt_2\right| \\
& + \left|\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i \sinh(-(t_2+t_1) + s_{i+1} + s_i) \right. \\
& \qquad \left. .G_{\mathcal{P}}(t_1,t_2)(K\langle h'(t_1), J(t_2)\rangle + K\langle h(t_1), J'(t_2)\rangle) dt_1 dt_2 \right| \\
\leq\ & \left|\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i [\cosh(s_I - t_1 + s_{i+1} - t_2) G_{\mathcal{P}}(t_1,t_2) - a\cosh(\frac{1}{2})] \right. \\
& \left. \cdot K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 \right| \\
& + a\cosh(\frac{1}{2}) \left|\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \int_{s_i}^{s_{i+1}} a_i K\langle h'(t_1), J'(t_2)\rangle dt_1 dt_2 - \int_0^1 K\langle h'(t), J'(t)\rangle dt\right| \\
& + \frac{1}{2}\int_0^1\int_0^1 \left|G^2_{\mathcal{P}}(t_1,t_2) - G^2(t_1,t_2)\right| c\left|h'(t_1)\right| \left|J'(t_2)\right| dt_1 dt_2 \\
& + \frac{1}{4}\int_0^1\int_0^1 \left|G^2_{\mathcal{P}}(t_1,t_2) - G^2(t_1,t_2)\right| c\left|h(t_1)\right| \left|J(t_2)\right| dt_1 dt_2 \\
& + E(h, J, \mathcal{P}) \\
\leq\ & \{\epsilon_1(\mathcal{P}) + \epsilon_2(\mathcal{P}) + \epsilon_3(\mathcal{P}) + \epsilon_4(\mathcal{P})\} \, \|h\| \, \|J\|
\end{aligned}
$$

where $\epsilon_1(\mathcal{P})$ is from Lemma 3.1.6, $\epsilon_2(\mathcal{P})$ is from Lemma 3.1.7,

$$\epsilon_3(\mathcal{P}) \equiv \frac{3}{4} c \sup_{(t_1,t_2)\in[0,1]^2} \{G_{\mathcal{P}}(t_1,t_2) - G(t_1,t_2)\}, E(h,J,\mathcal{P})$$

(c depends on K), $E(h,J,\mathcal{P})$ is from Lemma 3.1.5, and $\epsilon_4(\mathcal{P}) \equiv \sup_{\|k\|,\|l\|\leq 1}\{E(k,l,\mathcal{P})\}$. Noting that $\epsilon_3(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ (by Lemma 3.1.4, then from the above lemmas it is clear that $\epsilon(\mathcal{P}) \equiv \{\epsilon_1(\mathcal{P}) + \epsilon_2(\mathcal{P}) + \epsilon_3(\mathcal{P}) + \epsilon_4(\mathcal{P})\} \to 0$ as $|\mathcal{P}| \to 0$. Using this definition of $\epsilon(\mathcal{P})$ in (3.3), we have now completed the proof for Theorem 3.1.1. $\square$

3.2 Convergence of Finite Dimensional Ricci Tensor

Theorem 3.2.1 $\|P_{\mathcal{P}}(Ric - Ric_{\mathcal{P}})P_{\mathcal{P}}\|_{op} \to 0$ *as* $|\mathcal{P}| \to 0$.

Proof: Our goal is to find, for each partition $\mathcal{P}$, an $\epsilon(\mathcal{P})$ such that $\epsilon(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and, for each $h \in H_{\mathcal{P}}(\mathfrak{g})$,

$$\begin{aligned} |((Ric - Ric_{\mathcal{P}})h, h)| &\equiv |Ric\langle h,h\rangle - Ric_{\mathcal{P}}\langle h,h\rangle| \\ &\leq \epsilon(\mathcal{P}) \|h\|^2 . \end{aligned}$$

We first note that for a good orthonormal basis, S, of $H(\mathfrak{g})$, we have

$$\begin{aligned} \mathrm{Ric}\langle h,h\rangle &= \sum_{k\in S}(R\langle h,k\rangle k,h) \\ &= \sum_{k\in S}(\nabla_h\nabla_k k - \nabla_k\nabla_h k - \nabla_{[h,k]}k, h) \\ &= \sum_{k\in S} -(\nabla_k\nabla_h k, h) - (\nabla_{[h,k]}k, h) \\ &= \sum_{k\in S}(\nabla_h k, \nabla_k h) - (\nabla_{[h,k]}k, h) \\ &= \sum_{k\in S}(\nabla_k h + [h,k], \nabla_k h) - (\nabla_{[h,k]}k, h) \\ &= \sum_{k\in S} -(\nabla_k\nabla_k h, h) - (\nabla_k[h,k] + \nabla_{[h,k]}k, h) \\ &= -(\Delta h, h) - \sum_{k\in S}(\nabla_k[h,k] + \nabla_{[h,k]}k, h). \end{aligned}$$

Similarly,

$$\begin{aligned} &Ric_{\mathcal{P}}\langle h,h\rangle \\ = \; & -(\Delta_{\mathcal{P}}h,h) - \sum_{k\in S_{\mathcal{P}}}(\nabla_k[h,k]_{\mathcal{P}} + \nabla_{[h,k]_{\mathcal{P}}}k, h) \\ = \; & -(\Delta_{\mathcal{P}}h,h) - \sum_{k\in S_{\mathcal{P}}}(\nabla_k[h,k] + \nabla_{[h,k]}k, h) + \sum_{k\in S_{\mathcal{P}}}(\nabla_k(Q_{\mathcal{P}}[h,k]) + \nabla_{Q_{\mathcal{P}}[h,k]}k, h) \\ = \; & -(\Delta_{\mathcal{P}}h,h) - \sum_{k\in S_{\mathcal{P}}}(\nabla_k[h,k] + \nabla_{[h,k]}k, h) + \sum_{k\in S_{\mathcal{P}}}(2\nabla_k(Q_{\mathcal{P}}[h,k]) + [Q_{\mathcal{P}}[h,k],k], h) \end{aligned}$$

where $Q_{\mathcal{P}} = I - P_{\mathcal{P}}$ and $I =$ identity on $H(\mathfrak{g})$.

Let us compare $Ric\langle h,h\rangle$ and $Ric_{\mathcal{P}}\langle h,h\rangle$. By Theorem 3.1.1, we know there exists an $\epsilon_1(\mathcal{P})$ such that $|(\Delta h,h)-(\Delta_{\mathcal{P}}h,h)|\le\epsilon_1(\mathcal{P})\|h\|^2$ with $\epsilon_1(\mathcal{P})\to 0$ as $|\mathcal{P}|\to 0$.

Next, note that for $k\in S$

$$
\begin{aligned}
& (\nabla_k[h,k]+\nabla_{[h,k]}k,h) \\
= & (-\int_0^1 F(\cdot,t)[k,[h,k]'](t)dt+\frac{1}{2}\int_0^1 G(\cdot,t)[k,[h,k]](t)dt \\
& -\int_0^1 F(\cdot,t)[[h,k],k'](t)dt+\frac{1}{2}\int_0^1 G(\cdot,t)[[h,k],k](t)dt,h) \\
= & (-\int_0^1 F(\cdot,t)\{[k,[h,k]'](t)+[[h,k],k'](t)\}dt,h) \\
= & (-\int_0^1 F(\cdot,t)\{[k,[h',k]](t)+[k,[h,k']](t)+[[h,k],k'](t)\}dt,h) \\
= & (-\int_0^1 F(\cdot,t)\{[k,[h',k]](t)-[k',[k,h]](t)-[h,[k',k]](t)+[[h,k],k'](t)\}dt,h) \\
= & (-\int_0^1 F(\cdot,t)[k,[h',k]](t)dt,h) \\
= & \int_0^1\{\langle-\int_0^1 G_{ss}(s,t)[k,[h',k]](t)dt,h'(s)\rangle \\
& +\langle-\int_0^1 F(s,t)[k,[h',k]](t)dt,h(s)\rangle\}ds \\
= & \int_0^1\{\langle-\int_0^1(\delta(t-s)-G(s,t))[k,[h',k]](t)dt,h'(s)\rangle \\
& +\langle-\int_0^1 G_s(s,t)[k,[h',k]](t)dt,h(s)\rangle\}ds \\
= & \int_0^1\langle[k,[h',k]](s),h'(s)\rangle ds
\end{aligned}
$$

where we have used integration by parts for the last equation and the fact that S is a "good" basis (i.e., $k\in S\Longrightarrow[k',k]=0$) for the fifth equation.

Thus,

$$
\begin{aligned}
\sum_{k\in S}(\nabla_k[h,k]+\nabla_{[h,k]}k,h) &= \sum_{k\in S}\int_0^1\langle[k,h'](s),[k,h'](s)\rangle ds \\
&= \int_0^1 G(s,s)K\langle h'(s),h'(s)\rangle ds.
\end{aligned}
$$

Similarly,

$$\sum_{k\in S_{\mathcal{P}}} (\nabla_k[h,k] + \nabla_{[h,k]}k, h) = \int_0^1 G_{\mathcal{P}}(s,s)K\langle h'(s), h'(s)\rangle ds.$$

It follows then that

$$\begin{aligned} & \left|\sum_{k\in S}(\nabla_k[h,k] + \nabla_{[h,k]}k, h) - \sum_{k\in S_{\mathcal{P}}} (\nabla_k[h,k] + \nabla_{[h,k]}k, h)\right| \\ \leq & \int_0^1 \left|G(s,s) - G_{\mathcal{P}}(s,s)\right| K\langle h'(s), h'(s)\rangle ds \\ \leq & \ \epsilon_2(\mathcal{P}) \int_0^1 \left|\langle h'(s), h'(s)\rangle\right| ds \\ \leq & \ \epsilon_2(\mathcal{P}) \left\|h\right\|^2 \end{aligned}$$

where $\epsilon_2(\mathcal{P}) = (\sup_{s\in[0,1]} |G(s,s) - G_{\mathcal{P}}(s,s)|) \cdot C$ and C depends on $K\langle\cdot,\cdot\rangle$. Since we know by Lemma 3.1.4 that $G_{\mathcal{P}} \to 0$ uniformly as $|\mathcal{P}| \to 0$, then $\epsilon_2(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$.

We next claim that for each partition, $\mathcal{P}$, there exists an $\epsilon_3(\mathcal{P})$ such that $\epsilon_3(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and for $h \in H_{\mathcal{P}}(\mathfrak{g})$,

$$\left|\sum_{k\in S_{\mathcal{P}}} ([Q_{\mathcal{P}}[h,k],k],h)\right| \leq \epsilon_3(\mathcal{P}) \left\|h\right\|^2.$$

Since $h \in H_{\mathcal{P}}(\mathfrak{g})$, for an interval, $[s_i, s_{i+1}]$, of the partition $\mathcal{P}$, h must be of the form $h(t) = ce^t + de^{-t}$ for all $t \in [s_i, s_{i+1}]$. It is easy to check that in fact

$$h(t) = h(\bar{s}_i)\cosh(t - \bar{s}_i) + \frac{\Delta_i h}{\eta_i}\sinh(t - \bar{s}_i)\ \forall t \in [s_i, s_{i+1}],$$

where $\bar{s}_i \equiv \frac{s_i + s_{i+1}}{2}$, $\Delta_i h \equiv h(s_{i+1}) - h(s_i)$, and $\eta_i \equiv 2\sinh(\frac{s_{i+1}-s_i}{2})$.

Letting $\Delta_i \equiv s_{i+1} - s_i$, we may then write, for $t \in [s_i, s_{i+1}]$,

$$\begin{aligned} & [h,k](t) \\ = & \left[h(\bar{s}_i)\cosh(t - \bar{s}_i) + \frac{\Delta_i h}{\eta_i}\sinh(t - \bar{s}_i), k(\bar{s}_i)\cosh(t - \bar{s}_i) + \frac{\Delta_i k}{\eta_i}\sinh(t - \bar{s}_i)\right) \\ = & [h(\bar{s}_i), k(\bar{s}_i)]\cosh^2(t - \bar{s}_i) + \frac{1}{\eta_i}([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i), \Delta_i k])\cosh(t - \bar{s}_i)\sinh(t - \bar{s}_i) \\ & + \frac{1}{\eta_i^2}[\Delta_i h, \Delta_i k]\sinh^2(t - \bar{s}_i), \end{aligned}$$

and

$$\Delta_i[h,k] = ([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i), \Delta_i k])\cosh(\frac{\Delta_i}{2}).$$

Using Proposition 3.0.3 we then have

$$\begin{aligned}
P_{\mathcal{P}}[h,k](t) &= \frac{[h,k](s_i) + [h,k](s_{i+1})}{\varsigma_i}\cosh(t - \bar{s}_i) + \frac{\Delta_i[h,k]}{\eta_i}\sinh(t - \bar{s}_i) \\
&= \frac{\{2[h(\bar{s}_i), k(\bar{s}_i)]\cosh^2(\frac{\Delta_i}{2}) + \frac{2}{\eta_i^2}[\Delta_i h, \Delta_i k]\sinh^2(\frac{\Delta_i}{2})\}}{\varsigma_i}\cosh(t - \bar{s}_i) \\
&\quad + \frac{2}{\eta_i^2}([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i), \Delta_i k])\cosh(\frac{\Delta_i}{2})\sinh(\frac{\Delta_i}{2})\sinh(t - \bar{s}_i) \\
&= \{[h(\bar{s}_i), k(\bar{s}_i)]\cosh(\frac{\Delta_i}{2}) + \frac{1}{2\varsigma_i}[\Delta_i h, \Delta_i k]\}\cosh(t - \bar{s}_i) \\
&\quad + \frac{1}{\eta_i}([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i), \Delta_i k])\cosh(\frac{\Delta_i}{2})\sinh(t - \bar{s}_i),
\end{aligned}$$

where $\varsigma_i = 2\cosh(\frac{\Delta_i}{2})$. Thus, for $t \in [s_i, s_{i+1}]$,

$$\begin{aligned}
& Q_{\mathcal{P}}[h,k](t) \\
=\ & [h,k](t) - P_{\mathcal{P}}[h,k](t) \\
=\ & [h(\bar{s}_i), k(\bar{s}_i)](\cosh^2(t - \bar{s}_i) - \cosh(\frac{\Delta_i}{2})\cosh(t - \bar{s}_i)) \\
& + ([\frac{\Delta_i h}{\eta_i}, k(\bar{s}_i)] + [h(\bar{s}_i), \frac{\Delta_i k}{\eta_i}])(\cosh(t - \bar{s}_i) - \cosh(\frac{\Delta_i}{2}))\sinh(t - \bar{s}_i) \\
& + [\frac{\Delta_i h}{\eta_i}, \frac{\Delta_i k}{\eta_i}](\sinh^2(t - \bar{s}_i) - \frac{1}{2\varsigma_i}\eta_i^2\cosh(t - \bar{s}_i)). \qquad (3.13)
\end{aligned}$$

Next note that since $Q_{\mathcal{P}}[h,k] \in Ker(\tilde{\pi}_{\mathcal{P}})$, then

$$[Q_{\mathcal{P}}[h,k], k](s_i) = [Q_{\mathcal{P}}[h,k](s_i), k(s_i)] = 0$$

for all $s_i \in \mathcal{P}$. Thus, $[Q_{\mathcal{P}}[h,k], k] \in Ker(\tilde{\pi}_{\mathcal{P}})$ implying that

$$\sum_{k \in S_{\mathcal{P}}} ([Q_{\mathcal{P}}[h,k], k], h) = 0$$

since $h \in H_{\mathcal{P}}(\mathfrak{g})$.

We now prove some general inequalities that will be useful for the remainder of our proof of Theorem 3.2.1.

Lemma 3.2.2 *Suppose $f : [0,1] \to R$ is a positive function such that $\frac{f(x)}{x}$ is increasing and $f(x) = o(x)$, where we are using the notation $o(x)$ to mean a function with the property that $\frac{|f(x)|}{|x|} \to 0$ as $|x| \to 0$. Then for each partition, $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$ of $[0,1]$, $\exists\, \epsilon(\mathcal{P})$, with $\epsilon(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$, such that for all $h \in H_{\mathcal{P}}(\mathfrak{g})$*

$$\sum_{i=0}^{n-1} |K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle f(\Delta_i)| \leq \epsilon(\mathcal{P}) \|h\|^2$$

and

$$\sum_{i=0}^{n-1} |K\langle h(\bar{s}_i), \Delta_i h\rangle f(\Delta_i)| \leq \epsilon(\mathcal{P}) \|h\|^2 .$$

Proof:

$$\begin{aligned}
\sum_{i=0}^{n-1} |K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle f(\Delta_i)| &\leq \sum_{i=0}^{n-1} \Delta_i \left|K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle\right| \frac{f(\Delta_i)}{\Delta_i} \\
&\leq \left(\sum_{i=0}^{n-1} \Delta_i\right) \left(\sup_i \left|K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle\right|\right) \left(\sup_i \frac{f(\Delta_i)}{\Delta_i}\right) \\
&\leq \left(\sup_{s\in[0,1]} c\,|h(s)|^2\right) \frac{f(|\mathcal{P}|)}{|\mathcal{P}|} \\
&\leq \epsilon(\mathcal{P}) \|h\|^2 \qquad (3.14)
\end{aligned}$$

where c depends on K and $\epsilon(\mathcal{P}) \equiv 2c\frac{f(|\mathcal{P}|)}{|\mathcal{P}|}$.

To see why the last inequality holds in (3.14), note that for $s \in [0,1]$, $h(s) = h(t) + \int_t^s h'(x)dx\ \forall t \in [0,1]$. Thus

$$\begin{aligned}
h(s) &= \int_0^1 h(s)dt \\
&= \int_0^1 h(t)dt + \int_0^1 \left(\int_t^s h'(x)dx\right) dt
\end{aligned}$$

which implies

$$\begin{aligned}
|h(s)|^2 &\leq \left(\int_0^1 |h(t)|\,dt + \int_0^1 \left(\int_t^s \left|h'(x)\right| dx\right) dt\right)^2 \\
&\leq 2\left(\left(\int_0^1 |h(t)|\,dt\right)^2 + \left(\int_0^1 \left|h'(x)\right| dx\right)^2\right) \\
&\leq 2\left(\int_0^1 |h(t)|^2\,dt + \int_0^1 \left|h'(x)\right|^2 dx\right) \\
&= 2\,\|h\|^2
\end{aligned}$$

where we have used the Cauchy Schwartz inequality in the second line and Jensen's inequality in the third line.

