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Hypoelliptic heat kernel inequalities on H-type groups

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by

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2009

The dissertation of Nathaniel Gilbert Bartsch Eldredge is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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EPIGRAPH

I wish to God these calculations had been executed by steam.

—Charles Babbage

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ABSTRACT OF THE DISSERTATION

Hypoelliptic heat kernel inequalities on H-type groups

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Nathaniel Gilbert Bartsch Eldredge
Doctor of Philosophy in Mathematics

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Professor Bruce K. Driver, Chair

We study inequalities related to the heat kernel for the hypoelliptic sublaplacian on an H-type Lie group. Specifically, we obtain precise pointwise upper and lower bounds on the heat kernel function itself. We then apply these bounds to derive an estimate on the gradient of solutions of the heat equation, which is known to have various significant consequences including logarithmic Sobolev inequalities. We also present a computation of the heat kernel, and a discussion of the geometry of H-type groups including their geodesics and Carnot-Carathéodory distance functions.

Chapter 1

Introduction

1.1 Two trivial examples

In recent years, there has been considerable interest in the study of hypoelliptic operators and associated problems. The purpose of this dissertation is to address two specific questions, regarding estimates for the heat kernel, in the context of H-type Lie groups.

To introduce and motivate these problems, we begin with a simpler example.

Example 1.1.1. Let

$$\Delta^{(3)} := \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$$

be the usual Laplacian on $\mathbb{R}^3$. (We decorate various objects in this example with the superscript $^{(3)}$ to contrast with examples to come.) Consider the Cauchy problem for the associated heat equation:

$$\begin{aligned} \left(\Delta^{(3)} - \frac{\partial}{\partial t}\right) u(t, \mathbf{x}) = 0 \quad &\text{for all } t > 0, \mathbf{x} \in \mathbb{R}^3 \\ u(0, \mathbf{x}) = f(\mathbf{x}) \quad &\text{for all } \mathbf{x} \in \mathbb{R}^3 \end{aligned} \tag{1.1.1}$$

where, for instance, $f \in C_c(\mathbb{R}^3)$ (i.e. f is an continuous real-valued function on $\mathbb{R}^3$ with compact support). Then (1.1.1) has a unique bounded solution $u(t, \mathbf{x})$ (see [15]). To emphasize the dependence on the initial condition f, we write $u(t, \mathbf{x}) = P_t^{(3)} f(\mathbf{x})$. ($P_t^{(3)}$ is really the heat semigroup $P_t^{(3)} = e^{t\Delta^{(3)}}$). It is well-known that $P_t^{(3)}$ has a convolution kernel, which is the heat kernel $p_t^{(3)}$, i.e.

$$P_t^{(3)} f(\mathbf{x}) = f(\mathbf{x}) * p_t^{(3)} = \int_{\mathbb{R}^3} f(\mathbf{x}-\mathbf{u}) p_t^{(3)}(\mathbf{u})\, d\mathbf{u} \tag{1.1.2}$$

where

$$p_t^{(3)}(\mathbf{x}) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{1}{4t}|\mathbf{x}|^2}. \tag{1.1.3}$$

$p_t^{(3)}$ can also be viewed as the fundamental solution to (1.1.1), with initial condition a delta distribution supported at the origin, i.e. $p_t^{(3)} = P_t^{(3)}\delta_0$.

A useful interpretation of the heat equation (1.1.1) is as a model for diffusion . Imagine that $\mathbb{R}^3$ is filled with air, which is contaminated unevenly with perfume . If $f(\mathbf{x})$ represents the concentration of perfume at the point $\mathbf{x}$ at time $t = 0$, then the solution $u(t,\mathbf{x})$ gives the concentration at later times t as the perfume diffuses throughout $\mathbb{R}^3$. The heat kernel $p_t^{(3)}(\mathbf{x})$ corresponds to the concentration following an initial configuration with a unit mass of perfume concentrated at the origin. This model can also be viewed probabilistically, if the perfume is considered to consist of a large number of tiny particles moving randomly through $\mathbb{R}^3$ (specifically, moving according to Brownian motion; see Figure 1.1). Then $p_t^{(3)}(\mathbf{x})$ gives the probability density that a particle which begin at the origin at time $t = 0$ winds up "near" the point $\mathbf{x}$ at time t. It is then plausible that $p_t^{(3)}$ should be a smooth function, and $P_t^{(3)}$ a "smoothing" operator, since particles will immediately spread out from any area where they may be highly concentrated.

For comparison with following examples, we note the trivial fact that the vector fields $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$, the sum of whose squares gives $\Delta^{(3)}$, are translation invariant with respect to the usual vector addition $+$ on $\mathbb{R}^3$ (by the chain rule), and the same is true for $\Delta^{(3)}$. That is, if $L_{\mathbf{u}}(\mathbf{x}) = \mathbf{u} + \mathbf{x}$ denotes translation by $\mathbf{u} \in \mathbb{R}^3$, we have by the chain rule that $\Delta^{(3)}(f \circ L_{\mathbf{u}}) = (\Delta^{(3)} f) \circ L_{\mathbf{u}}$. Moreover, these vector fields have the property that they span the tangent space to $\mathbb{R}^3$ at every point; every differentiable curve is tangent at each point to some linear combination of $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$. For each fixed $t > 0$, the heat kernel behaves at infinity like $e^{-\frac{1}{4t}|\mathbf{x}|^2}$, where the distance $|\mathbf{x}|$ can be interpreted as the length of the shortest path from $\mathbf{0}$ to $\mathbf{x}$ which is tangent to the span of $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ (namely, a straight line, since the latter condition holds trivially).

Another property of interest is that the gradient in $\mathbf{x}$ of a solution $u = P_t^{(3)} f$ can be controlled in terms of the usual gradient $\nabla^{(3)}$ of the initial condition f, and we have

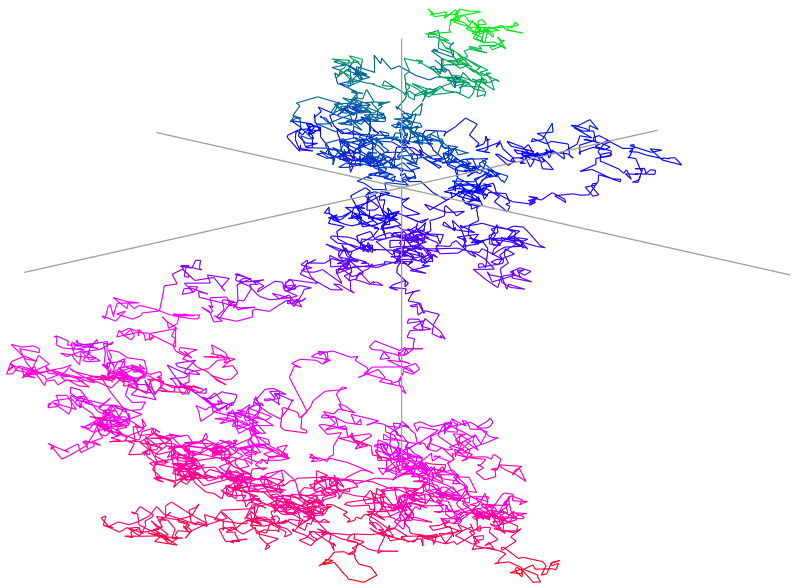

Figure 1.1: Brownian motion in $\mathbb{R}^3$. The colors vary with the z coordinate.

the inequality

$$\left|\left(\nabla^{(3)} P_t^{(3)} f\right)(\mathbf{x})\right| \leq K(t) P_t^{(3)} \left(\left|\nabla^{(3)} f\right|\right)(\mathbf{x}) \tag{1.1.4}$$

for some constant $K(t)$ depending on t. Indeed, by differentiating under the integral sign we have

$$\begin{aligned}
\left|\left(\nabla^{(3)} P_t^{(3)} f\right)(\mathbf{x})\right| &= \left|\nabla^{(3)} \int_{\mathbb{R}^3} f(\mathbf{x}-\mathbf{u}) p_t^{(3)}(\mathbf{u})\, d\mathbf{u}\right| \\
&= \left|\int_{\mathbb{R}^3} \nabla^{(3)} \left[f(\mathbf{x}-\mathbf{u}) p_t^{(3)}(\mathbf{u})\right] d\mathbf{u}\right| \\
&= \left|\int_{\mathbb{R}^3} \nabla^{(3)} f(\mathbf{x}-\mathbf{u}) p_t^{(3)}(\mathbf{u})\, d\mathbf{u}\right| \\
&\leq \int_{\mathbb{R}^3} \left|\nabla^{(3)} f(\mathbf{x}-\mathbf{u})\right| p_t^{(3)}(\mathbf{u})\, d\mathbf{u} \\
&= P_t^{(3)} \left(\left|\nabla^{(3)} f\right|\right)(\mathbf{x})
\end{aligned}$$

so that (1.1.4) holds with $K(t) \equiv 1$.

Much of the nice behavior of the operator $\Delta^{(3)}$ is related to the fact that it is an elliptic operator (see Definition 2.3.6). A particular consequence of this is "elliptic regularity," which guarantees that even weak solutions of (1.1.1) (in the sense of distributions) must actually be smooth functions. The following example shows what can go wrong if this condition is relaxed too far.

Example 1.1.2. We continue to work in $\mathbb{R}^3$, and let

$$\Delta^{(2)} := \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$$

be the Laplacian in the x and y variables only. For $\Delta^{(2)}$, the Cauchy problem

$$\begin{aligned} \left(\Delta^{(2)} - \frac{\partial}{\partial t}\right) u(t, \mathbf{x}) = 0 \quad \text{for all } t > 0, \mathbf{x} \in \mathbb{R}^3 \\ u(0, \mathbf{x}) = f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \end{aligned} \tag{1.1.5}$$

reduces to a one-parameter family of Cauchy problems in $\mathbb{R}^2$, indexed by z. In some sense, the degenerate operator $\Delta^{(2)}$ is not "using" all three dimensions. (1.1.5) still has a unique bounded solution $P_t^{(2)} f$ given the initial values f, namely

$$P_t^{(2)} f(x, y, z) = \iint_{\mathbb{R}^2} f(x - x', y - y', z) p_t^{(2)}(x', y')\, dx'\, dy'$$

where

$$p_t^{(2)}(x, y) = \frac{1}{4\pi t} e^{-\frac{1}{4t}(x^2+y^2)}.$$

However, it is obvious that no smoothness is imposed on the z dependence of $P_t^{(2)} f$. Indeed, the fundamental solution of (1.1.5) must be interpreted as the distribution $p_t^{(2)}(x, y)\delta_0(z)$.

The vector fields $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ obviously do not span the tangent space of $\mathbb{R}^3$ at any point, and paths which are everywhere tangent to this subspace must be horizontal, i.e. have constant z coordinate. $p_t^{(2)}(x, y)$ behaves at infinity like $e^{-\frac{1}{4t}|(x,y)|^2}$, where the "horizontal distance" $|(x, y)| = \sqrt{x^2 + y^2}$ could be interpreted as the length of the shortest horizontal path from the origin to (x, y). Off the x-y plane, this distance should be considered infinite.

The inequality

$$\left|\left(\nabla^{(2)} P_t^{(2)} f\right)(\mathbf{x})\right| \leq K(t) P_t^{(2)} \left(\left|\nabla^{(2)} f\right|\right)(\mathbf{x}) \tag{1.1.6}$$

again holds with $K(t) \equiv 1$, if we take $\nabla^{(2)} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ to be the gradient in the x and y variables only. However, this is really a one-parameter family of inequalities indexed by z, and provides very limited control over the behavior of the solution. For instance, a function f satisfying $\nabla^{(2)} f \equiv 0$ need not be constant on $\mathbb{R}^3$.

In terms of diffusion, (1.1.5) describes a system in which perfume diffuses horizontally, but not vertically. Probabilistically, particles move according to a two-dimensional "horizontal" Brownian motion, keeping their z coordinate fixed. This can be viewed as a three-dimensional Brownian motion which has been "constrained" to only follow horizontal paths. Of course, it is natural that concentration smoothing occurs within planes of constant z coordinate, but not between them.

1.2 One nontrivial example: the Heisenberg group

The foregoing examples represent two extremes of behavior. This dissertation focuses on a class of operators which occupy a middle ground.

Example 1.2.1. We work again on $\mathbb{R}^3$. Let X, Y be the vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z} \tag{1.2.1}$$

and take L to be the operator

$$L = X^2 + Y^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z} + \frac{1}{4}\left(x^2 + y^2\right)\left(\frac{\partial}{\partial z}\right)^2. \tag{1.2.2}$$

We immediately note that X, Y, L are not translation invariant with respect to vector addition on $\mathbb{R}^3$. However, if we equip $\mathbb{R}^3$ with the binary operation

$$(x, y, z) \star (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right), \tag{1.2.3}$$

then $(\mathbb{R}^3, \star)$ becomes a Lie group, known as the **Heisenberg group** $\mathbb{H}_1$. This is the prototype for the *H-type groups* which are the subject of this dissertation. Then X, Y, L are invariant with respect to *left* translation under $\star$ (or simply **left-invariant**), i.e. if $L_h(g) = h \star g$ is left translation by h, we have $L(f \circ L_h) = (Lf) \circ L_h$. (We shall begin using the letters g, h, k instead of $\mathbf{x}, \mathbf{u}$ to represent elements of $\mathbb{H}_1$, to emphasize its group structure, but shall not forget that $\mathbb{H}_1 = \mathbb{R}^3$ as a set and as a smooth manifold.)

We again consider the Cauchy problem

$$\begin{aligned}\left(L - \frac{\partial}{\partial t}\right)u(t,g) &= 0 \quad \text{for all } t > 0,\, g \in \mathbb{H}_1 \\ u(0,g) &= f(g) \quad \text{for all } g \in \mathbb{H}_1.\end{aligned} \tag{1.2.4}$$

As before, a unique bounded solution exists, given by

$$P_t f(g) = \int_{\mathbb{H}_1} f(g \star k^{-1}) p_t(k)\, dm(k)$$

where the heat kernel p_t is given by the more complicated formula

$$p_t(x,y,z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda z - \frac{1}{4}\lambda \coth(t\lambda)(x^2+y^2} \frac{\lambda}{4\pi \sinh(t\lambda)}\, d\lambda. \tag{1.2.5}$$

Here m is Lebesgue measure, which is also the Haar measure for $\mathbb{H}_1$ as it is invariant under left and right translation. A derivation of a generalization of (1.2.5) appears in Section 2.4.

The operator L is not elliptic; the matrix of coefficients of second-order partials is

$$Q(x,y,z) = \begin{pmatrix} 1 & 0 & -\frac{1}{2}y \\ 0 & 1 & \frac{1}{2}x \\ -\frac{1}{2}y & \frac{1}{2}x & \frac{1}{4}(x^2+y^2) \end{pmatrix}$$

which is easily shown to be positive semidefinite but degenerate for all (x,y,z). However, the heat kernel p_t is actually smooth, and hence so are bounded solutions to the Cauchy problem (1.2.4). Thus the operator L retains some regularity; specifically, it is hypoelliptic (see Definition 1.3.1).

The reason for this regularity is related to the following observation. Although the vector fields $\{X,Y\}$ clearly do not span the tangent space of $\mathbb{H}_1 = \mathbb{R}^3$ at any point (since there are only two of them), yet their Lie bracket $[X,Y] := XY - YX = \frac{\partial}{\partial z} =: Z$ is linearly independent, so that $\{X,Y,Z\}$ *does* span the tangent space at each point. (The relations $[X,Y] = Z$ and $[X,Z] = [Y,Z] = 0$, which define the **Heisenberg Lie algebra**, arise in quantum mechanics and are the reason for the use of the name of Heisenberg.) By analogy with Example 1.1.2, we call a path $\gamma(t) = (x(t), y(t), z(t))$ **horizontal** if it is tangent to some linear combination of X, Y (but not Z) at each point. See Figure 1.2. The relationship $[X,Y] = Z$ then suggests that a horizontal curve which travels a distance ϵ in

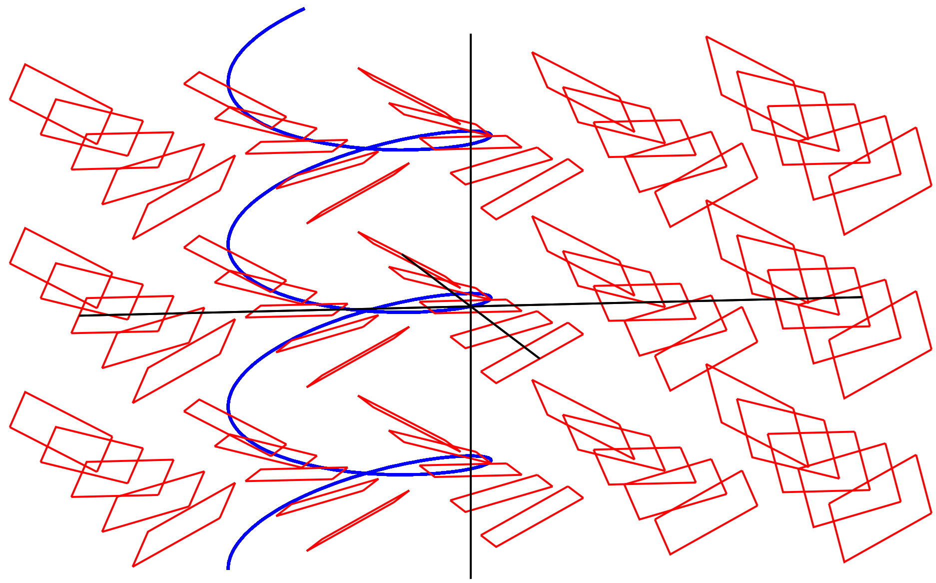

Figure 1.2: Horizontal planes in $\mathbb{H}_1$. Each of the planes in red represents the two-dimensional subspace of the tangent space $T_g\mathbb{H}_1$ at a point g which is spanned by $X(g), Y(g)$. A sample horizontal curve, which is tangent at each point to the corresponding subspace, is shown in blue.

the directions of $+X, +Y, -X, -Y$ successively will make ϵ^2 progress in the "forbidden" Z direction. Thus it is plausible that, unlike in Example 1.1.2, horizontal paths may be able to join arbitrary pairs of points of $\mathbb{H}_1$.

Our probabilistic diffusion model suggests how this relates to the regularity property of L. Perfume particles should move according to a "horizontal Brownian motion" that has been constrained to follow horizontal paths. See Figure 1.3. (Our definition of "horizontal" as "tangent to some linear combination of X, Y" requires reinterpretation, since Brownian motion paths are nowhere differentiable. This can be done in the language of stochastic calculus.) Since, unlike in Example 1.1.2, horizontal paths do not remain stuck in a submanifold, but are able to reach arbitrary points (see below), it seems reasonable that Brownian particles should be able to diffuse throughout space. Locally, their motion is horizontal to first order, but also vertical to second order. Thus high concentrations of perfume spread out in all directions, giving rise to a smoothing effect.

Let us see what such paths look like. It is clear that a horizontal path must satisfy

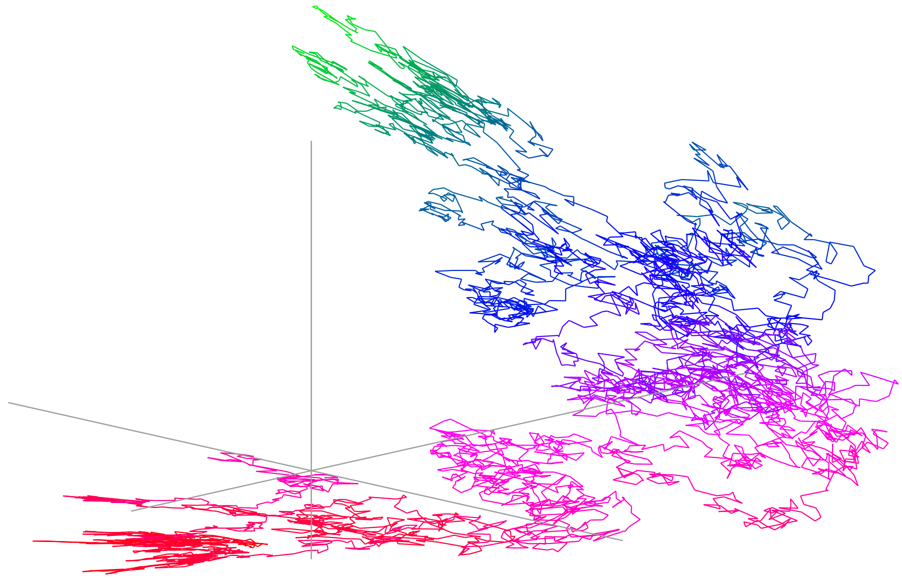

Figure 1.3: Horizontal Brownian motion in $\mathbb{H}_1$. The colors vary with the z coordinate.

$\dot{\gamma}(t) = \dot{x}(t)X(\gamma(t)) + \dot{y}(t)Y(\gamma(t))$, so that solving for $\dot{z}(t)$ and integrating we find

$$z(t) = z(0) + \frac{1}{2}\int_0^t x(t)(\dot{y}(t) - \dot{x}(t)y(t))\,dt. \tag{1.2.6}$$

By Green's theorem, this says that $z(t) - z(0)$ is equal to the (signed) area enclosed by the two-dimensional curve $(x(t), y(t))$. (If the curve is not closed, one may close it by adjoining straight lines from the origin to $(x(0), y(0))$ and $(x(t), y(t))$, since such lines do not contribute to the integral in (1.2.6). See Figure 1.4.) It is intuitively clear that one may connect any pair of points in $\mathbb{R}^2$ by a curve which encloses any prescribed signed area; thus any pair of points in $\mathbb{H}_1$ can indeed be joined by a horizontal path.

The regularity of L and the connectedness of $\mathbb{H}_1$ by horizontal paths are both strongly related to the fact that the vector fields X, Y, together with their bracket $[X, Y] = Z$, span the tangent space at each point. This *bracket generating condition* is what separates $\mathbb{H}_1$ from degenerate situations like Example 1.1.2. We shall say more about this condition in Section 1.3.

If we consider the **length** of a horizontal path to be the length of its horizontal projection $(x(t), y(t))$ (because $(\dot{x}(t), \dot{y}(t))$ are the coefficients of the horizontal vector fields X, Y, which we may consider to be "orthonormal"), the problem of finding the

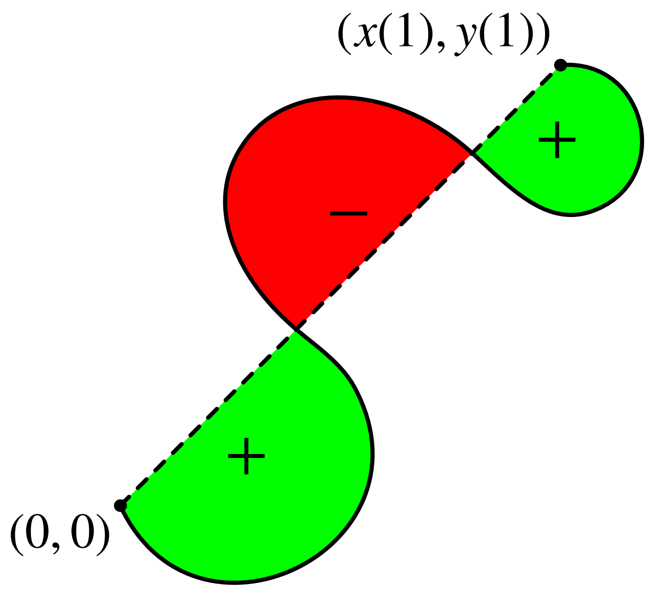

Figure 1.4: Signed area, as used to describe horizontal curves in $\mathbb{H}_1$. The plane curve from $(0, 0)$ to $(x(1), y(1))$ (solid black) may be closed by the straight line back to the origin (dashed). The areas thus enclosed contribute positively or negatively according to a "right-hand rule" based on the orientation of the curve.

shortest horizontal curve joining two points is just the problem of finding the shortest plane curve enclosing a given area in the previous sense. This is a classic problem in the calculus of variations[1] whose solution is the arc of a circle. Thus the "horizontal distance" (or **Carnot-Carathéodory distance**) $d(g, h)$ between any two points $g, h \in \mathbb{H}_1$ is finite. It is this distance that, as in the previous examples, we might expect to describe the behavior of the heat kernel p_t at infinity. Indeed, it was shown in [29] that

$$\frac{C_1}{1 + \sqrt{|(x, y)|\, d(0, g)}} e^{-\frac{1}{4} d(0,g)^2} \le p_1(g) \le \frac{C_2}{1 + \sqrt{|(x, y)|\, d(0, g)}} e^{-\frac{1}{4} d(0,g)^2} \tag{1.2.7}$$

for constants C_1, C_2. Chapter 4 of this dissertation is concerned with extending estimates like (1.2.7) to general H-type groups.

[1]The problem is commonly called Dido's Problem after the legendary [47] queen of Carthage. It seems she and her followers found themselves shipwrecked in North Africa after fleeing the murderous King Pygmalion of Tyre. She pled with the local authorities for some land on which to settle, but was offered only as much land as she could cover with an ox-hide. Interpreting the word "cover" creatively, she cut the hide into thin strips and used them to bound a large region of land, with the ocean as the other boundary. On this prime waterfront property she founded the city of Carthage. There she lived happily with her companions until the arrival of Aeneas from Troy, who caused for Dido an entirely different sort of problem [36], less easily solved by mathematics.

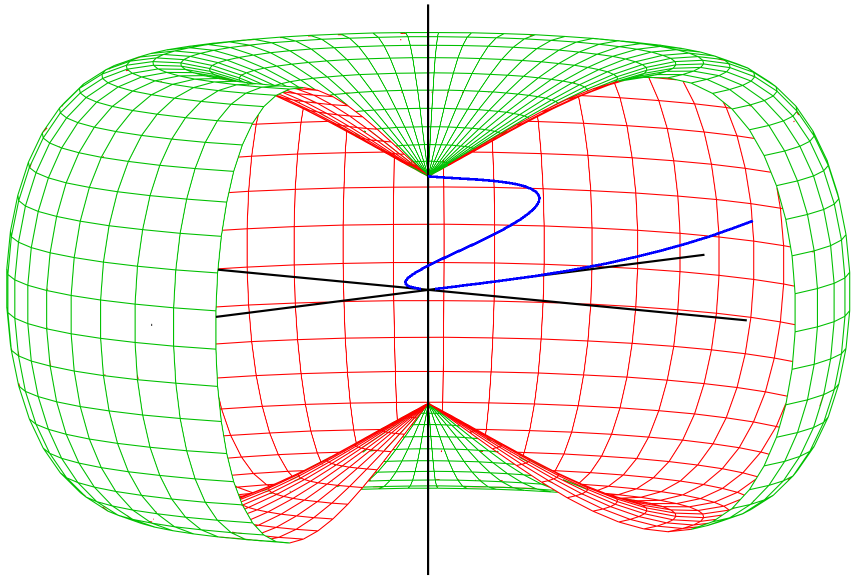

Figure 1.5: The unit ball of $\mathbb{H}_1$, with respect to the Carnot-Carathéodory distance d. A section has been removed to show geodesics (shortest horizontal paths, blue) emanating from the origin.

Regarding the problem of gradient estimates, it was first shown in [28] that there exists a constant K (independent of t!) such that

$$|\nabla P_t f| \leq K P_t(|\nabla f|) \tag{1.2.8}$$

where $\nabla = (X, Y)$ is the "horizontal gradient." The proof makes extensive use of the heat kernel bounds (1.2.7). (1.2.8 is a considerably more informative statement than its analogue (1.1.6) in Example 1.1.2. For instance, because $[X, Y] = Z$, we have that $\nabla f \equiv 0$ implies that f is constant. Chapter 5 of this dissertation follows a proof appearing in [5] to show that (1.2.7) also holds for general H-type groups.

1.3 Hypoelliptic operators and Lie groups

As suggested in the previous section, some sort of regularity condition is needed on an operator L to avoid the degeneracy of Example 1.1.2 without requiring ellipticity as in Example 1.1.1. We therefore confine our attention to hypoelliptic operators.

Definition 1.3.1. A partial differential operator L on a manifold M is said to be **hypoelliptic** if, for every distribution u on M, $Lu \in C^\infty(M)$ if and only if $u \in C^\infty(M)$.

By standard elliptic regularity results, every elliptic operator is hypoelliptic; its corresponding parabolic heat operator is hypoelliptic as well. See, for instance, Section 3.4 of [43]. The Heisenberg sublaplacian is an example of a hypoelliptic operator that is not elliptic. More examples are supplied by the following theorem, of which a simplified proof can be found in Chapter 7 of [43].

Theorem 1.3.2 (Hörmander [18])**.** *Let $X_1, \dots, X_n$ be smooth vector fields on a manifold M satisfying the following* ***bracket generating condition****: for each $m \in M$ there exists an integer r (the* ***rank*** *of $\{X_1, \dots, X_n\}$ at m) such that*

$$T_m M = \operatorname{span}\{X_{i_1}(m), [X_{i_1}, X_{i_2}](m), [X_{i_1}, [X_{i_2}, X_{i_3}]](m), \cdots : 1 \le i_1, \dots, i_r \le n\} \tag{1.3.1}$$

where at most $r-1$ brackets are taken. Let Y be another smooth vector field on M. Then the second-order operator $L := X_1^2 + \cdots + X_n^2 + Y$ is hypoelliptic.

In particular, this implies that "harmonic" functions on M (satisfying $Lf = 0$) are automatically smooth. Also, if we replace M by $M \times (0, \infty) = \{(m, t) : m \in M, t > 0\}$ and set $Y = \frac{\partial}{\partial t}$, we see that solutions u to the heat equation $\left(L - \frac{\partial}{\partial t}\right) u = 0$ are also smooth functions of m and t.

The bracket generating condition (1.3.1) requires that L be built out of enough vector fields to fill out the tangent space to M at each point, when their brackets are included. This serves to rule out situations like Example 1.1.2. In particular, if $X_1, \dots, X_n$ satisfy (1.3.1), it is easy to see that if $X_i f \equiv 0$ for all i (i.e. its "gradient" is identically zero) then f must be constant.

As in the case of the Heisenberg group, the bracket generating condition is also related to a geometric fact about horizontal paths.

Theorem 1.3.3 (Chow)**.** *If M is a connected manifold with vector fields $X_1, \dots, X_n$ satisfying (1.3.1), then any pair of points in M can be joined by a path which is tangent at each point to some linear combination of $X_1, \dots, X_n$.*

This also allows a reasonable Carnot-Carathéodory distance to be defined, as in $\mathbb{H}_1$. More will be said about this idea in Section 3.1.

Hörmander's theorem supplies a very large class of hypoelliptic operators; indeed, too large for present purposes. It is difficult to say much about an operator on a general smooth manifold without having some structure on the manifold. Lie groups provide such structure while at the same time not giving up too much generality, as we shall see.

Example 1.3.4. Let G be a Lie group with group operation $\star$, and let $X_1, \ldots, X_n$ be left-invariant vector fields on G. The bracket-generating condition (1.3.1) is then equivalent to the condition that the vector fields $\{X_1, \ldots, X_n\}$ generate the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ of all left-invariant vector fields on G. (Note that in this case, the rank of $\{X_1, \ldots, X_n\}$ is the same at every point of G.) The left-invariant operator $L = X_1^2 + \cdots + X_n^2$, called a **sublaplacian** is thus hypoelliptic.