The second inequality in the statement of the Lemma follows by the same reasoning used for the first. □

Lemma 3.2.3 *For* $h \in H_{\mathcal{P}}(\mathfrak{g})$,

$$\sum_{i=0}^{n-1} |K\langle \Delta_i h, \Delta_i h\rangle| \leq \epsilon(\mathcal{P}) \|h\|^2$$

where $\epsilon(\mathcal{P}) \to 0$ *as* $|\mathcal{P}| \to 0$.

Proof: Using Jensen's inequality, we get

$$\begin{aligned}
\sum_{i=0}^{n-1} |K\langle \Delta_i h, \Delta_i h\rangle| &\leq c\sum_{i=0}^{n-1} \Delta_i^2 \left|\frac{\Delta_i h}{\Delta_i}\right|^2 \\
&= c\sum_{i=0}^{n-1} \Delta_i^2 \left|\int_{s_i}^{s_{i+1}} h'(s)\frac{1}{\Delta_i} ds\right|^2 \\
&\leq c\sum_{i=0}^{n-1} \Delta_i^2 \int_{s_i}^{s_{i+1}} |h'(s)|^2 \frac{1}{\Delta_i} ds \\
&\leq c|\mathcal{P}| \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} |h'(s)|^2 ds \\
&\leq c|\mathcal{P}| \|h\|^2,
\end{aligned}$$

where c depends on K. The inequality follows by letting $\epsilon(\mathcal{P}) \equiv c|\mathcal{P}|$. □

We now need to consider the final term in $Ric_{\mathcal{P}}\langle h, h\rangle$:

$$\begin{aligned}
&\sum_{k\in S_{\mathcal{P}}} (2\nabla_k(Q_{\mathcal{P}}[h,k]), h) \\
= &\sum_{k\in S_{\mathcal{P}}} 2\left(-\int_0^1 F(\cdot,t)[k,(Q_{\mathcal{P}}[h,k])'](t)dt + \frac{1}{2}\int_0^1 G(\cdot,t)[k,Q_{\mathcal{P}}[h,k]](t)dt, h\right)
\end{aligned} \quad (3.15)$$

Using our previous representation of $Q_{\mathcal{P}}[h,k]$ in Equation (3.13), we can split the summand in (3.15) into two inner products as follows. The first inner product is

$$
\begin{aligned}
& \left(-\int_0^1 F(\cdot,t)[k,(Q_{\mathcal{P}}[h,k])'](t)dt, h\right) \\
= & \int_0^1 \left\{\left\langle -\int_0^1 G_{ss}(s,t)[k,(Q_{\mathcal{P}}[h,k])'](t)dt, h'(s)\right\rangle \right. \\
& \left. + \left\langle -\int_0^1 G_s(s,t)[k,(Q_{\mathcal{P}}[h,k])'](t)dt, h(s)\right\rangle\right\} ds \\
= & \int_0^1 \left\langle [k(s),(Q_{\mathcal{P}}[h,k])'(s)], h'(s)\right\rangle ds \\
= & \int_0^1 \left\langle (Q_{\mathcal{P}}[h,k])'(s), [h',k](s)\right\rangle ds \\
= & \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \langle [h(\bar{s}_i),k(\bar{s}_i)](2C_i(s)S_i(s) - \cosh(\frac{\Delta_i}{2})S_i(s)) \\
& + \frac{1}{\eta_i}([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i),\Delta_i k])(S_i^2(s) + (C_i(s) - \cosh(\frac{\Delta_i}{2})))C_i(s)) \\
& + \frac{1}{\eta_i^2}[\Delta_i h, \Delta_i k](2S_i(s)C_i(s) - \frac{1}{2\varsigma_i}\eta_i^2 S_i(s)), \\
& [h(\bar{s}_i),k(\bar{s}_i)]C_i(s)S_i(s) \\
& + \frac{1}{\eta_i}([\Delta_i h, k(\bar{s}_i)]C_i^2(s) + [h(\bar{s}_i),\Delta_i k]S_i^2(s)) + \frac{1}{\eta_i^2}[\Delta_i h, \Delta_i k]C_i(s)S_i(s)\rangle ds, \quad (3.16)
\end{aligned}
$$

where $C_i(s) \equiv \cosh(s- \bar{s}_i)$ and $S_i(s) \equiv \sinh(s- \bar{s}_i)$. For the second inner product in (3.15) we have

$$
\begin{aligned}
& \left(\int_0^1 G(\cdot,t)[k,Q_{\mathcal{P}}[h,k]](t)dt, h\right) \\
= & \int_0^1 \langle [k,Q_{\mathcal{P}}[h,k]](s), h(s)\rangle ds \\
= & \int_0^1 \langle Q_{\mathcal{P}}[h,k](s), [h,k](s)\rangle ds \\
= & \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \langle [h(\bar{s}_i),k(\bar{s}_i)](C_i^2(s) - \cosh(\frac{\Delta_i}{2})C_i(s)) \\
& + \frac{1}{\eta_i}([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i),\Delta_i k])(C_i(s) - \cosh(\frac{\Delta_i}{2}))S_i(s) \\
& + \frac{1}{\eta_i^2}[\Delta_i h, \Delta_i k](S_i^2(s) - \frac{1}{2\varsigma_i}\eta_i^2 C_i(s)), \\
& [h(\bar{s}_i),k(\bar{s}_i)]C_i^2(s) \\
& + \frac{1}{\eta_i}([\Delta_i h, k(\bar{s}_i)] + [h(\bar{s}_i),\Delta_i k])C_i(s)S_i(s) + \frac{1}{\eta_i^2}[\Delta_i h, \Delta_i k]S_i^2(s)\rangle ds. \quad (3.17)
\end{aligned}
$$

Before combining the above two results to get an expression for (3.15), we make the following definitions for brevity:

$$
\begin{aligned}
A_1^i(s) &= 4C_i(s)S_i(s) - 2\cosh(\frac{\Delta_i}{2})S_i(s) \\
A_2^i(s) &= 2S_i^2(s) + (2C_i(s) - \cosh(\frac{\Delta_i}{2}))C_i(s) \\
A_3^i(s) &= 4C_i(s)S_i(s) - \frac{1}{\varsigma_i}\eta_i^2 S_i(s) \\
A_4^i(s) &= C_i(s)S_i(s) \\
A_5^i(s) &= C_i^2(s) \\
A_6^i(s) &= S_i^2(s) \\
A_7^i(s) &= C_i(s)S_i(s) \\
A_8^i(s) &= C_i^2(s) - \cosh(\frac{\Delta_i}{2})C_i(s) \\
A_9^i(s) &= (C_i(s) - \cosh(\frac{\Delta_i}{2}))S_i(s) \\
A_{10}^i(s) &= S_i^2(s) - \frac{1}{2\varsigma_i}\eta_i^2 C_i(s) \\
A_{11}^i(s) &= C_i^2(s) \\
A_{12}^i(s) &= C_i(s)S_i(s) \\
A_{13}^i(s) &= C_i(s)S_i(s) \\
A_{14}^i(s) &= S_i^2(s)
\end{aligned}
$$

Upon expanding the two expressions in (3.16) and (3.17), adding them together, and then summing over the basis $S_{\mathcal{P}}$, we get the following inequality for (3.15):

$$
\begin{aligned}
&\left| \sum_{k \in S_{\mathcal{P}}} (2\nabla_k(Q_{\mathcal{P}}[h,k]), h) \right| \\
\leq\ & \sum_{i=0}^{n-1} \left\{ \left| K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle G_{\mathcal{P}}(\bar{s}_i, \bar{s}_i) \int_{s_i}^{s_{i+1}} \{A_1^i(s)A_4^i(s) + A_8^i(s)A_{11}^i(s)\} ds \right| \right. \\
& + \left| K\langle h(\bar{s}_i), \Delta_i h\rangle G_{\mathcal{P}}(\bar{s}_i, \bar{s}_i) \frac{1}{\eta_i} \int_{s_i}^{s_{i+1}} \left\{ A_1^i(s)A_5^i(s) + A_8^i(s)A_{12}^i(s) \right\} ds \right|
\end{aligned}
$$

$$+\left|K\langle h(\bar{s}_i),h(\bar{s}_i)\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i}\int_{s_i}^{s_{i+1}}\left\{A_1^i(s)A_6^i(s)+A_8^i(s)A_{13}^i(s)\right\}ds\right|$$
$$+\left|K\langle h(\bar{s}_i),\Delta_i h\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i}\int_{s_i}^{s_{i+1}}\left\{A_1^i(s)A_7^i(s)+A_8^i(s)A_{14}^i(s)\right\}ds\right|$$
$$+\left|K\langle \Delta_i h,h(\bar{s}_i)\rangle G_{\mathcal{P}}(\bar{s}_i,\bar{s}_i)\frac{1}{\eta_i}\int_{s_i}^{s_{i+1}}\{A_2^i(s)A_4^i(s)+A_9^i(s)A_{11}^i(s)\}ds\right|$$
$$+\left|K\langle \Delta_i h,\Delta_i h\rangle G_{\mathcal{P}}(\bar{s}_i,\bar{s}_i)\frac{1}{\eta_i^2}\int_{s_i}^{s_{i+1}}\{A_2^i(s)A_5^i(s)+A_9^i(s)A_{12}^i(s)\}ds\right|$$
$$+\left|K\langle \Delta_i h,h(\bar{s}_i)\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i^2}\int_{s_i}^{s_{i+1}}\left\{A_2^i(s)A_6^i(s)+A_9^i(s)A_{13}^i(s)\right\}ds\right|$$
$$+\left|K\langle \Delta_i h,\Delta_i h\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i^3}\int_{s_i}^{s_{i+1}}\left\{A_2^i(s)A_7^i(s)+A_9^i(s)A_{14}^i(s)\right\}ds\right|$$
$$+\left|K\langle h(\bar{s}_i),h(\bar{s}_i)\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i}\int_{s_i}^{s_{i+1}}\left\{A_2^i(s)A_4^i(s)+A_9^i(s)A_{11}^i(s)\right\}ds\right|$$
$$+\left|K\langle h(\bar{s}_i),\Delta_i h\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i^2}\int_{s_i}^{s_{i+1}}\left\{A_2^i(s)A_5^i(s)+A_9^i(s)A_{12}^i(s)\right\}ds\right|$$
$$+|K\langle h(\bar{s}_i),h(\bar{s}_i)\rangle((G_{\mathcal{P}}(s_{i+1},s_{i+1})-G_{\mathcal{P}}(s_i,s_{i+1}))-(G_{\mathcal{P}}(s_{i+1},s_i)-G_{\mathcal{P}}(s_i,s_i))$$
$$\cdot\frac{1}{\eta_i^2}\int_{s_i}^{s_{i+1}}\{A_2^i(s)A_6^i(s)+A_9^i(s)A_{13}^i(s)\}ds\Big|$$
$$+|K\langle h(\bar{s}_i),\Delta_i h\rangle((G_{\mathcal{P}}(s_{i+1},s_{i+1})-G_{\mathcal{P}}(s_i,s_{i+1}))-(G_{\mathcal{P}}(s_{i+1},s_I)-G_{\mathcal{P}}(s_i,s_i))$$
$$\cdot\frac{1}{\eta_i^3}\int_{s_i}^{s_{i+1}}\{A_2^i(s)A_7^i(s)+A_9^i(s)A_{14}^i(s)\}ds\Big|$$
$$+\left|K\langle \Delta_i h,h(\bar{s}_i)\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i^2}\int_{s_i}^{s_{i+1}}\left\{A_3^i(s)A_4^i(s)+A_{10}^i(s)A_{11}^i(s)\right\}ds\right|$$
$$+\left|K\langle \Delta_i h,\Delta_i h\rangle(G_{\mathcal{P}}(s_{i+1},\bar{s}_i)-G_{\mathcal{P}}(s_i,\bar{s}_i))\frac{1}{\eta_i^3}\int_{s_i}^{s_{i+1}}\left\{A_3^i(s)A_5^i(s)+A_{10}^i(s)A_{12}^i(s)\right\}ds\right|$$
$$+|K\langle \Delta_i h,h(\bar{s}_i)\rangle((G_{\mathcal{P}}(s_{i+1},s_{i+1})-G_{\mathcal{P}}(s_i,s_{i+1}))-(G_{\mathcal{P}}(s_{i+1},s_i)-G_{\mathcal{P}}(s_i,s_i))$$
$$\cdot\frac{1}{\eta_i^3}\int_{s_i}^{s_{i+1}}\{A_3^i(s)A_6^i(s)+A_{10}^i(s)A_{13}^i(s)\}ds\Big|$$
$$+|K\langle \Delta_i h,\Delta_i h\rangle((G_{\mathcal{P}}(s_{i+1},s_{i+1})-G_{\mathcal{P}}(s_i,s_{i+1}))-(G_{\mathcal{P}}(s_{i+1},s_i)-G_{\mathcal{P}}(s_i,s_i))$$
$$\cdot\frac{1}{\eta_i^4}\int_{s_i}^{s_{i+1}}\{A_3^i(s)A_7^i(s)+A_{10}^i(s)A_{14}^i(s)\}ds\Big|\Bigg\}$$

For each $i \in \{0, 1, \ldots, n-1\}$, let us label the above sixteen terms by $B_1^i, \ldots, B_{16}^i$, respectively, so that the inequality can be written more briefly as follows:

$$\left|\sum_{k\in S_{\mathcal{P}}}(2\nabla_k(Q_{\mathcal{P}}[h,k]),h)\right| \leq \sum_{i=0}^{n-1}\sum_{j=1}^{16} B_j^i.$$

Since $\sinh(x)$ and $\cosh(x)$ are odd and even functions, respectively, then

$$\begin{aligned} & A_1^i(s)A_5^i(s) + A_8^i(s)A_{12}^i(s) \\ = \quad & 4C_i^3(s)S_i(s) - 2\cosh(\frac{\Delta_i}{2})C_i^2(s)S_i(s) + C_i^3(s)S_i(s) - \cosh(\frac{\Delta_i}{2})C_i^2(s)S_i(s) \end{aligned}$$

is anti-symmetric about $\bar{s}_i = \frac{s_i+s_{i+1}}{2}$. Thus $B_2^i = 0$. Similarly, $B_j^i = 0$ for $j = 3, 5, 8, 9, 12, 14,$ and 15..

For the remaining terms, we will use the fact that there exists a bound M such that for all partitions $\mathcal{P}$, and all $(s,t) \in [0,1]^2$, $|G_{\mathcal{P}}(s,t)| < M$ and $|(G_{\mathcal{P}})_s(s,t)| < M$ (this can be verified by the definition of $G_{\mathcal{P}}$ and the continuity of sinh and cosh as in (**??**)). We also use the fact that $S_i(s)$ and $C_i(s)$ are analytic everywhere with the following Taylor series about $s = \bar{s}_i$:

$$\begin{aligned} S_i(s) &= \sinh(s - \bar{s}_i) = \sum_{n=0}^{\infty} \frac{(s - \bar{s}_i)^{2n+1}}{(2n+1)!} \\ C_i(s) &= \cosh(s - \bar{s}_i) = \sum_{n=0}^{\infty} \frac{(s - \bar{s}_i)^{2n}}{(2n)!}. \end{aligned}$$

For the B_1^i term we note that

$$\begin{aligned} & A_1^i(s)A_4^i(s) + A_8^i(s)A_{11}^i(s) \\ = \quad & 4C_i^2(s)S_i^2(s) - 2\cosh(\frac{\Delta_i}{2})C_i(s)S_i^2(s) + C_i^4(s) - \cosh(\frac{\Delta_i}{2})C_i^3(s). \end{aligned}$$

By symmetry about $\bar{s}_i = \frac{s_i+s_{i+1}}{2}$ and the Taylor expansion of the above term, it is clear that

$$\int_{s_i}^{s_{i+1}} \{A_1^i(s)A_4^i(s) + A_8^i(s)A_{11}^i(s)\}ds = f(\Delta_i),$$

where $f(x) = o(x)$ and $\frac{f(x)}{x}$ is increasing. Thus, by Lemma 3.2.2, there exists an $\epsilon_{4,2}(\mathcal{P})$ such that $\epsilon_{4,2}(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and

$$\begin{aligned} \sum_{i=0}^{n-1} B_2^i &\leq M \sum_{i=0}^{n-1} \left| K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle \int_{s_i}^{s_{i+1}} \{A_1^i(s)A_4^i(s) + A_8^i(s)A_{11}^i(s)\}ds \right| \\ &= M \sum_{i=0}^{n-1} |K\langle h(\bar{s}_i), h(\bar{s}_i)\rangle f(\Delta_i)| \\ &\leq \epsilon_{4,2}(\mathcal{P}) \left\| h \right\|^2. \end{aligned}$$

For the B_6^i term we use Lemma 3.2.3 to get

$$\begin{aligned}
\sum_{i=0}^{n-1} B_6^i &= \sum_{i=0}^{n-1} \left| K\langle \Delta_i h, \Delta_i h\rangle G_{\mathcal{P}}(\bar{s}_i, \bar{s}_i) \frac{1}{\eta_i^2} \int_{s_i}^{s_{i+1}} \{A_2^i(s)A_5^i(s) + A_9^i(s)A_{12}^i(s)\} ds \right| \\
&\leq (\sum_{i=0}^{n-1} |K\langle \Delta_i h, \Delta_i h\rangle|) M \sup_i \left\{ \left| \frac{1}{\Delta_i^2} \int_{s_i}^{s_{i+1}} \{A_2^i(s)A_5^i(s) + A_9^i(s)A_{12}^i(s)\} ds \right| \right\} \\
&\leq |\mathcal{P}| \, \|h\|^2 \, M \sup_i \left\{ \left| \frac{1}{\Delta_i^2} \int_{s_i}^{s_{i+1}} \{A_2^i(s)A_5^i(s) + A_9^i(s)A_{12}^i(s)\} ds \right| \right\}
\end{aligned}$$

Considering the Taylor series expansion of $S_i(s)$ and $C_i(s)$, one can see that

$$\begin{aligned}
& \int_{s_i}^{s_{i+1}} \{A_2^i(s)A_5^i(s) + A_9^i(s)A_{12}^i(s)\} ds \\
= & \int_{s_i}^{s_{i+1}} \{2C_i^2(s)S_i^2(s) + 2C_i^4(s) - 2C_i^3(s)\cosh(\frac{\Delta_i}{2}) \\
& + C_i^2(s)S_i^2(s) - C_i(s)S_i^2(s)\cosh(\frac{\Delta_i}{2})\} ds \\
= & \; O(\Delta_i^3)
\end{aligned} \tag{3.18}$$

where $O(x)$ is a function with the property that there exist $C > 0$ and $r > 0$ such that $\frac{|O(x)|}{|x|} < C$ for all $|x| < r$. It then follows that there exists an $N > 0$ such that for $|\mathcal{P}|$ sufficiently small,

$$\sup_i \left\{ \left| \frac{1}{\Delta_i^2} \int_{s_i}^{s_{i+1}} \{A_2^i(s)A_5^i(s) + A_9^i(s)A_{12}^i(s)\} ds \right| \right\} \leq N.$$

Thus, $\sum_{i=0}^{n-1} B_6^i \leq \epsilon_{4,6}(\mathcal{P}) \|h\|^2$, where $\epsilon_{4,6}(\mathcal{P}) = |\mathcal{P}| MN$.

For the B_7^i term we note that

$$A_2^i(s)A_6^i(s) + A_9^i(s)A_{13}^i(s) = 2S_i^4(s) - 2C_i(s)S_i^2(s) + 3C_i^2(s)S_i^2(s) - 3C_i(s)S_i^2(s)\cosh\left(\frac{\Delta_i}{2}\right)$$

By symmetry about $\bar{s}_i = \frac{s_i + s_{i+1}}{2}$ and the Taylor expansion of the above term, it is clear that

$$\frac{1}{\Delta_i} \int_{s_i}^{s_{i+1}} \{A_2^i(s)A_6^i(s) + A_9^i(s)A_{13}^i(s)\} ds = g(\Delta_i),$$

where $g(x) = o(x)$ and $\frac{g(x)}{x}$ is increasing. Next, by the Mean Value Theorem, for each $i \in \{0, 1, \ldots, n-1\}$ there exists an $\hat{s}_i \in (s_i, s_{i+1})$ such that $\frac{G_{\mathcal{P}}(s_{i+1}, \bar{s}_i) - G_{\mathcal{P}}(s_i, \bar{s}_i)}{\Delta_i} =$

$(G_{\mathcal{P}})_s(\hat{s}_i, \bar{s}_i)$. We then have

$$\begin{aligned}
\sum_{i=0}^{n-1} B_7^i &= \left| \sum_{i=0}^{n-1} K\langle \Delta_i h, h(\bar{s}_i)\rangle (G_{\mathcal{P}}(s_{i+1}, \bar{s}_i) - G_{\mathcal{P}}(s_i, \bar{s}_i)) \right. \\
&\quad \left. \cdot \frac{1}{\eta_i^2} \int_{s_i}^{s_{i+1}} \left\{ A_2^i(s) A_6^i(s) + A_9^i(s) A_{13}^i(s) \right\} ds \right| \\
&\leq \sum_{i=0}^{n-1} \left| K\langle \Delta_i h, h(\bar{s}_i)\rangle (G_{\mathcal{P}})_s(\hat{s}_i, \bar{s}_i) \frac{1}{\Delta_i} \int_{s_i}^{s_{i+1}} \left\{ A_2^i(s) A_6^i(s) + A_9^i(s) A_{13}^i(s) \right\} ds \right| \\
&\leq M \sum_{i=0}^{n-1} \left| K\langle \Delta_i h, h(\bar{s}_i)\rangle \frac{1}{\Delta_i} \int_{s_i}^{s_{i+1}} \left\{ A_2^i(s) A_6^i(s) + A_9^i(s) A_{13}^i(s) \right\} ds \right|.
\end{aligned}$$

It thus follows by Lemma 3.2.2 that there exists an $\epsilon_{4,7}(\mathcal{P})$ such that $\epsilon_{4,7}(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and $\sum_{i=0}^{n-1} B_7^i \leq \epsilon_{4,7}(\mathcal{P}) \|h\|^2$.

For the B_{10}^i term, from (3.18) and the reasoning for the B_7^i term it follows that there exists an $\epsilon_{4,10}(\mathcal{P})$ such that $\epsilon_{4,10}(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and $\sum_{i=0}^{n-1} B_{10}^i \leq \epsilon_{4,10}(\mathcal{P}) \|h\|^2$.

For the B_{11}^i term, we note that the same reasoning for the B_7^i term applies to show that there exists an $\epsilon_{4,11}(\mathcal{P})$ such that $\epsilon_{4,11}(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and $\sum_{i=0}^{n-1} B_{11}^i \leq \epsilon_{4,11}(\mathcal{P}) \|h\|^2$.

Going back to the B_4^i term, we note that

$$A_1^i(s) A_7^i(s) + A_8^i(s) A_{13}^i(s) = 5C_i^2(s) S_i^2(s) - 3\cosh(\frac{\Delta_i}{2}) C_i(s) S_i^2(s),$$

and thus the reasoning for the B_7^i term applies to show that there exists an $\epsilon_{4,4}(\mathcal{P})$ such that $\epsilon_{4,4}(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and $\sum_{i=0}^{n-1} B_4^i \leq \epsilon_{4,4}(\mathcal{P}) \|h\|^2$.

For the B_{13}^i term we note that

$$A_3^i(s) A_4^i(s) + A_{10}^i(s) A_{11}^i(s) = 5C_i^2(s) S_i^2(s) - \frac{1}{\varsigma_i} \eta_i^2 C_i(s) S_i^2(s) - \frac{1}{2\varsigma_i} \eta_i^2 C_i^3(s),$$

and so again the reasoning for the B_7^i term applies to show that there exists an $\epsilon_{4,13}(\mathcal{P})$ such that $\epsilon_{4,13}(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$ and $\sum_{i=0}^{n-1} B_{13}^i \leq \epsilon_{4,13}(\mathcal{P}) \|h\|^2$.

Finally, for the B_{16}^i term we note that, as in (3.18),

$$\begin{aligned}
&\int_{s_i}^{s_{i+1}} \{A_3^i(s) A_7^i(s) + A_{10}^i(s) A_{14}^i(s)\} ds \\
&= \int_{s_i}^{s_{i+1}} \{4C_i^2(s) S_i^2(s) + S_i^4(s) - \frac{3}{\varsigma_i} \eta_i^2 C_i(s) S_i^2(s)\} ds \\
&= O(\Delta_i^3).
\end{aligned}$$

So there exists an $N > 0$ such that for $|\mathcal{P}|$ sufficiently small,

$$\sup_i \left| \frac{1}{\Delta_i^3} \int_{s_i}^{s_{i+1}} \{A_3^i(s)A_7^i(s) + A_{10}^i(s)A_{14}^i(s)\} ds \right| \leq N.$$

Applying the Mean Value Theorem to $G_{\mathcal{P}}(\cdot, s_{i+1})$ and $G_{\mathcal{P}}(\cdot, s_i)$ and using Lemma 3.2.3, we get $\sum_{i=0}^{n-1} B_{16}^i \leq \epsilon_{4,16}(\mathcal{P}) \|h\|^2$, where $\epsilon_{4,16}(\mathcal{P}) = 2MN |\mathcal{P}|$.

We have thus shown that

$$\left| \sum_{k \in S_{\mathcal{P}}} (2\nabla_k(Q_{\mathcal{P}}[h,k]), h) \right| \leq \epsilon_4(\mathcal{P}) \|h\|^2,$$

where

$$\begin{aligned} \epsilon_4(\mathcal{P}) \quad \equiv \quad & \epsilon_{4,1}(\mathcal{P}) + \epsilon_{4,4}(\mathcal{P})\epsilon_{4,6}(\mathcal{P}) + \epsilon_{4,7}(\mathcal{P}) \\ & + \epsilon_{4,10}(\mathcal{P}) + \epsilon_{4,11}(\mathcal{P}) + \epsilon_{4,13}(\mathcal{P}) + \epsilon_{4,16}(\mathcal{P}). \end{aligned}$$

The proof of Theorem 3.2.1 is now complete by noting that

$$|((Ric - Ric_{\mathcal{P}})h, h)| \leq \epsilon(\mathcal{P}) \|h\|^2$$

where $\epsilon(\mathcal{P}) = \epsilon_1(\mathcal{P}) + \epsilon_2(\mathcal{P}) + \epsilon_3(\mathcal{P}) + \epsilon_4(\mathcal{P})$.□

Chapter 4

The Heat Kernel Measure

For each partition, $\mathcal{P}$, of $[0,1]$, let $\mathcal{B}_{\mathcal{P}}$ be the Borel σ-algebra on $G^{\mathcal{P}}$. We define $\mathcal{G}$ to be the smallest σ-algebra on $\mathcal{L}(G)$ such that the projections $\pi_{\mathcal{P}} : \mathcal{L}(G) \to G^{\mathcal{P}}$ are $\mathcal{G}/\mathcal{B}_{\mathcal{P}}$-measurable for each partition, $\mathcal{P}$. Our goal in this section is to show that for each fixed $g_0 \in \mathcal{L}(G)$ and each $t > 0$, there exists a unique heat kernel probability measure $\nu_t(g_0, \cdot)$ on $(\mathcal{L}(G), \mathcal{G})$, i.e., with the property that

$$(e^{t\frac{\Delta}{2}} f)(g_0) = \int_{\mathcal{L}(G)} f(g)\nu_t(g_0, dg)$$

for all bounded $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$, where Δ is defined below in (4.3) and $\mathcal{F}C^\infty(\mathcal{L}(G))$ are the cylinder functions defined in Section 1.2. We define $(e^{t\frac{\Delta}{2}} f)(g_0)$ to be the unique function, $u(t, g_0)$, on $[0,\infty) \times \mathcal{L}(G)$ that solves the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, g_0) &= \frac{1}{2}\Delta_{g_0} u(t, g_0) \\ \lim_{t \searrow 0} u(t, g_0) &= f(g_0). \end{aligned}$$

We note that $\{P_t \equiv e^{t\frac{\Delta}{2}}\}_{t\geq 0}$ is the semigroup of operators on $\mathcal{F}C^\infty(\mathcal{L}(G))$ with generator $L \equiv \frac{\Delta}{2}$ (i.e., $Lf = \lim_{t\searrow 0} \frac{P_t f - f}{t}$ $\forall\, f \in \mathcal{F}C^\infty(\mathcal{L}(G))$).

The heat kernel measure will be derived from an $\mathcal{L}(G)$-valued Brownian motion. The existence of such a process is due to Malliavin [17]. The procedure needed to construct the process for the free loop group is virtually the same as that for the pinned loop group as described by Driver [3], so we will only summarize it and verify any steps that would be different in the free loop case.

The procedure begins by constructing an $\mathcal{L}(\mathfrak{g})$-valued Brownian motion, $\{ \beta_t\}_{t\geq 0}$, where $\mathcal{L}(\mathfrak{g}) \equiv \{k \in C([0,1],\mathfrak{g}) : k(0) = k(1)\}$. We must first say what we mean by a "Brownian motion" on this space. Let

$$H^{BV} \equiv \{h \in H(\mathfrak{g}) : \exists \text{ rt. cts. function, } \lambda_h, \text{ of bounded variation with } h'(s) = \lambda_h(s) a.e.\} .$$

For each $h \in H^{BV}$, define $\alpha_h \in \mathcal{L}(\mathfrak{g})^*$ by

$$\alpha_h(k) \equiv -\int_0^1 \langle k(s), d\lambda_h(s)\rangle + \int_0^1 \langle k(s), h(s)\rangle ds \;\forall\, k \in \mathcal{L}(\mathfrak{g}), \tag{4.1}$$

where the first integral is defined in the Lebesgue-Stieljes sense. Note that for $h, k \in H^{BV} \subseteq H(\mathfrak{g})$, $\alpha_h(k) = (h,k)_{H(\mathfrak{g})}$.

For a given filtered probability space $(\mathcal{W}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$ satisfying the usual conditions (i.e., complete and right-continuous), we say that $\{\beta_t\}_{t\geq 0}$ is an $\mathcal{L}(\mathfrak{g})$-valued Brownian motion on $\mathcal{W}$ provided

1. β_t is $\mathcal{F}_t/\mathcal{G}$-measurable $\forall\, t > 0$,
2. for all $\omega \in \mathcal{W}$, the map $t \mapsto \beta_t(\omega)$ is continuous from $[0,\infty)$ to $\mathcal{L}(\mathfrak{g})$, and
3. $\{\beta_t\}_{t\geq 0}$ is a mean-zero Gaussian process with covariance

$$E[\alpha_h(\beta_t)\alpha_k(\beta_s)] = t \wedge \tau(h,k)$$

for all $h, k \in H^{BV}$ and $t, \tau \in [0,\infty)$.

It is a well know fact that such a Brownian motion exists so we will not prove it here (one method of proof uses Kolmogorov's extension theorem and the observation that $(H(\mathfrak{g}), \mathcal{L}(\mathfrak{g}))$ is an abstract Wiener space.

We will next show that an $\mathcal{L}(G)$-valued Brownian motion on $\mathcal{W}$ may be derived from $\{\beta_t\}_{t\geq 0}$. First, let us recall some definitions from the theory of stochastic integration. In general, if H and K are Hilbert spaces, let $B(H,K)$ be the space of bounded linear mappings from $H \to K$. If M and X are H and $B(H,K)$-valued semimartingales, respectively, then the following are the forward, backwards, and Stratonovich stochastic integrals, respectively, of X with respect to M:

$$\begin{aligned}\int_0^t XdM &\equiv \lim_{|\pi|\to 0}\sum_{i=0}^{\infty} X_{t_i^\pi}(M_{t\wedge t_{i+1}^\pi}-M_{t\wedge t_i^\pi}),\\ \int_0^t X\overleftarrow{dM} &\equiv \lim_{|\pi|\to 0}\sum_{i=0}^{\infty} X_{t_{i+1}^\pi}(M_{t\wedge t_{i+1}^\pi}-M_{t\wedge t_i^\pi}),\\ \int_0^t X\delta M &\equiv \lim_{|\pi|\to 0}\sum_{i=0}^{\infty} \frac{X_{t_i^\pi}+X_{t_{i+1}^\pi}}{2}(M_{t\wedge t_{i+1}^\pi}-M_{t\wedge t_i^\pi}),\end{aligned}$$

where the limits are in probability and each $\pi = \{0 = t_0^\pi < t_1^\pi < t_2^\pi < \cdots \infty\}$ is a partition of $[0,\infty)$.

The next theorem is due to Malliavin [17]. Assume $\mathfrak{g}_0$ is an orthonormal basis of $\mathfrak{g} = Lie(G)$, and $(\mathcal{W}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$ is the filtered probability space from above.

Theorem 4.0.1 *Suppose that G is a compact Lie group with Ad_G -invariant inner product, $\langle\cdot,\cdot\rangle$, on $\mathfrak{g}$. Then for each $g_0 \in \mathcal{L}(G)$, there exists a continuous adapted $\mathcal{L}(G)$ -valued process, $\{\Sigma_t\}_{t\leq 0}$, on $\mathcal{W}$ which solves the following family of Stratonovich differential equations:*

$$\begin{aligned}\Sigma_{\delta t}(s) &= \sum_{A\in\mathfrak{g}_0}\tilde{A}(\Sigma_t(s))\beta_{\delta t}^A(s)\\ \Sigma_0(s) &= g_0(s) \quad \forall\, s\in[0,1],\end{aligned} \tag{4.2}$$

where $\tilde{A}$ is the left-invariant vector field on G with $\tilde{A}(e) = A$, and $\beta_t^A(s) \equiv \langle A, \beta_t(s)\rangle$.

The interpretation of (4.2), as defined in Chapter IV of Ikeda and Watanabe [12], is that for any $f \in C^\infty(G)$,

$$f(\Sigma_t(s)) - f(\Sigma_0(s)) = \sum_{A\in\mathfrak{g}_0}\int_0^t \tilde{A}f(\Sigma_\tau(s))\beta_{\delta\tau}^A(s)$$

for all $s \in [0,1]$. The existence of a solution to (4.2) is actually proved in Ikeda and Watanabe [12]. It is the existence of a jointly continuous version of the solution that requires further work. The proof supplied by Driver [3] for the existence of such a jointly continuous version in the pinned loop case can be applied verbatim for the free loop case. The key to the proof is the application of Kolmogorov's continuity criterion. The only difference in our case is that we are using a different Green's function, $G(s,t)$, as defined in (2.4). We only need to note that $G(s,t)$ and $G_s(s,t)$ are uniformly bounded for all $(s,t) \in [0,1]^2$, then the argument in [3] goes through.

We now explain why $\{\Sigma_t\}_{t\geq 0}$ can be called a "Brownian motion" on $\mathcal{L}(G)$. First note that the Laplacian, Δ, on $\mathcal{L}(G)$ is defined by

$$\Delta f \equiv \sum_{h\in S} \nabla_h^2 f \equiv \sum_{h\in S} \nabla_h(\nabla_h f), \tag{4.3}$$

for all $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$, where S is an orthonormal basis for $H(\mathfrak{g})$. We define $\nabla_h f$ by

$$\begin{aligned} (\nabla_h f)(g) &\equiv \tilde{h} f(g) \\ &= \frac{d}{dt} \mid_{t=0} f(ge^{th}) \ \forall g \in \mathcal{L}(G), \end{aligned}$$

where $t \longmapsto ge^{th}$ is a path in $\mathcal{L}(G)$ with

$$\frac{d}{dt} \mid_{t=0} ((ge^{th})(s)) = \widetilde{h(s)} \mid_{g(s)} \in T_{g(s)}G$$

for each $s \in [0,1]$. Note that if $f = F \circ \pi_{\mathcal{P}}$ with, $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$, then

$$(\nabla_h f)(g) = \sum_{i=0}^{n-1} (h(s_i)^{(i)} F)(\pi_{\mathcal{P}} g),$$

where for any $A \in \mathfrak{g}$, $A^{(i)}$ is defined to be the left invariant vector field on $G^{\mathcal{P}}$ given by

$$(A^{(i)}F)(g_0,\ldots,g_{n-1}) \equiv \frac{d}{dr} \mid_{r=0} F(g_0,\ldots,g_{i-1},g_i e^{rA},g_{n+1},\ldots,g_{n-1}), \tag{4.4}$$

where e is the exponential map from $\mathfrak{g}$ to G (see Section 3.2 in do Carmo [7]).