Definition 1.3.5. A Lie algebra $\mathfrak{g}$ is **nilpotent** of step r if all r-fold Lie brackets vanish, i.e. $[X_1, [X_2, \ldots [X_r, X_{r+1}] \ldots]] = 0$ for all $X_1, \ldots, X_r, X_{r+1} \in \mathfrak{g}$. A nilpotent Lie algebra $\mathfrak{g}$ is **stratified** if there is a decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ such that $[V_1, V_i] = V_{i+1}$ for $1 \le i < r$ and $[V_1, V_r] = 0$. A Lie group is nilpotent (respectively, stratified) if its Lie algebra is.

A theorem of Rothschild and Stein [40] states, informally speaking, that a bracket-generating set of vector fields $\{X_i\}$ on a manifold M can be locally approximated in a neighborhood of a point $m \in M$ by a bracket-generating set of left-invariant vector fields $\{Y_i\}$ on some nilpotent Lie group G. This approach involves first lifting the vector fields $\{X_i\}$ to vector fields $\{\tilde{X}_i\}$ on $M \times \mathbb{R}^k$ for some k (effectively adding additional variables, to handle the possibility that the $\{X_i\}$ are not linearly independent at m). Then, $M \times \mathbb{R}^k$ is identified with the free nilpotent Lie group G with n generators, and under this identification the lifted vector fields $\{\tilde{X}_i\}$ differ from left-invariant vector fields only to small order. Thus, it makes sense to study Hörmander-type hypoelliptic operators by studying nilpotent Lie groups.

With regard to the heat kernel inequalities we study in this dissertation, much less is known for general nilpotent Lie groups than for the Heisenberg group. The known pointwise bounds on the heat kernel corresponding to the left-invariant hypoelliptic operator L are in general much less sharp than those in (1.2.7); see Section 4.2. Gradient bounds like (1.2.8) are also not known to hold with much generality, although a weaker L^p-type estimate has been shown in [34]; see Section 5.2.

It is not clear at this stage what sort of tools are appropriate to attack these problems in a general Lie group setting. Therefore, for this dissertation, we restrict our attention to a smaller class of nilpotent Lie groups, the so-called H-type or Heisenberg-type groups, which generalize in a more limited way the Heisenberg group $\mathbb{H}_1$ of Section 1.2. In this setting it is possible to carry out more explicit computations involving heat kernels and thereby obtain stronger results, analogous to (1.2.7) and (1.2.8) which are known for $\mathbb{H}_1$.

Chapter 2

H-type Groups

2.1 Definition and elementary properties

The objects of central study in this dissertation are the so-called H-type or Heisenberg-type groups. H-type groups were first introduced by Kaplan in [24]. Chapter 18 of [8] is an excellent reference for basic facts about these groups.

We begin with the definition.

Definition 2.1.1. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. We say $\mathfrak{g}$ is an **H-type Lie algebra** if there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that:

1. $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$; and

2. For each $Z \in \mathfrak{z}$, the map $J_Z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle \quad \text{for } X, Y \in \mathfrak{z}^\perp \tag{2.1.1}$$

   is an orthogonal map when $\|Z\|^2 := \langle Z, Z \rangle = 1$.

We will say that such an inner product is **admissible**, and that $\mathfrak{g}$ is an H-type Lie algebra **under** $\langle \cdot, \cdot \rangle$. (This should not be interpreted as a restrictive statement; if one particular inner product will do, certainly others will do as well.)

Notation 2.1.2. If G is a connected finite-dimensional Lie group, we write $\operatorname{Lie} G$ for the Lie algebra of left-invariant smooth vector fields on G, under the bracket operation $[X, Y] = XY - YX$.

Definition 2.1.3. An **H-type group** is a connected, simply connected Lie group G such that $\operatorname{Lie} G$ is an H-type Lie algebra.

Example 2.1.4. As in Example 1.2.1, the classical Heisenberg group $\mathbb{H}_1$ is the Lie group consisting of $\mathbb{R}^3$ with the following group operation:

$$(x, y, z) \star (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right). \tag{2.1.2}$$

The Heisenberg Lie algebra $\mathfrak{h}_1 = \operatorname{Lie} \mathbb{H}_1$ is spanned by the vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}. \tag{2.1.3}$$

We note that $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$. Then $\mathfrak{h}_1$ is an H-type Lie algebra under an inner product such that $\{X, Y, Z\}$ are orthonormal. (In particular, the center of $\mathfrak{h}_1$ is one-dimensional and spanned by Z, and $J_Z X = Y$, $J_Z Y = -X$.) Thus $\mathbb{H}_1$ is an H-type group.

Example 2.1.5. For $n \geq 1$, the (isotropic) **Heisenberg-Weyl group** $\mathbb{H}_n$ is the Lie group consisting of $\mathbb{R}^{2n+1}$ with the following group operation:

$$\begin{aligned}(x_1, \dots, x_{2n}, z) \star (x_1', \dots, x_{2n}', z') &= (x_1 + x_1', \dots, x_{2n} + x_{2n}', \\ z + z' + \frac{1}{2}((x_1 x_2' - x_1' x_2) &+ (x_3 x_4' - x_3' x_4) + \cdots + (x_{2n-1} x_{2n}' - x_{2n-1}' x_{2n}))).\end{aligned} \tag{2.1.4}$$

The Lie algebra $\mathfrak{h}_n = \operatorname{Lie} \mathbb{H}_n$ is spanned by

$$X_{2i-1} = \frac{\partial}{\partial x_{2i-1}} - \frac{1}{2} x_{2i} \frac{\partial}{\partial z}, \quad X_{2i} = \frac{\partial}{\partial x_{2i}} + \frac{1}{2} x_{2i-1} \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z} \tag{2.1.5}$$

where i ranges from 1 to n. Note that $[X_{2i-1}, X_{2i}] = Z$ and all other independent brackets are zero. Then $\mathfrak{h}_n$ is an H-type Lie algebra under an inner product such that $\{X_1, \dots, X_{2n}, Z\}$ are orthonormal. The center of $\mathfrak{h}_n$ is one-dimensional and spanned by Z, and $J_Z X_{2i-1} = X_{2i}$, $J_Z X_{2i} = -X_{2i-1}$. Thus $\mathbb{H}_n$ is an H-type group.

Example 2.1.6. The **complex Heisenberg group** is the Lie group G consisting of $\mathbb{C}^3$ with the group operation

$$(x, y, z) \star (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right). \tag{2.1.6}$$

If we write $x = x_1 + ix_2$, $y = y_1 + iy_2$, $z = z_1 + iz_2$, we find that $\operatorname{Lie} G$ is spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}y_1\frac{\partial}{\partial z_1} - \frac{1}{2}y_2\frac{\partial}{\partial z_2} \qquad X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}y_2\frac{\partial}{\partial z_1} - \frac{1}{2}y_1\frac{\partial}{\partial z_2}$$
$$Y_1 = \frac{\partial}{\partial y_1} + \frac{1}{2}x_1\frac{\partial}{\partial z_1} + \frac{1}{2}x_2\frac{\partial}{\partial z_2} \qquad Y_2 = \frac{\partial}{\partial y_2} - \frac{1}{2}x_2\frac{\partial}{\partial z_1} + \frac{1}{2}x_1\frac{\partial}{\partial z_2}$$
$$Z_1 = \frac{\partial}{\partial z_1} \qquad Z_2 = \frac{\partial}{\partial z_2}$$

We have $[X_1, Y_1] = -[X_2, Y_2] = Z_1$, $[X_2, Y_1] = [X_1, Y_2] = Z_2$, and all other independent brackets vanish. $\operatorname{Lie} G$ is an H-type Lie algebra under an inner product so that $\{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$ are orthonormal. The center is two-dimensional and spanned by Z_1, Z_2. We have

$$J_{Z_1}X_1 = Y_1 \qquad J_{Z_1}X_2 = -Y_2 \qquad J_{Z_1}Y_1 = -X_1 \qquad J_{Z_1}Y_2 = X_2$$
$$J_{Z_2}X_1 = Y_2 \qquad J_{Z_2}X_2 = Y_1 \qquad J_{Z_2}Y_1 = -X_2 \qquad J_{Z_2}Y_2 = -X_1$$

Thus G is an H-type group. This example shows explicitly that the H-type groups consist of more than the Heisenberg-Weyl groups, and may have centers with dimension larger than 1.

We now list a number of elementary algebraic properties of H-type Lie algebras, which are useful in computations.

Proposition 2.1.7. *Let $\mathfrak{g}$ be an H-type Lie algebra under the inner product $\langle\cdot,\cdot\rangle$, with center $\mathfrak{z}$ and J_Z defined by (2.1.1). If $Z, W \in \mathfrak{z}$, $X, Y \in \mathfrak{z}^\perp$, we have:*

1. *J_Z is linear in Z, i.e. $J_{aZ+W} = aJ_Z + J_W$.*
2. *J_Z is skew-adjoint, i.e. $\langle J_Z X, Y\rangle = -\langle X, J_Z Y\rangle$. In particular $\langle J_Z X, X\rangle = 0$.*
3. *$J_Z^2 = -\|Z\|^2 I$.*
4. *If $Z \neq 0$, J_Z is invertible and $J_Z^{-1} = -\|Z\|^{-2} J_Z$.*
5. *$J_Z J_W + J_W J_Z = -2\langle Z, W\rangle I$. (This is the fundamental relation that defines Clifford algebras, and suggests a connection between H-type Lie algebras and Clifford algebras. This connection is more fully explored in Section 2.2.)*

6. $\langle J_Z X, J_W X\rangle = \langle Z, W\rangle \|X\|^2$.

7. $\langle J_Z X, J_Z Y\rangle = \langle X, Y\rangle \|Z\|^2$.

8. $[X, J_Z X] = \|X\|^2 Z$.

9. *Define* $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ *as usual by* $\mathrm{ad}_X Y = [X, Y]$. *If* $\|X\| = 1$ *then* ad_X *maps* $(\ker \mathrm{ad}_X)^\perp$ *isometrically onto* $\mathfrak{z}$. *(This is sometimes taken as part of the definition of an H-type Lie algebra, in place of item 2 of Definition 2.1.1.)*

10. $\dim \mathfrak{z}^\perp$ *is even.*

11. $\dim \mathfrak{z}^\perp \geq \dim \mathfrak{z} + 1$. *(This bound is far from sharp; see Theorem 2.2.6 below.)*

Proof. 1. We have

$$\begin{aligned}\langle J_{aZ+W} X, Y\rangle &= \langle aZ + W, [X, Y]\rangle = a\,\langle Z, [X, Y]\rangle + \langle W, [X, Y]\rangle \\ &= a\,\langle J_Z X, Y\rangle + \langle J_W X, Y\rangle = \langle (aJ_Z + J_W)X, Y\rangle\,.\end{aligned}$$

2. $\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle = -\langle Z, [Y, X]\rangle = -\langle J_Z Y, X\rangle$.

3. If $\|Z\| = 1$, then J_Z is orthogonal by definition and skew-adjoint by item 2, so that

$$\left\langle J_Z^2 X, Y\right\rangle = -\langle J_Z X, J_Z Y\rangle = -\langle X, Y\rangle\,.$$

The general case follows by linearity (item 1).

4. An obvious consequence of item 3.

5. Using polarization, we have

$$\begin{aligned}J_Z J_W + J_W J_Z &= (J_Z + J_W)^2 - J_Z^2 - J_W^2 \\ &= J_{Z+W}^2 - J_Z^2 - J_W^2 \\ &= -(\|Z + W\|^2 - \|Z\|^2 - \|W\|^2)I \\ &= -2\,\langle Z, W\rangle\, I.\end{aligned}$$

6. By skew-adjointness, we have $\langle J_Z X, J_W X\rangle = -\langle J_Z J_W X, X\rangle = -\langle J_W J_Z X, X\rangle$, so that

$$\langle J_Z X, J_W X\rangle = -\frac{1}{2}\langle (J_Z J_W + J_W J_Z)X, X\rangle = -\frac{1}{2}\langle -2\langle Z, W\rangle X, X\rangle = \langle Z, W\rangle \|X\|^2$$

using item 5.

7. An obvious consequence of items 2 and 3.

8. Given $W \in \mathfrak{z}$, we have $\langle W, [X, J_Z X]\rangle = \langle J_W X, J_Z X\rangle = \langle W, Z\rangle \|X\|^2$ by definition of J_Z and item 6. Given $Y \in \mathfrak{z}^\perp$, we have $\langle Y, [X, J_Z X]\rangle = 0 = \langle Y, Z\rangle$. Thus $\langle U, [X, J_Z X]\rangle = \langle U, Z\rangle \|X\|^2$ for all $U \in \mathfrak{g}$, so that $[X, J_Z X] = \|X\|^2 Z$.

9. Suppose $\|X\| = 1$. It is obvious that the restriction of ad_X to $(\ker \operatorname{ad}_X)^\perp$ is injective. We next show the restriction maps onto $\mathfrak{z}$. Given $Z \in \mathfrak{z}$, let $Y = J_Z X$. Then $\operatorname{ad}_X Y = [X, Y] = [X, J_Z X] = Z$ by item 8. Moreover, suppose $U \in \ker \operatorname{ad}_X$; then $\langle Y, U\rangle = \langle J_Z X, U\rangle = \langle Z, [X, U]\rangle = \langle Z, \operatorname{ad}_X U\rangle = 0$, so $Y \in (\ker \operatorname{ad}_X)^\perp$.

   To show isometry, suppose $\operatorname{ad}_X Y = Z$. By injectivity $Y = J_Z X$. Then $\|Y\|^2 = \|J_Z X\|^2 = \|Z\|^2 \|X\|^2 = \|Z\|^2$.

10. For any nonzero $Z \in \mathfrak{z}$, $J_Z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ is a nonsingular skew-adjoint linear transformation. It follows that $\dim \mathfrak{z}^\perp$ must be even. (In particular, all the eigenvalues of J_Z are imaginary and must come in conjugate pairs.)

11. Let $\{Z_1, \ldots, Z_m\}$ be an orthonormal basis for $\mathfrak{z}$, and $X \in \mathfrak{z}^\perp$ be a unit vector. The vectors $\{J_{Z_1} X, \ldots, J_{Z_m} X\} \subset \mathfrak{z}^\perp$ are unit vectors, which are mutually orthogonal by item 6, and all are orthogonal to X by item 2. Thus $\{X, J_{Z_1} X, \ldots, J_{Z_m} X\}$ is a set of $m + 1$ orthonormal vectors in $\mathfrak{z}^\perp$.

□

Proposition 2.1.8. *If G is an H-type group, then G is nilpotent of step* 2 *and stratified.*

Proof. This is obvious from the condition that $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g} = \operatorname{Lie} G$. An appropriate stratification is $V_1 = \mathfrak{z}^\perp$, $V_2 = \mathfrak{z}$. □

Not all step 2 stratified Lie groups are H-type.

Example 2.1.9. The abelian Lie group $G = \mathbb{R}^n$ is step 1 stratified, but not H-type. (The Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ has center $\mathfrak{z} = \mathfrak{g}$, so $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = [0, 0] \neq \mathfrak{z}$).

Example 2.1.10. For $n \geq 2$, let $a_1, \ldots, a_n \in \mathbb{R}$ be constants, and let G be the **anisotropic Heisenberg-Weyl group** G consisting of $\mathbb{R}^{2n+1}$ with the following group operation:

$$(x_1, \ldots, x_{2n}, z) \star (x_1', \ldots, x_{2n}', z') = (x_1 + x_1', \ldots, x_{2n} + x_{2n}',$$
$$z + z' + \frac{1}{2}(a_1(x_1 x_2' - x_1' x_2) + a_2(x_3 x_4' - x_3' x_4) + \cdots + a_n(x_{2n-1} x_{2n}' - x_{2n-1}' x_{2n}))). \tag{2.1.7}$$

The Lie algebra $\mathfrak{h}_n = \operatorname{Lie} \mathbb{H}_n$ is spanned by

$$X_{2i-1} = \frac{\partial}{\partial x_{2i-1}} - \frac{a_i}{2} x_{2i} \frac{\partial}{\partial z}, \quad X_{2i} = \frac{\partial}{\partial x_{2i}} + \frac{a_i}{2} x_{2i-1} \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z} \tag{2.1.8}$$

where i ranges from 1 to n. Note that $[X_{2i-1}, X_{2i}] = a_i Z$ and all other independent brackets are zero, so that G is step 2 stratified (with $V_1 = \operatorname{span}\{X_j : 1 \leq j \leq 2n\}$, $V_2 = \operatorname{span} Z$). If the a_i are not all equal, then there is no inner product on $\mathfrak{g} = \operatorname{Lie} G$ under which it is an H-type Lie algebra. If there were, then we would have $J_Z X_{2i-1} = a_i X_{2i}$, $J_Z X_{2i} = -a_i X_{2i-1}$, so that $J_Z^2 X_{2i-1} = -a_i^2 X_{2i-1}$. Since the a_i are not all equal, this contradicts item 3 of Proposition 2.1.7. Thus G is not an H-type group.

Any H-type group can be realized in terms of Euclidean space, as we now show.

Proposition 2.1.11. *Let G be an H-type group, with $\operatorname{Lie} G = (\mathfrak{g}, [\cdot, \cdot])$ an H-type Lie algebra under some inner product $\langle \cdot, \cdot \rangle$), and $\mathfrak{z}$ the center of $\mathfrak{g}$. There exists $n, m \geq 0$, and a bijective linear isometry $\phi : (\mathfrak{g}, \langle \cdot, \cdot \rangle) \to (\mathbb{R}^{2n+m}, \langle \cdot, \cdot \rangle_e)$, where $\langle \cdot, \cdot \rangle_e$ is the usual Euclidean inner product on $\mathbb{R}^{2n+m}$, such that $\phi(\mathfrak{z}) = 0 \oplus \mathbb{R}^m$.*

Define a bracket $[\cdot, \cdot]'$ on $\mathbb{R}^{2n+m}$ via $[\phi(X), \phi(Y)]' = [X, Y]$. Then $(\mathbb{R}^{2n+m}, [\cdot, \cdot]')$ is an H-type Lie algebra under the Euclidean inner product, and $\phi : (\mathfrak{g}, [\cdot, \cdot]) \to (\mathbb{R}^{2n+m}, [\cdot, \cdot]')$ is an isomorphism of Lie algebras.

Define a group operation $\star'$ on $\mathbb{R}^{2n+m}$ via the Baker-Campbell-Hausdorff formula:

$$v \star' w := v + w + \frac{1}{2}[v, w]'. \tag{2.1.9}$$

Then $(\mathbb{R}^{2n+m}, \star')$ is an H-type Lie group which is isomorphic to G, and whose Lie algebra is canonically isomorphic to $(\mathbb{R}^{2n+m}, [\cdot, \cdot]')$.

Proof. Let $m = \dim \mathfrak{z}$ and $2n = \dim \mathfrak{z}^\perp$ (recall from Proposition 2.1.7 item 10 that $\dim \mathfrak{z}^\perp$ is even). By selecting an orthonormal basis for $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ which is adapted to the decomposition $\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z}$, we can construct a (non-canonical) bijective linear isometry $\phi : (\mathfrak{g}, \langle \cdot, \cdot \rangle) \to (\mathbb{R}^{2n+m}, \langle \cdot, \cdot \rangle_e)$ such that $\phi(\mathfrak{z}) = 0 \oplus \mathbb{R}^m$. If $[\cdot, \cdot]'$ is constructed as specified, it is obvious that $(\mathbb{R}^{2n+m}, [\cdot, \cdot]')$ is an H-type Lie algebra under the Euclidean inner product, and $\phi : (\mathfrak{g}, [\cdot, \cdot]) \to (\mathbb{R}^{2n+m}, [\cdot, \cdot]')$ is an isomorphism of Lie algebras.

Since G is a step 2 nilpotent Lie group which is connected and simply connected, if we define a group operation $\star$ on $\mathfrak{g}$ via

$$X \star Y := X + Y + \frac{1}{2}[X, Y] \tag{2.1.10}$$

then the exponential map $\mathrm{Exp} : (\mathfrak{g}, \star) \to G$ is a Lie group isomorphism. Since $\phi : (\mathfrak{g}, [\cdot, \cdot]) \to (\mathbb{R}^{2n+m}, [\cdot, \cdot]')$ is a Lie algebra isomorphism, and $\star'$ is defined appropriately in terms of $[\cdot, \cdot]'$, we have that $\phi : (\mathfrak{g}, \star) \to (\mathbb{R}^{2n+m}, \star')$ is a Lie group isomorphism. Thus G is isomorphic to $(\mathbb{R}^{2n+m}, \star')$. □

Thus, henceforth we can, and will, assume that our H-type group G is $\mathbb{R}^{2n+m}$ with a group operation $\star$ defined by (2.1.10) for some Lie bracket $[\cdot, \cdot]$, and that $\mathrm{Lie}\, G \cong (\mathbb{R}^{2n+m}, [\cdot, \cdot]$ is an H-type Lie algebra under the Euclidean inner product.

Notation 2.1.12. We will write elements of $G = \mathbb{R}^{2n+m}$ as $g = (x, z)$, where $x \in \mathbb{R}^{2n}$, $z \in \mathbb{R}^m$. $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^{2n+m}$. $\{e_1, \ldots, e_{2n}\}$ is the standard orthonormal basis for $\mathbb{R}^{2n} \oplus 0$, and $\{u_1, \ldots, u_m\}$ is the standard orthonormal basis for $0 \oplus \mathbb{R}^m$. We will use the coordinates $x_i(g) = \langle g, e_i \rangle$, $z_j(g) = \left\langle g, u_j \right\rangle$.

We now have several distinct structures on the single set $\mathbb{R}^{2n+m}$:

- A vector space, under the usual addition and scalar multiplication;
- An inner product space, under the usual Euclidean inner product;
- A Lie algebra, under the bracket $[\cdot, \cdot]$;
- A Lie group, under the group operation $\star$.

These structures do not necessarily interact nicely. In particular, addition, scalar multiplication, and the Euclidean inner product are not left- or right-invariant with respect to $\star$; the maps $v \mapsto v + w$ and $v \mapsto cv$ are not Lie group homomorphisms; and $\star$ is not bilinear. Nevertheless, we will sometimes view these structures, especially the inner product and the operators $J_z \in \mathrm{End}(\mathbb{R}^{2n})$ for $z \in \mathbb{R}^m$, as functions on $\mathbb{R}^{2n+m}$ viewed as a group. The reader who prefers a more intrinsic view may insert the exponential map (which is the identity on $\mathbb{R}^{2n+m}$) where appropriate.

Notation 2.1.13. Note that $\mathbb{R}^m = 0 \oplus \mathbb{R}^m$ is the center of the Lie algebra and also of the Lie group. Thus we will sometimes think of the Lie bracket as a map $[\cdot,\cdot] : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^m$. We can thus write the group operation as

$$(x, z) \star (x', z') = \left(x + x', z + z' + \frac{1}{2}[x, x']\right) \tag{2.1.11}$$

Note that 0 is the group identity, and the inverse is $(x, z)^{-1} = -(x, z)$.

Notation 2.1.14. For each $g \in G$, let $L_g, R_g : G \to G$ be the maps of left- and right-translation by g, and $j : G \to G$ be the inverse map. That is, $L_g h = g \star h$, $R_g h = h \star g$, $jh = h^{-1}$. Of course, L_g, R_g, j are diffeomorphisms of the smooth manifold G.

Proposition 2.1.15. *Lebesgue measure m on $G = \mathbb{R}^{2n+m}$ is invariant under left and right translation and inverses with respect to $\star$. Thus, G is unimodular, and we may take m to be a bi-invariant Haar measure on G.*

Proof. By inspection of (2.1.11), it is clear that for any g the differentials of L_g (and likewise R_g) is lower triangular, and thus the Jacobian determinant is 1. The Jacobian determinant of j is obviously 1 in absolute value. □

Given this measure, we can define convolution on G.

Definition 2.1.16. If $f_1, f_2 : G \to \mathbb{R}$, their convolution $f_1 * f_2$ is the function

$$(f_1 * f_2)(g) := \int_G f_1(g \star k^{-1}) f_2(k)\, dm(k) = \int_G f_1(k) f_2(k^{-1} \star g)\, dm(k).$$

for all g such that the integral makes sense. We may also take f_1, f_2 to be appropriate distributions.

The following properties are typical and we omit the proofs.

Proposition 2.1.17. *1.* $(f_1 \circ j) * (f_2 \circ j) = (f_2 * f_1) \circ j$.

2. *If f is a distribution on G and $\psi \in C_c^\infty(G)$, then $f * \psi, \psi * f \in C^\infty(G)$. The same holds if f is a tempered distribution and ψ is a Schwartz function on $G = \mathbb{R}^{2n+m}$.*

3. *(Young's inequality) If $f_1 \in L^1(G)$ and $f_2 \in L^p(G)$, then $f_1 * f_2 \in L^p(G)$ and $\|f_1 * f_2\|_{L^p} \le \|f_1\|_{L^1} \|f_2\|_{L^2}$.*

4. *If $(\cdot, \cdot)$ denotes the inner product on $L^2(G)$, $f_1, f_2 \in L^2(G)$ and $\psi \in L^1(G)$, then $(\psi * f_1, f_2) = \left(f_1, \tilde{\psi} * f_2\right)$ and $(f_1 * \psi, f_2) = \left(f_1, f_2 * \tilde{\psi}\right)$, where $\tilde{\psi}(g) = \overline{\psi(g^{-1})}$.*

An important operation on H-type groups is the following dilation.

Definition 2.1.18. For $\alpha \in (0, \infty)$, define $\varphi_\alpha : G \to G$ by $\varphi_\alpha(x, z) = (\alpha x, \alpha^2 z)$.

Observe that φ_α is a group automorphism of G, $\varphi_\alpha \circ \varphi_\beta = \varphi_{\alpha\beta}$, and $\varphi_\alpha^{-1} = \varphi_{\alpha^{-1}}$. We also note the change of variables $dm(\varphi_\alpha(g)) = \alpha^{2(n+m)}\, dm(g)$.

2.2 Algebraic properties

Algebraically, H-type Lie algebras (and hence also H-type groups) correspond to representations of Clifford algebras. In this section, we describe this correspondence, and classify the possible dimensions of H-type groups.

Definition 2.2.1. Let V be a real vector space and $B : V \times V \to \mathbb{R}$ be a bilinear form. The **Clifford algebra** $C\ell(V, B)$ is the algebra (with identity 1) freely generated by V subject to the relation

$$uv + vu = -2B(u, v). \tag{2.2.1}$$

(Some authors omit the negative sign.)

Definition 2.2.2. A **representation** of an algebra $\mathcal{A}$ is an algebra homomorphism $\pi : \mathcal{A} \to \mathrm{End}(W)$ for some finite-dimensional vector space W; the **dimension** of π is the dimension of W. (We require that $\pi(1) = I$.) If $\pi : \mathcal{A} \to \mathrm{End}(W)$, $\pi' : \mathcal{A}' \to \mathrm{End}(W')$ are representations of two algebras, we say they are **equivalent** if there is an algebra isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ and a vector space isomorphism $T : W \to W'$ such that

$$\pi(a) = T^{-1} \pi'(\psi(a)) T \tag{2.2.2}$$

for all $a \in \mathcal{A}$. If the vector spaces V, W are equipped with inner products and T is unitary, we say that π, π' are **unitarily equivalent**.

We note that if $\pi : \mathcal{A} \to \mathrm{End}(W)$ is a representation of $\mathcal{A}$, then the left action $v \cdot w = \pi(v)w$ turns W into a left $\mathcal{A}$-module. This is an alternate, and indeed more common, way to study representations of algebras.

One direction of the correspondence between H-type Lie algebras and representations of Clifford algebras is rather easy.

Proposition 2.2.3. *1. If $\mathfrak{g}$ is an H-type Lie algebra under the inner product $\langle \cdot, \cdot \rangle$, then the map*

$$\mathfrak{z} \ni z \mapsto \pi(z) := J_z \in \mathrm{End}(\mathfrak{z}^\perp)$$

extends uniquely to a representation of $C\ell(\mathfrak{z}, \langle \cdot, \cdot \rangle)$.

2. If $\mathfrak{g}'$ is another H-type Lie algebra, isomorphic to $\mathfrak{g}$, then there is an admissible inner product $\langle \cdot, \cdot \rangle'$ on $\mathfrak{g}'$ such that the map

$$\mathfrak{z}' \ni z' \mapsto \pi'(z') := J_{z'} \in \mathrm{End}(\mathfrak{z}'^\perp)$$

defines a representation of $C\ell(\mathfrak{z}', \langle \cdot, \cdot \rangle')$ which is unitarily equivalent to π.

3. If $(\mathfrak{g}, \langle \cdot, \cdot \rangle), (\mathfrak{g}', \langle \cdot, \cdot \rangle')$ are two H-type Lie algebras with admissible inner products, and the corresponding representations ϕ, ϕ' are unitarily equivalent, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.

Proof. 1. This follows directly from item 5 of Proposition 2.1.7.

2. If $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra isomorphism, then define the inner product $\langle \cdot, \cdot \rangle'$ on $\mathfrak{g}'$ via $\langle \phi(v), \phi(w) \rangle' := \langle v, w \rangle$. It is clear that $\mathfrak{z}' = \phi(\mathfrak{z})$, $\mathfrak{z}'^\perp = \phi(\mathfrak{z}^\perp)$. We also note that for $z \in \mathfrak{z}$ and $x, y \in \mathfrak{z}^\perp$, we have

$$\begin{aligned}
\left\langle J_{\phi(z)}\phi(x), \phi(y) \right\rangle' &= \langle \phi(z), [\phi(x), \phi(y)]' \rangle' \\
&= \langle \phi(z), \phi([x, y]) \rangle' \\
&= \langle z, [x, y] \rangle \\
&= \langle J_z x, y \rangle \\
&= \langle \phi(J_z x), \phi(y) \rangle'
\end{aligned}$$

so that $J_{\phi(z)}\phi(x) = \phi(J_z x)$. If $\|\phi(z)\|' = \|z\| = 1$, we have

$$\left\langle J_{\phi(z)}\phi(x), J_{\phi(z)}\phi(y)\right\rangle' = \langle \phi(J_z x), \phi(J_z y)\rangle' = \langle J_z x, J_z y\rangle = \langle x, y\rangle = \langle \phi(x), \phi(y)\rangle'$$

so that $J_{\phi(z)}$ is indeed orthogonal. Thus $\langle\cdot,\cdot\rangle'$ is admissible for $\mathfrak{g}$.

Moreover, the restriction of ϕ to $\mathfrak{z}$ extends uniquely to an isomorphism of algebras $\psi : C\ell(\mathfrak{z}, \langle\cdot,\cdot\rangle) \to C\ell(\mathfrak{z}', \langle\cdot,\cdot\rangle')$, and the restriction to $\mathfrak{z}^\perp$ is a unitary map $T : \mathfrak{z}^\perp \to \mathfrak{z}'^\perp$.

Let π, π' be the representations corresponding to $(\mathfrak{g}, \langle\cdot,\cdot\rangle), (\mathfrak{g}', \langle\cdot,\cdot\rangle')$. To show they are equivalent, it suffices to verify (2.2.2) for $a = z \in \mathfrak{z}$. But if $z \in \mathfrak{z}$, $x \in \mathfrak{z}^\perp$, we have

$$T^{-1}\pi'(\psi(z))Tx = \phi^{-1}(J_{\phi(z)}\phi(x)) = J_z x = \pi(z)x$$

since $J_{\phi(z)}\phi(x) = \phi(J_z x)$ as shown above.