Theorem 4.0.2 *Suppose $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$ is a given partition and $T > 0$. Suppose $f : [0,T] \times \mathcal{L}(G) \to R$ is a function of the form $f(t,g) = F(t, g_{\mathcal{P}})$, where $g_{\mathcal{P}} = \pi_{\mathcal{P}} g \in G^{\mathcal{P}}$, such that $F : [0,T] \times G^{\mathcal{P}} \to R$ is continuous, $F \mid_{(0,T)\times G^{\mathcal{P}}}$ is smooth with first and second order derivatives extending to continuous functions on $[0,T] \times G^{\mathcal{P}}$. Then $\{M_t\}_{t\in[0,T]}$, defined by*

$$M_t \equiv f(t,\Sigma_t) - f(0,\Sigma_0) - \int_0^t ((\frac{\partial}{\partial t} + \frac{1}{2}\Delta) f(\tau,\cdot))(\Sigma_\tau) d\tau \ \forall t \in [0,T], \tag{4.5}$$

is a martingale. In analogy with the finite dimensional case, we take this to be what defines $\{\Sigma_t\}_{t\geq 0}$ as a Brownian motion.

Proof: Let $(\Sigma_{\mathcal{P}})_t \equiv (\Sigma_t(s_0), \ldots, \Sigma_t(s_{n-1})) \in G^{\mathcal{P}}$ for all $t \in [0,1]$. By the fact that $\Sigma_t(s_i)$ solves Equation (4.2) for each $s_i \in \mathcal{P}$, it can be shown that $(\Sigma_{\mathcal{P}})_t$ satisfies

$$\delta(\Sigma_{\mathcal{P}})_t = \sum_{i=0}^{n-1} \sum_{A \in \mathfrak{g}_0} A^{(i)}((\Sigma_{\mathcal{P}})_t)\beta^A_{\delta t}(s_i).$$

By an application of Ito's formula, we get

$$d[f(t, \Sigma_t) = \frac{\partial}{\partial \tau} f(\tau, \Sigma_t) \mid_{\tau=t} + \sum_{i=0}^{n-1} \sum_{A \in \mathfrak{g}_0} (A^{(i)} F(t, \cdot))((\Sigma_{\mathcal{P}})_t)\beta^A_{dt}(s_i) + \frac{1}{2}\Delta f(t, \Sigma_t)dt.$$

Thus,

$$M_t = \sum_{i=0}^{n-1} \sum_{A \in \mathfrak{g}_0} \int_0^t (A^{(i)} F(\tau, \cdot))((\Sigma_{\mathcal{P}})_\tau)\beta^A_{\delta\tau}(s_i).$$

Since $\{\beta^A_t(s_i)\}_{t \geq 0}$ is a martingale for each s_i and $(A^{(i)}F(\tau, \cdot))((\Sigma_{\mathcal{P}})_\tau)$ is bounded for each i and A, then it is well known that this implies $\{M_t\}_{t \geq 0}$ is a martingale (see Theorem 20 on p. 20 in Protter [18]). □

We are now ready to define the heat kernel measure on $(\mathcal{L}(G), \mathcal{G})$. For each $t > 0$, let $\nu_t(e, \cdot)$ be the law of Σ_t, where $e \in \mathcal{L}(G)$ is the identity loop. In other words,

$$\nu_t(e, A) \equiv P(\Sigma_t \in A) \;\forall\; \in \mathcal{G}.$$

And for any $g_0 \in \mathcal{L}(G)$, define $\nu_t\,(g_0, \cdot)$ by

$$\begin{aligned} \nu_t(g_0, A) &= (L_{g_0 *}\nu_t(e, \cdot))(A) \\ &\equiv \nu_t(e, L^{-1}_{g_0}(A)) \\ &= P(L_{g_0} \circ \Sigma_t \in A). \end{aligned} \tag{4.6}$$

Theorem 4.0.3 *For all bounded $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$,*

$$(e^{t\frac{\Delta}{2}} f)(g_0) = \int_{\mathcal{L}(G)} f(g)\nu_t(g_0, dg) \;\forall\, g_0 \in \mathcal{L}(G).$$

Proof: Let $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$ be of the form $f(g) = F(g_{\mathcal{P}})$, $F \in C^\infty(G^{\mathcal{P}})$, for a given partition $\mathcal{P}$, and let $g_0 \in \mathcal{L}(G)$. Define

$$u(t, g_0) \equiv \int_{\mathcal{L}(G)} f(g)\nu_t(g_0, dg) \;\forall\, t \geq 0.$$

Then by (4.6),

$$u(t, g_0) = E[f(L_{g_0} \circ \Sigma_t)] = E[\tilde{f}(g_0, \Sigma_t)] = E[\tilde{F}(g_0, (\Sigma_{\mathcal{P}})_t)],$$

where $\tilde{f}(g_0, \cdot) \equiv f \circ L_{g_0}(\cdot)$, $\tilde{F}(g_0, \cdot) \equiv F \circ L_{(\pi_{\mathcal{P}} \circ g_0)}(\cdot) \in C^\infty(G^{\mathcal{P}})$, and $(\Sigma_{\mathcal{P}})_t \equiv \pi_{\mathcal{P}}(\Sigma_t)$. From (4.5) it follows that

$$\begin{aligned} u(t, g_0) &= E[M_t + \tilde{f}(g_0, \Sigma_0) + \int_0^t \frac{1}{2}\Delta_2 \tilde{f}(g_0, \Sigma_\tau) d\tau] \\ &= E[\tilde{f}(g_0, \Sigma_0)] + \int_0^t E(\frac{1}{2}\Delta_2 \tilde{f}(g_0, \Sigma_\tau)) d\tau, \end{aligned}$$

where we have used the fact that M_t is a martingale with $M_0 = 0$ a.e.. We will use the notation Δ_2 and $\Delta_{\mathcal{P},2}$, where appropriate, to indicate that we are applying Δ or $\Delta_{\mathcal{P}}$ to the second variable of the function on which they are operating. For example, $\Delta_2 \tilde{f}(g_0, \Sigma_\tau) \equiv \Delta(\tilde{f}(g_0, \cdot))(\Sigma_\tau)$. It is easy to check that $\Delta_2 \tilde{f}(g_0, \Sigma_\tau) = \Delta_{\mathcal{P},2} \tilde{F}(g_0, (\Sigma_{\mathcal{P}})_\tau)$, where

$$\Delta_{\mathcal{P},2} \tilde{F}(g_0, g_{\mathcal{P}}) = \sum_{A \in \mathfrak{g}_0} \sum_{i,j=0}^{n-1} G(s_i, s_j)(A^{(i)} A^{(j)} \tilde{F}(g_0, \cdot))(g_{\mathcal{P}})$$

for all $g_{\mathcal{P}} \in G^{\mathcal{P}}$, and

$$(\Sigma_{\mathcal{P}})_\tau \equiv \pi_{\mathcal{P}}(\Sigma_\tau) = (\Sigma_\tau(s_0), \ldots, \Sigma_\tau(s_{n-1})) \in G^{\mathcal{P}}.$$

$A^{(i)}$ is the notation we introduced in (4.4). We thus have

$$u(t, g_0) = E[\tilde{F}(g_0, (\Sigma_{\mathcal{P}})_0)] + \int_0^t E[\frac{1}{2}\Delta_{\mathcal{P},2} \tilde{F}(g_0, (\Sigma_{\mathcal{P}})_\tau)] d\tau.$$

Intuitively, we would like for $E[\frac{1}{2}\Delta_{\mathcal{P},2} \tilde{F}(g_0, (\Sigma_{\mathcal{P}})_\tau)] = \frac{1}{2}\Delta_{\mathcal{P},g_0} E[\tilde{F}(g_0, (\Sigma_{\mathcal{P}})_\tau)]$. This is indeed a true statement, but not trivial. One may refer to the chapter on diffusion processes on manifolds (Chapter V, p. 254) in Ikeda and Watanabe [12] for a proof. We can then write

$$u(t, g_0) = \tilde{F}(g_0, (\Sigma_{\mathcal{P}})_0) + \int_0^t \frac{1}{2}\Delta_{\mathcal{P},g_0} E[\tilde{F}(g_0, (\Sigma_{\mathcal{P}})_\tau)] d\tau,$$

and thus

$$\begin{aligned} \frac{\partial u}{\partial t}(t, g_0) &= \frac{1}{2}\Delta_{\mathcal{P},g_0} E[\tilde{F}(g_0, (\Sigma_{\mathcal{P}})_t)] = \frac{1}{2}\Delta_{g_0} E[\tilde{f}(g_0, \Sigma_t)] \\ &= \frac{1}{2}\Delta_{g_0} u(t, g_0). \end{aligned}$$

And by the dominated convergence theorem and the fact that $\Sigma_0 = e$ a.e., we have

$$\begin{aligned} \lim_{t \searrow 0} u(t, g_0) &= \lim_{t \searrow 0} E[f(L_{g_0} \circ \Sigma_t)] \\ &= E[f(L_{g_0} \circ e)] \\ &= E[f(g_0)] = f(g_0). \end{aligned}$$

Therefore, we have

$$(e^{t\frac{\Delta}{2}}f)(g_0) = \int_{\mathcal{L}(G)} f(g)\nu_t(g_0, dg). \ \square$$

For a fixed $g_{0\mathcal{P}} \in G^{\mathcal{P}}$ and $t > 0$, define the probability measure $\nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, \cdot)$ on $G^{\mathcal{P}}$ by

$$\nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, A) \equiv P(L_{g_{0\mathcal{P}}} \circ (\Sigma_{\mathcal{P}})_t \in A) \ \forall A \in \mathcal{G}^{\mathcal{P}}.$$

It then follows from the above discussion that $\{(\Sigma_{\mathcal{P}})_t\}_{t\geq 0}$ is a diffusion process on $G^{\mathcal{P}}$ with generator $\Delta_{\mathcal{P}}$ and that

$$(e^{t\frac{\Delta_{\mathcal{P}}}{2}}F)(g_{0\mathcal{P}}) = \int_{G^{\mathcal{P}}} F(g_{\mathcal{P}})\nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}) \tag{4.7}$$

for all $F \in C^\infty(G^{\mathcal{P}})$ and $g_{0\mathcal{P}} \in G^{\mathcal{P}}$.

The following lemma will be useful in proving the logarithmic Sobolev inequality.

Lemma 4.0.4 *For a partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$,*

$$\Delta_{\mathcal{P}} = \sum_{h\in\Gamma} \tilde{h}^2,$$

where Γ is an orthonormal basis of $(\mathfrak{g}^{\mathcal{P}}, (\cdot,\cdot)_{\mathcal{P}})$ and $(\cdot,\cdot)_{\mathcal{P}}$ is defined as in (3.1).

Proof: For each $s_i \in \mathcal{P}$, let $G_{\mathcal{P}}(s_i) \equiv (G_{\mathcal{P}}(s_i, s_0), \ldots, G_{\mathcal{P}}(s_i, s_{n-1})) \in R^n$, and for $A \in \mathfrak{g}$, $G_{\mathcal{P}}(s_i)A \equiv (G_{\mathcal{P}}(s_i, s_0)A, \ldots, G_{\mathcal{P}}(s_i, s_{n-1})A) \in \mathfrak{g}^{\mathcal{P}}$. For $F \in C^\infty(G^{\mathcal{P}})$ and $g_{\mathcal{P}} \in G^{\mathcal{P}}$, we then have

$$\begin{aligned} \sum_{h\in\Gamma} \tilde{h}^2 F(g_{\mathcal{P}}) &= \sum_{h\in\Gamma} \tilde{h}(\sum_{A\in\mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)}F(g_{\mathcal{P}}))\langle A, h_i\rangle \\ &= \sum_{h\in\Gamma} \sum_{A,B\in\mathfrak{g}_0} \sum_{i,j=0}^{n-1} (B^{(j)}A^{(i)}F(g_{\mathcal{P}}))\langle A, h_i\rangle\langle B, h_j\rangle, \end{aligned}$$

where each $h \in \Gamma$ is denoted by $h = (h_0, h_1, \ldots, h_{n-1})$. Recalling that $(Q_{ij}) = (G_{ij})^{-1}$,

$$\begin{aligned} \langle A, h_i\rangle &= \sum_{k,l=0}^{n-1} Q_{kl}\langle G(s_i, s_k)A, h_l\rangle \\ &= (G_{\mathcal{P}}(s_i)A, h)_{\mathcal{P}}. \end{aligned}$$

Using the same calculations for $\langle B, h_j \rangle$, we see that

$$\begin{aligned}\sum_{h\in\Gamma} \tilde{h}^2 F(g_{\mathcal{P}}) &= \sum_{h\in\Gamma} \sum_{A,B\in\mathfrak{g}_0} \sum_{i,j=0}^{n-1} (B^{(j)} A^{(i)} F(g_{\mathcal{P}}))(G_{\mathcal{P}}(s_i)A, h)_{\mathcal{P}} (G_{\mathcal{P}}(s_j)B, h)_{\mathcal{P}} \\ &= \sum_{A,B\in\mathfrak{g}_0} \sum_{i,j=0}^{n-1} (B^{(j)} A^{(i)} F(g_{\mathcal{P}}))(G_{\mathcal{P}}(s_i)A, G_{\mathcal{P}}(s_j)B)_{\mathcal{P}} \\ &= \sum_{A,B\in\mathfrak{g}_0} \sum_{i,j=0}^{n-1} (B^{(j)} A^{(i)} F(g_{\mathcal{P}}))\{\sum_{k,l=0}^{n-1} Q_{kl} G(s_i, s_k) G(s_j, s_l) \langle A, B\rangle\} \\ &= \sum_{A\in\mathfrak{g}_0} \sum_{i,j=0}^{n-1} (A^{(j)} A^{(i)} F(g_{\mathcal{P}})) G(s_i, s_j) = \Delta_{\mathcal{P}} F(g_{\mathcal{P}}). \ \square\end{aligned}$$

We next wish to observe that since $(\cdot,\cdot)_{\mathcal{P}}$ is left invariant on $G^{\mathcal{P}}$, then the Riemannian volume measure, $\lambda_{\mathcal{P}}$, on $G^{\mathcal{P}}$ is a left Haar measure (i.e., $\lambda_{\mathcal{P}}$ is a left invariant, nonzero Radon measure). And since $G^{\mathcal{P}}$ is compact, it is thus unimodular, i.e., $\lambda_{\mathcal{P}}$ is both a left and right Haar measure (see Proposition 10.16 in Folland [8]]). It is well known that in this case, $\sum_{h\in\Gamma} \tilde{h}^2$, and thus $\Delta_{\mathcal{P}}$, is the Laplace Beltrami operator on $G^{\mathcal{P}}$ (see Proposition 5.1 in Driver [3] and Remark (2.2) in Driver and Gross [6]. Using this fact, it then follows by Theorem 2.1 in Driver [3] that for $t > 0$, there exists a smooth function $p_t(x_{\mathcal{P}}, g_{\mathcal{P}})$ on $G^{\mathcal{P}} \times G^{\mathcal{P}}$ such that

$$(e^{t\frac{\Delta_{\mathcal{P}}}{2}} F)(x_{\mathcal{P}}) = \int_{G^{\mathcal{P}}} p_t(x_{\mathcal{P}}, g_{\mathcal{P}}) F(g_{\mathcal{P}}) \lambda_{\mathcal{P}}(dg_{\mathcal{P}}) \ \forall F \in C^\infty(G^{\mathcal{P}}).$$

By (4.7), we then have

$$\int_{G^{\mathcal{P}}} F(g_{\mathcal{P}}) \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}) = \int_{G^{\mathcal{P}}} F(g_{\mathcal{P}}) p_t(g_{0\mathcal{P}}, g_{\mathcal{P}}) \lambda_{\mathcal{P}}(dg_{\mathcal{P}})$$

for all $g_{0\mathcal{P}} \in G^{\mathcal{P}}$ and $F \in C^\infty(G^{\mathcal{P}})$.

Chapter 5

Integration By Parts on $\mathcal{L}(G)$

The motivation for developing an integration by parts formula on $\mathcal{L}(G)$ is related to the task of making our logarithmic Sobolev inequality a general as possible. In other words, we want the inequality to hold for a large class of functions. This will be accomplished by showing the closability of a certain symmetric, quadratic form that appears in the inequality.

5.1 Closed and Closable Quadratic Forms

Let us take a moment to review some facts about quadratic forms. One may refer to Ma and Röckner [16] or Fukushima, Oshima, and Takeda [9] for more on this subject. Let H be a general Hilbert space (H is not necessarily the same as $H(\mathfrak{g})$ from above). Let $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbf{R}$ be a bilinear form on a linear subspace $\mathcal{D}(\mathcal{E}) \to H$. We will assume $\mathcal{D}(\mathcal{E})$ is dense in H with respect to the Hilbert norm on H, and $\mathcal{E}$ is positive symmetric. Let $\mathcal{E} : \mathcal{D}(\mathcal{E}) \to \mathbf{R}$ also denote the associated quadratic form defined by

$$\mathcal{E}(v) \equiv \mathcal{E}(v, v) \ \ \forall\, v \in \mathcal{D}(\mathcal{E}).$$

Define the inner product, $(\cdot, \cdot)_1$, on $\mathcal{D}(\mathcal{E})$ by

$$(v, w)_1 \equiv (v, w)_H + \mathcal{E}(v, w) \ \ \forall\, v, w \in \mathcal{D}(\mathcal{E}),$$

where $(\cdot, \cdot)_H$ is the Hilbert inner product on H. Define $\|\cdot\|_1$ by $\|v\|_1 \equiv \sqrt{(v, v)_1}$ for all $v \in \mathcal{D}(\mathcal{E})$.

Definition 5.1.1 *$\mathcal{E}$ is closed if and only if $(\mathcal{D}(\mathcal{E}), (\cdot,\cdot)_1)$ is a Hilbert space.*

The following proposition is easy to verify.

Proposition 5.1.2 *$\mathcal{E}$ is closed if and only if for all sequences $\{v_n\} \subseteq \mathcal{D}(\mathcal{E})$ such that $\|v_n - v\|_H \to 0$ as $n \to \infty$ for some $v \in H$ and $\mathcal{E}(v_n - v_m) \to 0$ as $m, n \to \infty$, we have $v \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(v_n - v) \to 0$ as $m, n \to \infty$.*

If $(\mathcal{D}(\mathcal{E}), (\cdot,\cdot)_1)$ is not a Hilbert space, it may be possible to extend $\mathcal{E}$ to a closed quadratic form, $\overline{\mathcal{E}}$, on some larger space $H_1 \supset \mathcal{D}(\mathcal{E})$. This is possible when $\mathcal{E}$ is closable.

Definition 5.1.3 *$\mathcal{E}$ is closable if and only if for all sequences $\{v_n\} \subseteq \mathcal{D}(\mathcal{E})$ such that $\|v_n\|_H \to 0$ as $n \to \infty$ and $\mathcal{E}(v_n - v_m) \to 0$ as $m, n \to \infty$, we have $\mathcal{E}(v_n) \to 0$ as $n \to \infty$.*

If $\mathcal{E}$ is closable, we can define H_1 to be $H_1 \equiv \mathcal{C}/\sim$, where $\mathcal{C}$ is the set of Cauchy sequences in $(\mathcal{D}(\mathcal{E}), (\cdot,\cdot)_1)$, i.e.,

$$\mathcal{C} \equiv \{\{v_n\} \subseteq \mathcal{D}(\mathcal{E}) : \lim_{m,n\to\infty} \|v_n - v_m\|_1 = 0\},$$

and $\sim$ is the equivalence relation

$$\{v_n\} \sim \{w_n\} \Leftrightarrow \lim_{n\to\infty} \|v_n - w_n\|_1 = 0.$$

H_1 is thus the set of all equivalence classes in $\mathcal{C}$.

Given $\bar{v}, \bar{w} \in H_1$, where $\bar{v}$ and $\bar{w}$ are the equivalence classes of $\{v_n\}$ and $\{w_n\}$, define $((\cdot,\cdot))_1$ on H_1 by

$$((\bar{v}, \bar{w}))_1 \equiv \lim_{n\to\infty} (v_n, w_n)_1.$$

One can check that the definition of $\sim$ implies that $((\cdot,\cdot))_1$ is well-defined.

We can see that $\mathcal{D}(\mathcal{E})$ is imbedded in H_1 by the map

$$\iota : \mathcal{D}(\mathcal{E}) \to H_1$$

$$v \mapsto \bar{v},$$

where $\bar{v}$ is the equivalence class of $\{v_n\}$ and $v_n = v \; \forall n \geq 1$. Clearly, ι is an isometry.

We can also imbed H_1 in H by

$$\hat{\iota} : H_1 \to H$$

$$\bar{v} \mapsto \lim_{n\to\infty} v_n,$$

where $\{v_n\}$ is a representative of the equivalence class of $\bar{v}$ and the limit is with respect to $\|\cdot\|_H$. Since $\{v_n\}$ is Cauchy in $\|\cdot\|_1$ and $\|\cdot\|_H \le \|\cdot\|_1$, then $\{v_n\}$ is Cauchy in $\|\cdot\|_H$ and thus $\lim_{n\to\infty} v_n$ is guaranteed to exist.

Note that $\hat{\imath}$ is injective precisely due to the fact that $\mathcal{E}$ is closable. Indeed, if $\{v_n\} \in \bar{v} \in H_1$ and $\{w_n\} \in \bar{w} \in H_1$ with $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n$, then $\|v_n - w_n\|_H \to 0$ as $n \to \infty$ and

$$\mathcal{E}((v_n - w_n) - (v_m - w_m)) \le \mathcal{E}(v_n - v_m) - \mathcal{E}(w_n - w_m) \to 0 \text{ as } m, n \to \infty.$$

Hence, the closability of $\mathcal{E}$ gives $\mathcal{E}(v_n - w_n) \to 0$ as $n \to \infty$ and thus $\|v_n - w_n\|_1 \to 0$ as $n \to \infty$.

The form $\mathcal{E}$ can be extended to H_1 by defining $\overline{\mathcal{E}} : H_1 \times H_1 \to \mathbf{C}$ by

$$\overline{\mathcal{E}}(\bar{v}, \bar{w}) \equiv \lim_{n\to\infty} \mathcal{E}(v_n, w_n) \;\; \forall\, \bar{v}, \bar{w} \in H_1, \tag{5.1}$$

where $\{v_n\} \in \bar{v}$ and $\{w_n\} \in \bar{w}$. (One may use the Cauchy-Schwartz inequality and the fact that $\{v_n\}$ is $\|\cdot\|_1$-Cauchy to show the limit on the right in (5.1) exists.)

If we identify H_1 with $\hat{\imath}(H_1)$, i.e., let

$$\begin{aligned} H_1 \;\; &\equiv \;\; \{v \in H : \exists \text{ a } (\cdot,\cdot)_1 \text{ -Cauchy sequence } \{v_n\} \subseteq \mathcal{D}(\mathcal{E}) \\ &\qquad \text{with } \|v_n - v\|_H \to 0 \text{ as } n \to \infty\}, \end{aligned}$$

then we can extend $\mathcal{E}$ to $\overline{\mathcal{E}}$ on $H_1 \subseteq H$ in the same way as above, i.e.,

$$\mathcal{E}(v, w) \equiv \lim_{n\to\infty} \mathcal{E}(v_n, w_n) \;\; \forall\, v, w \in H_1,$$

where $\{v_n\}, \{w_n\} \subseteq \mathcal{C}$ and $\|v_n - v\|_H \to 0$, $\|w_n - w\|_H \to 0$ as $n \to \infty$. This allows us to view H_1 as an actual subspace of H and not a set of equivalence classes.

Now let H and K both be Hilbert spaces. It will be useful for us to consider the closability of a particular type of form defined in terms of a linear operator $A : H \to K$ as follows:

$$\begin{aligned} \mathcal{E}_A \;\; &: \;\; D(\mathcal{E}_A) \times D(\mathcal{E}_A) \to \mathbf{C} \\ \mathcal{E}_A(v, w) \;\; &\equiv \;\; (Av, Aw)_K \;\; \forall\, v, w \in D(A), \end{aligned}$$

where $D(\mathcal{E}_A) = D(A)$, the domain of A, is assumed to be dense in H. As we will show below, it turns out that $\mathcal{E}_A$ is closable if and only if A is a closable operator.

Letting $L(H,K)$ denote the space of linear operators from H to K, and $\Gamma(A)$ denote the graph of $A \in L(H,K)$, i.e.,

$$\Gamma(A) \equiv \{(v, Av) : v \in D(A)\} \subseteq H \times K,$$

recall the following definitions.

Definition 5.1.4 *$A \in L(H,K)$ is* closed *iff $\Gamma(A)$ is closed in $H \times K$. $A \in L(H,K)$ is* closable *iff $\overline{\Gamma(A)}$ is the graph of an operator in $L(H,K)$.*

If A is closable, we will let $\overline{A}$ be its closure, i.e., $\overline{A}$ is the linear operator with the property $\Gamma(\overline{A}) = \overline{\Gamma(A)}$. It is not hard to check that A is closable if and only if given $\{v_n\} \subseteq D(A)$ with$\|v_n\|_H \to 0$ and $\|Av_n - u\|_K \to 0$ as $n \to \infty$ for some $u \in K$, then $u = 0$. The next lemma gives a very convenient necessary and sufficient condition for closability of a linear operator which will be used in Theorem 5.6.1.

Lemma 5.1.5 *$A \in L(H,K)$ is closable if and only if $D(A^*)$ is dense in K.*

Proof: Recall that $A^* : D(A^*) \subseteq K \to H$ is the adjoint of A where $u \in D(A^*)$ if and only if $\exists\, w \in H$ such that

$$\langle Av, u \rangle = \langle v, w \rangle \;\; \forall\, v \in D(A),$$

in which case $A^*u \equiv w$.

We first introduce the following notation. Let $(\cdot,\cdot)_{K\times H}$ be the inner product on $K \times H$ defined by

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{K\times H} \equiv \langle u_1, u_2 \rangle_K + \langle v_1, v_2 \rangle_H \;\; \forall\, (u_1, v_1), (u_2, v_2) \in K \times H.$$

Let $T : H \times K \to K \times H$ be the unitary operator

$$T : (v, u) \longmapsto (u, -v).$$

Note that $\Gamma(A^*) = [T\Gamma(A)]^\perp$ by the following:

$$\begin{aligned}
(u,v) &\in \Gamma(A^*) \Leftrightarrow \langle Aw, u\rangle_K = \langle w, v\rangle_H \ \ \forall w \in D(A) \\
&\Leftrightarrow \langle Aw, u\rangle_K + \langle -w, v\rangle_H = 0 \ \ \forall w \in D(A) \\
&\Leftrightarrow \langle (Aw, -w), (u,v)\rangle_{K\times H} = 0 \ \ \forall w \in D(A) \\
&\Leftrightarrow \langle T(w, Aw), (u,v)\rangle_{K\times H} = 0 \ \ \forall w \in D(A) \\
&\Leftrightarrow (u,v) \perp T\Gamma(A).
\end{aligned}$$

It is a well-known fact that since $\overline{\Gamma(A)}$ is closed, then $\overline{\Gamma(A)} = (\overline{\Gamma(A)}^\perp)^\perp$ (see Lemma 3.3-6 in Kreyszig [14]). But by the continuity of the inner product on $H \times K$, $\overline{\Gamma(A)}^\perp = \Gamma(A)^\perp$, so $\overline{\Gamma(A)} = (\Gamma(A)^\perp)^\perp$. The fact that T is unitary then gives

$$\begin{aligned}
\overline{\Gamma(A)} &= (\Gamma(A)^\perp)^\perp \\
&= T^{-1}((T\Gamma(A)^\perp)^\perp) \\
&= T^{-1}(\Gamma(A^*)^\perp).
\end{aligned}$$

We can now finish the proof by contradiction:

$$\begin{aligned}
A \text{ is not closable} &\Leftrightarrow \exists\, u \neq 0 \text{ in } K \text{ and } \{v_n\} \subseteq D(A) \text{ with} \\
&\qquad v_n \to 0 \text{ and } Av_n \to u \text{ as } n \to \infty \\
&\Leftrightarrow (0,u) \in \overline{\Gamma(A)} = T^{-1}(\Gamma(A^*)^\perp) \\
&\Leftrightarrow T(0,u) = (u,0) \in \Gamma(A^*)^\perp \\
&\Leftrightarrow \langle (u,0), (w, A^*w)\rangle_{K\times H} = 0 \ \ \forall w \in D(A^*) \\
&\Leftrightarrow \langle u, w\rangle_K = 0 \ \ \forall w \in D(A^*) \\
&\Leftrightarrow u \perp D(A^*) \text{ for some } u \neq 0 \text{ in } K \\
&\Leftrightarrow D(A^*)^\perp \neq \{0\} \\
&\Leftrightarrow K \backslash \overline{D(A^*)} \neq \phi.
\end{aligned}$$

See Lemma 3.3-7 in Kreyszig [14] for an explanation of the last equivalence above. □

We can now make the following link between A and $\mathcal{E}_A$.

Lemma 5.1.6 *$\mathcal{E}_A$ is closable if and only if A is closable.*

Proof: Suppose $\mathcal{E}_A$ is closable. By definition, this means that if $\{v_n\} \subseteq D(\mathcal{E}_A)$ with $\|v_n\|_H \to 0$ and $\mathcal{E}_A(v_n - v_m) \to 0$ as $m, n \to \infty$, then $\mathcal{E}_A(v_n, v_n) = \|Av_n\|_K \to 0$. To show A is closable, we assume $\{v_n\} \subseteq D(A) = D(\mathcal{E}_A)$ with $\|v_n\|_H \to 0$ and $\|Av_n - u\|_K \to 0$ for some $u \in K$ and show this implies $u = 0$. But $\|Av_n - u\|_K \to 0$ implies $\{Av_n\}$ is Cauchy in K and thus $\mathcal{E}_A(v_n - v_m) = \|Av_n - Av_m\|_K \to 0$ as $m, n \to \infty$. By the closability of $\mathcal{E}_A$, we see that $\|Av_n\|_K = \mathcal{E}_A(v_n, v_n) \to 0$ as $n \to \infty$. By uniqueness of limits, it must be that $u = 0$.

Conversely, suppose A is closable and $\{v_n\} \subseteq D(\mathcal{E}_A) = D(A)$ with $\|v_n\|_H \to 0$ and $\mathcal{E}_A(v_n - v_m) \to 0$ as $m, n \to \infty$. Then $\|Av_n - Av_m\|_K \to 0$ as $m, n \to \infty$, and since K is complete, $\exists\, u \in K$ with $\|Av_n - u\|_K \to 0$. Since A is closable, then $u = 0$ and thus $\mathcal{E}_A(v_n, v_n) = \|Av_n\|_K \to 0$, proving that $\mathcal{E}_A$ is closable. $\square$

The quadratic form appearing in the logarithmic Sobolev inequality will be denoted by $\mathcal{E}_{g_0,t} : \mathcal{D}(\mathcal{E}_{g_0,t}) \times \mathcal{D}(\mathcal{E}_{g_0,t}) \to \mathbf{R}$, for each $t \geq 0$ and $g_0 \in \mathcal{L}(G)$, where

$$\begin{aligned} \mathcal{E}_{g_0,t}(f,g) &\equiv \int_{\mathcal{L}(G)} (\vec{\nabla} f(x), \vec{\nabla} g(x))_{H(\mathfrak{g})} \nu_t(g_0, dx), \\ \mathcal{E}_{g_0,t}(f) &\equiv \int_{\mathcal{L}(G)} \left\| \vec{\nabla} f(x) \right\|^2_{H(\mathfrak{g})} \nu_t(g_0, dx), \end{aligned}$$

and $\mathcal{D}(\mathcal{E}_{g_0,t}) \equiv \mathcal{F}C^\infty(\mathcal{L}(G))$. $\mathcal{E}_{g_0,t}$ is clearly symmetric and positive. By definition, if $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$ is of the form $f(g) = F(g_{\mathcal{P}})$,where $F \in C^\infty(G^{\mathcal{P}})$, and if $g \in \mathcal{L}(G)$, we define $\vec{\nabla} f(g)$ to be the unique element in $H(\mathfrak{g})$ such that

$$(\vec{\nabla} f(g), h) = (\tilde{h} f)(g) \;\forall\, h \in H(\mathfrak{g}). \tag{5.2}$$

In our case, for all $h \in H(\mathfrak{g})$, the fact that our Green's function, $G(s,t)$, is a reproducing kernel gives

$$\begin{aligned} (\sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(t,\cdot))(g_{\mathcal{P}}) G(s_i, \cdot) A, h) &= \sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(t,\cdot))(g_{\mathcal{P}}) \langle A, h(s_i) \rangle \\ &= \sum_{i=0}^{n-1} (h(s_i)^{(i)} F(t,\cdot))(g_{\mathcal{P}}) = (\tilde{h} f(t,\cdot))(g), \end{aligned}$$

where $h(s_i)^{(i)}$ is defined as in (4.4). Thus,

$$(\vec{\nabla} f(t,\cdot))(g) = \sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(t,\cdot))(g_{\mathcal{P}}) G(s_i, \cdot) A. \tag{5.3}$$

The Hilbert space in our case is $L^2(\mathcal{L}(G), \nu_t(g_0, \cdot)) \supset \mathcal{D}(\mathcal{E}_{g_0,t})$. The task of proving $\mathcal{E}_{g_0,t}$ is closable will require proving an integration by parts formula on $\mathcal{L}(G)$. The same proof given by Driver [3] for the case of the pinned loop group will hold in our case. The one important key to the proof which requires separate verification in our case is the finite dimensional approximations to the Laplacian and Ricci, which we have already proved in Theorems (3.1.1) and (3.2.1). Although we will not give the complete proof for the integration by parts formula here, we will review how the formula implies the closability of $\mathcal{E}_{g_0,t}$. It will also be important to explain the combination of probability and geometry that are implicit in the formula.

5.2 The Orthonormal Frame Bundle

We would like to introduce the notion of parallel translation along a Brownian motion valued on a Lie group. Because of the nature of the Brownian motion, it will be convenient to work in the context of the orthonormal frame bundle.

Let M be a d-dimensional compact, connect Lie group with left invariant metric, $\langle\cdot,\cdot\rangle$ (the following discussion will apply also when M is a general manifold). The *orthonormal frame bundle*, $O(M)$, of M is defined in the following way. For each $m \in M$, let $O_m(M)$ be the collection of linear isometries from T_eM to T_mM (where T_mM is the tangent space of M at m). Then let $O(M) \equiv \cup_{m\in M} O(M)$. One may think of $O_m(M)$ as being the set of all orthonormal bases of T_mM and $O(M)$ as the set of all orthonormal frames on M.