3. Suppose there exists an algebra isomorphism $\psi : C\ell(\mathfrak{z}, \langle\cdot,\cdot\rangle) \to C\ell(\mathfrak{z}', \langle\cdot,\cdot\rangle')$ and a unitary $T : \mathfrak{z}^\perp \to \mathfrak{z}'^\perp$ such that (2.2.2) holds. Given $x \in \mathfrak{z}^\perp$, $z \in \mathfrak{z}$, and set $\phi(x+z) = Tx + \psi(z)$. Clearly $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a well defined linear bijection; we verify it is a Lie algebra homomorphism. First, we note that $[\phi(x+z), \phi(y+w)]' = [\phi(x), \phi(y)]'$ by linearity and the fact that $\phi(\mathfrak{z}) = \mathfrak{z}'$. Next, if $x, y \in \mathfrak{z}^\perp$ and $z \in \mathfrak{z}$, we have

$$\begin{aligned}
\langle \phi(z), [\phi(x), \phi(y)]'\rangle' &= \left\langle J_{\phi(z)}\phi(x), \phi(y)\right\rangle' \\
&= \langle \pi'(\phi(z))Tx, Ty\rangle' \\
&= \langle T\pi(z)x, Ty\rangle' \\
&= \langle \pi(z)x, y\rangle \\
&= \langle J_z x, y\rangle \\
&= \langle z, [x, y]\rangle \\
&= \langle \phi(z), \phi([x, y])\rangle' .
\end{aligned}$$

Thus $[\phi(x), \phi(y)]' = \phi([x, y])$, so that ϕ is indeed a Lie algebra homomorphism.

□

The converse is only a little more involved.

Theorem 2.2.4. *If V is a finite-dimensional vector space equipped with an inner product $\langle\cdot,\cdot\rangle_V$ and $\pi : C\ell(V,\langle\cdot,\cdot\rangle_V) \to \mathrm{End}(W)$ is a representation of the corresponding Clifford algebra, then there exist a bracket $[\cdot,\cdot]$ and an inner product $\langle\cdot,\cdot\rangle$ on $W\oplus V$ under which it is an H-type Lie algebra with center V, and $J_z = \pi(z)$ for $z \in V$.*

Proof. We begin by constructing an inner product on W by a standard averaging technique. Let $u_1,\ldots,u_m$ be an orthonormal basis for V, so that $\left\langle u_i, u_j\right\rangle_V = \delta_{ij}$. Notice that $u_i^2 = -1$, $u_iu_j = -u_ju_i$ for $i \neq j$. Then the finite subset H of $C\ell(V,\langle\cdot,\cdot\rangle_V)$ defined by

$$H = \{\pm 1, \pm u_{i_1}u_{i_2}\ldots u_{i_n} : 1 \leq i_1,\ldots,i_n \leq m, n \geq 1\} \tag{2.2.3}$$

is a group under the algebra multiplication. Let $\langle\cdot,\cdot\rangle_{W,1}$ be any inner product on W. For $w, w' \in W$, let

$$\langle w, w'\rangle_{W,2} := \frac{1}{|H|}\sum_{g\in H}\langle \pi(g)w, \pi(g)w'\rangle_{W,1}\,. \tag{2.2.4}$$

Clearly $\langle\cdot,\cdot\rangle_{W,2}$ is again an inner product on W.

Define an inner product $\langle\cdot,\cdot\rangle$ on $W\oplus V$ by

$$\langle w+v, w'+v'\rangle := \langle w,w'\rangle_{W,2} + \langle v,v'\rangle_V\,.$$

Observe that with respect to this inner product, we have for each basis vector u_j that

$$\begin{aligned}\left\langle \pi(u_j)w, w'\right\rangle &= \frac{1}{|H|}\sum_{g\in H}\left\langle \pi(gu_j)w, \pi(g)w'\right\rangle_{W,1}\\ &= \frac{1}{|H|}\sum_{g'\in H}\left\langle \pi(g')w, \pi(g'u_j^{-1})w'\right\rangle_{W,1}\\ &= \left\langle w, \pi(u_j^{-1})w'\right\rangle\\ &= -\left\langle w, \pi(u_j)w'\right\rangle\end{aligned}$$

by making the change of dummy variables $g' = gu_j$, and noticing that $u_j^{-1} = -u_j$ (the inverse taken in H). Thus $\pi(u_j)$ is skew-adjoint on W with respect to $\langle\cdot,\cdot\rangle$, and by linearity this is also true of $\pi(v)$ for any $v \in V$. From the Clifford algebra relation, we also have $\pi(u)^2 = -I$, for any unit vector $u \in V$, from which it follows that $\pi(u)$ is an orthogonal linear transformation with respect to $\langle\cdot,\cdot\rangle$.

Define a bracket $[\cdot,\cdot]$ on $W\oplus V$ by

$$[w+v, w'+v'] := \sum_{j=1}^{m}\left\langle \pi(u_j)w, w'\right\rangle u_j \in V.$$

It is obvious that this bracket is bilinear, and its skew-symmetry follows from the skew-symmetry of $\pi(u_j)$ established above. The Jacobi identity is trivial, since $[W \oplus V, V] = 0$. Thus $\mathfrak{g} = (W \oplus V, [\cdot, \cdot])$ is a Lie algebra.

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. We already have that $V \subset \mathfrak{z}$. To see the reverse inclusion, suppose there exists w, v such that $[w, w'] = 0$ for all $w' \in W$. In particular,

$$0 = \langle u_1, [w, \pi(u_1)w] \rangle = \sum_{j=1}^{m} \left\langle \pi(u_j)w, \pi(u_1)w \right\rangle \left\langle u_1, u_j \right\rangle = \langle \pi(u_1)w, \pi(u_1)w \rangle = \langle w, w \rangle$$

so that $w = 0$. Thus $W \cap \mathfrak{z} = 0$, so that $\mathfrak{z} = V$.

To see that $\mathfrak{g}$ is an H-type Lie algebra under $\langle \cdot, \cdot \rangle$, we first note that $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = [W, W] \subset V = \mathfrak{z}$. Next, $v \in V$, $w, w' \in W$ we have

$$\langle v, [w, w'] \rangle = \sum_{j=1}^{m} \left\langle \pi(u_j)w, w' \right\rangle \left\langle v, u_j \right\rangle = \langle \pi(v)w, w' \rangle$$

so that $J_v = \pi(v)$. We have already shown that $\pi(v)$ is orthogonal when $\|v\| = 1$. Finally, we must show $[W, W] = V$. For nonzero $w \in W$, we have $\langle u_k, [w, \pi(u_l)w] \rangle = \langle \pi(u_k)w, \pi(u_l)w \rangle$. If $k = l$ then the isometry of $\pi(u_k)$ gives $\langle u_l, [w, \pi(u_l)w] \rangle = \langle w, w \rangle$. If $k \neq l$ then u_k, u_l anticommute, so by skew-symmetry

$$\langle \pi(u_k)w, \pi(u_l)w \rangle = -\langle \pi(u_l)\pi(u_k)w, w \rangle = \langle \pi(u_k)\pi(u_l)w, w \rangle = -\langle \pi(u_l)w, \pi(u_k)w \rangle$$

so that $\langle u_k, [w, \pi(u_l)w] \rangle = 0$. Thus $[w, \pi(u_l)w] = u_l \langle w, w \rangle$. It follows by linearity that $[w, \pi(v)w] = v$ for any $v \in V$. □

From this theorem we can immediately derive some consequences regarding the possible dimensions of H-type Lie algebras and their centers.

Corollary 2.2.5. *For any $m \geq 1$, there exists an H-type Lie algebra $\mathfrak{g}$ with center $\mathfrak{z}$ such that $\dim \mathfrak{z} = m$ and $\dim \mathfrak{g}$ is arbitrarily large.*

Proof. Take $V = \mathbb{R}^m$ with the Euclidean inner product. Let $\pi : C\ell(V, \langle \cdot, \cdot \rangle) \to \operatorname{End}(W)$ be any nontrivial representation of the corresponding Clifford algebra. (Note that the group H defined in (2.2.3) forms a basis for $C\ell(V, \langle \cdot, \cdot \rangle_V)$, so any group representation of H extends by linearity to an algebra representation of $C\ell(V, \langle \cdot, \cdot \rangle_V)$.) Then Theorem 2.2.4 gives an H-type Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{z} = \dim V = m$ and $\dim \mathfrak{g} = m + \dim W$. To make $\dim \mathfrak{g}$ larger, replace π with $\pi \oplus \pi$, et cetera. □

Necessary and sufficient conditions on the dimension of a Clifford algebra and its representations are given by the Hurwitz-Radon-Eckmann theorem [14]. The corresponding statement for H-type Lie algebras was given in [24]; we restate it here.

Theorem 2.2.6. *For any nonnegative integer k, we can uniquely write $k = a2^{4p+q}$ where a is odd and $0 \le q \le 3$; let $\rho(k) := 8p + 2^q$. (ρ is sometimes called the **Hurwitz-Radon function**.) There exists an H-type Lie algebra of dimension $2n + m$ with center of dimension m if and only if $m < \rho(2n)$.*

In particular, suppose $m < \rho(2n)$ where $2n = a2^{4p+q}$ as above. Since $2^q \le 2q + 2$ for $0 \le q \le 3$, we have

$$2^m < 2^{8p+2^q} \le 2^{8p+2q+2} = 4(2^{4p+q})^2 \le 4(2n)^2$$

so that $n > \frac{1}{4}2^{m/2}$. Thus, in order for an H-type Lie algebra to have a large-dimensional center, the complement of the center must be of very large dimension.

Much more information about Clifford algebras, including a classification of their representations, can be found in [2]. This in particular could be useful in constructing examples of H-type groups for computations.

2.3 The sublaplacian L

In this section, we construct the sublaplacian operator which will be the focus of this dissertation. G denotes an H-type Lie group identified with $\mathbb{R}^{2n+m}$.

Notation 2.3.1. For $i = 1, \ldots, 2n$, let $X_i, \hat{X}_i$ be respectively the unique left- and right-invariant vector fields on G with $X_i(0) = \hat{X}_i(0) = \frac{\partial}{\partial x_i}$. For $j = 1, \ldots, m$, let Z_j be the bi-invariant vector field $Z_j = \frac{\partial}{\partial u_j}$.

Proposition 2.3.2. *The vector fields $\{X_1, \ldots, X_{2n}\}$ are bracket generating in the sense of 1.3.1.*

Proof. $\operatorname{span}\{X_1, \ldots, X_{2n}\} = \mathfrak{z}^\perp$, and we have $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$. Thus any element of $\mathfrak{z}$ can be written as a linear combination of brackets of pairs of the X_i. □

We can write

$$X_i f(g) = \left.\frac{d}{ds}\right|_{s=0} f(g \star (se_i, 0)), \quad \hat{X}_i f(g) = \left.\frac{d}{ds}\right|_{s=0} f((se_i, 0) \star g). \tag{2.3.1}$$

A straightforward calculation shows

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{m} \left\langle J_{u_j} x, e_i \right\rangle \frac{\partial}{\partial z_j} \\ \hat{X}_i &= \frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{j=1}^{m} \left\langle J_{u_j} x, e_i \right\rangle \frac{\partial}{\partial z_j} \end{aligned} \tag{2.3.2}$$

We note that $[X_i, \hat{X}_j] = 0$ for all i, j.

The vector fields X_i interact with the dilations $\varphi_\alpha, \alpha > 0$ of Definition 2.1.18 via

$$X_i(f \circ \varphi_\alpha) = \alpha(X_i f) \circ \varphi_\alpha. \tag{2.3.3}$$

We also note that

$$X_i(f \circ j) = -(\hat{X}_i f) \circ j \tag{2.3.4}$$

where $j(g) = g^{-1}$.

Definition 2.3.3. The left-invariant **gradient** (or "subgradient") ∇ on G is given by $\nabla f = (X_1 f, \dots, X_{2n} f)$, with the right-invariant $\hat{\nabla}$ defined analogously. We shall also use the notation $\nabla_x f := \left(\frac{\partial}{\partial x_i} f, \dots, \frac{\partial}{\partial x_{2n}} f\right)$ and $\nabla_z f := \left(\frac{\partial}{\partial z_1} f, \dots, \frac{\partial}{\partial z_m} f\right)$ to denote the usual Euclidean gradients in the x and z variables, respectively. Note that ∇_z is both left- and right-invariant.

From (2.3.2) it is easy to verify that

$$\begin{aligned} \nabla f(x, z) &= \nabla_x f(x, z) + \frac{1}{2} J_{\nabla_z f(x,z)} x \\ \hat{\nabla} f(x, z) &= \nabla_x f(x, z) - \frac{1}{2} J_{\nabla_z f(x,z)} x. \end{aligned} \tag{2.3.5}$$

As shorthand, we write

$$\nabla = \nabla_x + \frac{1}{2} J_{\nabla_z} x. \tag{2.3.6}$$

In particular, since J_z depends linearly on z and is orthogonal for $|z| = 1$, we have

$$\left|(\nabla - \hat{\nabla}) f(x, z)\right| = |x| \, |\nabla_z f(x, z)| \, . \tag{2.3.7}$$

The gradient has an even nicer form when applied to functions with appropriate symmetry.

Definition 2.3.4. A function $f : G \to \mathbb{R}$ is **radial** if $f(x,z) = \tilde{f}(|x|, |z|)$ for some $\tilde{f} : [0,\infty) \times [0,\infty) \to \mathbb{R}$. By abuse of notation, we will also write $f(x,z) = f(|x|, |z|)$.

For a radial function f, we have

$$\nabla f(x,z) = f_{|x|}(|x|, |z|)\hat{x} + \frac{1}{2} f_{|z|}(|x|, |z|)\, |x|\, J_{\hat{z}}\hat{x} \tag{2.3.8}$$

where we use the notation $\hat{u} := \frac{u}{|u|}$ to denote the unit vector in the u direction. We draw attention to the fact that $\hat{x}$ and $J_{\hat{z}}\hat{x}$ are orthogonal unit vectors in $\mathbb{R}^{2n}$ for any nonzero x, z.

Definition 2.3.5. The left-invariant **sublaplacian** L is the second-order differential operator defined by

$$L := X_1^2 + \cdots + X_{2n}^2. \tag{2.3.9}$$

For convenience in computations involving L, we adopt the following notation. If $\mathbf{A}, \mathbf{B}$ are k-tuples of operators, e.g. $\mathbf{A} = (A_1, \ldots, A_k)$, we let $(\mathbf{A}, \mathbf{B}) := \sum_{i=1}^k A_i B_i$. (Note that in general $(\mathbf{A}, \mathbf{B}) \neq (\mathbf{B}, \mathbf{A})$.) We can write L in terms of the gradient ∇ as $L = (\nabla, \nabla)$, which by (2.3.6) gives

$$L = \left(\nabla_x + \frac{1}{2} J_{\nabla_z} x, \nabla_x + \frac{1}{2} J_{\nabla_z} x\right) = \Delta_x + \left(\nabla_x, J_{\nabla_z} x\right) + \frac{1}{4} |x|^2 \Delta_z. \tag{2.3.10}$$

We used the fact that $\left(\nabla_x, J_{\nabla_z} x\right) = \left(J_{\nabla_z} x, \nabla_x\right)$, because $\frac{\partial}{\partial x_i} \langle J_u x, e_i \rangle = \langle J_u e_i, e_i \rangle = 0$ for any $u \in \mathbb{R}^m$ by item 2 of Proposition 2.1.7.

By Hörmander's theorem (Theorem 1.3.2), the bracket generating condition on the vector fields $\{X_i\}$ implies that L is hypoelliptic. However, L is not elliptic, as we now verify. We give a definition here to ensure that there is no doubt about terminology.

Definition 2.3.6. Let

$$A = \sum_{i,j=1}^n a_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^n b_i(\mathbf{x}) \frac{\partial}{\partial x_i} + c(\mathbf{x})$$

be a general second-order partial differential operator. The **principal symbol** of A is the quadratic form-valued function

$$Q_A(\mathbf{x})(\xi) = \sum_{i,j=1}^n a_{ij}(\mathbf{x}) \xi_i \xi_j.$$

We will say A is **elliptic** if $Q_A(\mathbf{x})$ is (strictly) positive definite for all $\mathbf{x} \in \mathbb{R}^n$, i.e. $Q_A(\mathbf{x})(\xi) > 0$. A is **degenerate elliptic** if $Q_A(\mathbf{x})$ is positive semidefinite for all $\mathbf{x} \in \mathbb{R}^n$, i.e. $Q_A(\mathbf{x})(\xi) \geq 0$.

The principal symbol of our sublaplacian L is then given by

$$Q_L(x,z)(\xi,\eta) = \left|\xi + \frac{1}{2}J_\eta x\right|^2 \geq 0$$

so that $Q_L(x,z)$ is positive semidefinite for all (x,z). However, $Q(x,z)$ is also degenerate for all (x,z), which can be seen by taking $\xi = -\frac{1}{2}J_\eta x$. Therefore L is degenerate elliptic, but not elliptic.

We record at this point some convolution formulas involving X_i, which will be used later.

Proposition 2.3.7. *1. If f is a distribution on G and $\psi \in C_c^\infty(G)$, then $X_i(f * \psi) = f * X_i\psi$. Hence also $L(f * \psi) = f * L\psi$. The same holds if f is a tempered distribution on G and ψ is a Schwartz function.*

2. (Integration by parts) If $f \in C^\infty(G)$, $\psi \in C_c^\infty(G)$ then $(X_i f, \psi) = -(f, X_i\psi)$, and $\left(\hat{X}_i f, \psi\right) = -\left(f, \hat{X}_i\psi\right)$. Hence also $(Lf, \psi) = (f, L\psi)$.

We now mention some (rather weak) positivity-preserving properties of the heat equation.

Theorem 2.3.8. *Suppose $u \in C^{2,1}(G \times [0,T])$ has the following properties:*

1. u solves the heat equation $\left(L - \frac{\partial}{\partial t}\right)u = 0$;

2. $u(\cdot,t)$ vanishes at infinity uniformly in t. That is, for any $\epsilon > 0$ there exists a compact set $K \subset G$ such that $\sup_{K^C \times [0,T]} |u| < \epsilon$.

Then $u \geq \inf_G u(\cdot, 0)$.

The proof is adapted from an argument in [21].

Proof. Let $\bar{G} = G \cup \{\infty\}$ be the one-point compactification of G. By condition 2, u extends continuously to $\bar{G} \times [0,T]$ by setting $u(\infty,t) = 0$.

Let $A := \inf_G u(\cdot,0)$. Let $c > 0$ be a constant, and let $v(g,t) = e^{ct}(u(g,t) - A + 1)$. We show that $v \geq 1$. Suppose the contrary; then $v(g_0,t_0) = a$ for some $0 < a < 1$, $(g_0,t_0) \in G \times (0,T]$. Since v is a continuous function on the compact set $\bar{G} \times [0,T]$, there exists $h \in G$, $s > 0$ such that $v(h,s) = a$, and $v(g,t) > a$ for all $g \in G$, $t < s$. (The set $v^{-1}((-\infty, a]) \subset \bar{G} \times [0, T]$ is compact and disjoint from $\bar{G} \times \{0\}$, hence there is some point (h, s) of $v^{-1}((-\infty, a])$ whose distance from $\bar{G} \times \{0\}$ is minimum.) In particular, $v_t(h, s) \le 0$.

On the other hand, we must have $v(\cdot, s) \ge a$, so that $v(\cdot, s)$ has a global minimum at h; hence so does u. Since L is an operator with positive semidefinite principal symbol, we must have $0 \le Lu(h, s) = u_t(h, s)$. Then

$$v_t(h, s) = ce^{ct}(u(h, s) - A + 1) + e^{ct}u_t(h, s) \ge ce^{ct}(u(h, s) - A + 1) = ca > 0$$

which is a contradiction. Therefore $v \ge 1$. By letting c tend to 0, it follows that $u \ge A$. □

Moreover, nonnegative solutions of the heat equation immediately become positive everywhere. This is a manifestation of the idea that "heat propagates at infinite speed." One way to show this is by using the following Harnack inequality, which is proved in [46]. This seems to be a rather larger hammer than should be needed to crush this insect, but the author is not presently aware of a simpler approach.

Theorem 2.3.9 (Special case of [46] Theorem III.2.1)**.** *Let M be a smooth manifold, $\{X_1, \dots, X_k\}$ a bracket generating set of vector fields on M, K a compact subset of M, and $0 < s < t < \infty$. Then there exists a constant C such that if $u \in C^{2,1}(M \times [0, \infty))$ is a positive solution of $\left(\sum X_i^2 - \frac{\partial}{\partial t}\right) u = 0$, then*

$$\sup_{x \in K} |u(x, s)| \le C \inf_{x \in K} |u(x, t)| .$$

We observe that the word "positive" in this theorem can immediately be replaced with the word "nonnegative," by replacing u with $u + \epsilon$ and letting $\epsilon \downarrow 0$. (The constant C is independent of u.)

Corollary 2.3.10. *If $u \in C^{2,1}(M \times [0, \infty))$ is a nonnegative solution of $\left(\sum X_i^2 - \frac{\partial}{\partial t}\right) u = 0$, and $u(\cdot, 0)$ is not identically zero, then $u(\cdot, t) > 0$ for all $t > 0$.*

Proof. Fix $y \in M$, $t > 0$. Since $u(\cdot, 0)$ is not identically zero, there exists $y_0 \in M$ with $u(y_0, 0) > 0$. Then $u(y_0, s) > 0$ for some $0 < s < t$. Let K be a compact set containing both y and y_0. Then by Theorem 2.3.9,

$$0 < u(y_0, s) \le \sup_{x \in K} |u(x, s)| \le C \inf_{x \in K} |u(x, t)| \le Cu(y, t).$$

□

To conclude this section, we remark that the operators L and ∇ are not intrinsic to the group G, since they were constructed in terms of the vector fields $\{X_i\}$ which in turn depend on the choice of orthonormal basis $\{e_i\}$ for $\mathfrak{z}^\perp$. Actually, L and $f \mapsto |\nabla f|$ depend only on the choice of admissible inner product $\langle \cdot, \cdot \rangle$.

Thus, given an abstract H-type group G, there is no canonical sublaplacian L unless further choices are made. Selecting a specific admissible inner product on $\mathfrak{g}$ will suffice. (In our treatment, with G realized as $\mathbb{R}^{2n+m}$, we have selected the Euclidean inner product, without loss of generality.)

More generally, if G is replaced with a subriemannian manifold M, the subriemannian metric can be used to construct a canonical sublaplacian. The construction proceeds along similar lines to the construction of the Laplace-Beltrami operator on a Riemannian manifold, with the role of the Riemannian volume form taken instead by the so-called Popp measure. See sections 10.5–10.6 of [35] for more details. In the case of an H-type group, the Popp measure corresponds to (a multiple of) the Haar measure, and the sublaplacian thus obtained is the same as the one used here.

2.4 The heat kernel p_t

The purpose of this section is to derive an explicit integral formula for the heat kernel p_t which is the fundamental solution of the heat equation $\left(L - \frac{\partial}{\partial t}\right) u = 0$. Loosely speaking, p_t should be a solution with initial condition $p_0 = \delta_0$ a delta distribution on $\mathbb{R}^{2n+m}$, so that solutions with other initial conditions can be found via convolution.

We begin with an informal computation that yields a formula for a candidate function p_t. Afterwards, we verify that this function has the properties that one would expect of a heat kernel.

Our computation proceeds by obtaining the heat kernel as the Fourier transform of the Mehler kernel, similar in spirit to the computation in [11]. For general step 2 nilpotent groups, [17] derived a formula probabilistically from a formula in [27] regarding the Lévy area process; [30] extended it to the Cayley Heisenberg group. [44] has a similar computation. [37] obtains the formula for H-type groups as the Radon transform of the heat kernel for the Heisenberg group. In the case of the Heisenberg group itself, [25] has a computation using magnetic field heat kernels. [6] gives a short, direct proof by assuming *a priori* that the function should be Gaussian in form; [31] is similar in spirit but covers a broader class of groups. [20] proceeds via approximation of Brownian motion by random walks; [1] extends this to a broader class of nilpotent Lie groups by using noncommutative Fourier transforms. [7] uses complex Hamiltonian mechanics.

First, we record a couple of formal algebraic identities that will help in the computations. A, B, C should be interpreted as operators; bold indicates k-tuples of operators, e.g. $\mathbf{B} = (B_1, \ldots, B_k)$.

$$\begin{aligned}
[A, BC] &= ABC - BCA + BAC - BAC = [A, B]\, C - B\, [C, A] \\
[A, (\mathbf{B}, \mathbf{C})] &= \left[A, \sum B_i C_i\right] = \sum [A, B_i C_i] \\
&= \sum ([A, B_i]\, C_i - B_i\, [C_i, A]) = ([A, \mathbf{B}]\, , \mathbf{C}) - (\mathbf{B}, [\mathbf{C}, A])\, .
\end{aligned}$$

We also note that if Δ is the usual Laplacian on $\mathbb{R}^k$ and f is a smooth function, we have

$$[\Delta, f] = \Delta f + 2\, (\nabla f, \nabla)$$

Formally, we begin with the expression $p_t = e^{tL}\delta_0$. Using (2.3.10) we write $L = \Delta_x + M + \frac{1}{4}\, |x|^2\, \Delta_z$, where $M := \left(\nabla_x, J_{\nabla_z} x\right)$ is an "angular momentum" operator. We first note that M commutes with Δ_x and with $|x|^2\, \Delta_z$:

$$\begin{aligned}
[\Delta_x, M] &= -\left[\Delta_x, \left(x, J_{\nabla_z}\nabla_x\right)\right] \\
&= -\left([\Delta_x, x]\, , J_{\nabla_z}\nabla_x\right) + \left(x, \cancel{\left[\Delta_x, J_{\nabla_z}\nabla_x\right]}\right) \\
&= -\left(2\nabla_x, J_{\nabla_z}\nabla_x\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\left[\left(J_{\nabla_z} x, \nabla_x\right), \frac{1}{4}\, |x|^2\, \Delta_z\right] &= \frac{1}{4}\left[\left(J_{\nabla_z} x, \nabla_x\right), |x|^2\right] \Delta_z \\
&= \frac{1}{4}\left(\left(J_{\nabla_z} x, \left[\nabla_x, |x|^2\right]\right) - \left(\cancel{\left[|x|^2, J_{\nabla_z} x\right]}, \nabla_x\right)\right) \Delta_z \\
&= \frac{1}{4}\left(J_{\nabla_z} x, 2x\right) \Delta_z = 0.
\end{aligned}$$

Thus by the Baker-Campbell-Hausdorff formula we have

$$e^{tL}\delta_0 = e^{t(\Delta_x + \frac{1}{4}|x|^2\Delta_z)}e^{tM}\delta_0.$$

However, note that M annihilates radial functions. Indeed, if $f \in C^\infty(\mathbb{R}^{2n+m})$ is radial in x, so that $f(x,z) = f(|x|, z)$, then $\nabla_x f = \hat{x} f_{|x|}$, where $\hat{x} = \frac{1}{|x|}x$, so

$$\left\langle J_{\nabla_z f} x, \nabla_x f \right\rangle = \left\langle J_{\nabla_z f} x, \hat{x} f_{|x|} \right\rangle = |x| f_{|x|} \left\langle J_{\nabla_z f} \hat{x}, \hat{x} \right\rangle = 0.$$

Thus, since δ_0 can be approximated by smooth radial functions, it is reasonable to write $M\delta_0 = 0$ and thus $e^{tM}\delta_0 = \delta_0$.

Now we have $p_t = e^{t(\Delta_x + \frac{1}{4}|x|^2\Delta_z)}\delta_0$, i.e.

$$\left(\frac{\partial}{\partial t} - \left(\Delta_x + \frac{1}{4}|x|^2 \Delta_z\right)\right) p_t = 0, \quad p_0 = \delta_0.$$

Taking a Fourier transform in the z variables, we see that $\hat{p}_t(x,\lambda) := \int_{\mathbb{R}^m} e^{-i\langle \lambda, z\rangle} p_t(x,z)\,dz$ satisfies the **quantum harmonic oscillator** equation

$$\left(\frac{\partial}{\partial t} - \left(\Delta_x - \frac{1}{4}|x|^2 |\lambda|^2\right)\right) u_t = 0, \tag{2.4.1}$$

with initial condition $\hat{p}_0 = \delta_0^{(x)} \otimes 1$, where $\delta_0^{(x)}$ is the delta distribution on $\mathbb{R}^{2n}$.

(2.4.1) says that $\hat{p}_t$ is the **Mehler kernel** $m_{t,\lambda} := e^{t\left(\Delta_x - \frac{1}{4}|x|^2|\lambda|^2\right)}\delta_0$, the fundamental solution to the **quantum harmonic oscillator**. We now derive Mehler's formula for $m_{t,\lambda}$. Other derivations can be found in [42, pp. 38, 55], [41, p. 29], and references therein.

By the Trotter product formula, we expect to have

$$m_{t,\lambda} = \lim_{N\to\infty} \left(e^{-\frac{1}{4N}|x|^2|\lambda|^2} e^{\frac{t}{N}\Delta}\right)\delta_0.$$

Since $e^{\frac{t}{N}\Delta}\delta_0$ is a Gaussian (i.e. of the form $A(t)e^{-b(t)|x|^2}$), and the family of Gaussians is preserved by the operators $e^{\frac{t}{N}\Delta}\delta_0$ and $e^{-\frac{1}{4N}|x|^2|\lambda|^2}$, we expect that $m_{t,\lambda}$ should be a Gaussian as well. Thus we assume

$$m_{t,\lambda}(x) = A(t)e^{-b(t)|x|^2} \tag{2.4.2}$$

and solve for the functions $A(t), b(t)$.

Since $m_{t,\lambda}$ should solve the quantum harmonic oscillator equation (2.4.1), we have

$$(A'(t) - |x|^2 b'(t))e^{-b(t)|x|^2} = (4b(t)^2 |x|^2 - 4nb(t) - \frac{1}{4} |\lambda|^2 |x|^2)A(t)e^{-b(t)|x|^2} \tag{2.4.3}$$

whence, by equating like powers of $|x|$,

$$A'(t) = -4nA(t)b(t) \tag{2.4.4}$$

$$b'(t) = -4b(t)^2 + \frac{1}{4} |\lambda|^2 . \tag{2.4.5}$$

We first solve the separable ODE (2.4.5). Note that the initial condition $m_0 = \delta_0$ suggests the initial condition $b(0) = +\infty$, so we write

$$\int_\infty^{b(t)} \frac{d\beta}{-4\beta^2 + \frac{1}{4} |\lambda|^2} = \int_0^t d\tau$$

$$\frac{1}{|\lambda|} \coth^{-1}\left(\frac{4b(t)}{|\lambda|}\right) = t \quad \text{(see [39, p. 451])}$$

$$b(t) = \frac{1}{4} |\lambda| \coth(t |\lambda|).$$

We substitute into (2.4.4) and separate variables to find

$$A(t) = c(\lambda)e^{-n|\lambda| \int \coth(t|\lambda|)\, dt}$$

$$= c(\lambda) \sinh(t |\lambda|)^{-n}$$

since $\int \coth(t |\lambda|)\, dt = \frac{1}{|\lambda|} \ln \sinh(t |\lambda|)$ (see [39, end pages]). $c(\lambda)$ is some "constant" depending on λ but not on t.