Assume ∇ is a $\langle\cdot,\cdot\rangle$-compatible covariant derivative that is torsion skew symmetric (i.e., $\langle T\langle X,Y\rangle, Y\rangle \equiv 0$ for all vector fields X, Y on M, where T is the torsion tensor $T\langle X,Y\rangle = \nabla_X Y - \nabla_Y X - [X,Y]$). Recall that for a function $f \in C^\infty(M)$, the covariant derivative of the differential df is a degree-2 covariant tensor, ∇df, defined by

$$\nabla df\langle X,Y\rangle \equiv X(df\langle Y\rangle) - df\langle\nabla_X Y\rangle.$$

We define the Laplacian with respect to ∇ by

$$\Delta f \equiv tr\nabla df = \sum_{i=0}^{d}\{E_iE_if - df\langle\nabla_{E_i}E_i\rangle \ \ \forall\, f \in C^\infty(M), \tag{5.4}$$

where $\{E_i\}_{i=1}^d$ is a local orthonormal frame on M.

For a smooth path $\sigma : (a,b) \to M$ and a vector field $X(s) \in T_{\sigma(s)}M$ along σ, we say that X is *parallel* with respect to ∇ along σ provided

$$\nabla_{\dot{\sigma}(s)} X(s) = 0 \;\; \forall\, s \in (a,b). \tag{5.5}$$

Since Brownian motion on M will not be smooth, we need to generalize the meaning of "parallel".

For each $u \in O(M)$, let $T_uO(M)$ be the tangent space of $O(M)$ at u. Let $\pi : TO(M) \to O(M)$ be the projection defined by $\pi X = u$ for $X \in T_uO(M)$, and let $\tilde{\pi} : O(M) \to M$ be the projection defined by $\tilde{\pi} u \equiv m$ for $u \in O_m(M)$. Define the *canonical 1-form* $\theta : TO(M) \to T_eM$ by

$$\theta\langle X\rangle \equiv (\pi X)^{-1} \tilde{\pi}_* X \;\; \forall\, X \in O(M). \tag{5.6}$$

So if X is of the form $X = u'(0)$ for some smooth curve u in $O(M)$ with $u(0) = u \equiv \pi X$, then $\theta\langle X\rangle = u^{-1}\sigma'(0)$ where $\sigma(s) = \tilde{\pi} u(s)$. In other words, $\theta\langle X\rangle$ can be thought of as the "coordinates" of $\sigma'(0)$ with respect to u (if we think of u as corresponding to an orthonormal basis for T_mM where $m = \tilde{\pi} u$).

Let $O(T_eM)$ be the collection of linear isometries from T_eM to itself (i.e., $O(T_eM) \equiv O_e(M)$), and let $so(T_eM)$ be the Lie algebra of $O(T_eM)$ (i.e., the skew-symmetric linear transformations on T_eM). The *connection 1-form* $\omega : TO(M) \to so(T_eM)$ is defined by

$$(\omega\langle X\rangle)\langle a\rangle \equiv u(0)^{-1} \left.\frac{\nabla}{ds}\right|_{s=0} (u(s)a) \;\forall\, X \in TO(M), a \in T_eM, \tag{5.7}$$

where u is any smooth curve in $O(M)$ with $u'(0) = X$.

For each $a \in T_eM$, we let B_a denote the *horizontal vector field* on $TO(M)$. By this we mean that for each $u \in O(M)$, $B_a(u)$ is the unique vector in $T_uO(M)$ such that $\omega\langle B_a(u)\rangle = 0$ and $\tilde{\pi}_* B_a(u) = ua$. $B_a(u)$ is called the *horizontal lift* of a to u.

5.3 Parallel Translation

We have already defined the meaning of parallel translation in Equation (5.5) along a smooth path. We are now ready to define the notion of parallel translation along an M-valued Brownian motion. Assume $\{\Sigma_t\}_{t\geq 0}$ is an M-valued Brownian motion on a given filtered probability space, $(\mathcal{W}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$, satisfying the usual conditions. By definition, this means that $\{\Sigma_t\}_{t\geq 0}$ is an adapted continuous process such that $\Sigma_0 = e$ almost everywhere, and

$$f(\Sigma_t) - f(\Sigma_0) - \frac{1}{2}\int_0^t (\Delta f)(\Sigma_\tau)d\tau \tag{5.8}$$

is a martingale $\forall\, f \in\ C^\infty(M)$. We say that an $O(M)$ -valued process, $\{//_t\}_{t\geq 0}$, is *parallel translation* along $\{\Sigma_t\}_{t\geq 0}$ provided $//_0$ is the identity in $O(T_eM)$, $\tilde{\pi}\circ //_t = \Sigma_t\, \forall\, t \geq 0$, and

$$\int_0^t \omega\langle \delta //_\tau\rangle = 0\ \ \forall\, t \geq 0. \tag{5.9}$$

One may interpret the integral in (5.9) as follows. Assuming $//_\tau \in U$ for $\tau \in [s_1, s_2] \subset [0,t]$ and a local chart, (U,x), of $O(M)$, then we can write ω as

$$\omega = \sum_{i=1}^n f_i dx_i,$$

where $n \equiv \dim(O(M))$ and $f_i : O(M) \to so(T_eM)$ for $i \in \{1, \ldots, n\}$. We then define

$$\int_{s_1}^{s_2} \omega\langle \delta //_\tau\rangle \equiv \sum_{i=1}^n \int_{s_1}^{s_2} f_i(//_\tau)\delta(x_i(//_\tau)). \tag{5.10}$$

If $//_\tau$ lies in more than one chart of $O(M)$, one may use a partition of unity to define the entire integral $\int_0^t \omega\langle \delta //_\tau\rangle$ as a combination of integrals of the type in (5.10) (see Chapter 8 in Spivak [20]).

We can see how (5.9) generalizes the notion of parallel translation as given in Equation (5.5), for if u is a deterministic smooth path in $O(M)$ satisfying (5.9), then we would have

$$\begin{aligned}\int_0^t \omega\langle \delta u(\tau)\rangle \ &= \int_0^t \omega\langle u'(\tau)\rangle d\tau = 0\ \ \forall\, t \geq 0 \\ &\Rightarrow\ \omega\langle u'(t)\rangle = 0\ \ \forall\, t \geq 0.\end{aligned} \tag{5.11}$$

By (5.7) this would imply $u(t)^{-1}\frac{\nabla}{dt}(u(t)a) = 0\, \forall\, a \in T_eM$. Viewing u as a path in the orthonormal frame bundle, i.e., $u(t)$ corresponds to an orthonormal basis in $T_{\tilde{\pi}u(t)}M$ for each t, we see that (5.11) implies that u_t is the parallel translation, in the sense of (5.5), of an orthonormal bases $u(0) \in O(T_eM)$ along the path $\sigma \equiv \tilde{\pi}u$ in M. Since $\{\Sigma_t\}_{t\geq 0}$ is not a smooth process, we see that it is necessary to use a stochastic integral, as in (5.9), to define parallel translation.

5.4 Brownian Motion Via The Development Map

In this section we discuss how Brownian motions on the Lie group, M, and on the Lie algebra, T_eM, are related via the development map and its inverse (sometimes called the "rolling" map). Although we did not refer to it by name, a rolling map was actually used in Theorem 4.0.1 to derive a Brownian motion on the free loop group, $\mathcal{L}(G)$.

It will be helpful at this point to give a general outline of what we will do in this section as it relates to our anticipated integration by parts formula. We will first start with a T_eM-valued Brownian motion, $\{\beta_t\}_{t\geq 0}$, and apply the rolling map with respect to a " left-covariant" derivative to get an M-valued Brownian motion, $\{\Sigma_t\}_{t\geq 0}$. The reason we use the left-covariant and not the Levi-Civita covariant derivative is that it is much easier to solve $\{\Sigma_t\}_{t\geq 0}$ in this context since parallel translation turns out to be equivalent to left-translation.

We then use the development map with respect to the Levi-Civita covariant derivative to derive a different Brownian motion, $\{b_t\}_{t\geq 0}$, in T_eM. The reason for this will be apparent when we see how the integration by parts formula involves the Levi-Civita Ricci tensor.

Let us now start with a T_eM-valued Brownian motion, $\{\beta_t\}_{t\geq 0}$, on the filtered probability space $(\mathcal{W}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$. Assume $\{\beta_t\}_{t\geq 0}$ has covariance

$$E[\langle \beta_t, h\rangle, \langle \beta_\tau, k\rangle] = t \wedge \tau \langle h, k\rangle \;\; \forall\, t, \tau \geq 0 \text{ and } h, k \in T_eM.$$

Let ∇^L be the unique *left-covariant derivative* on M, i.e., such that $\nabla^L_{\tilde{X}}\tilde{Y} = 0$ for all left-invariant vector fields $\tilde{X}$ and $\tilde{Y}$ on M (see Chapter II, section 3 in Helgason [11]). For a smooth path, σ, in M with $\sigma(0) = e$, it is easy to see that for $X \in T_eM$, $L_{\sigma(t)*}X = \tilde{X}_{\sigma(t)}$ is the parallel translation of X along σ with respect to ∇^L. Indeed, if $\{E_i\}_{i=1}^d$ is an orthonormal basis of T_eM and $X = \sum\limits_{i=0}^{d} x^i E_i$, then

$$\begin{aligned} \nabla^L_{\dot{\sigma}(t)}(L_{\sigma(t)*}X) &= \nabla^L_{\dot{\sigma}(t)}(\sum_{i=0}^{d} x^i (\tilde{E}_i)_{\sigma(t)}) \\ &= \sum_{i=0}^{d} x^i \nabla^L_{\dot{\sigma}(t)}(\tilde{E}_i)_{\sigma(t)} = 0. \end{aligned}$$

Let $\{\Sigma_t\}_{t\geq 0}$ be the unique solution to the following Stratonovich stochastic differential equation:

$$\delta\Sigma_t = \sum_{i=0}^{d} \tilde{E}_i(\Sigma_t)\delta\beta_t^{E_i}, \quad \Sigma_0 = e, \tag{5.12}$$

where $\{E_i\}_{i=1}^d$ is an orthonormal basis of T_eM and $\beta_t^{E_i} \equiv \langle E_i, \beta_t\rangle$ for all $t \geq 0$. As in Theorem 4.0.1, we interpret (5.12) to mean

$$f(\Sigma_t) - f(\Sigma_0) = \sum_{i=0}^{d} \int_0^t \tilde{E}_i f(\Sigma_\tau)\delta\beta_\tau^{E_i} \;\; \forall\, f \in C^\infty(M).$$

The mapping from $\{\beta_t\}_{t\geq 0}$ to $\{\Sigma_t\}_{t\geq 0}$ is what we call the *rolling map.* The existence of a solution to (5.12) is discussed in Chapter V of Ikeda and Watanaba [12]. One may solve such an equation by using Whiteney's theorem to imbed M into $\mathbf{R}^{2d+1}$. It is the fact that we have used a Stratonovich differential equation that guarantees that the solution to (5.12) will be back in M.

Proposition 5.4.1 *$\{\Sigma_t\}_{t\geq 0}$ is an M-valued Brownian motion. In other words,*

$$f(\Sigma_t) - f(\Sigma_0) - \frac{1}{2}\int_0^t (\Delta f)(\Sigma_\tau)d\tau$$

is a martingale for all $f \in C^\infty(M)$, where Δ is defined as in (5.4).

Proof: As we have noted at the end of Chapter 4, since M is unimodular, then $\Delta = \sum_{i=1}^{d} \tilde{E}_i^2$, where $\{E_i\}_{i=1}^d$ is again an orthonormal basis for T_eM.

By (5.12) and the definition of the Stratonovich integral, for each $f \in C^\infty(M)$

$$f(\Sigma_t) - f(\Sigma_0) = \sum_{i=1}^{d}\{\int_0^t (\tilde{E}_i f)(\Sigma_\tau)d\beta_\tau^{E_i} + \frac{1}{2} \prec \tilde{E}_i f(\Sigma), \beta^{E_i} \succ_t\},$$

where $\prec \cdot, \cdot \succ$ denotes quadratic covariation. Note that

$$\begin{aligned}
\sum_{i=1}^{d} \prec & \tilde{E}_i f(\Sigma), \beta^{E_i} \succ_t = \sum_{i=1}^{d} \int_0^t d(\tilde{E}_i f(\Sigma_\tau))d\beta_\tau^{E_i} \\
&= \sum_{i=1}^{d} \int_0^t \sum_{j=1}^{d}\{\tilde{E}_j(\tilde{E}_i f)(\Sigma_\tau)d\beta_\tau^{E_i} + \frac{1}{2}d \prec \tilde{E}_j(\tilde{E}_i f)(\Sigma), \beta^{Ej} \succ_\tau\}d\beta_\tau^{E_i} \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}\{\int_0^t \tilde{E}_j(\tilde{E}_i f)(\Sigma_\tau)d \prec \beta^{E_i}, \beta^{Ej} \succ_\tau + \frac{1}{2} \prec\prec \tilde{E}_j(\tilde{E}_i f)(\Sigma), \beta^{Ej} \succ, \beta^{E_i} \succ_t\},
\end{aligned}$$

where we have used (5.12) again for the second equation above. For the third equation, see Theorem 5.7 in Chung and Williams [1]. Since $\prec \tilde{E}_j(\tilde{E}_i f)(\Sigma), \beta^{Ej} \succ_\tau$ is of bounded variation, then $\prec\prec \tilde{E}_j(\tilde{E}_i f)(\Sigma), \beta^{Ej} \succ, \beta^{E_i} \succ_t = 0$. And since $\{\beta_t\}_{t\geq 0}$ is a Brownian motion, then $\prec \beta^{E_i}, \beta^{Ej} \succ_\tau = \delta_{ij}\tau$ (see Theorem 3.16 in Karatzas and Shreve [13]). We thus have

$$f(\Sigma_t) - f(\Sigma_0) = \sum_{i=1}^{d} \int_0^t \tilde{E}_i f(\Sigma_\tau) d\beta_\tau^{E_i} + \sum_{i=1}^{d} \frac{1}{2} \int_0^t \tilde{E}_i^2 f(\Sigma_\tau) d\tau.$$

Since $\tilde{E}_i f(\Sigma_\tau)$ is bounded, then $\int_0^t \tilde{E}_i f(\Sigma_\tau) d\beta_\tau^{E_i}$ is a martingale for each. $\square$

Now that we have derived a Brownian motion, $\{\Sigma_t\}_{t\geq 0}$, on M from the Brownian motion $\{\beta_t\}_{t\geq 0}$ on T_eM, we will derive yet another Brownian motion, $\{b_t\}_{t\geq 0}$, on T_eM by applying the development map to $\{\Sigma_t\}_{t\geq 0}$ with respect to the Levi-Civita covariant derivative.

From now on we will let $\{//_t\}_{t\geq 0}$ denote parallel translation along $\{\Sigma_t\}_{t\geq 0}$ with respect to the Levi-Civita covariant derivative, which we will denote by ∇. Thus, $\{//_t\}_{t\geq 0}$ satisfies Equation (5.9). Let $\{b_t\}_{t\geq 0}$ be the T_eM -valued Brownian motion defined by

$$b_t \equiv \int_0^t \theta\langle \delta //_\tau \rangle \ \ \forall\, t \geq 0, \tag{5.13}$$

where θ is the canonical 1-form in (5.6). This mapping sending from $\{\Sigma_t\}_{t\geq 0}$ to $\{b_t\}_{t\geq 0}$ is what we call the *development map*. With $\{b_t\}_{t\geq 0}$ defined in this way, one can see that $\{//_t\}_{t\geq 0}$ satisfies the Stratonovich stochastic differential equation

$$\delta //_t = B_{\delta b_t}(//_t), \tag{5.14}$$

where $B_{\delta b_t}(//_t) \equiv \sum_{i=1}^{n-1} B_{E_i}(//_t)\delta b_t^{E_i}$ and $b_t^{E_i} \equiv \langle b_t, E_i \rangle$. In a sense, (5.14) says that b_t is the process in T_eM whose " velocity vector" at time t, when parallel translated along Σ, is mapped to the "velocity vector" of Σ at time t. The relationship between $\{b_t\}_{t\geq 0}$ and $\{\beta_t\}_{t\geq 0}$ is given in the theorem below.

Theorem 5.4.2 *Define the $O(T_eM)$-valued process U_t by*

$$U_t \equiv L_{\Sigma_t^{-1}*} //_t \ . \tag{5.15}$$

Then

$$b_t = \int_0^t U_\tau^{-1} \delta\beta_\tau, \tag{5.16}$$

and

$$b_t = \int_0^t U_\tau^{-1} d\beta_\tau \ \ \forall\, t \geq 0. \tag{5.17}$$

Also, U_t satisfies the Stratonovich differential equation

$$\delta U_t + D_{\delta\beta_t} U_t = 0, \ \ U_0 = I_{T_e M}, \tag{5.18}$$

where D is the Levi-Civita derivative, ∇, restricted to $T_e M$.

Proof: We refer to Driver [4] for the proof of (5.18). For (5.16), recall that $b_t \equiv \int_0^t \theta\langle \delta //_\tau \rangle$. By (5.6) and the fact that $\Sigma_t = \tilde{\pi} \circ //_t$, we get

$$\begin{aligned} b_t &= \int_0^t (\pi\delta //_\tau)^{-1} \tilde{\pi}_* \delta //_\tau \\ &= \int_0^t //_\tau^{-1} \delta\Sigma_\tau. \end{aligned}$$

Applying (5.12) gives

$$\begin{aligned} b_t &= \int_0^t //_\tau^{-1} \{ \sum_{i=0}^d \tilde{E}_i(\Sigma_\tau) \delta\beta_\tau^{E_i} \} \\ &= \int_0^t //_\tau^{-1} L_{\Sigma_\tau *} \{ \sum_{i=0}^d E_i(\Sigma_\tau) \delta\beta_\tau^{E_i} \} \\ &= \int_0^t U_\tau^{-1} \delta\beta_\tau, \end{aligned}$$

which proves (5.16).

For (5.17), the definition of the Stratonovich stochastic integral in (5.16) gives

$$\begin{aligned} b_t &= \int_0^t U_\tau^{-1} d\beta_\tau + \frac{1}{2} \prec U^{-1}, \beta \succ_t \\ &= \int_0^t U_\tau^{-1} d\beta_\tau + \frac{1}{2} \int_0^t d(U_\tau^{-1}) d\beta_\tau. \end{aligned}$$

We claim that $\{\int_0^t U_\tau^{-1} d\beta_\tau\}_{t \geq 0}$ is a Brownian motion. Since $\{b_t\}_{t \geq 0}$ is also a Brownian motion in $T_e M$ and $\prec U^{-1}, \beta \succ_t$ is a process of bounded variation, it must be that $\frac{1}{2} \prec U^{-1}, \beta \succ_t = 0$ for all $t \geq 0$, and thus $b_t = \int_0^t U_\tau^{-1} d\beta_\tau$.

To see that $\{\int_0^t U_\tau^{-1} d\beta_\tau\}_{t\geq 0}$ is a $O(T_eM)$-valued Brownian motion, let (A_τ^{ij}) be the matrix corresponding to U_τ^{-1} with respect to a fixed basis of $O(T_eM)$ and let β_τ^i be the ith component of β_τ with respect to this basis. Since $L_{\Sigma_\tau^{-1}*}$ and $//_\tau$ are isometries, then by definition of U_τ it is clear that (A_τ^{ij}) is orthogonal. We may thus calculate the quadratic covariation of the ith and kth components of $\int_0^t U_\tau^{-1} d\beta_\tau$ as follows:

$$\begin{aligned}
&\prec \int_0^\cdot A_\tau^{ij} d\beta_\tau^j, \int_0^\cdot A_\tau^{kl} d\beta_\tau^l \succ_t = \int_0^t A_\tau^{ij} A_\tau^{kl} d \prec \beta^j, \beta^l \succ_\tau \\
&= \int_0^t A_\tau^{ij} A_\tau^{kl} \delta_{jl} d\tau \\
&= \int_0^t A_\tau^{ij} (A_\tau^{-1})^{jk} d\tau \\
&= \delta_{ik} t.
\end{aligned}$$

By Lévy's martingale characterization of Brownian motion (see Theorem 3.16 in [13]), it follows that $\{\int_0^t U_\tau^{-1} d\beta_\tau\}_{t\geq 0}$ is indeed a Brownian motion. □

Note that since $U_\tau^{-1} U_\tau = I_{T_eM}$, Stratonovich integration by parts gives

$$\delta(U_\tau^{-1} U_\tau) = (\delta U_\tau^{-1}) U_\tau + U_\tau^{-1} \delta U_\tau = 0.$$

Thus,

$$\begin{aligned}
\delta U_\tau^{-1} &= -U_\tau^{-1} (\delta U_\tau) U_\tau^{-1} \\
&= U_\tau^{-1} (D_{\delta\beta_\tau} U_\tau) U_\tau^{-1} \\
&= U_\tau^{-1} D_{\delta\beta_\tau},
\end{aligned}$$

where we have used (5.18) for the last equation. This gives

$$\begin{aligned}
\int_0^t d(U_\tau^{-1}) d\beta_\tau &= \int_0^t \delta(U_\tau^{-1}) d\beta_\tau - \frac{1}{2} \int_0^t d \prec U^{-1}, 1 \succ_\tau d\beta_\tau \\
&= \int_0^t \delta(U_\tau^{-1}) d\beta_\tau \\
&= \int_0^t U_\tau^{-1} D_{\delta\beta_\tau} d\beta_\tau \\
&= \int_0^t U_\tau^{-1} D_{d\beta_\tau} d\beta_\tau - \frac{1}{2} \int_0^t U_\tau^{-1} D_{d\prec\beta,1\succ_\tau} d\beta_\tau \\
&= \int_0^t U_\tau^{-1} D_{d\beta_\tau} d\beta_\tau
\end{aligned}$$

$$= \int_0^t U_\tau^{-1} \sum_{i,j=1}^d (D_{E_i} E_j) d\beta_\tau^{E_i} d\beta_\tau^{E_j}$$

$$= \int_0^t U_\tau^{-1} \sum_{i=1}^d (D_{E_i} E_i) d\tau.$$

By the last part of the proof of Theorem 5.4.2, we know that $\prec U^{-1}, \beta \succ_t = 0$ for all $t \geq 0$, so by the above calculation,

$$\int_0^t U_\tau^{-1} \sum_{i=1}^d (D_{E_i} E_i) d\tau = 0 \;\; \forall \, t \geq 0.$$

Since U_τ^{-1} is obviously invertible, then this implies that $\sum_{i=1}^d (D_{E_i} E_i) = 0$. We may thus write the Laplacian in (5.4) as

$$\Delta f = \sum_{i=1}^d \tilde{E}_i(\tilde{E}_i f),$$

which agrees with the Laplace Beltrami operator as in Proposition 5.4.1.

5.5 Integration By Parts

We are now ready to give our integration by parts formula for left invariant vector fields on $\mathcal{L}(G)$. Let β and Σ be the Brownian motions on $\mathcal{L}(\mathfrak{g})$ and $\mathcal{L}(G)$ as in Theorem 4.0.1. For what follows, if $h \in H(\mathfrak{g})$, we define (h, β_t) to mean a continuous version of

$$\lim_{n \to \infty} \alpha_{h_n}(\beta_t), \tag{5.19}$$

where α is defined as in (4.1), $\{h_n\} \subset H^{BV}$ with $h_n \to h$ in $H(\mathfrak{g})$, and the limit in (5.19) is in L^2 with respect to the probability measure P on the space $\mathcal{W}$ on which β is defined.

Theorem 5.5.1 *Let $t > 0$ and $h \in H(\mathfrak{g})$. Assume $l : [0, t] \to \mathbf{R}$ is an absolutely continuous function with $l(0) = 0$, $l(t) = 1$, and $\int_0^t \dot{l}^2(\tau) d\tau < \infty$. Let Ric be the Levi-Civita Ricci tensor on $H(\mathfrak{g})$ as defined in (2.21). Then for all $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$,*

$$E[(\tilde{h} f)(\Sigma_t)] = E[f(\Sigma_t) \int_0^t (\{\dot{l}(\tau) - \frac{1}{2} l(\tau) Ric\} H(\tau), \overleftarrow{d\beta_\tau})], \tag{5.20}$$

where $H(t)$ $H(\mathfrak{g})$ is the unique $H(\mathfrak{g})$-valued process that solves the backwards Stochastic differential equation

$$h - H(s) + \int_s^t \nabla_{\overleftarrow{d\beta_\tau}} H(\tau) + \frac{1}{2}\int_0^t \Delta H(\tau) d\tau = 0. \tag{5.21}$$

Here, ∇ is our Levi-Civita covariant derivative on $H(\mathfrak{g})$ (as in Definition 2.1.13 and Δ is the Laplacian $\Delta \equiv \sum_{k\in S} \nabla_k^2$ (as in Definition 3.0.9. By abuse of notation, we take the inner product in the right side of (5.20) to mean

$$(\{\dot{l}(\tau) - \frac{1}{2}l(\tau)Ric\}H(\tau), \overleftarrow{d\beta_\tau}) \equiv (\dot{l}(\tau)H(\tau), \overleftarrow{d\beta_\tau}) - \frac{1}{2}l(\tau)Ric\langle H(\tau), \overleftarrow{d\beta_\tau}\rangle.$$

Sketch of Proof: We will discuss the key steps in the proof as it is presented in Driver [3]. Assume $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_{n-1} < 1\}$ is a partition of $[0,1]$, and $f = F \circ \pi_{\mathcal{P}}$, where $F \in C^\infty(G^{\mathcal{P}})$. Let $(\Sigma_{\mathcal{P}})_t \equiv \pi_{\mathcal{P}} \circ \Sigma_t = (\Sigma_t(s_0), \ldots, \Sigma_t(s_{n-1}))$ and $(\beta_{\mathcal{P}})_t \equiv \tilde{\pi}_{\mathcal{P}} \circ \beta_t = (\beta_t(s_0), \ldots, \beta_t(s_{n-1}))$. As in Theorem 4.0.1, $\Sigma_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ are related by the equation

$$\begin{aligned}(\Sigma_{\mathcal{P}})_{\delta t} &= \sum_{j=0}^{dn} \tilde{E}_j((\Sigma_{\mathcal{P}})_t)\delta(\beta_{\mathcal{P}})_t^{E_j} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{dn} E_j^{(i)}((\Sigma_{\mathcal{P}})_t)\langle \beta_{\delta t}(s_i), E_j\rangle,\end{aligned}$$

where $d = \dim(G)$ and $\{E_i\}_{i=1}^d$ is an orthonormal basis for $T_e M$.

For the compact, finite dimensional Lie group, $M \equiv G^{\mathcal{P}}$, the following finite dimensional version of (5.21) holds by Corollary (6.5) in Driver [4]:

$$E[(\tilde{h}f)(\Sigma_t)] = E[(\tilde{h}_{\mathcal{P}}F)((\Sigma_{\mathcal{P}})_t)] = E[f((\Sigma_{\mathcal{P}})_t)\int_0^t (\{\dot{l}(\tau) - \frac{1}{2}l(\tau)Ric_{\mathcal{P}}\}H_{\mathcal{P}}(\tau), \overleftarrow{d(\beta_{\mathcal{P}})_\tau})], \tag{5.22}$$

where $h_{\mathcal{P}} \equiv \pi_{\mathcal{P}} h$, and $Ric_{\mathcal{P}}$ is the Ricci tensor with respect to the Levi-Civita covariant derivative, $\nabla^{\mathcal{P}}$, on $\mathfrak{g}^{\mathcal{P}}$. We define the $\mathfrak{g}^{\mathcal{P}}$ -valued process by

$$(H_{\mathcal{P}})_\tau \equiv U_{\mathcal{P}}(\tau, t)h_{\mathcal{P}} \equiv U_{\mathcal{P}}(\tau)U_{\mathcal{P}}(t)^{-1}h_{\mathcal{P}},$$

where $U_{\mathcal{P}}(t)$ is the $O(\mathfrak{g}^{\mathcal{P}}) - valued$ process,

$$U_{\mathcal{P}}(t) \equiv L_{(\Sigma_{\mathcal{P}})_t^{-1}*}(//_{\mathcal{P}})_t,$$

satisfying the Stratonovich differential equation

$$dU_{\mathcal{P}}(t) + \nabla^{\mathcal{P}}_{\delta(\beta_{\mathcal{P}})_t} U_{\mathcal{P}}(t) = 0, \quad U_{\mathcal{P}}(0) = I_{\mathfrak{g}^{\mathcal{P}}},$$

and $(//_{\mathcal{P}})_t$ is parallel translation along $(\Sigma_{\mathcal{P}})_t$.

We should note that the proof for (5.22) relies on the use of the $\mathfrak{g}^{\mathcal{P}}$-valued Brownian motion $\{(b_{\mathcal{P}})_t\}_{t\geq 0}$ resulting from the development map applied to $\{(\Sigma_{\mathcal{P}})_t\}_{t\geq 0}$ as in (5.13).

The next key to the proof of (5.20) is passing to the limit as $|\mathcal{P}| \to 0$ in (5.22). This requires a detailed argument for which we refer to the one offered by Driver [3]. The reason the same argument applies in our case is due to the crucial fact that our finite dimensional Ricci tensors and Laplacians converge to their infinite dimensional counterparts as we have shown in Theorems (3.1.1) and (3.2.1). $\square$

5.6 Closability of the Quadratic Form

Recall that the quadratic form we are interested in for the logarithmic Sobolev inequality is given by

$$\mathcal{E}_{g_0,t}(f,g) \equiv \int_{\mathcal{L}(G)} (\vec{\nabla} f(x), \vec{\nabla} g(x))_{H(\mathfrak{g})} \nu_t(g_0, dx)$$

for each $t \geq 0$ and $g_0 \in \mathcal{L}(G)$, where $\vec{\nabla} : D(\vec{\nabla}) = \mathcal{F}C^\infty(\mathcal{L}(G)) \to L^2(\mathcal{L}(G), \nu_t(g_0,\cdot); H(\mathfrak{g}))$ is the gradient operator satisfying the condition in (5.2). As in (4.6), $\nu_t(e,\cdot)$ is the law at time t of the Brownian motion $\{\Sigma_t\}_{t\geq 0}$ constructed in (5.12) and $\nu_t(g_0,\cdot) = L_{g_0 *}\nu_t(e,\cdot)$.

Theorem 5.6.1 *For each $t > 0$ and $g_0 \in \mathcal{L}(G)$, $\mathcal{E}_{g_0,t}$ is closable, where $D(\mathcal{E}_{g_0,t}) = \mathcal{F}C^\infty(\mathcal{L}(G))$.*

Proof: It is sufficient to prove $\mathcal{E}_t \equiv \mathcal{E}_{e,t}$ is closable for every $t \geq 0$ since for each $g_0 \in \mathcal{L}(G)$,

$$\begin{aligned}
\mathcal{E}_{g_0,t}(f,g) &= \int_{\mathcal{L}(G)} (\vec{\nabla} f(x), \vec{\nabla} g(x))_{H(\mathfrak{g})} \nu_t(g_0, dx) \\
&= \int_{\mathcal{L}(G)} (\vec{\nabla} f(L_{g_0}x), \vec{\nabla} g(L_{g_0}x))_{H(\mathfrak{g})} \nu_t(e, dx) \\
&= \int_{\mathcal{L}(G)} (\vec{\nabla} (f\circ L_{g_0})(x), \vec{\nabla} (g\circ L_{g_0})(x))_{H(\mathfrak{g})} \nu_t(e, dx) \\
&= \mathcal{E}_t(f\circ L_{g_0}, g\circ L_{g_0})
\end{aligned}$$

for all $f, g \in D(\mathcal{E}_t) = \mathcal{F}C^\infty(\mathcal{L}(G))$. Thus, if we knew $\mathcal{E}_t$ were closable, then given $\{f_n\} \subseteq D(\mathcal{E}_t)$ with $\|f_n\|_{L^2(\mathcal{L}(G),\nu_t(g_0,\cdot))} \to 0$ and $\mathcal{E}_{g_0,t}(f_n - f_m) \to 0$ as $m, n \to \infty$, then

$$\begin{aligned} \|f_n \circ L_{g_0}\|_{L^2(\mathcal{L}(G),\nu_t(e,\cdot))} &\to 0 \\ \text{and } \mathcal{E}_{g_0,t}(f_n \circ L_{g_0} - f_m \circ L_{g_0}) &\to 0 \text{ as } m, n \to \infty. \end{aligned}$$

So $\mathcal{E}_{g_0,t}(f_n) = \mathcal{E}_t(f_n \circ L_{g_0}) \to 0$ as $n \to \infty$.

We next show that $\vec{\nabla}$ has a densely defined adjoint. Using our integration by parts formula in (5.20) with $l(\tau) \equiv \frac{\tau}{t}$, for example, we see that for all $h \in H(\mathfrak{g})$ and $f, g \in \mathcal{F}C^\infty(\mathcal{L}(G))$,

$$\begin{aligned} \langle \vec{\nabla}(fg), h\rangle_{L^2(\mathcal{L}(G),\nu_t(e,\cdot);H(\mathfrak{g}))} &= \int_{\mathcal{L}(G)} (\vec{\nabla}(fg)(x), h)_{H(\mathfrak{g})}\nu_t(e,dx) \\ &= \int_{\mathcal{L}(G)} \tilde{h}(fg)(x)\nu_t(e,dx) \\ &= E[\tilde{h}(fg)(\Sigma_t)] \\ &= E[(fg)(\Sigma_t)\frac{1}{t}\int_0^t (\{I - \frac{1}{2}\tau Ric\}H(\tau), \overleftarrow{d\beta}_\tau)] \\ &= E[(fg)(\Sigma_t)\frac{1}{t}E[\int_0^t (\{I - \frac{1}{2}\tau Ric\}H(\tau), \overleftarrow{d\beta}_\tau) \mid \sigma(\Sigma_t)]] \\ &= E[(fg)(\Sigma_t)\varphi_h(\Sigma_t)], \end{aligned}$$

where $\varphi_h : \mathcal{L}(G) \to \mathbf{R}$ is a Borel measurable function such that

$$\varphi_h(\Sigma_t) = \frac{1}{t}E[\int_0^t (\{I - \frac{1}{2}\tau Ric\}H(\tau), \overleftarrow{d\beta}_\tau)_{H(\mathfrak{g})} \mid \sigma(\Sigma_t)] \;\; a.s..$$

Thus, letting $L^2 \equiv L^2(\mathcal{L}(G), \nu_t(e,\cdot); H(\mathfrak{g}))$, we have

$$\begin{aligned} \langle \vec{\nabla} f, gh\rangle_{L^2} + \langle \vec{\nabla} g, fh\rangle_{L^2} &= \langle \vec{\nabla}(fg), h\rangle_{L^2} \\ &= \int_{\mathcal{L}(G)} (fg)(x)\varphi_h(x)\nu_t(e,dx) \\ &= \langle f, g\varphi_h\rangle_{L^2(\mathcal{L}(G),\nu_t(e,\cdot))}, \end{aligned}$$

and thus

$$\begin{aligned} \langle \vec{\nabla} f, gh\rangle_{L^2} &= \langle f, g\varphi_h\rangle_{L^2(\mathcal{L}(G),\nu_t(e,\cdot))} - \int_{\mathcal{L}(G)} f(x)\tilde{h}g(x)\nu_t(e,dx) \\ &= \langle f, g\varphi_h - \tilde{h}g\rangle_{L^2(\mathcal{L}(G),\nu_t(e,\cdot))}. \end{aligned} \tag{5.23}$$

Letting $\mathcal{F}C^\infty(\mathcal{L}(G), H(\mathfrak{g})) \equiv \{gh : g \in \mathcal{L}(G), h \in H(\mathfrak{g})\}$, we see by (5.23) that for $gh \in \mathcal{F}C^\infty(\mathcal{L}(G), H(\mathfrak{g}))$

$$\vec{\nabla}^*(gh) = g\varphi_h - \tilde{h}g.$$

Thus, $\mathcal{F}C^\infty(\mathcal{L}(G), H(\mathfrak{g})) \subset \mathcal{D}(\vec{\nabla}^*)$.