Hence

$$m_{t,\lambda}(x) = c(\lambda) \sinh(t |\lambda|)^{-n} e^{-\frac{1}{4}|\lambda| \coth(t|\lambda|)|x|^2}. \tag{2.4.6}$$

To determine the constant $c(\lambda)$, we assert that since $m_0 = \delta_0$, we should have

$$\lim_{t\to 0} \int_{\mathbb{R}^{2n}} m_{t,\lambda}(x)\, dx = 1.$$

We compute

$$\int_{\mathbb{R}^{2n}} m_{t,\lambda}(x)\, dx = c(\lambda) \sinh(t |\lambda|)^{-n} \int_{\mathbb{R}^{2n}} e^{-\frac{1}{4}|\lambda| \coth(t|\lambda|)|x|^2}\, dx$$

$$= c(\lambda) \left(\frac{4\pi}{|\lambda| \cosh(t |\lambda|)}\right)^n$$

$$\to c(\lambda) \left(\frac{4\pi}{|\lambda|}\right)^n$$

as $t \to 0$. Thus we take $c(\lambda) = \left(\frac{|\lambda|}{4\pi}\right)^n$ to obtain

$$m_{t,\lambda}(x) = \left(\frac{|\lambda|}{4\pi \sinh(t\,|\lambda|)}\right)^n e^{-\frac{1}{4}|\lambda| \coth(t|\lambda|)|x|^2}. \tag{2.4.7}$$

By construction, $m_{t,\lambda}(x)$ as defined by (2.4.7) solves (2.4.1). We remark immediately on the scaling property

$$m_{t,\lambda}(x) = t^{-n} m_{1,t\lambda}(x/\sqrt{t}) \tag{2.4.8}$$

Proposition 2.4.1. *$m(t, x, \lambda) = m_{t,\lambda}(x) \in C^\infty((0, \infty) \times \mathbb{R}^{2n+m})$, and for all multi-indices α, β and all $t > 0$, $m(t, \cdot, \cdot)$ is a Schwartz function on $\mathbb{R}^{2n+m}$.*

Proof. By (2.4.8) it suffices to take $t = 1$.

Set $a(s) = \left(\frac{s}{4\pi \sinh s}\right)^n$, $b(s) = \frac{1}{4} s \coth s$, so that

$$m(1, x, \lambda) = a(|\lambda|)e^{-b(|\lambda|)|x|^2}. \tag{2.4.9}$$

Note that a, b are entire even functions, and thus $a(|\lambda|), b(|\lambda|)$ are entire functions of $|\lambda|^2$, making them smooth functions of λ.

We observe that a is a Schwartz function. Next, noting that $\coth' s = -\operatorname{csch} s$ and $\operatorname{csch}' s = -\operatorname{csch} s \coth s$, we see that the derivatives of $b(s)$ are of the form

$$b^{(k)}(s) = sP_k(\coth s, \operatorname{csch} s) + Q_k(\coth s, \operatorname{csch} s)$$

for polynomials P_k, Q_k. Since $\lim_{s\to\infty} \coth s = 1$ and $\lim_{s\to\infty} \operatorname{csch} s = 0$, it follows that $b^{(k)}(s)$ is of at most linear growth. In particular, if $A(s)$ is a Schwartz function, so are $A(s)b(s)$ and $A(s)b'(s)$. By induction, it follows that for any β, k we have

$$\partial_s^k \partial_\rho^l a(s)e^{-b(s)\rho^2} = \sum_{i=0}^{l} A_i(s)\rho^i e^{-b(s)\rho^2}$$

where the A_i are Schwartz functions. Also, $b(s) \geq \frac{1}{4}$,[1] so $\rho^i e^{-b(s)\rho^2} \leq \rho^i e^{-\frac{1}{4}\rho^2}$, which is also rapidly decaying. Thus $a(s)e^{-b(s)\rho^2}$ is a Schwartz function of (ρ, s). By (2.4.9), the proof is complete. □

We now define the heat kernel to be the inverse Fourier transform of the Mehler kernel.

[1]To see this, set $y(s) = s\cosh s - \sinh s$ and write $b(s) - \frac{1}{4} = \frac{1}{4}\operatorname{csch}(s)y(s)$. We have $y(0) = 0$ and $y'(s) = s\sinh s \geq 0$, so $y(s) \geq 0$.

Definition 2.4.2. The **heat kernel** on an H-type group $G = \mathbb{R}^{2n+m}$ is the function p_t given by

$$\begin{aligned} p_t(x,z) &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{i\langle\lambda,z\rangle} m_{t,\lambda}(x)\,d\lambda \\ &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{i\langle\lambda,z\rangle-\frac{1}{4}|\lambda|\coth(t|\lambda|)|x|^2}\left(\frac{|\lambda|}{4\pi\sinh(t\,|\lambda|)}\right)^n d\lambda. \end{aligned} \tag{2.4.10}$$

The next proposition suggests that p_t deserves to be called the heat kernel, since it is the fundamental solution to the heat equation.

Proposition 2.4.3. *1. $p_t(x,z) \in C^\infty((0,\infty)\times\mathbb{R}^{2n+m})$, and p_t is a Schwartz function on $\mathbb{R}^{2n+m}$ for each $t > 0$.*

2. $\left(L - \frac{\partial}{\partial t}\right)p_t = 0$.

3. $\int_G p_t(x,z)\,dm = 1$ for all $t > 0$.

4. For $\alpha > 0$, $p_t(x,z) = \alpha^{2(n+m)} p_{\alpha^2 t}(\varphi_\alpha(x,z))$. In particular, $\lim_{t\to 0} p_t(x,z) = 0$ for all $(x,z) \neq 0$.

5. p_t is a radial function (i.e. $p_t(x,z)$ depends only on $|x|, |z|$) as is $\frac{\partial}{\partial t}p_t = Lp_t$. In particular, $p_t(g^{-1}) = p_t(g)$ and $Lp_t(g^{-1}) = Lp_t(g)$.

6. p_t vanishes uniformly at infinity. That is, for any $\epsilon > 0$ there exists a compact set $K \subset G$ such that $\sup_{t>0,\,g\in K^C} |p_t(g)| < \epsilon$.

Proof. 1. Clear from Proposition 2.4.1 and properties of the Fourier transform.

2. Clear by properties of the Fourier transform, since $m_{t,\lambda}$ solves (2.4.1).

3. By Fourier inversion, we have

$$\begin{aligned} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^m} p_t(x,z)\,dz\,dx &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i\langle\lambda,z\rangle} m_{t,\lambda}(x)\,d\lambda\,dz\,dx \\ &= \int_{\mathbb{R}^{2n}} m_{t,0}(x)\,dx \\ &= \int_{\mathbb{R}^{2n}}\left(\frac{1}{4\pi t}\right)^n e^{-\frac{1}{4t}|x|^2}.\,dx = 1. \end{aligned}$$

4. To see the scaling $p_t(x,z) = \alpha^{2(n+m)} p_{\alpha^2 t}(\varphi_\alpha(x,z))$, make the change of variables $\lambda = \alpha^2 \lambda'$ in (2.4.10). Taking $\alpha = t^{-1/2}$ we get

$$\lim_{t\to 0} p_t(x,z) = \lim_{\alpha\to\infty} \alpha^{2(n+m)} p_1(\alpha x, \alpha^2 z) = 0$$

by item 1.

5. It is clear from (2.4.10) that p_t is radial in x. To see it is radial in z, suppose $|z'| = |z|$, so that $z' = Tz$ for some orthogonal matrix T. Then making the change of variables $\lambda = T\lambda'$ in (2.4.10) shows that $p_t(x,z) = p_t(x,z')$.

6. Suppose first that $t \le 1$. By item 4 and the fact that p_1 is a Schwartz function, we have

$$\begin{aligned}
|p_t(x,z)| &= t^{-(n+m)} \left|p_1(t^{-1/2}x, t^{-1}z)\right| \\
&\le Ct^{-(n+m)}(\left|t^{-1/2}x\right|^2 + \left|t^{-1}z\right|^2)^{-(n+m)} \\
&= C(|x|^2 + t^{-1}|z|^2)^{-(n+m)} \\
&\le C(|x|^2 + |z|^2)^{-(n+m)}.
\end{aligned}$$

When $t \ge 1$, we have

$$\begin{aligned}
|p_t(x,z)| &= t^{-(n+m)} \left|p_1(t^{-1/2}x, t^{-1}z)\right| \\
&\le C' t^{-(n+m)}(\left|t^{-1/2}x\right|^2 + \left|t^{-1}z\right|^2)^{-(n+m)/2} \\
&= C'(t|x|^2 + |z|^2)^{-(n+m)/2} \\
&\le C'(|x|^2 + |z|^2)^{-(n+m)/2}.
\end{aligned}$$

Thus, taking $K = \{|x|^2 + |z|^2 \le \min\{(C^{-1}\epsilon)^{n+m}, (C'^{-1}\epsilon)^{(n+m)/2}\}\}$ suffices.

□

Definition 2.4.4. The **heat semigroup** $P_t, t \ge 0$ is the one-parameter family of operators given by the convolution (see Definition 2.1.16)

$$\begin{aligned}
P_t f(g) &:= (f * p_t)(g), \quad t > 0 \\
P_0 f &:= f
\end{aligned} \tag{2.4.11}$$

for any distribution $f \in \mathcal{D}(G)$ such that the integral makes sense for all $g \in G$ and all $t > 0$. Note that we can make the change of variables $k = k^{-1}$ in Definition 2.1.16 and use the fact that $p_t(k) = p_t(k^{-1})$ (item 5 of Proposition 2.4.3) to write

$$P_t f(g) = \int_G f(gk) p_t(k)\, dm(k). \tag{2.4.12}$$

The following facts about P_t are almost immediate.

Proposition 2.4.5. *1. If f is a tempered distribution on G, then $P_t f \in C^\infty(G)$ and $P_t f$ satisfies the heat equation $\left(L - \frac{\partial}{\partial t}\right) P_t f = 0$.*

2. If f is a (tempered distribution, $L^p(G)$ function, uniformly continuous function, bounded continuous function), then as $t \to 0$, $P_t f \to f$ (in the sense of distributions, in L^p, uniformly, uniformly on compact subsets of G) respectively.

Proof. 1. That $P_t f \in C^\infty(G)$ follows from item 2 of Proposition 2.1.17. By differentiating under the integral sign with respect to t and using item 1 of Proposition 2.3.7, we have

$$\frac{\partial}{\partial t} P_t f = f * \frac{\partial}{\partial t} p_t = f * L p_t = L P_t f.$$

2. This is a standard "approximate delta function" argument, making use of items 3 and 4 of Proposition 2.4.3.

□

Theorem 2.4.6. *$p_t > 0$ for all $t > 0$.*

Proof. Let $f \in C_c^\infty(G)$ be nonnegative, with compact support K. We first show that $u(g, t) := P_t f(g)$ satisfies the hypotheses of Theorem 2.3.8. Certainly u solves the heat equation, and $u(\cdot, 0) = f \geq 0$. To show u vanishes uniformly at infinity, fix $\epsilon > 0$. Let K be a compact set such that $|p_t(g)| \leq \epsilon / \|f\|_\infty$ for all $t > 0$, $g \in K^C$ whose existence is guaranteed by item 6 of Proposition; note that $0 \in K$. Let $K' = K \star \operatorname{supp} f = \{k \star h : k \in K, h \in \operatorname{supp} f\}$; note that $\operatorname{supp} f \subset K'$. Suppose $g \notin K'$. If $t = 0$ we have $P_t f(g) = f(g) = 0$ since $g \notin \operatorname{supp} f$. If $t > 0$ we have

$$|P_t f(g)| = |p_t * f(g)| \leq \|f\|_\infty \sup_{k \in \operatorname{supp} f} \left| p_t(g \star k^{-1}) \right|.$$

But for $k \in \operatorname{supp} f$, we have $g \star k^{-1} \notin K$. (If $g \star k^{-1} \in K$, then $g \in K \star k \subset K \star \operatorname{supp} f = K'$, contrary to our choice $g \notin K'$.) Therefore by definition of K we have $\left|p_t(g \star k^{-1})\right| \leq \epsilon / \|f\|_\infty$, so that $|P_t f(g)| \leq \epsilon$.

Thus by Theorem 2.3.8 we have $P_t f \geq 0$ for $t \in [0, T]$ for any $T > 0$. We now replace f with a sequence of nonnegative approximate delta functions to see that $p_t \geq 0$ for all $t > 0$. Corollary 2.3.10 does not apply directly to p_t, because p_t is not continuous up to $t = 0$; however, it does apply to p_{t+t_0} for arbitrary fixed $t_0 > 0$. We conclude that $p_{t+t_0} > 0$ for all t, and since t_0 was arbitrary, $p_t > 0$ for all t. □

The reader unfamiliar with the functional analysis machinery in the following theorem may omit it, as it is not essential to the remainder of the dissertation, or refer to Chapter VIII of [38].

Theorem 2.4.7. *P_t is a strongly continuous self-adjoint contraction semigroup on $L^2(G)$, and if L is viewed as a unbounded operator on $L^2(G)$ with domain $\mathcal{D}(L) = C_c^\infty(G)$, then the infinitesimal generator of P_t is $\bar{L}$, the closure of L, which is self-adjoint. That is, $P_t = e^{t\bar{L}}$.*

Proof. It is trivial that $P_0 = I$ is a self-adjoint contraction.

It follows from Theorem 2.4.6 and item 3 of Proposition 2.4.3 that $\int_G |p_t| \, dm = \int_G p_t \, dm = 1$, so by Young's inequality (item 3 of Proposition 2.1.17) P_t is a contraction on $L^2(G)$ for each $t > 0$.

By item 4 of Proposition 2.1.17, we have for any $f_1, f_2 \in L^2(G)$ that

$$(P_t f_1, f_2) = (f_1 * p_t, f_2) = (f_1, f_2 * \tilde{p}_t).$$

But p_t is a real-valued radial function (see item 5 of Proposition 2.4.3), so $\tilde{p}_t = p_t$ and thus $(P_t f_1, f_2) = (f_1, P_t f_2)$. P_t is self-adjoint for each $t > 0$.

To show P_t is a semigroup, it suffices to verify that $p_s * p_t = p_{s+t}$. One approach is to notice that for any $s > 0$, $u(g, t) := p_s * p_t - p_{s+t}$ is a solution of the heat equation which vanishes uniformly at infinity and satisfies $u(\cdot, 0) \equiv 0$. By applying Theorem 2.3.8 to u and $-u$, we must have $u \equiv 0$.

That P_t is strongly continuous in t follows from item 2 of Proposition 2.4.5.

Since P_t is a strongly continuous self-adjoint contraction semigroup, it has a self-adjoint infinitesimal generator A. To show that $A = \bar{L}$, we show first that $L \subset A$, so that

$$\bar{L} \subset A = A^* \subset L^* \tag{2.4.13}$$

(since A is closed). Next, we show that L is essentially self-adjoint, i.e. $\bar{L} = L^*$, so that equality must hold in (2.4.13). (Note that item 2 of Proposition 2.3.7 verifies explicitly that L is symmetric, i.e. $L \subset L^*$, which also follows from $L \subset A$ since A is self-adjoint.)

To see that $L \subset A$, let $f \in C_c^\infty(G)$. First, we observe that for $t > 0$, $\frac{1}{\epsilon}(p_{t+\epsilon} - p_t) \to \frac{\partial}{\partial t} p_t = L p_t$ as $\epsilon \downarrow 0$, not only pointwise but also in $L^1(G)$. To verify the latter, we note by the usual combination of the mean value theorem and dominated convergence that it suffices to show

$$\int_G \sup_{\tau \in [t, t+\epsilon]} \left| \frac{\partial}{\partial \tau} p_\tau(g) \right| dm(g) < \infty.$$

By item 4 of Proposition 2.4.3, we have for $\tau \in [t, t+\epsilon]$:

$$\begin{aligned}
\left| \frac{\partial}{\partial \tau} p_\tau(x,z) \right| &= \left| \frac{\partial}{\partial \tau} \tau^{-(n+m)} p_1(\tau^{-1/2} x, \tau^{-1} z) \right| \\
&\leq \left| -(n+m) \tau^{-(n+m+1)} p_1(\tau^{-1/2} x, \tau^{-1} z) \right| \\
&\quad + \left| \tau^{-(n+m)} \left\langle \nabla_x p_1(\tau^{-1/2} x, \tau^{-1} z), x \right\rangle \left(-\frac{1}{2} \tau^{-3/2} \right) \right| \\
&\quad + \left| \tau^{-(n+m)} \left\langle \nabla_z p_1(\tau^{-1/2} x, \tau^{-1} z), z \right\rangle (-\tau^{-2}) \right| \\
&\leq C \left| -(n+m) \tau^{-(n+m+1)} - \frac{1}{2} \tau^{-(n+m+3/2)} - \tau^{-(n+m+2)} \right| \\
&\quad \times \left(1 + \tau^{-1/2} |x| + \tau^{-1} |z| \right)^{-(2n+m+2)} \\
&\leq C' t^{-(n+m+1)} \left(1 + (t+\epsilon)^{-1/2} |x| + (t+\epsilon)^{-1} |z| \right)^{-(2n+m+2)}
\end{aligned}$$

whose integral over $G = \mathbb{R}^{2n+m}$ is finite, where we used the fact that p_1 is a Schwartz function to bound p_1 and its derivatives in terms of the integrable function

$$\left(1 + \tau^{-1/2} |x| + \tau^{-1} |z| \right)^{-(2n+m+2)}.$$

Therefore, by using Young's inequality and item 1 of Proposition 2.3.7, we have that

$$\frac{1}{\epsilon}(P_{t+\epsilon} - P_t) f \to f * (L p_t) = L P_t f \text{ in } L^2(G). \tag{2.4.14}$$

Next, we note that

$$\begin{aligned} LP_t f(g) = f * (Lp_t) &= \int_G f(g \star k^{-1}) Lp_t(k)\, dm(k) \\ &= \int_G f(g \star k) Lp_t(k)\, dm(k) \end{aligned}$$

(where we have made the change of variables $k \to k^{-1}$ and used item 5 of Proposition 2.4.3 to see $Lp_t(k^{-1}) = Lp_t(k)$)

$$\begin{aligned} &= \left(f \circ L_g, Lp_t\right) && \text{where } L_g \text{ is left translation} \\ &= \left(L(f \circ L_g), p_t\right) && \text{by item 2 of Proposition 2.3.7} \\ &= \left((Lf) \circ L_g, p_t\right) \\ &= \int_G Lf(g \star k) p_t(k)\, dm(k) \\ &= \int_G Lf(g \star k^{-1}) p_t(k)\, dm(k) && \text{as before, since } p_t(k^{-1}) = p_t(k) \\ &= P_t Lf(g). \end{aligned}$$

Thus P_t commutes with L, as one would expect.

Now for any $t > 0$ we have

$$(P_{s+t} - P_s) f = \int_s^t LP_\tau f\, d\tau = \int_s^t P_\tau Lf\, d\tau$$

where the integral is a Riemann integral of an $L^2(G)$-valued continuous function of τ, and we have used a corresponding version of the fundamental theorem of calculus thanks to (2.4.14). The integral is L^2-continuous in s, so letting $s \downarrow 0$ and using the semigroup property and the strong continuity of P_t, we have

$$(P_t - I) f = \int_0^t P_\tau Lf\, d\tau.$$

So

$$\begin{aligned} \left\| \frac{P_t f - f}{t} - Lf \right\| &= \frac{1}{t} \left\| \int_0^t (P_\tau - I) Lf\, d\tau \right\| \\ &\le \sup_{\tau \in [0,t]} \|(P_\tau - I) Lf\| \to 0 \end{aligned}$$

by the strong continuity of P_t. Therefore L agrees with A, the generator of P_t, on $\mathcal{D}(L) = C_c^\infty(G)$, so $L \subset A$.

To conclude, we show that $\bar{L} = L^*$, so that L is essentially self-adjoint. I learned the following standard argument from L. Gross. Consider the vector fields X_i as acting on $C^\infty(G)$, and likewise $L_0 := \sum X_i^2$, which is an extension of L. Suppose first that $f \in \mathcal{D}(L^*) \cap C^\infty(G)$. Then if $h \in C_c^\infty(G)$, we have

$$(L^* f, h) = (f, Lh) = (L_0 f, h)$$

by item 2 of Proposition 2.3.7.

Next, we show that $|\nabla f| \in L^2(G)$. Use the Urysohn lemma to choose $\psi \in C_c^\infty(G)$ such that $\psi \equiv 1$ on some neighborhood of 0, and let $\psi_n = \psi \circ \varphi_{1/n}$. We then have $\psi_n \to 1$ boundedly, and by (2.3.3) we have $\nabla \psi_n = \frac{1}{n}(\nabla \psi) \circ \varphi_{1/n} \to 0$ uniformly and $L\psi_n = \frac{1}{n^2}(L\psi) \circ \varphi_{1/n} \to 0$ uniformly. Then, integrating by parts as in item 2 of Proposition 2.3.7, we have

$$\begin{aligned}
\int_G \psi_n \, |\nabla f|^2 \, dm &= \sum_i \int_G \psi_n (X_i f)^2 \, dm \\
&= -\sum_i \int_G f X_i(\psi_n X_i f) \, dm \\
&= -\sum_i \int_G f \psi_n X_i^2 f \, dm - f X_i \psi_n X_i f \, dm \\
&= -\int_G \psi_n f L_0 f + \sum_i X_i \psi_n X_i (f^2) \, dm \\
&= -\int_G \psi_n f L^* f + f^2 L \psi_n \, dm.
\end{aligned}$$

Letting $n \to \infty$, so that $\psi_n \to 1$ and $L\psi_n \to 0$ boundedly, the dominated convergence theorem gives

$$\|\nabla f\| = -(f, L^* f) < \infty.$$

Now we have $\psi_n f \in C_c^\infty$ and $\psi_n f \to f$ by dominated convergence. We also have

$$L^*(\psi_n f) = L_0(\psi_n f) = (L_0 \psi_n) f + \psi_n L_0 f + 2 \langle \nabla \psi_n, \nabla f \rangle .$$

As $n \to \infty$, we find $L(\psi_n f) = L^*(\psi_n f) \to L_0 f = L^* f$. Thus, $\mathcal{D}(L^*) \cap C^\infty \subset \mathcal{D}(\bar{L})$.

Finally, suppose $f \in \mathcal{D}(L^*)$, and let $\phi_n \in C_c^\infty(G)$ be a sequence of "approximate delta functions," so that $\phi_n * f \to f$ and $\phi_n * L^* f \to L^* f$. For any $h \in C_c^\infty(G)$ we then have

$$\begin{aligned}
(\phi_n * L^* f, h) &= \left(L^* f, \tilde{\phi}_n * h\right) && \text{(item 4 of Proposition 2.1.17)} \\
&= \left(f, L(\tilde{\phi}_n * h)\right) && \text{as } \tilde{\phi}_n, h, \tilde{\phi}_n * h \in C_c^\infty \\
&= \left(f, \tilde{\phi}_n * Lh\right) \\
&= (\phi_n * f, Lh)
\end{aligned}$$

so that $\phi_n * f \in \mathcal{D}(L^*) \cap C^\infty(G)$ and $\bar{L}(\phi_n * f) = L^*(\phi_n * f) = \phi_n * L^* f \to L^* f$. Thus $\mathcal{D}(L^*) \subset \mathcal{D}(\bar{L})$, which completes the proof.

□

Chapter 2, in part, is adapted from material awaiting publication as Eldredge, Nathaniel, "Precise Estimates for the Subelliptic Heat Kernel on H-type Groups," to appear, *Journal de Mathématiques Pures et Appliquées*, 2009and Eldredge, Nathaniel, "Gradient Estimates for the Subelliptic Heat Kernel on H-type Groups," submitted, *Journal of Functional Analysis*, 2009. The dissertation author was the sole author of these papers.

Chapter 3

Subriemannian Geometry

H-type groups lend themselves naturally to the structure of a subriemannian manifold. The geometry that arises in this sense will be crucial in the sequel, particularly its geodesics and the corresponding Carnot-Carathéodory distance function. The goal of this chapter will be to describe the necessary theory and obtain explicit formulas for the geodesics and the distance function. The computation is a straightforward application of Hamiltonian mechanics, but we have not previously seen it appear in the literature in the case of H-type groups. The corresponding computation for the Heisenberg groups (where the center has dimension $m = 1$) appeared in [7] as well as [9]; a computation for $m \leq 7$, which could be extended without great difficulty, can be found in the preprint [10].

3.1 Subriemannian manifolds, geodesics and Hamiltonian mechanics

Definition 3.1.1. A **subriemmanian manifold** is a smooth manifold Q together with a subbundle $\mathcal{H}$ of TQ (the **horizontal bundle** or **horizontal distribution**, whose elements are **horizontal vectors**) and a metric $\langle \cdot, \cdot \rangle_q$ on each fiber $\mathcal{H}_q$, depending smoothly on $q \in Q$. $\mathcal{H}$ is **bracket-generating** at q if there is a local frame $\{X_i\}$ for $\mathcal{H}$ near q such that

$$\operatorname{span}\{X_i(q), [X_i, X_j](q), [X_i, [X_j, X_k]](q), \dots\} = T_q Q.$$

An H-type group G can naturally be equipped as a subriemannian manifold, by letting $\mathcal{H}_g := \{X(g) : X \in \mathfrak{z}^\perp\}$, and using the inner product on $\mathfrak{g}$ as the metric on $\mathcal{H}$. In other words, $\mathcal{H}_g$ is spanned by $\{X_1(g), \ldots, X_{2n}(g)\}$, which give it an orthonormal basis. The bracket generating condition is obviously satisfied, since $\mathfrak{g} = \mathfrak{z}^\perp \oplus [\mathfrak{z}^\perp, \mathfrak{z}^\perp]$.

Definition 3.1.2. Let $\gamma : [0,1] \to Q$ be an absolutely continuous path. We say γ is **horizontal** if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ for almost every $t \in [0,1]$. In such a case we define the *length* of γ as $\ell(\gamma) := \int_0^1 \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}}\, dt$. The **Carnot-Carathéodory distance** $d : Q \times Q \to [0, \infty]$ is defined by

$$d(q_1, q_2) = \inf\{\ell(\gamma) : \gamma(0) = q_1, \gamma(1) = q_2, \gamma \text{ horizontal}\}. \tag{3.1.1}$$

Under the bracket generating condition, the Carnot-Carathéodory distance is well behaved. We refer the reader to Chapter 2 and Appendix D of [35] for proofs of the following two theorems, the first of which is largely a restatement of Chow's theorem (Theorem 1.3.3).

Theorem 3.1.3 (Chow)**.** *If $\mathcal{H}$ is bracket generating and Q is connected, then any two points $q_1, q_2 \in Q$ are joined by a horizontal path whose length is finite. Thus $d(q_1, q_2) < \infty$, and d is easily seen to be a distance function on Q. The topology induced by d is equal to the manifold topology for Q.*

Theorem 3.1.4. *If Q is complete under the Carnot-Carathéodory distance d, then the infimum in the definition of d is achieved; that is, any two points $q_1, q_2 \in Q$ are joined by at least one shortest horizontal path.*

One way to compute the Carnot-Carathéodory distance is to find such a shortest path and compute its length. To find a shortest path, we use Hamiltonian mechanics, following Chapters 1 and 5 of [35]. Roughly speaking, it can be shown that a length minimizing path also minimizes the energy $\frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|\, dt$, and as such should solve Hamilton's equations of motion. The argument uses the method of Lagrange multipliers, and requires that the endpoint map taking horizontal paths to their endpoints has a surjective differential. This always holds in the Riemannian setting, but is not generally true in subriemannian geometry; the Martinet distribution (see Chapter 3 of [35]) is a counterexample in which some shortest paths do not satisfy Hamilton's equations.

Additional assumptions on $\mathcal{H}$ are needed. One which is sufficient (but certainly not necessary) is that the distribution be *fat*:

Definition 3.1.5. Let Θ be the canonical 1-form on the cotangent bundle T^*Q , $\omega = d\Theta$ the canonical symplectic 2-form, and let $\mathcal{H}^0 := \{p_q \in T^*Q : p_q(\mathcal{H}_q) = 0\}$ be the annihilator of $\mathcal{H}$. (Note $\mathcal{H}^0$ is a sub-bundle, and hence also a submanifold, of T^*Q.) We say $\mathcal{H}$ is **fat** if $\mathcal{H}^0$ is symplectic away from the zero section. That is, if $p_q \in \mathcal{H}^0$ is not in the zero section, $v \in T_{p_q}\mathcal{H}^0$, and $\omega(v, w) = 0$ for all other $w \in T_{p_q}\mathcal{H}^0$, then $v = 0$.

Definition 3.1.6. If $(Q, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is a subriemannian manifold, the subriemannian **Hamiltonian** $H : T^*Q \to \mathbb{R}$ is defined by

$$H(p_q) = \sum_i p_q(v_i)^2 \tag{3.1.2}$$

where $\{v_i\}$ is an orthonormal basis for $(\mathcal{H}_q, \langle\cdot,\cdot\rangle_q)$. It is clear that this definition is independent of the chosen basis. Let the **Hamiltonian vector field** X_H on T^*Q be the unique vector field satisfying $dH + \omega(X_H, \cdot) = 0$ (as elements of T^*T^*Q). X_H is well defined because ω is symplectic. **Hamilton's equations of motion** are the ODEs for the integral curves of X_H.

The following theorem summarizes (a special case of) the argument of Chapters 1 and 5 of [35].

Theorem 3.1.7. *If $\mathcal{H}$ is fat, then any length minimizing path $\sigma : [0, 1] \to Q$, when parametrized with constant speed, is also energy minimizing and is the projection onto Q of a path $\gamma : [0, 1] \to T^*Q$ which satisfies Hamilton's equations of motion: $\dot\gamma(t) = X_H(\gamma(t))$.*

3.2 Geodesics for H-type groups

In this section, we will verify that Theorem 3.1.7 applies to H-type groups, and then find a formula for the geodesics (shortest paths) by solving Hamilton's equations. To begin, we adopt a coordinate system for the cotangent bundle T^*G.

Notation 3.2.1. Let $(x, z, \xi, \eta) : T^*G \to \mathbb{R}^{2n} \times \mathbb{R}^m \times \mathbb{R}^{2n} \times \mathbb{R}^m$ be the coordinate system on T^*G such that $x^i(p_g) = x^i(g)$, $z^j(p_g) = z^j(g)$, $\xi_i(p_g) = p(\frac{\partial}{\partial x^i})$, $\eta_j(p_g) = p(\frac{\partial}{\partial z^j})$. That is,

$$p_g = \left(x(g), z(g), \sum_i \xi_i dx^i + \sum_j \eta_j dz^j \right).$$

In these coordinates, the canonical 2-form ω has the expression $\omega = \sum_i d\xi_i \wedge dx^i + \sum_j d\eta_j \wedge dz^j$.

Proposition 3.2.2. *If G is an H-type group with horizontal distribution $\mathcal{H}$ spanned by the vector fields X_i, then $\mathcal{H}$ is fat.*

Proof. For an H-type group G, we have $p_g \in \mathcal{H}^0$ iff $p_g(X_i(g)) = 0$ for all i. We can thus form a basis for $\mathcal{H}_g^0 \subset T_g^*G$ by

$$\begin{aligned} w^j &= dz^j - \sum_i dz^j(X_i(g))dx^i \\ &= dz^j - \frac{1}{2} \sum_i \left(J_{u_j} x(g), e_i \right) dx^i. \end{aligned}$$

Expressing p_g in this basis as $p_g = \sum_j \theta_j w^j$ yields a system of coordinates (x, z, θ) for $\mathcal{H}^0$, where θ can be identified with the element $(\theta^1, \ldots, \theta^m)$ of $\mathbb{R}^m$. In terms of the coordinates (x, z, ξ, η) for T^*G, we have $\eta = \theta$, $\xi = -\frac{1}{2} J_\theta x$.

So let $\gamma : (-\epsilon, \epsilon) \to \mathcal{H}^0$ be a curve in $\mathcal{H}^0$ which avoids the zero section. $\dot{\gamma}(0)$ is thus a generic element of $T\mathcal{H}^0$. We write $\gamma(t)$ in coordinates as $(x(t), z(t), \theta(t))$, where $\theta(t) \neq 0$. In terms of the coordinates (x, z, ξ, η) on T^*G, we have $\eta(t) = \theta(t), \xi(t) = -\frac{1}{2} J_{\theta(t)} x(t)$. Differentiating the latter gives

$$\dot{\xi}(t) = -\frac{1}{2}(J_{\dot{\theta}(t)} x(t) + J_{\theta(t)} \dot{x}(t)).$$

Suppose that for all other such curves γ' which avoid the zero section and satisfy

$\gamma'(0) = \gamma(0)$, we have $\omega(\dot\gamma(0), \dot\gamma'(0)) = 0$. In terms of coordinates,

$$\begin{aligned}
0 = \omega(\dot\gamma(0), \dot\gamma'(0)) &= \sum_i (\dot\xi_i(0)\dot x'^i(0) - \dot\xi_i'(0)\dot x^i(0)) + \sum_j (\dot\eta_j(0)\dot z'^j(0) - \dot\eta_j'(0)\dot z^j(0)) \\
&= \left\langle \dot\xi(0), \dot x'(0) \right\rangle - \left\langle \dot\xi'(0), \dot x(0) \right\rangle + \langle \dot\eta(0), \dot z'(0)\rangle - \langle \dot\eta'(0), \dot z(0)\rangle \\
&= -\frac{1}{2}\left\langle J_{\dot\theta(0)}x(0) + J_{\theta(0)}\dot x(0), \dot x'(0) \right\rangle + \frac{1}{2}\left\langle J_{\dot\theta'(0)}x'(0) + J_{\theta'(0)}\dot x'(0), \dot x(0) \right\rangle \\
&\quad + \left\langle \dot\theta(0), \dot z'(0) \right\rangle - \left\langle \dot\theta'(0), \dot z(0) \right\rangle \\
&= \frac{1}{2}\left\langle x(0), J_{\dot\theta(0)}\dot x'(0) + J_{\dot\theta'(0)}\dot x(0) \right\rangle + \langle J_{\theta(0)}\dot x'(0), \dot x(0)\rangle \\
&\quad + \left\langle \dot\theta(0), \dot z'(0) \right\rangle - \left\langle \dot\theta'(0), \dot z(0) \right\rangle
\end{aligned}$$

For arbitrary $u \in \mathbb{R}^m$, take $\gamma'(t) = (x(0), z(0) + tu, \theta(0))$; then $0 = \omega(\dot\gamma(0), \dot\gamma'(0)) = \left\langle \dot\theta(0), u \right\rangle$, so we must have $\dot\theta(0) = 0$. Next, for arbitrary $v \in \mathbb{R}^{2n}$, take $\gamma'(t) = (x(0) + tv, z(0), \theta(0))$; then we have $0 = \langle J_{\theta(0)}u, \dot x(0)\rangle$. But $\theta(0) \neq 0$ by assumption, so J_{θ_0} is nonsingular and we must have $\dot x(0) = 0$. Finally, take $\gamma'(t) = (x(0), z(0), \theta(0) + tu)$; then $\langle u, \dot z(0)\rangle = 0$, so $\dot z(0) = 0$. Thus we have shown that if $\omega(\dot\gamma(0), \dot\gamma'(0)) = 0$ for all γ', we must have $\dot\gamma(0) = 0$, which completes the proof. □

We now proceed to compute and solve Hamilton's equations of motion for an H-type group G.

The subriemannian Hamiltonian on T^*G is defined by (cf. (3.1.2))

$$H(p_g) := \frac{1}{2}\sum_{i=1}^{2n} p_g(X_i(g))^2, \quad p_g \in T_g^*G. \tag{3.2.1}$$

In terms of the above coordinates, we may compute

$$p_g(X_i(g)) = p_g\left(\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_j \left\langle J_{u_j}x, e_i \right\rangle \frac{\partial}{\partial z^j}\right) = \xi_i(p_g) + \frac{1}{2}\left\langle J_{\eta(p_g)}x(g), e_i \right\rangle$$

so that

$$H(p_g) = \frac{1}{2}\left|\xi(p_g) + \frac{1}{2}J_{\eta(p_g)}x(g)\right|^2.$$

Recall that a path $\gamma : [0, T] \to T^*Q$ satisfies Hamilton's equations iff $\dot\gamma(t) = X_H(\gamma(t))$, i.e. $dH_{\gamma(t)} + \omega(\dot\gamma(t), \cdot) = 0$.

In an H-type group G, we write γ in coordinates as $\gamma(t) = (x(t), z(t), \xi(t), \eta(t)) : [0, T] \to T^*G$, so that we have

$$\omega(\dot{\gamma}(t), \cdot) = \sum_i (\dot{\xi}_i(t) dx^i - \dot{x}^i(t) d\xi_i) + \sum_j (\dot{\eta}_j(t) dz^j - \dot{z}^j(t) d\eta_j).$$

Thus Hamilton's equations of motion read

$$\dot{x}^i = \frac{\partial H}{\partial \xi^i}, \quad \dot{\xi}_i = -\frac{\partial H}{\partial x^i}, \quad \dot{z}_j = \frac{\partial H}{\partial \eta^j}, \quad \dot{\eta}_j = -\frac{\partial H}{\partial z^j}. \tag{3.2.2}$$

To compute the derivatives, we note that $\frac{1}{2}\nabla_x |Ax + y|^2 = A^*Ax + A^*y$. If we write $B_x \eta = J_\eta x$, then $\langle B_x \eta, y\rangle = \langle \eta, [x, y]\rangle$, so $B_x^* = [x, \cdot]$, and $B_x^* B_x = |x|^2 I$. So for a path $\gamma(t) = (x(t), z(t), \xi(t), \eta(t)) : [0, T] \to T^*G$, Hamilton's equations of motion read

$$\dot{x} = \nabla_\xi H = \xi + \frac{1}{2} J_\eta x \tag{3.2.3}$$

$$\dot{z} = \nabla_\eta H = \frac{1}{2} \nabla_\eta \left| \xi + \frac{1}{2} B_x \eta \right|^2 = \frac{1}{4} |x|^2 \eta + \frac{1}{2} [x, \xi] \tag{3.2.4}$$

$$\dot{\xi} = -\nabla_x H = -\frac{1}{4} |\eta|^2 x + \frac{1}{2} J_\eta \xi \tag{3.2.5}$$

$$\dot{\eta} = -\nabla_z H = 0. \tag{3.2.6}$$

Notation 3.2.3. Define the function $\nu : \mathbb{R} \to \mathbb{R}$ by

$$\nu(\theta) = \frac{2\theta - \sin 2\theta}{1 - \cos 2\theta} = \frac{\theta}{\sin^2 \theta} - \cot \theta = -\frac{d}{d\theta}[\theta \cot \theta] \tag{3.2.7}$$

where the alternate form comes from the double-angle identities.

Theorem 3.2.4. $(x(t), z(t))$ *is the projection of a solution to Hamilton's equations with* $x(0) = 0$*,* $z(0) = 0$ *and* $x(1)$*,* $z(1)$ *given if and only if:*

1. *If* $z(1) = 0$*, we have*

$$x(t) = t x(1), \quad z(t) = 0. \tag{3.2.8}$$

2. *If* $z(1) \neq 0$*, we have*

$$x(t) = \frac{1}{|\eta_0|^2} J_{\eta_0} (I - e^{t J_{\eta_0}}) \xi_0 \tag{3.2.9}$$

$$z(t) = \frac{|\xi_0|^2}{2 |\eta_0|^3} (|\eta_0| t - \sin(|\eta_0| t)) \eta_0 \tag{3.2.10}$$

where, if $x(1) \neq 0$ *we have*

$$\eta_0 = 2\theta \frac{z(1)}{|z(1)|} \tag{3.2.11}$$

$$\xi_0 = -|\eta_0|^2 \left(J_{\eta_0}(e^{J_{\eta_0}} - I)\right)^{-1} x(1). \tag{3.2.12}$$

where θ *is a solution to*

$$\nu(\theta) = \frac{4\,|z(1)|}{|x(1)|^2}; \tag{3.2.13}$$

with ν *as given by (3.2.7), and if* $x(1) = 0$ *we have*

$$\eta_0 = 2\pi k \frac{z(1)}{|z(1)|}$$

$$|\xi_0| = \sqrt{4k\pi\,|z(1)|}$$

for some integer $k \geq 1$.

Proof. We solve (3.2.3–3.2.6), assuming $x(0) = 0$, $z(0) = 0$. By (3.2.6) we have $\eta(t) \equiv \eta(0) = \eta_0$. If $\eta_0 = 0$, we can see by inspection that the solution is

$$\eta(t) = 0, \quad \xi(t) = \xi_0, \quad x(t) = t\xi_0, \quad z(t) = 0, \tag{3.2.14}$$

namely, a straight line from the origin, whose length is clearly $|x(1)|$. This is (3.2.8), which we shall see is forced when $z(1) = 0$.

Otherwise, assume $\eta_0 \neq 0$. We may solve (3.2.3) for ξ to see that

$$\xi = \dot{x} - \frac{1}{2} J_{\eta_0} x. \tag{3.2.15}$$

Notice that substituting (3.2.15) into (3.2.4) shows that

$$\dot{z} = \frac{1}{2}[x, \dot{x}], \tag{3.2.16}$$

from which an easy computation verifies that $(x(t), z(t))$ is indeed a horizontal path. (This is analogous to the formula (1.2.6) for $\mathbb{H}_1$.)

Substituting (3.2.15) into the right side of (3.2.5) shows that

$$\dot{\xi} = -\frac{1}{4}|\eta_0|^2 x + \frac{1}{2} J_{\eta_0}(\dot{x} - \frac{1}{2} J_{\eta_0} x) = \frac{1}{2} J_{\eta_0} \dot{x}$$

since $J_{\eta_0}^2 x = -|\eta_0|^2 x$. Thus

$$\xi = \frac{1}{2} J_{\eta_0} x + \xi_0 \tag{3.2.17}$$

where $\xi_0 = \xi(0)$. If $\xi_0 = 0$, it is easily seen that we have the trivial solution $x(t) = 0$, $z(t) = 0$, $\xi(t) = 0$, $\eta(t) = \eta_0$, so we assume now that $\xi_0 \neq 0$. (3.2.17) may be substituted back into (3.2.3) to get

$$\dot{x} = J_{\eta_0} x + \xi_0 \tag{3.2.18}$$

so that

$$x = (J_{\eta_0})^{-1}(e^{tJ_{\eta_0}} - I)\xi_0 = -\frac{1}{|\eta_0|^2} J_{\eta_0}(e^{tJ_{\eta_0}} - I)\xi_0. \tag{3.2.19}$$

Differentiation (or substitution) shows

$$\dot{x} = e^{tJ_{\eta_0}} \xi_0. \tag{3.2.20}$$

Note that

$$|x|^2 = \frac{1}{|\eta_0|^2} \left|(e^{tJ_{\eta_0}} - I)\xi_0\right|^2 = \frac{2}{|\eta_0|^2}(1 - \cos(|\eta_0|\, t))\, |\xi_0|^2\,. \tag{3.2.21}$$

It is easy to see from (3.2.19) that $x(t)$ lies in the plane spanned by ξ_0 and $J_{\eta_0}\xi_0$, and $x(t)$ sweeps out a circle centered at $\frac{1}{|\eta_0|^2} J_{\eta_0}\xi_0$ and passing through the origin. In particular, the radius of the circle is $|\xi_0|/|\eta_0|$.

Now substituting (3.2.19) and (3.2.20) into (3.2.16), we have

$$\begin{aligned}
\dot{z} &= -\frac{1}{2\,|\eta_0|^2}\left([J_{\eta_0} e^{tJ_{\eta_0}}\xi_0, e^{tJ_{\eta_0}}\xi_0] - [J_{\eta_0}\xi_0, e^{tJ_{\eta_0}}\xi_0]\right) \\
&= \frac{1}{2\,|\eta_0|^2}\left(\left|e^{tJ_{\eta_0}}\xi_0\right|^2 \eta_0 + [J_{\eta_0}\xi_0, e^{tJ_{\eta_0}}\xi_0]\right) \\
&= \frac{|\xi_0|^2}{2\,|\eta_0|^2}\,(1 - \cos(|\eta_0|\, t))\, \eta_0
\end{aligned}$$

By integration,

$$z = \frac{|\xi_0|^2}{2\,|\eta_0|^3}(|\eta_0|\, t - \sin(|\eta_0|\, t))\eta_0. \tag{3.2.22}$$

In particular,

$$|z| = \frac{|\xi_0|^2}{2\,|\eta_0|^2}(|\eta_0|\, t - \sin(|\eta_0|\, t)). \tag{3.2.23}$$

We note that inspection of (3.2.23) shows that $z(t) \neq 0$ for $t > 0$. Thus the only solution with $z(1) = 0$ is that of (3.2.8).

To make more sense of this, let $r = |\xi_0| / |\eta_0|$ be the radius of the arc swept out by $x(t)$, and $\phi = |\eta_0| t$ be the angle subtended by the arc. Then

$$|z| = \frac{1}{2} r^2 \phi - \frac{1}{2} r^2 \sin\phi$$

which is the area of the region between an arc of radius r subtending an angle ϕ and the chord which spans it.

We must determine ξ_0, η_0 in terms of $x(1), z(1)$. We have already ruled out the case $z(1) = 0$. If $x(1) = 0$, then (3.2.21) shows we must have $|\eta_0| = 2k\pi$ for some integer $k \geq 1$. (3.2.22, 3.2.23) then shows $\eta_0 = 2k\pi z(1)/ |z(1)|$, and $|\xi_0| = \sqrt{4k\pi |z(1)|}$, as desired. In this case the direction of ξ_0 is not determined and ξ_0 may be any vector with the given length.

On the other hand, if $x(1) \neq 0$, then $|\eta_0|$ is not an integer multiple of 2π, so we may divide (3.2.23) by (3.2.21) to obtain

$$\frac{|z(1)|}{|x(1)|^2} = \frac{|\eta_0| - \sin|\eta_0|}{4(1 - \cos|\eta_0|)} = \frac{1}{4}\nu(\theta) \tag{3.2.24}$$

taking $\theta = \frac{1}{2}|\eta_0|$, where ν is as in (3.2.7). Then by (3.2.21) we have

$$|\xi_0|^2 = \frac{1}{2} |x(1)|^2 \frac{|\eta_0|^2}{1 - \cos(|\eta_0|)} = |x(1)|^2 \frac{\theta^2}{\sin^2\theta}. \tag{3.2.25}$$

Note that once the magnitudes of η_0, ξ_0 are known, their directions are determined: $\eta_0 = z(1)\,|\eta_0| / |z(1)|$ by (3.2.22), while ξ_0 can be recovered from (3.2.19):

$$\xi_0 = -\eta_0^2 (J_{\eta_0}(e^{J_{\eta_0}} - I))^{-1} x(1).$$

So η_0, ξ_0 and hence $x(t), z(t)$ are all determined by a choice of $|\eta_0|$ satisfying (3.2.24). Writing $\theta = |\eta_0|$ gives (3.2.9–3.2.10).

The "if" direction of the theorem requires verifying that the given formulas in fact satisfy Hamilton's equations, which is routine. □

To compute the Carnot-Carathéodory distance function for G, we must decide which of the solutions given in Theorem 3.2.4 is the shortest, and compute its length. We collect, for future reference, some facts about the function ν of (3.2.7).

Lemma 3.2.5. *There is a constant $c > 0$ such that $\nu'(\theta) > c$ for all $\theta \in [0, \pi)$.*

Proof. By direct computation, $\nu'(\theta) = \frac{2(\sin\theta - \theta\cos\theta)}{\sin^3\theta}$. By Taylor expansion of the numerator and denominator we have $\nu'(0) = 2/3 > 0$. For all $\theta \in (0, \pi)$ we have $\sin^3\theta > 0$, so it suffices to consider $y(\theta) := \sin\theta - \theta\cos\theta$. Now $y(0) = 0$ and $y'(\theta) = \theta\sin\theta > 0$ for $\theta \in (0, \pi)$, so $y(\theta) > 0$ for $\theta \in (0, \pi)$. Thus $\nu'(\theta) > 0$ for $\theta \in [0, \pi)$, and continuity and the fact that $\lim_{\theta\uparrow\pi} \nu'(\theta) = +\infty$ establishes the existence of the constant c. □

Corollary 3.2.6. *$\nu(\theta) \geq c\theta$ for all $\theta \in [0, \pi)$, where c is the constant from Lemma 3.2.5.*

Proof. Integrate the inequality in Lemma 3.2.5. Note that $\nu(0) = 0$. □

For an H-type group, we obtain the following explicit formula for the distance. Note that since our notions of horizontal paths, length and distance are all defined in terms of the left-invariant vector fields X_i, these concepts are all left-invariant. That is, if γ is a horizontal path in G, then for any $k \in G$, $L_k\gamma$ is a horizontal path with $\ell(L_k\gamma) = \ell(\gamma)$, and therefore we have $d(g, h) = d(kg, kh)$ for all $g, h, k \in G$. Thus the distance function is completely determined by distance from the identity. We write this as $d_0(g) = d(0, g)$ for short.

Theorem 3.2.7. *In an H-type group, the Carnot-Carathéodory distance from the identity 0 to a point (x, z) is given by*

$$d_0(x,z) = d(0,(x,z)) = \begin{cases} |x| \frac{\theta}{\sin\theta}, & z \neq 0, x \neq 0 \\ |x|, & z = 0 \\ \sqrt{4\pi|z|}, & x = 0 \end{cases} \tag{3.2.26}$$

where θ is the unique solution in $[0, \pi)$ to $\nu(\theta) = \frac{4|z|}{|x|^2}$.

Proof of Theorem 3.2.7. We compute the lengths of the paths given in Lemma 3.2.4. The $z = 0$ case is obvious. Observe that for a horizontal path $\sigma(t) = (x(t), z(t))$, we have $\dot\sigma(t) = \sum_{i=1}^{2n} \dot x^i(t) X_i(\gamma(t))$, so that $\|\dot\sigma(t)\| = |\dot x(t)|$. For paths solving Hamilton's equations, (3.2.20) shows that $|\dot x(t)| = |\xi_0|$, so $\ell(\gamma) = |\xi_0|$. In the case $x = 0$, we have $|\xi_0| = \sqrt{4k\pi|z(1)|}$, where k may be any positive integer; clearly this is minimized by taking $k = 1$.

Now we must handle the case $x \neq 0$, $z \neq 0$. In this case we have $\ell(\gamma) = |\xi_0| = |x| \frac{\theta}{\sin\theta}$, by (3.2.25), where θ solves (3.2.13) (recall $\theta = \frac{1}{2}|\eta_0|$). The function ν has

$\nu(0) = 0$, $\nu(\pi) = +\infty$, and by Lemma 3.2.5 ν is strictly increasing on $[0, \pi)$. Thus among the solutions of (3.2.13) there is exactly one in $[0, \pi)$. We show this is the solution that minimizes $\left(\frac{\theta}{\sin\theta}\right)^2$ and hence also minimizes $\ell(\gamma)$.

For brevity, let $y = \frac{4|z|}{|x|^2}$. If $y \in [0, \pi/2]$ then $y = \nu(\theta)$ for a unique $\theta \in [0, \infty)$. This is because $\nu(\theta) > \nu(\pi/2) = \pi/2$ for $\theta > \pi/2$. Since θ is increasing on $[0, \pi)$ it suffices to show this for $\theta > \pi$. But for such θ we have

$$\nu(\theta) = \frac{\theta - \sin\theta\cos\theta}{\sin^2\theta} \geq \frac{\theta - \frac{1}{2}}{\sin^2\theta} \geq \theta - \frac{1}{2} > \pi - \frac{1}{2} > \frac{\pi}{2}$$

since $\sin\theta\cos\theta \leq \frac{1}{2}$ for all θ.

Otherwise, suppose $y > \pi/2$. Let

$$F(\theta) := \frac{\left(\frac{\theta}{\sin\theta}\right)^2}{\nu(\theta)} = \frac{\theta^2}{\theta - \sin\theta\cos\theta}$$

which is smooth on $(\pi/2, \infty)$ after removing the removable singularities. We will show that if $\pi/2 < \theta_1 < \pi < \theta_2$, then $F(\theta_1) < F(\theta_2)$. Thus if θ_1 is the unique solution to $y = \nu(\theta)$ in $(\pi/2, \pi)$ and $\theta_2 > \pi$ is another solution, we will have

$$\left(\frac{\theta_1}{\sin\theta_1}\right)^2 = \nu(\theta_1)F(\theta_1) = yF(\theta_1) < yF(\theta_2) = \nu(\theta_2)F(\theta_2) = \left(\frac{\theta_2}{\sin\theta_2}\right)^2$$

Toward this end, we compute

$$\begin{aligned} F'(\theta) &= \frac{2\theta(\theta - \sin\theta\cos\theta) - \theta^2(1 - \cos^2\theta + \sin^2\theta)}{(\theta - \sin\theta\cos\theta)^2} \\ &= \frac{2\theta\cos\theta(\theta\cos\theta - \sin\theta)}{(\theta - \sin\theta\cos\theta)^2}. \end{aligned}$$

For $\theta \in (\pi/2, \pi)$ we have $\cos\theta < 0$, $\sin\theta > 0$ and thus $F'(\theta) > 0$. So $F(\theta_1) < F(\pi)$ and it suffices to show $F(\pi) = \pi < F(\theta_2)$. We have $F'(\pi) = 2 > 0$ so this is true for θ_2 near π, and $F(+\infty) = +\infty$ so it is also true for large θ_2. To complete the argument we show that it holds at critical points of F. Suppose $F'(\theta_c) = 0$ where $\theta_c > \pi$; then either $\cos\theta_c = 0$ or $\theta_c\cos\theta_c - \sin\theta_c = 0$. If the former then $F(\theta_c) = \theta_c > \pi$. If the latter, then $\theta_c = \tan\theta_c$, so

$$F(\theta_c) = \frac{\theta_c^2}{\theta_c - \sin\theta_c\cos\theta_c} = \frac{\theta_c^2}{\theta_c - \tan\theta_c\cos^2\theta_c} = \frac{\theta_c^2}{\theta_c(1 - \cos^2\theta_c)} \geq \theta_c > \pi$$

which completes the proof. □

We note that it is apparent from (3.2.26) that we have the scaling property

$$d_0(\varphi_\alpha(x,z)) = \alpha d_0(x,z) \tag{3.2.27}$$

with φ as in Definition 2.1.18.

Notation 3.2.8. If $f, h : G \to \mathbb{R}$, we write $f(g) \asymp h(g)$ to mean there exist finite positive constants C_1, C_2 such that $C_1 h(g) \le f(g) \le C_2 h(g)$ for all $g \in G$, or some specified subset thereof.

Corollary 3.2.9. $d_0(x,z) \asymp |x| + |z|^{1/2}$. *Equivalently,* $d_0(x,z)^2 \asymp |x|^2 + |z|$.

Proof. (Based on [8, Proposition 5.4].) We know $d_0(x,z)$ is a continuous function (with respect to the manifold topology on G) which is positive except at $(0,0)$. $d'(x,z) := |x|+|z|^{1/2}$ is another such function, so the conclusion obviously holds on the unit sphere of d'. Now $d'(\varphi_\alpha(x,z)) = \alpha d'(x,z)$, and inspection of (3.2.26) shows that the same holds for d, so for general (x,z) it suffices to apply the previous statement with $\alpha = d'(x,z)^{-1}$. □

This can also be verified by direct computation. By continuity we can assume $x \neq 0$, $z \neq 0$. If θ is the unique solution in $[0,\pi)$ to $\nu(\theta) = \frac{4|z|}{|x|^2}$, we have $d_0(x,z)^2 = |x|^2 \left(\frac{\theta}{\sin\theta}\right)^2$, so if we let

$$F(\theta) := \frac{\left(\frac{\theta}{\sin\theta}\right)^2}{1+\nu(\theta)} = \frac{d_0(x,z)^2}{|x|^2 + 4\,|z|} \tag{3.2.28}$$

it will be enough to show there exist D_1, D_2 with $0 < D_1 \le F(\theta) \le D_2$ for all $\theta \in [0,\pi)$. F is obviously continuous and positive on $(0,\pi)$. We can simplify F as

$$F(\theta) = \frac{\theta^2}{\sin^2\theta + \theta - \sin\theta\cos\theta}$$

from which it is obvious that $\lim_{\theta\uparrow\pi} F(\theta) = \pi > 0$, and easy to compute that $\lim_{\theta\downarrow 0} F(\theta) = 1 > 0$, which is sufficient to establish the corollary.

Results of this form apply to general stratified Lie groups. A standard argument, paraphrased from [8], where many more details can be found, is as follows. Once it is known that d generates the Euclidean topology on G, then $d_0(x,z)$ is a continuous function which is positive except at $(0,0)$. $d'(x,z) := |x| + |z|^{1/2}$ is another such function, so the conclusion obviously holds on the unit sphere of d'. Now $d'(\varphi_\alpha(x,z)) = \alpha d'(x,z)$, and inspection of (3.2.26) shows that the same holds for d, so for general (x, z) it suffices to apply the previous statement with $\alpha = d'(x, z)^{-1}$.

Chapter 3, in part, is adapted from material awaiting publication as Eldredge, Nathaniel, “Precise Estimates for the Subelliptic Heat Kernel on H-type Groups,” to appear, *Journal de Mathématiques Pures et Appliquées*, 2009. The dissertation author was the sole author of this paper.

Chapter 4

Heat Kernel Estimates

4.1 Statement of results

The goal of this section is to establish pointwise upper and lower estimates on the heat kernel p_t on an H-type group G, as well as its gradient ∇p_t. See Corollary 4.1.2 and Theorems 4.1.3 and 4.1.4 below.

Theorem 4.1.1. *There exists* $D_0 > 0$ *such that*

$$p_1(x,z) \asymp \frac{d_0(x,z)^{2n-m-1}}{1 + (|x|\, d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4} d_0(x,z)^2}. \tag{4.1.1}$$

for $d_0(x,z) \geq D_0$.

Corollary 4.1.2.

$$p_t(x,z) \asymp t^{-m-n} \frac{1 + (t^{1/2} d_0(x,z))^{2n-m-1}}{1 + (t\,|x|\, d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4t} d_0(x,z)^2} \tag{4.1.2}$$

for $(x,z) \in G$*,* $t > 0$*, with the implicit constants independent of t as well as* (x,z)*.*

Proof. Theorem 4.1.1 establishes (4.1.2) for $t = 1$ and $d_0(x,z) \geq D_0$. For $d_0(x,z) \leq D_0$ the estimate follows from continuity and the fact that $p_t(x,z) > 0$ (Theorem 2.4.6).

Once (4.1.2) holds for all (x,z) and $t = 1$, item 4 of Proposition 2.4.3 and (3.2.27) show that it holds for all t, with the same constants. □

We also obtain precise upper and lower estimates on the gradient of the heat kernel. Again we work only on $d_0(x,z) \geq D_0$, and since ∇p_t vanishes for $x = 0$, it is not as clear how to extend to all of G. However, the upper bound is sufficient to establish (4.1.4), which is of interest itself.

Theorem 4.1.3. *There exists $D_0 > 0$ such that*

$$|\nabla p_1(x,z)| \asymp |x| \frac{d_0(x,z)^{2n-m+1}}{1 + (|x|\, d_0(x,z))^{n+\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.1.3}$$

for $d_0(x,z) \geq D_0$. In particular, we can combine this with the lower bound of Theorem 4.1.1 to see that there exists $C > 0$ such that

$$|\nabla p_1(x,z)| \leq C(1 + d_0(x,z))p_1(x,z). \tag{4.1.4}$$

By (2.3.8) and differentiation under the integral sign, we have

$$\nabla p_1(x,z) = -\frac{1}{2}(2\pi)^{-m}(4\pi)^{-n}\, |x|\, (q_1(x,z)\hat{x} + q_2(x,z)J_{\hat{z}}\hat{x}) \tag{4.1.5}$$

where

$$q_1(x,z) = -\frac{2}{|x|}\frac{\partial p_1(x,z)}{\partial\, |x|} = \int_{\mathbb{R}^m} e^{i\langle\lambda,z\rangle - \frac{1}{4}|\lambda| \coth|\lambda| |x|^2} \left(\frac{|\lambda|}{\sinh(|\lambda|)}\right)^{n+1} \cosh(|\lambda|)\, d\lambda \tag{4.1.6}$$

$$q_2(x,z) = \frac{\partial p_1(x,z)}{\partial\, |z|} = \int_{\mathbb{R}^m} e^{i\langle\lambda,z\rangle - \frac{1}{4}|\lambda| \coth|\lambda| |x|^2} \left(\frac{|\lambda|}{\sinh(|\lambda|)}\right)^{n} (-i)\, \langle\lambda, \hat{z}\rangle\, d\lambda \tag{4.1.7}$$

As before, (4.1.6) and (4.1.7) do not really depend on $\hat{z}$ but only on $|x|\,, |z|$.

The function q_2 is of interest in its own right, because it gives the norm of the "vertical gradient" of p_1: $|q_2| = |\nabla_z p_1|$. The proof of Theorem 4.1.3 includes estimates on q_2; we record here the upper bound.

Theorem 4.1.4. *There exists $D_0 \geq 0$ and a constant $C > 0$ such that*

$$|\nabla_z p_1(x,z)| = |q_2(x,z)| \leq C \frac{d_0(x,z)^{2n-m-1}}{1 + (|x|\, d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2}. \tag{4.1.8}$$

whenever $d_0(x,z) \geq D_0$. In particular, for all $(x,z) \in G$ we have

$$|\nabla_z p_1(x,z)| \leq C p(x,z). \tag{4.1.9}$$

Remark. Since our proof is based on analysis of the formula (2.4.10), we will henceforth treat (2.4.10) as the definition of a function p_1 on $\mathbb{R}^{2n+m}$. In particular, it makes sense for all n, m, whether or not an H-type group of the corresponding dimension actually exists (which can be ascertained via Theorem 2.2.6). The proofs of Theorems 4.1.1 and 4.1.3 do not depend on the values of n and m, so they likewise remain valid for all n, m. The estimates given are in terms of the distance function d, which likewise should be taken as a function defined by the formula (3.2.26). Indeed, the only place where we need p_1 to be a heat kernel is in the proof of Corollary 4.1.2, where we use the positivity of p_1 which follows from the general theory (Theorem 2.4.6).

In particular, in Section 4.5 we shall make use of estimates on p_1 for values of n, m not necessarily corresponding to H-type groups.

The proofs of these two theorems are broken into two cases, depending on the relative sizes of $|x|$ and $|z|$. Section 4.3 deals with the case when $|z| \lesssim |x|^2$; here we apply a steepest descent type argument to approximate the desired function by a Gaussian. Section 4.4 handles the case $|z| \gg |x|^2$ by a transformation to polar coordinates and a residue computation which only works for odd m. The result for m even can be deduced from that for m odd by a Hadamard descent approach, which is contained in Section 4.5.