We next show that $\mathcal{F}C^\infty(\mathcal{L}(G), H(\mathfrak{g}))$ is dense in $L^2 = L^2(\mathcal{L}(G), \nu_t(e, \cdot); H(\mathfrak{g}))$ by showing that $\mathcal{F}C^\infty(\mathcal{L}(G), H(\mathfrak{g}))^\perp = \{0\}$ (see Lemma 3.3-7 in Kreyszig [14]). Indeed, suppose that $J \in \mathcal{F}C^\infty(\mathcal{L}(G), H(\mathfrak{g}))$ and $(gh, J)_{L^2} = 0$ for $g \in \mathcal{L}(G)$ and $h \in H(\mathfrak{g})$. Then

$$\int_{\mathcal{L}(G)} g(x)(h, J(x))_{H(\mathfrak{g})} \nu_t(e, dx) = 0.$$

Since g was arbitrary, implies that for all $h \in H(\mathfrak{g})$, $(h, J(x))_{H(\mathfrak{g})}$ for almost all x. Thus, $J(x) = 0$ for almost all x since we can write

$$J(x) = \sum_{h \in S} (h, J(x))_{H(\mathfrak{g})}$$

where S is an orthonormal (countable) basis of S.

Finally, by Lemma 5.1.5, we can conclude that $\vec{\nabla}$ is closable. Lemma 5.1.6 then implies that $\mathcal{E}_t$ is closable. $\square$

Chapter 6

The Logarithmic Sobolev Inequality

We are now ready to state the finite dimensional version of our logarithmic Sobolev inequality. For any $F \in C^{\infty}(G^{\mathcal{P}})$, let

$$P_t F(g_{0\mathcal{P}}) \equiv \int_{G^{\mathcal{P}}} F(g_{\mathcal{P}}) \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}).$$

Theorem 6.0.1 *Let C be the constant in Theorem 2.3.11 such that $Ric\langle h,h\rangle \geq -C(h,h)$ $\forall\, h \in H(\mathfrak{g})$. Then $\exists\, M > 0$ such that for a fixed partition, $\mathcal{P}$, of $[0,1]$ and for $t \geq 0$,*

$$P_t(F^2 \log F^2) \leq \frac{2}{C_0}(e^{C_0 t} - 1) P_t(\left\|\vec{\nabla}_{\mathcal{P}} F\right\|_{\mathfrak{g}^{\mathcal{P}}}^2) + P_t(F^2)\log(P_t(F^2)) \tag{6.1}$$

on $G^{\mathcal{P}}$, where $C_0 \equiv C + M$ and $\tilde{\nabla}_{\mathcal{P}} F$ is the gradient of F on $G^{\mathcal{P}}$ with respect to $(\cdot,\cdot)_{\mathcal{P}}$. By convention, $0 \log 0 \equiv 0$.

Proof: Recall that by Theorem 3.2.1, for each partition $\mathcal{P}$, there exists an $\epsilon(\mathcal{P}) > 0$ such that

$$|Ric\langle h,h\rangle - Ric_{\mathcal{P}}\langle h,h\rangle| \leq \epsilon(\mathcal{P}) \left\|h\right\|_{H(\mathfrak{g})}^2 \ \ \forall\, h \in H_{\mathcal{P}}(\mathfrak{g}). \tag{6.2}$$

It is not difficult to see from the proof of Theorem 3.2.1 that $\epsilon(\mathcal{P})$ is uniformly bounded, i.e., $\exists\, M > 0$ such that $\epsilon(\mathcal{P}) \leq M$ for all partitions $\mathcal{P}$ of $[0,1]$. From (6.2), we thus have for each partition $\mathcal{P}$,

$$\begin{aligned} Ric_{\mathcal{P}}\langle h,h\rangle &\geq Ric\langle h,h\rangle - \epsilon(\mathcal{P}) \left\|h\right\|_{H(\mathfrak{g})}^2 \\ &\geq -C \left\|h\right\|_{H(\mathfrak{g})}^2 - M \left\|h\right\|_{H(\mathfrak{g})}^2 . \end{aligned}$$

Hence,

$$Ric_{\mathcal{P}}\langle h,h\rangle \geq -C_0 \|h\|^2_{H(\mathfrak{g})},$$

where $C_0 \equiv C + M > 0$.

Let $\nabla^{\mathcal{P}}$ be the Levi-Civita covariant derivative on $H_{\mathcal{P}}(\mathfrak{g})$ as described in Definition 3.0.8. Let $\tilde{\nabla}^{\mathcal{P}}$ denote the Levi-Civita covariant derivative on $\mathfrak{g}^{\mathcal{P}}$ with respect to the metric $(\cdot,\cdot)_{\mathcal{P}}$ defined in (3.1). Since $\tilde{\pi}_{\mathcal{P}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ is a Lie algebra isometric isomorphism, by Propositions 3.0.5 and 3.0.7 , it easy to check that for all $\vec{A}$ and $\vec{B}$ in $\mathfrak{g}^{\mathcal{P}}$,

$$\tilde{\nabla}^{\mathcal{P}}_{\vec{A}}\vec{B} = \tilde{\pi}_{\mathcal{P}}\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B}).$$

Now let $R_{\mathfrak{g}^{\mathcal{P}}}$ be the Levi-Civita curvature tensor and $Ric_{\mathfrak{g}^{\mathcal{P}}}$ be the Levi-Civita Ricci tensor with respect to $(\cdot,\cdot)_{\mathcal{P}}$ on $G^{\mathcal{P}}$. We can see then that for $\vec{A} \in \mathfrak{g}^{\mathcal{P}}$,

$$\begin{aligned}
& Ric_{\mathfrak{g}^{\mathcal{P}}}\langle \vec{A},\vec{A}\rangle \\
= & \sum_{\vec{B}\in\mathfrak{g}^{\mathcal{P}}_0} (R_{\mathfrak{g}^{\mathcal{P}}}\langle \vec{A},\vec{B}\rangle\vec{B},\vec{A})_{\mathcal{P}} \\
= & \sum_{\vec{B}\in\mathfrak{g}^{\mathcal{P}}_0} (\tilde{\pi}_{\mathcal{P}}\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}(\tilde{\pi}^{-1}_{\mathcal{P}}\tilde{\pi}_{\mathcal{P}}\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})) - \tilde{\pi}_{\mathcal{P}}\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})}(\tilde{\pi}^{-1}_{\mathcal{P}}\tilde{\pi}_{\mathcal{P}}\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})) \\
& \qquad -\tilde{\pi}_{\mathcal{P}}\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}[\vec{A},\vec{B}])}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B}),\vec{A})_{\mathcal{P}} \\
= & \sum_{\vec{B}\in\mathfrak{g}^{\mathcal{P}}_0} (\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}(\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})) - \nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})}(\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B})) \\
& -\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}[\vec{A},\vec{B}])}(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{B}),\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})_{H(\mathfrak{g})} \\
= & \sum_{k\in S_{\mathcal{P}}} (\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}(\nabla^{\mathcal{P}}_h h) - \nabla^{\mathcal{P}}_h(\nabla^{\mathcal{P}}_{(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})}h) - \nabla^{\mathcal{P}}_{[\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A},h]_{\mathcal{P}}}h,\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})_{H(\mathfrak{g})} \\
= & Ric_{\mathcal{P}}\langle \tilde{\pi}^{-1}_{\mathcal{P}}\vec{A},\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A}\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
Ric_{\mathfrak{g}^{\mathcal{P}}}\langle \vec{A},\vec{A}\rangle &= Ric_{\mathcal{P}}\langle \tilde{\pi}^{-1}_{\mathcal{P}}\vec{A},\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A}\rangle \\
&\geq -C_0(\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A},\tilde{\pi}^{-1}_{\mathcal{P}}\vec{A})_{H(\mathfrak{g})} \\
&= -C_0(\vec{A},\vec{A})_{\mathcal{P}}.
\end{aligned}$$

By left-translation invariance of $Ric_{\mathfrak{g}^{\mathcal{P}}}$ and $(\cdot,\cdot)_{\mathcal{P}}$, it follows that for all partitions $\mathcal{P}$,

$$Ric_{\mathfrak{g}^{\mathcal{P}}}\langle \vec{A},\vec{A}\rangle \geq -C_0(\vec{A},\vec{A})_{\mathcal{P}} \;\forall\, \vec{A} \in TG^{\mathcal{P}}.$$

Since $G^{\mathcal{P}}$ is a unimodular Lie group, then the inequality in (6.1) follows by Theorem 2.9 in Driver and Lohrenz [5] and is originally due to Bakry and Ledoux. $\square$

Since the constant, C_0, in Theorem 6.0.1 is independent of the partition $\mathcal{P}$, then the logarithmic Sobolev inequality holds in the context of $\mathcal{L}(G)$ as stated in the theorem below.

Theorem 6.0.2 *(Logarithmic Sobolev Inequality on Cylinder Functions) Let C_0 be the constant above and let $g_0 \in \mathcal{L}(G)$ be fixed. For all $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$ and $t > 0$,*

$$\int_{\mathcal{L}(G)} f^2 \log f^2 d\mu \leq \frac{2}{C_0}(e^{C_0 t} - 1) \int_{\mathcal{L}(G)} \left\| \vec{\nabla} f \right\|_{H(\mathfrak{g})}^2 d\mu + \int_{\mathcal{L}(G)} f^2 d\mu \log(\int_{\mathcal{L}(G)} f^2 d\mu), \tag{6.3}$$

where $\mu \equiv \nu_t(g_0, \cdot)$.

Proof: The inequality in (6.3) follows from Theorem 6.0.1 with the following observations. First, note that for $f \in \mathcal{F}C^\infty(\mathcal{L}(G))$ of the form $f = F \circ \pi_{\mathcal{P}}$, $F \in C^\infty(G^{\mathcal{P}})$,

$$\begin{aligned} \int_{\mathcal{L}(G)} f^2 \log f^2 d\mu &= \int_{\mathcal{L}(G)} F^2(\pi_{\mathcal{P}} g) \log F^2(\pi_{\mathcal{P}} g) \nu_t(g_0, dg) \\ &= \int_{\mathcal{W}} F^2(\pi_{\mathcal{P}} \circ L_{g_0} \circ \Sigma_t) \log F^2(\pi_{\mathcal{P}} \circ L_{g_0} \circ \Sigma_t) dP \\ &= \int_{\mathcal{W}} F^2(L_{g_{0\mathcal{P}}} \circ (\Sigma_{\mathcal{P}})_t) \log F^2(L_{g_{0\mathcal{P}}} \circ (\Sigma_{\mathcal{P}})_t) dP \\ &= \int_{G^{\mathcal{P}}} F^2(g_{\mathcal{P}}) \log F^2(g_{\mathcal{P}}) \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}). \end{aligned} \tag{6.4}$$

Similarly,

$$\int_{\mathcal{L}(G)} f^2 d\mu \log(\int_{\mathcal{L}(G)} f^2 d\mu) = \int_{G^{\mathcal{P}}} F^2(g_{\mathcal{P}}) \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}) \log(\int_{G^{\mathcal{P}}} F^2(g_{\mathcal{P}}) \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}})). \tag{6.5}$$

Next, note that using notation from the proof of Lemma 4.0.4, we have

$$\begin{aligned} \int_{\mathcal{L}(G)} \left\| \vec{\nabla} f \right\|_{H(\mathfrak{g})}^2 d\mu &= \int_{\mathcal{L}(G)} \left\| \sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(\pi_{\mathcal{P}} g)) G(s_i, \cdot) A \right\|_{H(\mathfrak{g})}^2 \nu_t(g_0, dg) \\ &= \int_{G^{\mathcal{P}}} \left\| \sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(g_{\mathcal{P}})) G(s_i, \cdot) A \right\|_{H(\mathfrak{g})}^2 \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}) \\ &= \int_{G^{\mathcal{P}}} \left\| \sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(g_{\mathcal{P}})) G_{\mathcal{P}}(s_i) A \right\|_{\mathfrak{g}^{\mathcal{P}}}^2 \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}), \end{aligned}$$

where, for the last equation, we have used the fact that $\pi_{\mathcal{P}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ is an isometry. But for any $h = (h_0, h_1, \ldots, h_{n-1}) \in \mathfrak{g}^{\mathcal{P}}$,

$$\begin{aligned}(\sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(g_{\mathcal{P}})) G_{\mathcal{P}}(s_i) A, h)_{\mathfrak{g}^{\mathcal{P}}} &= \sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(g_{\mathcal{P}})) \langle A, h_i \rangle \\ &= \tilde{h} F(g_{\mathcal{P}}).\end{aligned}$$

Hence, $\sum_{A \in \mathfrak{g}_0} \sum_{i=0}^{n-1} (A^{(i)} F(g_{\ \mathcal{P}})) G_{\mathcal{P}}(s_i) A = \vec{\nabla}_{\mathcal{P}} F(g_{\mathcal{P}})$. We may thus write

$$\int_{\mathcal{L}(G)} \left\| \vec{\nabla} f \right\|_{H(\mathfrak{g})}^2 d\mu = \int_{G^{\mathcal{P}}} \left\| \vec{\nabla}_{\mathcal{P}} F(g_{\mathcal{P}}) \right\|_{\mathfrak{g}^{\mathcal{P}}}^2 \nu_t^{\mathcal{P}}(g_{0\mathcal{P}}, dg_{\mathcal{P}}). \tag{6.6}$$

Using the equivalences in Equations (6.4), (6.5), and (6.6). as substitutions in (6.1) gives us (6.3). □

Since we have shown that $\mathcal{E}_{g_0,t}$ is closable for all $g_0 \in \mathcal{L}(G)$ and $t > 0$, we can extend the inequality in Theorem 6.0.2 to the Hilbert space $\mathcal{D}(\overline{\mathcal{E}}_{g_0,t})$ where $\overline{\mathcal{E}}_{g_0,t}$ is the closed extension of $\mathcal{E}_{g_0,t}$ as in (5.1).

Theorem 6.0.3 *(General Logarithmic Sobolev Inequality)Let C_0 be the constant in Theorem 6.0.1 and let $g_0 \in \mathcal{L}(G)$ be fixed. For all $t > 0$ and $f \in \mathcal{D}(\overline{\mathcal{E}}_{g_0,t})$,*

$$\int_{\mathcal{L}(G)} f^2 \log f^2 d\mu \leq \frac{2}{C_0}(e^{C_0 t} - 1)\overline{\mathcal{E}}_{g_0,t}(f) + \int_{\mathcal{L}(G)} f^2 d\mu \log(\int_{\mathcal{L}(G)} f^2 d\mu), \tag{6.7}$$

where $\mu \equiv \nu_t(g_0, \cdot)$.

Proof: Let $t > 0$ and $f \in \mathcal{D}(\overline{\mathcal{E}}_{g_0,t})$. By our discussion in Section 5.1, there exists a $(\cdot,\cdot)_1$-Cauchy sequence, $\{f_n\}$, in $\mathcal{D}(\mathcal{E}_{g_0,t}) = \mathcal{F}C^\infty(\mathcal{L}(G))$ such that $f_n \to f$ in $L^2(\mathcal{L}(G), \nu_t(g_0, \cdot))$ and $\overline{\mathcal{E}}_{g_0,t}(f_n) = \lim_{n\to\infty} \mathcal{E}_{g_0,t}(f_n)$. Since f_n is L^2-convergent, there exists a subsequence, $\{f_{n_k}\}$, which converges pointwise to f a.e.. Let $-M$, with $M > 0$, be a lower bound for the function $x \mapsto x \log x$ on $(0, \infty)$ so that $M + f_{n_k}^2 \log f_{n_k}^2 \geq 0$ for all k. Fatou's Lemma then implies

$$\int_{\mathcal{L}(G)} \{M + f^2 \log f^2\} d\mu \leq \liminf_{k\to\infty} \int_{\mathcal{L}(G)} \{M + f_{n_k}^2 \log f_{n_k}^2\} d\mu$$

and thus

$$\int_{\mathcal{L}(G)} f^2 \log f^2 d\mu \leq \liminf_{k\to\infty} \int_{\mathcal{L}(G)} f_{n_k}^2 \log f_{n_k}^2 d\mu.$$

Using Theorem 6.0.2, this give us

$$\begin{aligned}\int_{\mathcal{L}(G)} f^2 \log f^2 d\mu &\leq \liminf_{k\to\infty} \int_{\mathcal{L}(G)} f_{n_k}^2 \log f_{n_k}^2 d\mu \\ &\leq \lim_{k\to\infty} \{\frac{2}{C_0}(e^{C_0 t}-1)\mathcal{E}_{g_0,t}(f_{n_k}) \\ &\quad + \int_{\mathcal{L}(G)} f_{n_k}^2 d\mu \log(\int_{\mathcal{L}(G)} f_{n_k}^2 d\mu)\} \\ &= \frac{2}{C_0}(e^{C_0 t}-1)\overline{\mathcal{E}}_{g_0,t}(f) + \int_{\mathcal{L}(G)} f^2 d\mu \log(\int_{\mathcal{L}(G)} f^2 d\mu). \ \square\end{aligned}$$

In conclusion, one may refer to Davies [2] or Gross [10] for some general properties of logarithmic Sobolev inequalities. In particular, Gross discusses the relationships between logarithmic Sobolev inequalities, such as the one in (6.7) above, and the unique self-adjoint operator, $H_{g_0,t}$, associated to the closed form, $\overline{\mathcal{E}}_{g_0,t}$, appearing in the inequality (i.e., $\overline{\mathcal{E}}_{g_0,t}(f,g) = (H_{g_0,t}f,g)_{L^2}$ for all $f,g \in \mathcal{D}(\overline{\mathcal{E}}_{g_0,t})$). For example, by the Rothaus-Simon mass gap theorem, (6.7) implies that

$$\|f\|_{L^2} - (f,1)_{L^2}^2 \leq \frac{e^{C_0 t}-1}{C_0}\overline{\mathcal{E}}_{g_0,t}(f)$$

for all $f \in \mathcal{D}(\overline{\mathcal{E}}_{g_0,t})$. This implies that the spectrum of $H_{g_0,t}$ is contained in $\{0\} \cap [\frac{C_0}{e^{C_0 t}-1}, \infty)$. Gross also discusses an equivalence between the logarithmic Sobolev inequality and a particular hypercontractivity property of the operator semigroup, $P(T) \equiv e^{-TH_{g_0,t}}$, associated to $H_{g_0,t}$.

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