4.2 Previous work

Estimates of the form (4.1.1) for the classical Heisenberg group first appeared in [28], in the context of a gradient estimate for the heat semigroup, as did an estimate equivalent to (4.1.4). A proof for Heisenberg groups in all dimensions followed in [29]. Our proof is similar in spirit to the latter, in that it relies on the analysis of an explicit formula for p_t using steepest descent methods and elementary complex analysis.

Less precise versions of the inequalities (4.1.1) are known to hold in more general settings. Using Harnack inequalities one can show that for general nilpotent Lie groups,

$$C_1 R_1(t) e^{-\frac{d^2}{ct}} \leq p_t \leq C_2(\epsilon) R_2(t) e^{-\frac{d^2}{(4+\epsilon)t}} \tag{4.2.1}$$

for some constants c, C_1, C_2 and functions R_1, R_2, where C_2 depends on $\epsilon > 0$; see chapter IV of [46]. [12], among others, improves the upper bound to

$$p_t(g) \le CR_3(g,t)e^{-\frac{d_0(g)^2}{4t}}, \tag{4.2.2}$$

with R a polynomial correction, using logarithmic Sobolev inequalities, whereas [45] improves the lower bound to

$$p_t \ge C(\epsilon)R_4(t)e^{-\frac{d^2}{(4-\epsilon)t}}. \tag{4.2.3}$$

Similar but slightly weaker estimates were shown for more general sum-of-squares operators satisfying Hörmander's condition in [26] by means of Malliavin calculus, and in [23] by more elementary methods involving homogeneity and the regular dependence of p_t on t.

In the specific case of the classical Heisenberg group, asymptotic results similar to (4.1.1) had been previously obtained in [17] and [19], but without the necessary uniformity to translate them into pointwise estimates. A precise upper bound equivalent to that of (4.1.1) was given in [7] for Heisenberg groups of all dimensions. All three of these works, like [29] and the present proof, were based on an explicit formula for p_t and involved steepest descent type methods. In [16], similar techniques were used to obtain a Li-Yau-Harnack inequality for the heat equation on Heisenberg groups.

4.3 Steepest descent

We first handle the region where $|z| \le B_1 |x|^2$ for some constant B_1. If $\theta = \theta(x,z)$ is as in Theorem 3.2.7, this implies $\nu(\theta) \le 4B_1$; since ν increases on $[0,\pi)$ we have $0 \le \theta \le \theta_0$ in this region. Note also that by Corollary 3.2.9 we have $d_0(x,z)^2 \le D_2(1+B_1)|x|^2$, as well as $d_0(x,z)^2 \ge |x|^2$ which is clear from (3.2.26). Thus for this region the bounds of Theorems 4.1.1, 4.1.3 and 4.1.4 are implied by the following:

Theorem 4.3.1. *For each constant $B_1 > 0$ there exists $D_0 > 0$ such that*

$$p_1(x,z) \asymp \frac{1}{|x|^m} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.3.1}$$

$$|q_i(x,z)| \le \frac{C_2}{|x|^m} e^{-\frac{1}{4}d_0(x,z)^2}, \quad i = 1,2 \tag{4.3.2}$$

$$\frac{C_1}{|x|^m} e^{-\frac{1}{4}d_0(x,z)^2} \le \max\{|q_1(x,z)|, |q_2(x,z)|\} \tag{4.3.3}$$

for all x, z with $d_0(x,z) \geq D_0$ and $|z| \leq B_1 |x|^2$.

Our approach here will be a steepest descent argument. Very informally, the motivation is as follows: given a function $F(x) = \int_{\mathbb{R}} e^{-x^2 f(\lambda)} a(\lambda)\, d\lambda$, move the contour of integration to a new contour Γ which passes through a critical point λ_c of f, so that $f(\lambda) \approx f(\lambda_c) + \frac{1}{2} f''(\lambda_c)(\lambda - \lambda_c)^2$. Then we have

$$F(x) \approx e^{-x^2 f(\lambda_c)} \int_\Gamma e^{-x^2 f''(\lambda_c)(\lambda-\lambda_c)^2/2} a(\lambda)\, d\lambda.$$

For large x the integrand looks like a Gaussian concentrated near λ_c, so $F(x) \asymp e^{-x^2 f(\lambda_c)} \frac{a(\lambda_c)}{x\sqrt{f''(\lambda_c)}}$. Our proof essentially follows this line, in $\mathbb{R}^m$ instead of $\mathbb{R}$, but more care is required to establish the desired uniformity.

Our first task is to extend the integrand to a meromorphic function on $\mathbb{C}^m$, so that we may justify moving the contour of integration.

Let $\cdot$ denote the bilinear (not sesquilinear) dot product on $\mathbb{C}^m$, and for $\lambda \in \mathbb{C}^m$ write $\lambda^2 := \lambda \cdot \lambda$; this defines an analytic function from $\mathbb{C}^m$ to $\mathbb{C}$, and $\lambda^2 = |\lambda|^2$ iff $\lambda \in \mathbb{R}^m$. For $w \in \mathbb{C}$, let $\sqrt{w}$ denote the branch of the square root function satisfying $\operatorname{Im} \sqrt{w} \geq 0$ and $\sqrt{w} > 0$ for $w > 0$ (so the branch cut is the positive real axis). Thus if $g : \mathbb{C} \to \mathbb{C}$ is an analytic even function, $\lambda \mapsto g(\sqrt{\lambda^2})$ is analytic as well, and satisfies $g(\sqrt{\lambda^2}) = g(|\lambda|)$ for $\lambda \in \mathbb{R}^m$. This holds in particular for the function $\frac{\sinh w}{w}$, and thus the functions $\frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}}$ and $\sqrt{\lambda^2} \coth \sqrt{\lambda^2}$ are analytic away from points with $\sqrt{\lambda^2} = ik\pi$, $k = 1, 2, \dots$.

Using this notation, we let

$$\begin{aligned}
a_0(\lambda) &:= \left(\frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}} \right)^n \\
a_1(\lambda) &:= \cosh \sqrt{\lambda^2} \left(\frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}} \right)^{n+1} \\
a_2(\lambda) &:= -i \left(\frac{\sqrt{\lambda^2}}{\sinh \sqrt{\lambda^2}} \right)^n \lambda \cdot \hat{z} \in \mathbb{C}^{2n}.
\end{aligned}$$

As mentioned previously, $\hat{z}$ may be any unit vector in $\mathbb{R}^m$ without affecting the computation. Therefore we shall treat it as fixed, while $|z|$ is allowed to vary.

Also, for $\lambda \in \mathbb{C}^m, \theta \in [0, \theta_0], \hat{z} \in S^{m-1} \subset \mathbb{R}^m$, we define

$$f(\lambda, \theta, \hat{z}) := -i\nu(\theta)\lambda \cdot \hat{z} + \sqrt{\lambda^2} \coth \sqrt{\lambda^2} \tag{4.3.4}$$

so that

$$\frac{|x|^2}{4} f(\lambda, \theta(x,z), \frac{z}{|z|}) = -i\lambda \cdot z + \frac{1}{4}\sqrt{\lambda^2}\coth\sqrt{\lambda^2}\,|x|^2\,.$$

We henceforth write θ for $\theta(x,z)$. Thus we now have

$$p_1(x,z) = (4\pi)^{-m-n}\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4}f(\lambda,\theta,\hat{z})} a_0(\lambda)\,d\lambda \tag{4.3.5}$$

$$q_i(x,z) = (4\pi)^{-m-n}\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4}f(\lambda,\theta,\hat{z})} a_i(\lambda)\,d\lambda, \qquad i = 1,2 \tag{4.3.6}$$

Written thus, the integrands have obvious meromorphic extensions to $\lambda \in \mathbb{C}^n$, analytic away from the set $\{\sqrt{\lambda^2} = ik\pi,\ k = 1,2,\ldots\}$.

A simple calculation verifies that $\frac{d}{dw} w\coth w = i\nu(-iw)$, so we can compute the gradient of f with respect to λ as

$$\nabla_\lambda f(\lambda,\theta,\hat{z}) = -i\nu(\theta)\hat{z} + i\nu(-i\sqrt{\lambda^2})\hat{\lambda} \tag{4.3.7}$$

which vanishes when $\lambda = i\theta\hat{z}$. Thus $i\theta\hat{z}$ is the desired critical point. We observe that

$$f(i\theta\hat{z},\theta,\hat{z}) = \theta\nu(\theta) + i\theta\coth(i\theta) = \theta(\nu(\theta) + \cot(\theta)) = \frac{\theta^2}{\sin^2\theta} \tag{4.3.8}$$

so by (3.2.26),

$$|x|^2 f(i\theta\hat{z},\theta,\hat{z}) = d_0(x,z)^2. \tag{4.3.9}$$

Thus we define

$$\psi(\lambda,\theta,\hat{z}) := f(\lambda,\theta,\hat{z}) - f(i\theta\hat{z},\theta,\hat{z}) = -i\nu(\theta)\lambda\cdot\hat{z} + \sqrt{\lambda^2}\coth\sqrt{\lambda^2} - \frac{\theta^2}{\sin^2\theta}. \tag{4.3.10}$$

We then have

$$p_t(x,z) = (4\pi)^{-m-n} e^{-d_0(x,z)^2/4}\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4}\psi(\lambda,\theta,\hat{z})} a_0(\lambda)\,d\lambda \tag{4.3.11}$$

and analogous formulas for q_1, q_2. Thus let

$$h_i(x,z) := \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{4}\psi(\lambda,\theta,\hat{z})} a_i(\lambda)\,d\lambda. \tag{4.3.12}$$

It will now suffice to estimate h_i.

The first step in the steepest descent method is to move the “contour” of integration to pass through $i\theta\hat{z}$. Some preliminary computations are in order.

Lemma 4.3.2. *For $a, b \in \mathbb{R}^m$, we have*

$$|a| - |b| \leq \left|\mathrm{Re}\ \sqrt{(a+bi)^2}\right| \leq |a|\,, \quad 0 \leq \mathrm{Im}\ \sqrt{(a+bi)^2} \leq |b|\,. \tag{4.3.13}$$

Equality holds in the upper bounds if and only if a and b are parallel, i.e. $a = rb$ for some $r \in \mathbb{R}$.

Proof. First note that $(a+bi)^2 = |a|^2 - |b|^2 + 2ia\cdot b$. So by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left|(a+bi)^2\right|^2 &= (|a|^2 - |b|^2)^2 + (2a\cdot b)^2 \\ &\leq (|a|^2 - |b|^2)^2 + 4\,|a|^2\,|b|^2 \\ &= (|a|^2 + |b|^2)^2 \end{aligned} \tag{4.3.14}$$

so that $\left|(a+bi)^2\right| \leq |a|^2 + |b|^2$. Equality holds in the Cauchy-Schwarz inequality iff a and b are parallel. On the other hand,

$$\left|(a+bi)^2\right| \geq \mathrm{Re}(a+bi)^2 = |a|^2 - |b|^2\,. \tag{4.3.15}$$

Now we can write

$$\begin{aligned} \left(\mathrm{Re}\ \sqrt{(a+bi)^2}\right)^2 &= \frac{1}{4}\left(\sqrt{(a+bi)^2} + \overline{\sqrt{(a+bi)^2}}\right)^2 \\ &= \frac{1}{4}\left((a+bi)^2 + \overline{(a+bi)^2} + 2\left|\sqrt{(a+bi)^2}\right|^2\right) \\ &= \frac{1}{2}(|a|^2 - |b|^2 + \left|(a+bi)^2\right|). \end{aligned}$$

The upper bound for $\left|\mathrm{Re}\ \sqrt{(a+bi)^2}\right|$ then follows from (4.3.14). The lower bound is trivial if $|a| \leq |b|$, and otherwise we have by (4.3.15) that

$$\left(\mathrm{Re}\ \sqrt{(a+bi)^2}\right)^2 \geq |a|^2 - |b|^2 \geq (|a| - |b|)^2.$$

The lower bound for $\mathrm{Im}\ \sqrt{(a+bi)^2}$ holds by our definition of $\sqrt{\cdot}$, and the upper bound is similar to the previous one. □

Lemma 4.3.3. *For each $\theta_0 \in [0, \pi)$ there exists $c(\theta_0) > 0$ such that if $a, b \in \mathbb{R}^n$ with $|a| \geq c(\theta_0)$, $|b| \leq 2\pi$, we have*

$$\mathrm{Re}\,\psi(a+ib, \theta, \hat{z}) \geq |a|\,/2 \tag{4.3.16}$$

and

$$|a_i(a+ib)| \le 1 \tag{4.3.17}$$

for all $\theta \in [0,\theta_0]$, $\hat{z}, \hat{x} \in S^{m-1} \subset \mathbb{R}^m$.

Proof. Fix $\theta_0 \in [0,\pi)$. Note first that

$$\operatorname{Re}\psi(a+ib,\theta,\hat{z}) = \nu(\theta) b\cdot\hat{z} - \operatorname{Re} f(i\theta\hat{z},\theta,\hat{z}) + \operatorname{Re}\left[\sqrt{(a+bi)^2}\coth\sqrt{(a+bi)^2}\right]. \tag{4.3.18}$$

By continuity, $\nu(\theta)b\cdot\hat{z} - \operatorname{Re} f(i\theta\hat{z},\theta,\hat{z})$ is bounded below by some constant independent of a for all $\theta \in [0,\theta_0]$, $|b| \le 2\pi$. Thus it suffices to show that for sufficiently large $|a|$,

$$\operatorname{Re}\left[\sqrt{(a+bi)^2}\coth\sqrt{(a+bi)^2}\right] \ge \frac{2}{3}|a|\,. \tag{4.3.19}$$

Now for $\alpha \in \mathbb{R}, \beta \in [-2\pi, 2\pi]$ we have

$$\begin{aligned}
\operatorname{Re}((\alpha+i\beta)\coth(\alpha+i\beta)) &= \frac{\alpha\sinh\alpha\cosh\alpha + \beta\sin\beta\cos\beta}{\cosh^2\alpha - \cos^2\beta} \\
&\ge \alpha\coth\alpha - \frac{\beta}{\cosh^2\alpha} \\
&\ge \alpha\coth\alpha - \frac{2\pi}{\cosh^2\alpha} \\
&\ge \frac{3}{4}|\alpha|
\end{aligned}$$

for sufficiently large $|\alpha|$. (Recall that $\lim_{\alpha\to\pm\infty}\coth\alpha = \pm 1$.) Thus, since

$$\left|\operatorname{Re}\sqrt{(a+bi)^2}\right| \ge |a| - |b| \ge |a| - 2\pi$$

and

$$\left|\operatorname{Im}\sqrt{(a+bi)^2}\right| \le 2\pi,$$

it is clear that (4.3.19) holds for sufficiently large $|a|$.

For the bound on a_i, note that the sinh factor in the denominator of each a_i can be estimated by

$$|\sinh(\alpha+i\beta)| = \left|\frac{e^{\alpha+i\beta} - e^{-\alpha+i\beta}}{2}\right| \ge \left|\frac{\left|e^{\alpha+i\beta}\right| - \left|e^{-\alpha+i\beta}\right|}{2}\right| = |\sinh\alpha|$$

so that $\left|\sinh\sqrt{(a+bi)^2}\right| \ge \left|\sinh\operatorname{Re}\sqrt{(a+bi)^2}\right| \ge |\sinh(|a|-2\pi)|$ for $|a| \ge 2\pi$. This grows exponentially with $|a|$, so it certainly dominates the polynomial growth of the numerator, and we have $|a_i(a+ib)| \le 1$ for large enough $|a|$. □

Lemma 4.3.4. *Let* $F(\lambda) := e^{-\frac{|x|^2}{4}\psi(\lambda,\theta,\hat{z})}a_i(\lambda)$ *be the integrand in (4.3.12), where* x, z *are fixed. If* $\tau \in \mathbb{R}^m$ *with* $|\tau| < \pi$*, then*

$$h_i(x,z) = \int_{\mathbb{R}^m} F(\lambda)\, d\lambda = \int_{\mathbb{R}^m} F(\lambda + i\tau)\, d\lambda. \tag{4.3.20}$$

Proof. Note first that F is analytic at $\lambda + ib$ when $|b| < \pi$, by the second inequality in Lemma 4.3.2. Also, by Lemma 4.3.3, we have

$$|F(\lambda + ib)| \le e^{-|x|^2|\lambda|/8} \tag{4.3.21}$$

as soon as $|\lambda| > c(\theta)$.

We view $\int_{\mathbb{R}^m} F(\lambda)\, d\lambda$ as m iterated integrals and handle them one at a time. For $1 \le k \le m$, suppose we have shown that

$$\int_{\mathbb{R}^m} F(\lambda)\, d\lambda = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \dots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \dots, \lambda_m)\, d\lambda_1 \dots d\lambda_m. \tag{4.3.22}$$

Continuity of F and (4.3.21) show that F is integrable, so we may apply Fubini's theorem and evaluate the $d\lambda_k$ integral first:

$$\int_{\mathbb{R}^m} F(\lambda)\, d\lambda = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \dots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \dots, \lambda_m)\, d\lambda_k d\lambda_1 \dots d\lambda_m.$$

Now

$$\begin{aligned} &\int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \dots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \dots, \lambda_m)\, d\lambda_k \\ &= \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} F(\lambda_1 + i\tau_1, \dots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \dots, \lambda_m)\, d\lambda_k. \end{aligned}$$

Since $\lambda_k \mapsto F(\lambda_1 + i\tau_1, \dots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \dots, \lambda_m)$ is analytic for $|\mathrm{Im}\,\lambda_k| \le \tau_k$ (which holds because $|(\tau_1, \dots, \tau_k)| \le |\tau| < \pi$), we have

$$\int_{-\alpha}^{\alpha} F(\dots, \lambda_k, \dots)\, d\lambda_k = \int_{-\alpha}^{-\alpha+i\tau_k} F + \int_{-\alpha+i\tau_k}^{\alpha+i\tau_k} F + \int_{\alpha+i\tau_k}^{\alpha} F$$

where the contour integrals are taken along straight (horizontal or vertical) lines. But as soon as α exceeds $c(\theta)$ from Lemma 4.3.3, (4.3.21) gives

$$\begin{aligned} &\int_{-\alpha}^{-\alpha+i\tau_k} |F(\lambda_1 + i\tau_1, \dots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \dots, \lambda_m)|\, d\lambda_k \\ &\le \tau_k e^{-|x|^2|(\lambda_1, \dots, \lambda_{k-1}, -\alpha, \lambda_k, \dots, \lambda_m)|/8} \\ &\le \pi e^{-|x|^2|\alpha|/8} \to 0 \text{ as } \alpha \to \infty. \end{aligned}$$

A similar argument shows the same for $\int_{\alpha+i\tau_k}^{\alpha} F$, so we have

$$\begin{aligned}&\int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \ldots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \ldots, \lambda_m)\, d\lambda_k \\ &= \int_{-\infty+i\tau_k}^{\infty+i\tau_k} F(\lambda_1 + i\tau_1, \ldots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k, \ldots, \lambda_m)\, d\lambda_k \\ &= \int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \ldots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k + i\tau_k, \ldots, \lambda_m)\, d\lambda_k.\end{aligned}$$

Thus applying Fubini's theorem again, we have shown

$$\int_{\mathbb{R}^m} F(\lambda)\, d\lambda = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} F(\lambda_1 + i\tau_1, \ldots, \lambda_{k-1} + i\tau_{k-1}, \lambda_k + i\tau_k, \ldots, \lambda_m)\, d\lambda_1 \ldots d\lambda_m. \tag{4.3.23}$$

Applying this argument successively for $k = 1, 2, \ldots, m$ establishes the lemma. □

For the remainder of this section, we assume that $|z| \leq B_1 |x|^2$, so that $\theta \leq \theta_0(B_1)$. We next show that the contribution from λ far from the origin is negligible.

Lemma 4.3.5. *There exist* $r > 0$ *and a constant* $C > 0$ *such that*

$$\left| \int_{B(0,r)^C} e^{-\frac{|x|^2}{4}\psi(\lambda + i\theta\hat{z}, x, z)} a_i(\lambda + i\theta\hat{z})\, d\lambda \right| \leq \frac{C}{|x|^{2m}}. \tag{4.3.24}$$

Proof. From Lemma 4.3.3, if $r \geq c(\theta_0)$ we have

$$\begin{aligned}\int_{B(0,r)^C} \left| e^{-\frac{|x|^2}{4}\psi(\lambda + i\theta\hat{z}, x, z)} a_i(\lambda + i\theta\hat{z}) \right| d\lambda &\leq \int_{B(0,r)^C} e^{-\frac{|x|^2}{8}|\lambda|}\, d\lambda \\ &= \omega_{m-1} \int_r^\infty e^{-|x|^2\rho/8} \rho^{m-1}\, d\rho \\ &\leq \omega_{m-1} \int_0^\infty e^{-|x|^2\rho/8} \rho^{m-1}\, d\rho \\ &= \omega_{m-1} (b\,|x|^2)^{-m} \int_0^\infty e^{-\rho} \rho^{m-1}\, d\rho \\ &= \frac{C}{|x|^{2m}}\end{aligned}$$

where ω_{m-1} is the hypersurface measure of S^{m-1}. □

We can now apply a steepest descent argument. As a similar argument will be used later in this paper (see Proposition 4.4.7), we encapsulate it in the following lemma.

Lemma 4.3.6. *Let* $\Sigma \subset \mathbb{R}^k$ *for some* k*,* $r > 0$*,* $B(0,r)$ *the ball of radius* r *in* $\mathbb{R}^m$*, and* $g : B(0,r) \times \Sigma \to \mathbb{R}$*,* $k : \mathbb{R}^{2n} \times [-r,r] \times \Sigma \to \mathbb{C}$ *be measurable. Define* $F : \mathbb{R}^{2n} \times \Sigma \to \mathbb{C}$ *by*

$$F(x,\sigma) := \int_{B(0,r)} e^{-|x|^2 g(\lambda,\sigma)} k(x,\lambda,\sigma)\, d\lambda. \tag{4.3.25}$$

Suppose:

1. *There exists a positive constant* b_1 *such that* $g(\lambda,\sigma) \geq b_1 |\lambda|^2$ *for all* $\lambda \in B(0,r), \sigma \in \Sigma$*;*
2. k *is bounded, i.e.* $k_2 := \sup_{x\in\mathbb{R}^{2n},\lambda\in B(0,r),\sigma\in\Sigma} |k(x,\lambda,\sigma)| < \infty$*.*

Then there exists a positive constant C_2' *such that*

$$|F(x,\sigma)| \leq \frac{C_2'}{|x|^m} \tag{4.3.26}$$

for all $x > 0$*,* $\sigma \in \Sigma$*.*

If additionally we have:

3. *There exists a positive constant* b_2 *such that* $g(\lambda,\sigma) \leq b_2 |\lambda|^2$ *for all* $\lambda \in B(0,r), \sigma \in \Sigma$*;*
4. *There exists a function* $\epsilon : \mathbb{R}^+ \to [0,r]$ *such that* $\lim_{\rho\to+\infty} \rho\epsilon(\rho) = +\infty$*, and*

$$k_1 := \inf_{x\in\mathbb{R}^{2n},\lambda\in B(0,\epsilon(|x|)),\sigma\in\Sigma} \operatorname{Re} k(x,\lambda,\sigma) > 0. \tag{4.3.27}$$

Then there exist positive constants C_1' *and* x_0 *such that for all* $|x| \geq x_0$ *and* $\sigma \in \Sigma$ *we have*

$$\operatorname{Re} F(x,\sigma) \geq \frac{C_1'}{|x|^m}. \tag{4.3.28}$$

Proof. The upper bound is easy, since

$$\begin{aligned}
|F(x,\sigma)| &\leq k_2 \int_{B(0,r)} e^{-|x|^2 b_1 |\lambda|^2}\, d\lambda \\
&= \frac{k_2}{|x|^m} \int_{B(0,rx)} e^{-b_1|\lambda|^2}\, d\lambda \\
&\leq \frac{k_2}{|x|^m} \int_{\mathbb{R}^m} e^{-b_1|\lambda|^2}\, d\lambda \\
&= \frac{k_2(\pi/b_1)^{m/2}}{|x|^m}.
\end{aligned}$$

For the lower bound, let

$$F_1(x,\sigma) := \int_{B(0,r)\setminus B(0,\epsilon(|x|))} e^{-|x|^2 g(\lambda,\sigma)} k(x,\lambda,\sigma)\, d\lambda$$
$$F_2(x,\sigma) := \int_{B(0,\epsilon(|x|))} e^{-|x|^2 g(\lambda,\sigma)} k(x,\lambda,\sigma)\, d\lambda$$

so that $F = F_1 + F_2$. Now we have

$$\begin{aligned}|F_1(x,\sigma)| &\le k_2 \int_{B(0,r)\setminus B(0,\epsilon(|x|))} e^{-|x|^2 b_1 |\lambda|^2}\, d\lambda \\ &\le k_2 \int_{\mathbb{R}^m\setminus B(0,\epsilon(|x|))} e^{-|x|^2 b_1 |\lambda|^2}\, d\lambda \\ &\le \frac{k_2}{|x|^m} \int_{\mathbb{R}^m\setminus B(0,|x|\epsilon(|x|))} e^{-b_1 |\lambda'|^2}\, d\lambda'\end{aligned}$$

where we make the change of variables $\lambda' = |x|\,\lambda$. For F_2 we have

$$\begin{aligned}\operatorname{Re} F_2(x,\sigma) &\ge k_1 \int_{B(0,\epsilon(|x|))} e^{-|x|^2 b_2 |\lambda|^2}\, d\lambda \\ &= \frac{1}{|x|^m} k_1 \int_{B(0,|x|\epsilon(|x|))} e^{-b_2 |\lambda'|^2}\, d\lambda'.\end{aligned}$$

So we have

$$\begin{aligned}|x|^m \operatorname{Re} F(x,\sigma) &\ge |x|^m \operatorname{Re} F_2(x,\sigma) - \||x|^m F_1(x,\sigma)| \\ &\ge k_1 \int_{B(0,|x|\epsilon(|x|))} e^{-b_2|\lambda'|^2}\, d\lambda' - k_2 \int_{\mathbb{R}^m\setminus B(0,|x|\epsilon(|x|))} e^{-b_1|\lambda'|^2}\, d\lambda' \\ &\to k_1(\pi/b_2)^{m/2} - 0 > 0\end{aligned}$$

as $|x| \to \infty$. So there exists x_0 so large that for all $|x| \ge x_0$,

$$\operatorname{Re} F(x,\sigma) \ge \frac{1}{2} k_1 (\pi/b_2)^{m/2} \frac{1}{|x|^m} \tag{4.3.29}$$

as desired. □

We need another computation before being able to apply this lemma.

Lemma 4.3.7. Re $\sqrt{(\lambda + i\theta\hat{z})^2}\coth\sqrt{(\lambda + i\theta\hat{z})^2} \ge \theta\cot\theta$, *with equality iff* $\lambda = 0$.

Proof. We first note that the function $\beta\cot\beta$ is strictly decreasing on $[0,\pi)$. To see this, note $\frac{d}{d\beta}\beta\cot\beta = -\nu(\beta)$. By Corollary 3.2.6 $\nu(\beta) > 0$. In particular, $\beta\cot\beta \leq 1$.

Next we observe that for $\alpha \in \mathbb{R}, \beta \in [0,\pi)$ we have

$$\operatorname{Re}((\alpha + i\beta)\coth(\alpha + i\beta)) \geq \beta\cot\beta \tag{4.3.30}$$

with equality iff $\alpha = 0$. This can be seen by verifying that

$$\operatorname{Re}((\alpha + i\beta)\coth(\alpha + i\beta)) - \beta\cot\beta = \frac{\sinh^2\alpha(\alpha\coth\alpha - \beta\cot\beta)}{\cosh^2\alpha - \cos^2\beta} \tag{4.3.31}$$

which is a product of positive terms when $\alpha \neq 0$, since $\alpha\coth\alpha > 1 \geq \beta\cot\beta$ and $\cosh^2\alpha > 1 \geq \cos^2\beta$.

Therefore, we have

$$\operatorname{Re}\sqrt{(\lambda + i\theta\hat{z}^2}\coth\sqrt{(\lambda + i\theta\hat{z})^2} \geq \left(\operatorname{Im}\sqrt{(\lambda + i\theta\hat{z})^2}\right)\cot\left(\operatorname{Im}\sqrt{(\lambda + i\theta\hat{z})^2}\right) \tag{4.3.32}$$

$$\geq \theta\cot\theta \tag{4.3.33}$$

because $0 \leq \operatorname{Im}\sqrt{(\lambda + i\theta\hat{z})^2} \leq \theta < \pi$ by Lemma 4.3.2.

If equality holds in (4.3.33), it must be that $\operatorname{Im}\sqrt{(\lambda + i\theta\hat{z})^2} = \theta$. By Lemma 4.3.2 λ and $\hat{z}$ are parallel, so $\sqrt{(\lambda + i\theta\hat{z})^2} = \pm|\lambda| + i\theta$. If equality also holds in (4.3.32), we have

$$\operatorname{Re}(\pm|\lambda| + i\theta)\coth(\pm|\lambda| + i\theta) = \theta\cot\theta$$

so by (4.3.30) it must be that $|\lambda| = 0$. This proves the claim. □

Lemma 4.3.8. *Given $r > 0$, there exist constants $b_1, b_2, b_3 > 0$ depending only on r and θ_0 such that*

$$b_1|\lambda|^2 \leq \operatorname{Re}\psi(\lambda + i\theta\hat{z}, \theta, \hat{z}) \leq b_2|\lambda|^2 \tag{4.3.34}$$

and

$$|\operatorname{Im}\psi(\lambda + i\theta\hat{z}, \theta, \hat{z})| \leq b_3|\lambda|^3 \tag{4.3.35}$$

for all $\lambda \in B(0,r) \subset \mathbb{R}^m, \theta \in [0,\theta_0], \hat{z} \in S^{m-1} \subset \mathbb{R}^m$.

Proof. Note first that $\psi(\lambda + i\theta\hat{z}, \theta, \hat{z})$ is smooth for $\theta \in [0, \theta_0]$ since Im $\sqrt{(\lambda + i\theta\hat{z})} \le \theta \le \theta_0 < \pi$, so that we are avoiding the singularities of $w \coth w$.

We have $\psi(i\theta\hat{z}, \theta, \hat{z}) = 0$ and $\nabla_\lambda \psi(i\theta\hat{z}, \theta, \hat{z}) = 0$. We now show the Hessian $H(i\theta\hat{z})$ of ψ at $i\theta\hat{z}$ is real and uniformly positive definite.

By direct computation, we can find

$$\frac{\partial^2}{\partial\lambda_i \partial\lambda_j}\psi(\lambda, \theta, \hat{z}) = \nu'(-i\sqrt{\lambda^2})\frac{\lambda_i\lambda_j}{\lambda^2} + i\frac{\nu(-i\sqrt{\lambda^2})}{\sqrt{\lambda^2}}\left(\delta_{ij} - \frac{\lambda_i\lambda_j}{\lambda^2}\right) \tag{4.3.36}$$

so that for $u \in \mathbb{R}^m$,

$$H(\lambda)u \cdot u = \nu'(-i\sqrt{\lambda^2})\frac{(\lambda \cdot u)^2}{\lambda^2} + i\frac{\nu(-i\sqrt{\lambda^2})}{\sqrt{\lambda^2}}\left(|u|^2 - \frac{(\lambda \cdot u)^2}{\lambda^2}\right) \tag{4.3.37}$$

and in particular

$$\begin{aligned} H(i\theta\hat{z})u \cdot u &= \nu'(\theta)(\hat{z} \cdot u)^2 + \frac{\nu(\theta)}{\theta}\left(|u|^2 - \hat{z} \cdot u)^2\right) \\ &= |u|^2\left(s\nu'(\theta) + \frac{\nu(\theta)}{\theta}(1 - s)\right) \end{aligned}$$

where $s := \left(\frac{\hat{z} \cdot u}{|u|}\right)^2$, so $0 \le s \le 1$. Note this is a real number whenever $u \in \mathbb{R}^m$. Thus we have $H(i\theta\hat{z})u \cdot u$ written as a convex combination of two real functions of θ, so

$$H(i\theta\hat{z})u \cdot u \ge |u|^2 \min\{\frac{\nu(\theta)}{\theta}, \nu'(\theta)\} \ge c\,|u|^2 \tag{4.3.38}$$

where c is the lesser of the two constants provided by Lemma 3.2.5 and Corollary 3.2.6 respectively. This is valid for $\theta > 0$ and hence by continuity also for $\theta = 0$.

By Taylor's theorem, this shows that (4.3.34) and (4.3.35) hold for small λ. The upper bounds thus automatically hold for all $\lambda \in B(0, r)$ by continuity. To obtain the lower bound on $\operatorname{Re}\psi$, it will suffice to show $\operatorname{Re}\psi > 0$ for all $\lambda \ne 0$. But we have

$$\begin{aligned} \operatorname{Re}\psi(\lambda + i\theta\hat{z}, \theta, \hat{z}) &= \theta\nu(\theta) - \operatorname{Re} f(i\theta\hat{z}, \theta, \hat{z}) + \operatorname{Re}\left[\sqrt{(\lambda + i\theta\hat{z})^2} \coth \sqrt{((\lambda + i\theta\hat{z})^2}\right] \\ &= \theta\nu(\theta) - \frac{\theta^2}{\sin^2\theta} + \operatorname{Re}\left[\sqrt{(\lambda + i\theta\hat{z})^2} \coth \sqrt{((\lambda + i\theta\hat{z})^2}\right] \\ &= -\theta\cot\theta + \operatorname{Re}\left[\sqrt{(\lambda + i\theta\hat{z})^2} \coth \sqrt{((\lambda + i\theta\hat{z})^2}\right] \\ &\ge 0 \end{aligned}$$

by Lemma 4.3.7, with equality iff $\lambda = 0$. □

The proof of Theorem 4.3.1 can now be completed.

Proof of Theorem 4.3.1. We establish (4.3.1) first. We can apply Lemma 4.3.6 with $\Sigma := [0, \theta_0] \times S^{m-1}$, $\sigma = (\theta, \hat{z})$, r the value from Lemma 4.3.5, and

$$g(\lambda, (\theta, \hat{z})) := \frac{1}{4} \operatorname{Re} \psi(\lambda + i\theta\hat{z}, \theta, \hat{z})$$
$$k(x, \lambda, (\theta, \hat{z})) := e^{i \frac{|x|^2}{4} \operatorname{Im} \psi(\lambda + i\theta\hat{z}, \theta, \hat{z})} a_0(\lambda + i\theta\hat{z}).$$

The necessary bounds on g come from (4.3.34). For an upper bound on k, we have $|k(x, \lambda, (\theta, \hat{z}))| = |a_0(\lambda + i\theta\hat{z})|$, which is bounded by the fact that $(\lambda, \theta, \hat{z})$ ranges over the bounded region $B(0, r) \times [0, \theta_0] \times S^{m-1}$ which avoids the singularities of a_0.

Now for the lower bound on k. By direct computation, we have $a_0(i\theta\hat{z}) = \left(\frac{\theta}{\sin\theta}\right)^n \geq 1$; by continuity there exists δ such that $\operatorname{Re} e^{is} a_0(\lambda + i\theta\hat{z}) \geq \frac{1}{2}$ for all $|\lambda| \leq \delta$ and $|s| \leq \delta$, where $s \in \mathbb{R}$. If $|\lambda| \leq |x|^{-2/3} \delta / b_3$, where b_3 is as in (4.3.35), we will have $|x|^2 |\operatorname{Im} \psi(\lambda + i\theta\hat{z})| \leq \delta$. Thus set $\epsilon(x) := \min\{\delta, |x|^{-2/3} \delta / b_3\}$, so that $\operatorname{Re} k(x, \lambda, (\theta, \hat{z}) \geq \frac{1}{2}$ for all $|\lambda| \leq \epsilon(x)$ and all $(\theta, \hat{z}) \in \Sigma$, and $\lim_{\rho\to\infty} \rho\epsilon(\rho) = \lim_{\rho\to\infty} \rho^{1/3}\delta/b_3 = +\infty$.

Thus Lemma 4.3.6 applies, and so combining it with Lemmas 4.3.4 and 4.3.5 we have that there exist positive constants C, C_1', C_2', x_0 such that

$$\left(\frac{C_1'}{|x|^m} - \frac{C}{|x|^{2m}}\right) e^{-\frac{1}{4} d_0(x,z)^2} \leq p_t(x, z) \leq \left(\frac{C_2'}{|x|^m} + \frac{C}{|x|^{2m}}\right) e^{-\frac{1}{4} d_0(x,z)^2}. \tag{4.3.39}$$

whenever $|x| \geq x_0$. We can choose x_0 larger if necessary so that $|x|^{-m} \gg |x|^{-2m}$. Then taking $D_0 = x_0$ will establish (4.3.1).

For q_i, the upper bound is similar; $|a_i|$ is bounded above just like $|a_0|$, establishing (4.3.2).

For (4.3.3), we cannot necessarily bound both $|q_i|$ below simultaneously, but it suffices to take them one at a time. For $0 \leq \theta(x, z) \leq \frac{\pi}{4}$, we have $a_1(i\theta\hat{z}) = \cos\theta \left(\frac{\theta}{\sin\theta}\right)^{n+1} \geq \frac{1}{\sqrt{2}}$, so by the above logic we obtain the desired lower bound on $|q_1|$ for such θ. If $\frac{\pi}{4} \leq \theta \leq \theta_0$, we estimate q_2 in the same way, since we have $a_2(i\theta\hat{z}) = \left(\frac{\theta}{\sin\theta}\right)^n \theta \geq \frac{\pi}{4}$. □

4.4 Polar coordinates

In this section, we obtain estimates for $p_1(x,z)$ and $|\nabla p_1(x,z)|$ when $|z| \geq B_1 |x|^2$, where B_1 is sufficiently large. This means that $\theta(x,z) \geq \theta_0$ for some θ_0 near π. Note that by Corollary 3.2.9, we have $d_0(x,z) \asymp \sqrt{|z|}$ in this region.

We first consider p_1 and show the following.

Theorem 4.4.1. *For m odd, there exist constants B_1, D_0 such that*

$$p_1(x,z) \asymp \frac{|z|^{n-\frac{m+1}{2}}}{1 + (|x|\ \sqrt{|z|})^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.1}$$

or, equivalently,

$$p_1(x,z) \asymp \frac{d_0(x,z)^{2n-m-1}}{1 + (|x|\, d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.2}$$

for $|z| \geq B_1 |x|^2$ and $|z| \geq D_0$ (equivalently, $d_0(x,z) \geq D_0$).

The effect of the requirement that $|z| \leq B_1 |x|^2$ in the previous section was to ensure that the critical point $i\theta\hat{z}$ stayed away from the singularities of the integrand. As $B_1 \to \infty$, the critical point approaches the set of singularities, and the change of contour we used is no longer effective; the constants in the estimates of Theorem 4.3.1 blow up. In the case of the Heisenberg groups, where the center of G has dimension $m = 1$, the singularity is a single point, and the technique used in [19] and [7] is to move the contour past the singularity and concentrate on the resulting residue term. For $m > 1$, the singularities form a large manifold and this technique is not easy to use directly. However, by making a change to polar coordinates, we can reduce the integral over $\mathbb{R}^m$ to one over $\mathbb{R}$; this replaces the Fourier transform by the so-called Hankel transform. (A similar approach is used in [37] in the context of L^p estimates for the analytic continuation of p_t.) When m is odd, we recover a formula very similar to that for $m = 1$, and the above-mentioned technique is again applicable.

For the rest of this section, we assume that m is odd.

For $m \geq 3$, we write (2.4.10) in polar coordinates to obtain

$$p_1(x,z) = (2\pi)^{-m}(4\pi)^{-n} \int_0^\infty \int_{S^{m-1}} e^{i\rho\sigma\cdot z}\, d\sigma e^{-\frac{|x|^2}{4}\rho\coth\rho} \left(\frac{\rho}{\sinh\rho}\right)^n \rho^{m-1}\, d\rho \tag{4.4.3}$$

$$= \frac{(2\pi)^{-m}(4\pi)^{-n}}{2} \int_{-\infty}^\infty \int_{S^{m-1}} e^{i\rho\sigma\cdot z}\, d\sigma e^{-\frac{|x|^2}{4}\rho\coth\rho} \left(\frac{\rho}{\sinh\rho}\right)^n \rho^{m-1}\, d\rho \tag{4.4.4}$$

since the integrand is an even function of ρ. (To see this, make the change of variables $\sigma \to -\sigma$ in the $d\sigma$ integral. It is not true when m is even.)

The $d\sigma$ integral can be written in terms of a Bessel function. Using spherical coordinates, we can write, for arbitrary $\hat{v} \in S^{m-1}$ and $w \in \mathbb{C}$,

$$\begin{aligned}
\int_{S^{m-1}} e^{iw\sigma\cdot\hat{v}}\, d\sigma &= \frac{2\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} \int_0^{\pi} e^{iw\cos\varphi} \sin^{m-2}\varphi\, d\varphi \\
&= \frac{4\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(w\cos\varphi) \sin^{m-2}\varphi\, d\varphi \qquad \text{(by symmetry)} \\
&= \frac{(2\pi)^{m/2}}{w^{m/2-1}} J_{m/2-1}(w) \qquad \text{(see page 79 of [32])} \\
&= \operatorname{Re} \frac{(2\pi)^{m/2}}{w^{m/2-1}} H^{(1)}_{m/2-1}(w)
\end{aligned}$$

where $H_\nu(w)$ is the Hankel function of the first kind, defined by $H_\nu(w) = J_\nu(w) + iY_\nu(w)$, with Y_ν the Bessel function of the second kind. Page 72 of [32] has a closed-form expression for H_ν which yields

$$\int_{S^{m-1}} e^{iw\sigma\cdot\hat{v}}\, d\sigma = 2(2\pi)^{\frac{m-1}{2}} \operatorname{Re}\left[\frac{e^{iw}}{w^{m-1}} \sum_{k=1}^{\frac{m-1}{2}} c_{m,k}(-iw)^k \right] \tag{4.4.5}$$

where the coefficients are

$$c_{m,k} = \frac{(m-k-2)!}{2^{\frac{m-1}{2}-k}\left(\frac{m-1}{2}-k\right)!(k-1)!} > 0.$$

The reason for the use of the Hankel function is the appearance of the e^{iw} factor, which gives us an integrand looking much like that for p_t when $m = 1$. This will allow us to apply similar techniques to those which have been used previously for $m = 1$. We have

$$p_1(x,z) = (\operatorname{Re}) \sum_{k=1}^{(m-1)/2} c_{m,k}\, |z|^{k-m+1} \int_{-\infty}^{\infty} e^{i\rho|z| - \frac{|x|^2}{4}\rho\coth\rho} \frac{\rho^n}{\sinh^n\rho} (-i\rho)^k\, d\rho \tag{4.4.6}$$

$$= \sum_{k=1}^{(m-1)/2} c_{m,k}\, |z|^{k-m+1}\, e^{-\frac{1}{4}d_0(x,z)^2} \int_{-\infty}^{\infty} e^{-\frac{|x|^2}{4}\psi(\rho,\theta)} a_k(\rho)\, d\rho \tag{4.4.7}$$

where, using similar notation as before,

$$\psi(\rho,\theta) := -i\nu(\theta)\rho + \rho\coth\rho - \frac{\theta^2}{\sin^2\theta} \tag{4.4.8}$$

$$a_k(\rho) := \left(\frac{\rho}{\sinh\rho}\right)^n (-i\rho)^k \tag{4.4.9}$$

The constants and coefficients have all been absorbed into the $c_{m,k}$; we note that $c_{1,0} > 0$, $c_{m,k} > 0$ for $k \geq 1$, and $c_{m,0} = 0$ for $m > 1$. We dropped the (Re) because the imaginary part vanishes, being the integral of an odd function.

For $m = 1$, we can write

$$p_1(x,z) = (4\pi)^{-n} e^{-\frac{1}{4}d_0(x,z)^2} \int_{-\infty}^{\infty} e^{-\frac{|x|^2}{4}\psi(\rho,\theta)} a_0(\rho)\, d\rho \tag{4.4.10}$$

The integrals appearing in the terms of the sum in (4.4.7), as well as in (4.4.10), are all susceptible to the same estimate, as the following theorem shows.

Theorem 4.4.2. *Let $S \subset \mathbb{C}$ be the strip $S = \{0 \leq \operatorname{Im}\rho \leq 3\pi/2\}$. Suppose $a(\rho)$ is a function analytic on $S\backslash\{i\pi\}$, with a pole of order n at $\rho = i\pi$, $a(i\theta) \geq 1$ for $\theta_0 \leq \theta < \pi$, and $\int_{\mathbb{R}} |a(\rho + 3i\pi/2)|\, d\rho < \infty$. Let*

$$h(x,z) := \int_{-\infty}^{\infty} e^{-\frac{|x|^2}{4}\psi(\rho,\theta)} a(\rho)\, d\rho. \tag{4.4.11}$$

There exist B_1, D_0 such that

$$\operatorname{Re} h(x,z) \asymp \frac{|z|^{n-1}}{1 + (|x|\sqrt{|z|})^{n-\frac{1}{2}}} \tag{4.4.12}$$

for all (x,z) with $|z| \geq B_1 |x|^2$ and $|z| \geq D_0$.

The proof of Theorem 4.4.2 occupies the rest of this section. Theorem 4.4.1 follows, since Theorem 4.4.2 applies to each term of (4.4.7) (note each a_k satisfies the hypotheses), and the $k = (m-1)/2$ term will dominate for large $|z|$.

An argument similar to Lemma 4.3.4, using the fact that Lemma 4.3.3 applies for $|b| \leq 2\pi$, will allow us to move the contour to the line $\operatorname{Im}\rho = 3\pi/2$, accounting for the residue at $i\pi$:

$$h(x,z) := \underbrace{\int_{-\infty}^{\infty} e^{-\frac{|x|^2}{4}\psi(\rho+3i\pi/2,\theta)} a(\rho + 3i\pi/2)\, d\rho}_{h_l(x,z)} + \underbrace{\operatorname{Res}(e^{-\frac{|x|^2}{4}\psi(\rho,\theta)} a(\rho); \rho = i\pi)}_{h_r(x,z)}. \tag{4.4.13}$$

The following lemma shows that $h_l(x,z)$, the integral along the horizontal line, is negligible.

Lemma 4.4.3. *There exists $\theta_0 < \pi$ and a constant $C > 0$ such that for all (x,z) with $\theta(x,z) \in [\theta_0, \pi)$ we have*

$$|h_l(x,z)| \leq C e^{-d_0(x,z)^2/8}. \tag{4.4.14}$$

Proof. Observe that $\coth(\rho + 3i\pi/2) = \tanh\rho$. So

$$\operatorname{Re}\psi(\rho + 3i\pi/2, \theta) = \rho\tanh\rho + \frac{3\pi}{2}\nu(\theta) - \frac{\theta^2}{\sin^2\theta}$$

Therefore we have

$$\begin{aligned} |h_l(x,z)| &\le e^{-\frac{|x|^2}{4}\left(\frac{3\pi}{2}\nu(\theta) - \frac{\theta^2}{\sin^2\theta}\right)} \int_{\mathbb{R}} e^{-\frac{|x|^2}{4}\rho\tanh\rho} \left|a(\rho + 3i\pi/2)\right| d\rho \\ &\le e^{-\frac{|x|^2}{4}\left(\frac{3\pi}{2}\nu(\theta) - \frac{\theta^2}{\sin^2\theta}\right)} \int_{\mathbb{R}} \left|a(\rho + 3i\pi/2)\right| d\rho \end{aligned}$$

as $\tau\tanh\tau \ge 0$. The integral in the last line is a finite constant, since $a(\cdot + 3i\pi/2)$ is integrable by assumption.

However, for θ sufficiently close to π, we have $\nu(\theta) \ge \frac{1}{\pi}\frac{\theta^2}{\sin^2\theta}$. (If $\beta(\theta) := \nu(\theta)\left(\frac{\theta^2}{\sin^2\theta}\right)^{-1}$, we have $\lim_{\theta\uparrow\pi}\beta(\theta) = 1/\pi$ and $\lim_{\theta\uparrow\pi}\beta'(\theta) = -2/\pi^2 < 0$. Indeed, $\theta > 0.51$ suffices.) Thus for such θ we have

$$|h_l(x,z)| \le Ce^{-\frac{|x|^2}{8}\frac{\theta^2}{\sin^2\theta}} = Ce^{-d_0(x,z)^2/8}. \tag{4.4.15}$$

□

To handle the residue term h_r, write it as

$$h_r(x,z) = \oint_{\partial B(i\pi,r)} e^{-\frac{|x|^2}{4}\psi(\rho,\theta)/4}(\rho)\, d\rho. \tag{4.4.16}$$

We can choose any $r \in (0,\pi)$ because the integrand is analytic on the punctured disk. To facilitate dealing with the singularity at $\theta = \pi$, we adopt the parameters

$$\begin{aligned} s &:= \pi - \theta(x,z) \\ y &:= \pi\,|x|^2/s. \end{aligned} \tag{4.4.17}$$

Note that

$$y/s \asymp |z|, \quad y \asymp |x|\sqrt{|z|}. \tag{4.4.18}$$

If we let (compare (4.3.10))

$$\begin{aligned} \phi(w,s) &:= \frac{1}{4\pi}s\psi(i(\pi - w), \pi - s) \\ &= \frac{s}{4\pi}\left(\nu(\pi - s)(\pi - w) + (\pi - w)\cot(\pi - w) - \frac{(\pi - s)^2}{\sin^2 s}\right) \end{aligned} \tag{4.4.19}$$

$$F(y,s) := s^{n-1}\oint_{\partial B(0,r)} e^{-y\phi(w,s)}a(i(\pi - w))(-i)\, dw \tag{4.4.20}$$

we have

$$h_r(x,z) = s^{-(n-1)}F(y,s). \tag{4.4.21}$$

Note we have made the change of variables $\rho = i(\pi - w)$ from (4.4.16) to (4.4.20).

Observe that F is analytic in y and s for $s \neq k\pi$, $k \in \mathbb{Z}$, so we shall now consider y and s as complex variables. The factor of s^{n-1} in F was inserted to clear a pole of order $n-1$ at $s = 0$, whose presence will be apparent later.

Computing a Laurent series for ϕ about $(i\pi, \pi)$, which converges for $0 < |s| < \pi$, $0 < |w| < \pi$, we find

$$\phi(w,s) = \frac{1}{2} - \frac{w}{4s} - \frac{s}{4w} - sU(w,s) \tag{4.4.22}$$

with U analytic for $|s| < \pi$, $|w| < \pi$. Also, by the hypotheses on a,

$$a(i(\pi - w)) = w^{-n}V(w) \tag{4.4.23}$$

where V is analytic for $|w| < \pi/2$ and $V(0) > 0$. Thus we have

$$F(y,s) = s^{n-1}\oint_{\partial B(0,r)} e^{-y\left(\frac{1}{2}-\frac{w}{4s}-\frac{s}{4w}-sU(w,s)\right)}w^{-n}V(w)(-i)\,dw \tag{4.4.24}$$

The constant term in the expansion of ψ is slightly inconvenient, so let $G(y,s) = e^{y/2}F(y,s)$. Then:

$$\begin{aligned}
G(y,s) &= s^{n-1}\oint_{\partial B(0,r)} e^{y\left(\frac{w}{4s}+\frac{s}{4w}+sU(w,s)\right)}w^{-n}V(w)(-i)\,dw \\
&= s^{n-1}\oint\sum_{k=0}^{\infty}\frac{y^k}{k!}\left(\frac{w}{4s}+\frac{s}{4w}+sU(w,s)\right)^k w^{-n}V(w)\,dw(-i) && (4.4.25)\\
&= s^{n-1}\sum_{k=0}^{\infty}\frac{y^k}{k!}\oint\left(\frac{w}{4s}+\frac{s}{4w}+sU(w,s)\right)^k w^{-n}V(w)(-i)\,dw \\
&=: \sum_{k=0}\frac{y^k g_k(s)}{k!} && (4.4.26)
\end{aligned}$$

where we let

$$g_k(s) := s^{n-1}\oint\left(\frac{w}{4s}+\frac{s}{4w}+sU(w,s)\right)^k w^{-n}V(w)(-i)\,dw. \tag{4.4.27}$$

The interchange of sum and integral in (4.4.25) is justified by Fubini's theorem, since for fixed s $U(s,\cdot)$ and V are bounded on $B(0,r)$, and thus

$$\sum_{k=0}^{\infty}\oint_{B(0,r)}\left|\frac{y^k}{k!}\left(\frac{w}{4s}+\frac{s}{4w}+sU(w,s)\right)^k\left(\frac{\pi}{w}+V(w)\right)^n\right|dw$$
$$\leq\sum_{k=0}^{\infty}\frac{|y|^k}{k!}2\pi r\left(\frac{r}{4|s|}+\frac{|s|}{4r}+|s|\sup_{|w|=r}|U(w,s)|\right)^k\left(\frac{\pi}{r}+\sup_{|w|=r}|V(w)|\right)^n$$
$$=2\pi r\left(\frac{\pi}{r}+\sup_{|w|=r}|V(w)|\right)^n\exp\left(|y|\left(\frac{r}{4|s|}+\frac{|s|}{4r}+|s|\sup_{|w|=r}|U(w,s)|\right)\right)<\infty.$$

We now examine more carefully the terms g_k in (4.4.26–4.4.27).

Lemma 4.4.4. *If g_k is defined by (4.4.27), then:*

1. *g_k is analytic for $|s|\leq s_0$;*
2. *There exists $C=C(s_0)\geq 0$ independent of k such that $|g_k(s)|\leq C^k$ for each k and all $|s|\leq s_0$;*
3. *For $k\leq n-1$, $g_k(s)=s^{n-1-k}h_k(s)$, where h_k is analytic for $|s|\leq s_0$. In particular, $g_k(0)=0$ for $k<n-1$.*
4. *For $k\geq n-1$, $g_k(0)>0$ when $k+n$ is odd, and $g_k(0)=0$ when $k+n$ is even.*

Proof. By the multinomial theorem,

$$g_k(s)=\sum_{a+b+c=k}\binom{k}{a,b,c}s^{n-1}\oint_{\partial B(0,r)}\left(\frac{w}{4s}\right)^a\left(\frac{s}{4w}\right)^b(sU(w,s))^c\,w^{-n}V(w)(-i)\,dw \tag{4.4.28}$$
$$=\sum_{a+b+c=k}\binom{k}{a,b,c}4^{-(a+b)}\oint_{\partial B(0,r)}w^{a-b-n}s^{-(a-b-n)-1}(sU(w,s))^c\,V(w)(-i)\,dw \tag{4.4.29}$$
$$=\sum_{\substack{a+b+c=k\\ a-b-n\leq -1}}\binom{k}{a,b,c}4^{-(a+b)}\oint_{\partial B(0,r)}w^{a-b-n}s^{-(a-b-n)-1}(sU(w,s))^c\,V(w)(-i)\,dw \tag{4.4.30}$$

since for terms with $a-b-n\geq 0$, the integrand is analytic in w and the integral vanishes. Now the integrand of each term of (4.4.30) is clearly analytic in s, hence so is g_k itself, establishing item 1.

For item 2, let $U_0 := \sup_{|w|=r,|s|\le s_0} |U(w,s)|$, and $V_0 := \sup_{|w|=r} |V(w)|$. Then for $|s| \le s_0$,

$$\begin{aligned}
|g_k(s)| &\le \sum_{\substack{a+b+c=k\\ a-b-n\le -1}} \binom{k}{a,b,c} 4^{-(a+b)} (2\pi r) r^{a-b-n} s_0^{-(a-b-n)-1} (s_0 U_0)^c V_0 \\
&\le 2\pi r V_0 \frac{s_0^{n-1}}{r^n} \sum_{\substack{a+b+c=k\\ a-b-n\le -1}} \binom{k}{a,b,c} \left(\frac{r}{4s_0}\right)^a \left(\frac{s_0}{4r}\right)^b (s_0 U_0)^c \\
&\le 2\pi r V_0 \frac{s_0^{n-1}}{r^n} \sum_{a+b+c=k} \binom{k}{a,b,c} \left(\frac{r}{4s_0}\right)^a \left(\frac{s_0}{4r}\right)^b (s_0 U_0)^c \\
&\le 2\pi r V_0 \frac{s_0^{n-1}}{r^n} \left(\frac{r}{4s_0} + \frac{s_0}{4r} + s_0 U_0\right)^k
\end{aligned}$$

so that a constant C can be chosen with $g_k(s) \le C^k$, establishing item 2.

For item 3, suppose $k \le n-1$ and let $h_k(s) = s^{k-n+1} g_k(s)$, so that

$$h_k(s) = \sum_{\substack{a+b+c=k\\ a-b-n\le -1}} \binom{k}{a,b,c} 4^{-(a+b)} \oint_{\partial B(0,r)} w^{a-b-n} s^{-(a-b-k)} (sU(w,s))^c V(w)(-i)\, dw$$

But $a-b-k \le a-k \le 0$ since $a \le k$ by definition, so only positive powers of s appear, and h_k is analytic in s.

For item 4, we see that when $s = 0$, each term of (4.4.30) will vanish unless $c = 0$ and $a-b-n = -1$, i.e. $a+b = k$ and $a-b = n-1$. If k and n have the same parity, this happens for no term, so $g_k(0) = 0$. If k and n have opposite parity, this forces $a = (k+n-1)/2$, $b = (k-n+1)/2$, both of which are nonnegative integers. In this case

$$\begin{aligned}
g_k(s) &= \binom{k}{(k+n-1)/2} 4^{-k} \oint w^{-1} V(w)(-i)\, dw \\
&= \binom{k}{(k+n-1)/2} 4^{-k} 2\pi V(0) > 0
\end{aligned}$$

since $V(0) > 0$. □

From this we derive corresponding properties of the function F.

Corollary 4.4.5. *Let $F(y,s)$ be defined as in (4.4.20). Then for all $s_0 < \pi$:*

1. *F is analytic for all y and all $0 \le s \le s_0$.*

2. *We may write*

$$F(y,s) = e^{-y/2}\left[\sum_{k=0}^{n-1} \frac{y^k s^{n-1-k}}{k!} h_k(s) + y^n H(y,s)\right] \tag{4.4.31}$$

with h_k, H *analytic for all* y *and all* $0 \le s \le s_0$. *Furthermore,* $h_{n-1}(0) > 0$

3. $F(y,0) > 0$ *for all* $y > 0$.

Proof. We prove the corresponding facts about $G = e^{y/2}F$. By items 1 and 2 of Lemma 4.4.4, we have that G is analytic for $|s| \le s_0$ and all y, since the sum in (4.4.26) is a sum of analytic functions and converges uniformly. By item 3 we have that

$$G(y,s) = \sum_{k=0}^{n-1} \frac{y^k s^{n-1-k}}{k!} h_k(s) + y^n \sum_{k=0}^{\infty} \frac{y^k}{(n+k)!} g_{n+k}(s).$$

And by items 3 and 4, $G(y,0) = \sum_{k=n-1}^{\infty} \frac{y^k g_k(0)}{k!} > 0$ for all $y > 0$. □

Proposition 4.4.6. *For all* $y_1 > 0$*, there exist* $\delta > 0$*, and* $0 < C_1' \le C_2' < \infty$ *such that*

$$C_1' y^{n-1} \le \operatorname{Re} F(y,s) \le |F(y,s)| \le C_2' y^{n-1} \tag{4.4.32}$$

for all $0 \le y < y_1$, $0 \le s < \delta y$. *(Here we are treating* y *and* s *as real variables.)*

Proof. Let K be a positive constant so large that $|h_k(s)| \le K$ and $|H(y,s)| \le K$ for all $0 \le y < y_1$, $0 \le s < y_1$, $k \le n-1$. For any $\delta < 1$ and all $s \le \delta y < y_1$, we have

$$\begin{aligned}
\operatorname{Re} G(y,s) &= \frac{y^{n-1}}{(n-1)!} \operatorname{Re} h_{n-1}(s) + \sum_{k=0}^{n-2} \frac{y^k s^{n-1-k}}{k!} \operatorname{Re} h_k(s) + y^n \operatorname{Re} H(y,s) \\
&\ge \frac{y^{n-1}}{(n-1)!} \operatorname{Re} h_{n-1}(s) - \sum_{k=0}^{n-2} \frac{y^{n-1}\delta^{n-1-k} K}{k!} - y^n K \\
&= y^{n-1}\left[\frac{\operatorname{Re} h_{n-1}(s)}{(n-1)!} - K \sum_{k=0}^{n-2} \frac{\delta^{n-1-k}}{k!}\right] - y^n K.
\end{aligned}$$

Since $h_{n-1}(0) > 0$, we may now choose δ so small that the bracketed term is positive for all $0 \le s \le \delta y_1$. Then there exists $y_0 > 0$ so small that for all $0 \le y \le y_0$, we have

$\operatorname{Re} F(y,s) \geq e^{-y_0/2} \operatorname{Re} G(y,s) \geq C_1' y^{n-1}$ for some $C_1' > 0$. On the other hand,

$$\begin{aligned}
|F(y,s)| &\leq |G(y,s)| \\
&\leq \sum_{k=0}^{n-1} \frac{y^k s^{n-1-k}}{k!} |h_k(s)| + y^n \operatorname{Re} H(y,s) \\
&\leq y^{n-1} \sum_{k=0}^{n-1} \frac{K\delta^{n-1-k}}{k!} + y^n K.
\end{aligned}$$

Again, for small y (take y_0 smaller if necessary), we have $|F(y,s)| \leq C_2' y^{n-1}$.

It remains to handle $y_0 \leq y \leq y_1$. But this presents no difficulty; as $F(y,0) > 0$ for all $y > 0$, and F is continuous, there exists δ so small that

$$\inf_{y_0 \leq y \leq y_1, 0 \leq s \leq \delta y_1} \operatorname{Re} F(y,s) > 0.$$

This completes the proof. □

Proposition 4.4.7. *There exists $y_1 > 0$, $s_0 > 0$ and constants $C_1, C_2 > 0$ such that*

$$\frac{C_1}{\sqrt{y}} \leq \operatorname{Re} F(y,s) \leq |F(y,s)| \leq \frac{C_2}{\sqrt{y}} \tag{4.4.33}$$

for all $y > y_1$, $0 < s < s_0$.

Proof. Here the Gaussian approximation technique of Section 4.3 is again applicable. We will fix the contour in (4.4.20) as a circle of radius $r = s$, parametrize it, and examine the integrand directly. Thus let $w = se^{i\gamma}$ in (4.4.20) to obtain

$$F(y,s) = s^{n-1} \int_{-\pi}^{\pi} e^{-y\phi(se^{i\gamma},s)} a(i(\pi - se^{i\gamma})) se^{i\gamma} \, d\gamma. \tag{4.4.34}$$

We shall apply Lemma 4.3.6, with $m = 1$, $\lambda = \gamma$, $r = \pi$, $x = \sqrt{y}$. Let

$$g(\gamma, s) = \operatorname{Re} \phi(se^{i\gamma}, s) \tag{4.4.35}$$

$$k(\sqrt{y}, \gamma, s) = e^{-i\sqrt{y}^2 \operatorname{Im} \phi(se^{i\gamma},s)} s^n a(i(\pi - se^{i\gamma})) e^{i\gamma} \tag{4.4.36}$$

Since $\phi(s,s) = 0$ and $w = s$ is a critical point of $\phi(w,s)$, we have

$$\frac{\partial^2}{\partial^2 \gamma} \phi(se^{i\gamma}, s)\Big|_{\gamma=0} = \frac{s}{4\pi} \phi''(s,s)(is)^2 = \frac{s^3 \nu'(\pi - s)}{4\pi} \tag{4.4.37}$$

which is bounded and positive for all small s (recall $\nu(\pi - s) \sim s^{-2}$). Thus there exists s_0, ϵ small enough and constants b_1, b_2 such that

$$b_1\gamma^2 \leq g(\gamma, s) \leq b_2\gamma^2 \tag{4.4.38}$$

for $s < s_0$, $|\gamma| < \epsilon$. Also, we have from (4.4.22) that

$$\phi(se^{i\gamma}, s) = \frac{1}{2} - \frac{1}{2}\cos\gamma - sU(se^{i\gamma}, s) \tag{4.4.39}$$

so that by taking s_0 smaller if necessary, we can ensure $g(\gamma, s) > 0$ for all $s < s_0$ and $\epsilon \leq |\gamma| \leq \pi$. Thus (4.4.38) holds for $s < s_0$ and all $\gamma \in [-\pi, \pi]$, with possibly different constants b_1, b_2.

Boundedness of k follows from the fact that a has a pole of order n at $i\pi$, so $s^n a(i(\pi - se^{i\gamma})) = V(se^{i\gamma})$ is bounded for small s. Finally, since $\frac{\partial^2}{\partial^2\gamma}\phi(se^{i\gamma}, s)\Big|_{\gamma=0} > 0$ and $V(0) > 0$, the argument used in the proof of Theorem 4.3.1 shows that the necessary lower bound on k also holds. Then an application of Lemma 4.3.6 completes the proof.

□

Proof of Theorem 4.4.2. Choose y_1, s_0 so that Proposition 4.4.7 holds, and take B_1 large enough so that $\theta(x, z) \geq \pi - s$ when $|z| \geq B_1 |x|^2$. Use this value of y_1 and choose a δ such that Proposition 4.4.6 holds, and take D_0 large enough that $s < \delta y$ when $|z| \geq dD0$ (see (4.4.18)). So for such (x, z), either (4.4.32) or (4.4.33) holds; which one depends on the value of $y = y(x, z)$. We can combine them to get

$$C_1' \frac{y^{n-1}}{1 + y^{n-\frac{1}{2}}} \leq \operatorname{Re} F(y, s) \leq |F(y, s)| \leq C_2' \frac{y^{n-1}}{1 + y^{n-\frac{1}{2}}}. \tag{4.4.40}$$

Inserting this into (4.4.21) and using (4.4.18), we have (in more compact notation)

$$h_r(x, z) \asymp \left(\frac{y}{s}\right)^{n-1} \frac{1}{1 + y^{n-\frac{1}{2}}} \asymp \frac{|z|^{n-1}}{1 + (|x|\ \sqrt{|z|})^{n-\frac{1}{2}}}. \tag{4.4.41}$$

By Lemma (4.4.3), h_l is clearly negligible by comparison, so Theorem 4.4.2 is proved.

□

A similar argument will give us the estimates on ∇p_1 and q_2 which correspond to Theorems 4.1.3 and 4.1.4.

Theorem 4.4.8. *For m odd, there exist constants B_1, D_0, C such that*

$$|\nabla p_1(x,z)| \asymp \frac{|x|\, d_0(x,z)^{2n-m+1}}{1 + (|x|\, d_0(x,z))^{n+\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.42}$$

and

$$|q_2(x,z)| \le C \frac{d_0(x,z)^{2n-m-1}}{1 + (|x|\, d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.43}$$

whenever $|z| \ge B_1 |x|^2$ and $d_0(x,z) \ge D_0$.

Proof. Applying (4.1.5) to (4.4.6), we have

$$\nabla p_1(x,z) = -\frac{1}{2}(2\pi)^{-m}(4\pi)^{-n}\, |x|\, (q_1(x,z)\hat{x} + q_2(x,z) J_{\hat{z}}\hat{x})$$

where

$$\begin{aligned}
q_1(x,z) &= -\frac{2}{|x|}\frac{\partial p_1(x,z)}{\partial |x|} \\
&= -\sum_{k=0}^{(m-1)/2} c_{m,k}\, |z|^{k-m+1} \int_{-\infty}^{\infty} e^{i\rho|z| - \frac{|x|^2}{4}\rho \coth\rho} \left(\frac{\rho}{\sinh\rho}\right)^{n+1} (-\cosh\rho)(-i\rho)^k \, d\rho \\
q_2(x,z) &= \frac{\partial p_1(x,z)}{\partial |z|} \\
&= \sum_{k=0}^{(m-1)/2} \left[c_{m,k}(k-m+1)\, |z|^{k-m} \int_{-\infty}^{\infty} e^{i\rho|z| - \frac{|x|^2}{4}\rho\coth\rho} \left(\frac{\rho}{\sinh\rho}\right)^n (-i\rho)^k \, d\rho \right] \\
&\quad - \sum_{k=0}^{(m-1)/2} \left[c_{m,k}\, |z|^{k-m+1} \int_{-\infty}^{\infty} e^{i\rho|z| - \frac{|x|^2}{4}\rho\coth\rho} \left(\frac{\rho}{\sinh\rho}\right)^n (-i\rho)^{k+1} \, d\rho \right]
\end{aligned}$$

Each integral can be estimated by Theorem 4.4.2. For q_1, each integral is comparable to $e^{-\frac{1}{4}d_0(x,z)^2} \frac{|z|^n}{1+(|x|\sqrt{|z|})^{n+\frac{1}{2}}}$, and the $k = (m-1)/2$ term dominates, so

$$|q_1(x,z)| \asymp \frac{|z|^{n-(m-1)/2}}{1 + (|x|\sqrt{|z|})^{n+\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2}. \tag{4.4.44}$$

The appearance of the extra minus sign in q_1 is to account for the fact that $\cosh(i\pi) = -1$, but Theorem 4.4.2 requires that $a(\lambda)$ be positive near $\lambda = i\pi$.

For q_2, each integral is comparable to $\frac{|z|^{n-1}}{1+(|x|\sqrt{|z|})^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2}$, and the $k = (m-1)/2$ term of the second sum dominates, so

$$|q_2(x,z)| \asymp \frac{|z|^{n-1-(m-1)/2}}{1 + (|x|\sqrt{|z|})^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.45}$$

which in particular implies (4.4.43). To combine (4.4.44) and (4.4.45), note that for $|x|^2\,|z|$ bounded we have

$$|q_1(x,z)| \asymp |z|^{n-(m-1)/2}\, e^{-\frac{1}{4}d_0(x,z)^2}; \quad |q_2(x,z)| \asymp |z|^{n-1-(m-1)/2}\, e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.46}$$

so that the q_1 term dominates, and

$$|\nabla p_1(x,z)| \asymp |x|\,|z|^{n-(m-1)/2}\, e^{-\frac{1}{4}d_0(x,z)^2}. \tag{4.4.47}$$

For $|x|^2\,|z|$ bounded away from 0 we have

$$\begin{aligned} |q_1(x,z)| &\asymp |x|^{-n-\frac{1}{2}}\,|z|^{\frac{n}{2}-\frac{m}{2}+\frac{1}{4}}\, e^{-\frac{1}{4}d_0(x,z)^2} \\ |q_2(x,z)| &\asymp |x|^{-n+\frac{1}{2}}\,|z|^{\frac{n}{2}-\frac{m}{2}-\frac{1}{4}}\, e^{-\frac{1}{4}d_0(x,z)^2} \asymp \frac{|x|}{\sqrt{|z|}} q_1(x,z) \end{aligned} \tag{4.4.48}$$

so that the q_1 term dominates again ($\frac{|x|}{\sqrt{|z|}}$ is bounded by assumption). Thus

$$|\nabla p_1(x,z)| \asymp |x|\,\frac{|z|^{n-(m-1)/2}}{1+(|x|\,\sqrt{|z|})^{n+\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \tag{4.4.49}$$

which is equivalent to the desired estimate. □

4.5 Hadamard descent

In this section, we obtain estimates for $p_1(x,z)$ and $|\nabla p_1(x,z)|$ for $|z| \geq B_1\,|x|^2$, $|z| \geq D_0$, in the case where the center dimension m is even. The methods of the previous section are not directly applicable, but we can deduce an estimate for even m by integrating the corresponding estimate for $m+1$. As discussed in the remark at the end of Section 4.1, this is valid even though there may not exist an H-type group of dimension $2n+m+1$ with center dimension $m+1$, since the estimates we use are derived from the formula (2.4.10) and hold for all values of n, m.

We continue to assume that $|z| \geq B_1\,|x|^2$ and $|z| \geq D_0$ for some sufficiently large B_1, D_0. To emphasize the dependence on the dimension, we write $p^{(n,m)}$ for the function p_1 in (2.4.10).

In order to estimate $p^{(n,m)}$ for m even, we consider $p^{(n,m+1)}$. We can observe that

$$p^{(n,m)}(x,z) = \int_{\mathbb{R}} p^{(n,m+1)}(x,(z,z_{m+1}))\,dz_{m+1} \tag{4.5.1}$$

since $\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda_{m+1}z_{m+1}} f(\lambda_{m+1})\, d\lambda_{m+1}\, dz_{m+1} = 2\pi f(0)$. Note that $|(\lambda, 0)|_{\mathbb{R}^{m+1}} = |\lambda|_{\mathbb{R}^m}$. Now $p^{(n,m+1)}$ can be estimated by means of Theorem 4.4.1. Using the fact that $|(z, z_{m+1})| \geq |z|$, we have that for m even, there exist constants B_1, D_0 such that

$$p^{(n,m)}(x,z) \asymp Q^{(2n-m-2,n-\frac{1}{2})}(x,z) \tag{4.5.2}$$

whenever $|z| \geq B_1 |x|^2$ and $|z| \geq D_0$, where

$$Q^{(\alpha,\beta)}(x,z) := \int_{\mathbb{R}} \frac{d_0(x,(z,z_{m+1}))^\alpha}{1 + (|x|\, d_0(x,(z,z_{m+1})))^\beta} e^{-\frac{1}{4}d_0(x,(z,z_{m+1}))^2}\, dz_{m+1} \tag{4.5.3}$$

Thus it suffices to estimate the integrated bounds given by $Q^{(\alpha,\beta)}$.

Lemma 4.5.1. *For $|z| \geq B_1 |x|^2$ and $|z| \geq D_0$, we have*

$$Q^{(\alpha,\beta)}(x,z) \asymp \frac{d_0(x,z)^{\alpha+1}}{1 + (|x|\, d_0(x,z))^\beta} e^{-\frac{1}{4}d_0(x,z)^2}. \tag{4.5.4}$$

We will require two preliminary computations. Since $d_0(x,z)$ depends on z only through $|z|$, we will occasionally treat d_0 as a function on $\mathbb{R}^{2n} \times [0,\infty)$.

Lemma 4.5.2. *There exist positive constants c_1, c_2, B_1 such that for all $x \in \mathbb{R}^{2n}, u \in \mathbb{R}$ with $u \geq B_1 |x|^2$, we have $0 < c_1 \leq \frac{\partial}{\partial u} d_0(x,u)^2 \leq c_2 < \infty$.*

Proof. Let $\mu(\theta) = \frac{\theta^2}{\sin^2\theta}$, so that $d_0(x,u)^2 = |x|^2 \mu(\theta)$ with $\theta = \theta(x,z) = \nu^{-1}\left(\frac{2u}{|x|^2}\right)$. Then

$$\frac{\partial}{\partial u} d_0(x,u)^2 = 2\frac{\mu'(\theta)}{\nu'(\theta)}. \tag{4.5.5}$$

It is easily verified that $\mu'(\theta) > 0$, $\nu'(\theta) > 0$ for all $\theta \in (0,\pi)$, and $\frac{\mu'(\theta)}{\nu'(\theta)} \to \pi > 0$ as $\theta \to \pi$. □

Lemma 4.5.3. *For any $\alpha \in \mathbb{R}$, there exists $C_\alpha > 0$ such that for all $w_0 \geq 1$ we have*

$$\int_{w_0}^\infty w^\alpha e^{-w}\, dw \leq C_\alpha w_0^\alpha e^{-w_0}. \tag{4.5.6}$$

Proof. For $\alpha \leq 0$, w^α is decreasing for $w \geq 1$, so

$$\int_{w_0}^\infty w^\alpha e^{-w}\, dw \leq w_0^\alpha \int_{w_0}^\infty e^{-w}\, dw = w_0^\alpha e^{-w_0} \tag{4.5.7}$$

and this holds with $C_\alpha = 1$. Now, for a nonnegative integer n, suppose the lemma holds for all $\alpha \leq n$. Then if $n < \alpha \leq n+1$, we integrate by parts to obtain

$$\int_{w_0}^\infty w^\alpha e^{-w}\, dw = w_0^\alpha e^{-w_0} + \alpha \int_{w_0}^\infty w^{\alpha-1} e^{-w}\, dw \leq (1 + \alpha C_{\alpha-1}) w_0^\alpha e^{-w_0}$$

so that the lemma also holds for all $\alpha \leq n+1$. By induction the proof is complete. □

Proof of Lemma 4.5.1. We make the change of variables $u = |(z, z_{m+1})|$ so that $z_{m+1} = \sqrt{u^2 - |z|^2}$. By our previous abuse of notation, we can write $d_0(x,(z,z_{m+1})) = d_0(x,u)$. Thus

$$\begin{aligned} Q^{(\alpha,\beta)}(x,z) &= \int_{|z|}^{\infty} \frac{d_0(x,u)^\alpha}{1+(|x|\,d_0(x,u))^\beta} e^{-\frac{1}{4}d_0(x,u)^2} \frac{u}{\sqrt{u^2-|z|^2}}\,du \\ &\asymp \int_{|z|}^{\infty} \frac{1}{\sqrt{u-|z|}} \frac{1}{\sqrt{u+|z|}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^\beta} e^{-\frac{1}{4}d_0(x,u)^2}\,du. \end{aligned}$$

We used the fact that $u \asymp d_0(x,u)^2$ where $|z| \geq B_1 |x|^2$, by Corollary 3.2.9.

Now, noting that $u \mapsto d_0(x,u)$ is an increasing function, and $w \mapsto w^{\alpha+2}e^{-\frac{1}{4}w^2}$ is decreasing for large enough w, the lower bound can be obtained by

$$\begin{aligned} Q^{(\alpha,\beta)}(x,z) &\geq \int_{|z|}^{|z|+1} \frac{1}{\sqrt{u-|z|}} \frac{1}{\sqrt{u+|z|}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^\beta} e^{-\frac{1}{4}d_0(x,u)^2}\,du \\ &\geq \left(\int_{|z|}^{|z|+1} \frac{1}{\sqrt{u-|z|}}\,du\right) \frac{1}{\sqrt{2\,|z|+1}} \frac{d_0(x,|z|+1)^{\alpha+2}}{1+(|x|\,d_0(x,|z|+1))^\beta} e^{-\frac{1}{4}d_0(x,|z|+1)^2} \\ &= 2\frac{1}{\sqrt{2\,|z|+1}} \frac{d_0(x,|z|+1)^{\alpha+2}}{1+(|x|\,d_0(x,|z|+1))^\beta} e^{-\frac{1}{4}d_0(x,|z|+1)^2} \\ &\geq C\frac{1}{\sqrt{2\,|z|}} \frac{d_0(x,z)^{\alpha+2}}{1+(|x|\,d_0(x,z))^\beta} e^{-\frac{1}{4}d_0(x,z)^2} \end{aligned}$$

where the last line follows because $u \mapsto d_0(x,u)^2$ is Lipschitz, as shown by Lemma 4.5.2, with a constant independent of x.

Since $|z| \asymp d_0(x,z)^2$, we have that

$$Q^{(\alpha,\beta)}(x,z) \geq C' \frac{d_0(x,z)^{\alpha+1}}{1+(|x|\,d_0(x,z))^\beta} e^{-\frac{1}{4}d_0(x,z)^2}. \tag{4.5.8}$$

For an upper bound, we have

$$Q^{(\alpha,\beta)}(x,z) \leq C\left[\int_{|z|}^{|z|+1} \frac{1}{\sqrt{u-|z|}} \frac{1}{\sqrt{u+|z|}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^\beta} e^{-\frac{1}{4}d_0(x,u)^2}\,du + \int_{|z|+1}^{\infty} \ldots\right]$$

Now

$$\begin{aligned} &\int_{|z|}^{|z|+1} \frac{1}{\sqrt{u-|z|}} \frac{1}{\sqrt{u+|z|}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^\beta} e^{-\frac{1}{4}d_0(x,u)^2}\,du \\ &\leq \left(\int_{|z|}^{|z|+1} \frac{1}{\sqrt{u-|z|}}\,du\right) \frac{1}{\sqrt{2\,|z|}} \frac{d_0(x,z)^{\alpha+2}}{1+(|x|\,d_0(x,z))^\beta} e^{-\frac{1}{4}d_0(x,z)^2} \\ &= 2\frac{1}{\sqrt{2\,|z|}} \frac{d_0(x,z)^{\alpha+2}}{1+(|x|\,d_0(x,z))^\beta} e^{-\frac{1}{4}d_0(x,z)^2} \\ &\leq C\frac{d_0(x,z)^{\alpha+1}}{1+(|x|\,d_0(x,z))^\beta} e^{-\frac{1}{4}d_0(x,z)^2}. \end{aligned}$$

For the other term, we observe

$$\begin{aligned}
&\int_{|z|+1}^{\infty} \frac{1}{\sqrt{u-|z|}} \frac{1}{\sqrt{u+|z|}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^{\beta}} e^{-\frac{1}{4}d_0(x,u)^2}\,du \\
&\le \int_{|z|+1}^{\infty} \frac{1}{\sqrt{u+|z|}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^{\beta}} e^{-\frac{1}{4}d_0(x,u)^2}\,du \\
&\le \int_{|z|}^{\infty} \frac{1}{\sqrt{2u}} \frac{d_0(x,u)^{\alpha+2}}{1+(|x|\,d_0(x,u))^{\beta}} e^{-\frac{1}{4}d_0(x,u)^2}\,du \\
&\le C \int_{|z|}^{\infty} \frac{d_0(x,u)^{\alpha+1}}{1+(|x|\,d_0(x,u))^{\beta}} e^{-\frac{1}{4}d_0(x,u)^2}\,du
\end{aligned}$$

We now make the change of variables $w = \frac{1}{4}d_0(x,u)^2$. By the above lemma, du/dw is bounded, so

$$\int_{|z|}^{\infty} \frac{d_0(x,u)^{\alpha+1}}{1+(|x|\,d_0(x,u))^{\beta}} e^{-\frac{1}{4}d_0(x,u)^2}\,du \le C \int_{\frac{1}{4}d_0(x,z)^2}^{\infty} \frac{(4w)^{(\alpha+1)/2}}{1+(2\,|x|\,\sqrt{w})^{\beta}} e^{-w}\,dw.$$

If $d_0(x,z) \le 1/\,|x|$, we have

$$\begin{aligned}
\int_{\frac{1}{4}d_0(x,z)^2}^{\infty} \frac{(4w)^{(\alpha+1)/2}}{1+(2\,|x|\,\sqrt{w})^{\beta}} e^{-w}\,dw &\le \int_{\frac{1}{4}d_0(x,z)^2}^{\infty} (4w)^{(\alpha+1)/2} e^{-w}\,dw \\
&\le C d_0(x,z)^{\alpha+1} e^{-\frac{1}{4}d_0(x,z)^2} \\
&\le 2C \frac{d_0(x,z)^{\alpha+1}}{1+(|x|\,d_0(x,z))^{\beta}} e^{-\frac{1}{4}d_0(x,z)^2}
\end{aligned}$$

where we have used Lemma 4.5.3.

On the other hand, when $d_0(x,z) \ge 1/\,|x|$, we have

$$\begin{aligned}
\int_{\frac{1}{4}d_0(x,z)^2}^{\infty} \frac{(4w)^{(\alpha+1)/2}}{1+(2\,|x|\,\sqrt{w})^{\beta}} e^{-w}\,dw &\le (2\,|x|)^{-\beta} \int_{\frac{1}{4}d_0(x,z)^2}^{\infty} (4w)^{(\alpha+1-\beta)/2} e^{-w}\,dw \\
&\le C\,|x|^{-\beta}\, d_0(x,z)^{\alpha+1-\beta} e^{-\frac{1}{4}d_0(x,z)^2} \\
&\le 2C \frac{d_0(x,z)^{\alpha+1}}{1+(|x|\,d_0(x,z))^{\beta}} e^{-\frac{1}{4}d_0(x,z)^2}
\end{aligned}$$

Combining all this, we have as desired that

$$Q^{(\alpha,\beta)}(x,z) \asymp \frac{d_0(x,z)^{\alpha+1}}{1+(|x|\,d_0(x,z))^{\beta}} e^{-\frac{1}{4}d_0(x,z)^2}. \tag{4.5.9}$$

□

Corollary 4.5.4. *Theorems 4.4.1 and 4.4.8 also hold for m even.*

Proof. The heat kernel estimate of Theorem 4.4.1 is immediate, given (4.5.2) and Lemma 4.5.1.

To obtain an estimate on ∇p_1, we define $q_1^{(n,m)} := -\frac{2}{|x|}\frac{\partial}{\partial |x|} p_1^{(n,m)}(x,z)$, $q_2^{(n,m)} := \frac{\partial}{\partial |z|} p_1^{(n,m)}(x,z)$, as in (4.1.5).

For q_1, we simply differentiate (4.5.1) to see

$$\begin{aligned} q_1^{(n,m)}(x,z) &= \int_{\mathbb{R}} q_1^{(n,m+1)}(x,(z,z_{m+1}))\,dz_{m+1} \\ &\asymp Q^{(2n-m,n+\frac{1}{2})}(x,z) && \text{by (4.4.44)} \\ &\asymp \frac{d_0(x,z)^{2n-m+1}}{1+(|x|\,d_0(x,z))^{n+\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} && \text{by Lemma 4.5.1.} \end{aligned}$$

For q_2, we again differentiate (4.5.1). Here we obtain

$$\begin{aligned} q_2^{(n,m)}(x,z) &= \int_{\mathbb{R}} q_2^{(n,m+1)}(x,(z,z_{m+1}))\frac{|z|}{|(z,z_{m+1})|}\,dz_{m+1} \\ &\asymp |z|\,Q^{(2n-m-4,n-\frac{1}{2})} && \text{by (4.4.45)} \\ &\asymp d_0(x,z)^2 \frac{d_0(x,z)^{2n-m-3}}{1+(|x|\,d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2} \\ &\asymp \frac{d_0(x,z)^{2n-m-1}}{1+(|x|\,d_0(x,z))^{n-\frac{1}{2}}} e^{-\frac{1}{4}d_0(x,z)^2}. \end{aligned}$$

Repeating the computation from Theorem 4.4.8, we have the desired estimates on $|\nabla p_1|$ and $|q_2|$. □

Chapter 4, in large part, is adapted from material awaiting publication as Eldredge, Nathaniel, "Precise Estimates for the Subelliptic Heat Kernel on H-type Groups," to appear, *Journal de Mathématiques Pures et Appliquées*, 2009. The dissertation author was the sole author of this paper.

Chapter 5

Gradient Estimates

5.1 Statement of results

Notation 5.1.1. Let $\mathcal{C}$ be the class of $f \in C^1(G)$ for which there exist constants $M \geq 0$, $a \geq 0$, and $\epsilon \in (0, 1)$ such that

$$|f(g)| + |\nabla f(g)| + \left|\hat{\nabla} f(g)\right| \leq M e^{a d(0,g)^{2-\epsilon}}$$

for all $g \in G$. By the heat kernel bounds in Theorem 4.1.1, $P_t f$ as defined by (2.4.12) makes sense for all $f \in \mathcal{C}$.

The main theorem of this chapter is the following:

Theorem 5.1.2. *There exists a finite constant K such that for all $f \in \mathcal{C}$,*

$$|\nabla P_t f| \leq K P_t(|\nabla f|). \tag{5.1.1}$$

This theorem can be interpreted as a quantitative statement about the smoothing properties of P_t. As we shall see in Section 5.5, it has a number of significant consequences.

5.2 Previous work

The inequality (5.1.1) arose in the work of Bakry in the context of Riemannian manifolds [3], [4]. In this case, (5.1.1) is strongly related to the geometry of the manifold, and in particular to its Ricci curvature.

Theorem 5.2.1 (Bakry)**.** *If L is the Laplace-Beltrami operator on a Riemannian manifold M, then*

$$|\nabla P_t f| \leq e^{kt} P_t(|\nabla f|) \quad \forall f \in C_c^\infty(M) \tag{5.2.1}$$

if and only if

$$\mathrm{Ric}(v, v) \geq -k\,|v|^2 \quad \forall v \in TM. \tag{5.2.2}$$

To contrast this theorem with the situation of hypoelliptic operators, consider the Heisenberg group $\mathbb{H}_1$ as in Section 1.2. The operator $L = X^2 + Y^2$ cannot be the Laplace-Beltrami operator of any Riemannian metric, since L is not elliptic. One might try to approximate L by Laplace-Beltrami operators. For example, if for $\epsilon > 0$ we impose a metric $\langle \cdot, \cdot \rangle_\epsilon$ on $\mathbb{H}_1$ such that $\langle X, Y \rangle = \langle X, Z \rangle = \langle Y, Z \rangle = 0$, $\langle X, X \rangle = \langle Y, Y \rangle = 1$, and $\langle Z, Z \rangle = 1/\epsilon$, the corresponding Laplace-Beltrami operator is $L_\epsilon := X^2 + Y^2 + \epsilon Z^2$, which approximates L as $\epsilon \to 0$. However, the Ricci curvature tensor corresponding to $\langle \cdot, \cdot \rangle_\epsilon$ has $\mathrm{Ric}(X, X) = -\frac{1}{4\epsilon} \to -\infty$ as $\epsilon \to 0$, so the constant k in Theorem 5.2.1 blows up, and the gradient bound in (5.2.1) becomes useless. Therefore, Theorem 5.2.1 does not seem to be directly useful in the hypoelliptic Lie group setting.

We remark another distinction: in (5.2.1), the "constant" e^{kt} on the right side of the inequality depends on t and in particular tends to 1 as $t \to 0$. (e^{kt} is not known, or even believed, to be the best constant.) However, in (5.1.1) in the case of H-type groups, the constant K is independent of t. Indeed, it can be shown that the *best* constant is also independent of t, and is strictly greater than 1. In particular, the best constant does not tend to 1 as $t \to 0$. This is another indication of the significant differences between the hypoelliptic and Riemannian settings.

Progress in the hypoelliptic case was made by Driver and Melcher in [13], which obtained the following L^p-type estimate in the case of the Heisenberg group $\mathbb{H}_1$:

$$|\nabla P_t f|^p \leq K_p P_t(|\nabla f|^p). \tag{5.2.3}$$

Their argument proceeded probabilistically via methods of Malliavin calculus and did not depend on heat kernel estimates, but they also showed that their argument could not produce (5.1.1), which is the corresponding estimate with $p = 1$. [34] extended (5.2.3) to the case of a general Lie group, at the cost of replacing the constant K_p with a function

$K_p(t)$ which in general was shown only to be finite for all t. However, for nilpotent Lie groups, the constant $K_p(t)$ was shown to be bounded independent of t.

As mentioned in Section 1.2, the first extension of (5.1.1) itself to a hypoelliptic setting was due to H.-Q. Li in [28]. Like the argument in this dissertation, the proof relies on pointwise upper and lower estimates for the heat kernel, and a pointwise upper estimate for its gradient, shown in the case of Heisenberg-Weyl groups in [29]. [5] contains two alternate (and much simpler) proofs of (5.1.1) for the classical Heisenberg group, also depending on the pointwise heat kernel estimates from [29]. The proof of Theorem 5.1.2 in the case of H-type groups, which occupies the following section, follows rather closely the approach taken by the first proof in [5].

5.3 Proof of gradient estimate

Following an argument found in [13], by left-invariance of P_t and ∇, we see that in order to establish (5.1.1) it suffices to show that it holds at the identity, i.e. to show

$$|(\nabla P_t f)(0)| \leq K P_t(|\nabla f|)(0). \tag{5.3.1}$$

It also suffices to assume $t = 1$. This can be seen by taking $t = 1$ in (5.3.1) and replacing f by $f \circ \varphi_{s^{1/2}}$.

Therefore, in order to prove Theorem 5.1.2, it will suffice to show $|(\nabla P_1 f)(0)| \leq K P_1(|\nabla f|)(0)$. We may replace ∇ by $\hat{\nabla}$ on the left side, since $\nabla = \hat{\nabla}$ at 0. Since $[X_i, \hat{X}_j] = 0$, we expect that $\hat{\nabla}$ should commute with P_t, which we now verify.

Proposition 5.3.1. *For $f \in C$, $\hat{\nabla} P_t f(0) = (P_t \hat{\nabla} f)(0)$.*

Proof. By (2.3.1) and (2.4.12) we have

$$\begin{aligned} \hat{X}_i P_t f(0) &= \left.\frac{d}{ds}\right|_{s=0} P_t f(se_i, 0) \\ &= \left.\frac{d}{ds}\right|_{s=0} \int_G f((se_i, 0) \star k) p_t(k)\, dm(k). \end{aligned}$$

We now differentiate under the integral sign, which can be justified because

$$\begin{aligned}\left|\frac{d}{ds} f((se_i, 0) \star k)\right| &= \left|\frac{d}{d\sigma}\right|_{\sigma=0} f(((s+\sigma)e_i, 0) \star k)\Big| \\ &= \left|\frac{d}{d\sigma}\right|_{\sigma=0} f((\sigma e_i, 0) \star (se_i, 0) \star k)\Big| \\ &= \left|\hat{X}_i f((se_i, 0) \star k)\right| \\ &\leq M e^{a d(0,(se_i,0)\star k)^{2-\epsilon}}.\end{aligned}$$

But

$$\begin{aligned}d(0, (se_i, 0) \star k) &= d((se_i, 0)^{-1}, k) = d((-se_i, 0), k) \\ &\leq d(0, (-se_i, 0)) + d(0, k) = |s| + d(0, k).\end{aligned}$$

Thus for all $s \in [-1, 1]$ we have

$$\left|\frac{d}{ds} f((se_i, 0) \star k)\right| \leq M e^{a(1+d(0,k))^{2-\epsilon}} \leq M' e^{a' d(0,k)^{2-\epsilon}}$$

for some M', a', and therefore by the heat kernel bounds of Theorem 4.1.1 we have

$$\int_G \sup_{s\in[-1,1]} \left|\frac{d}{ds} f((se_i, 0) \star k)\right| p_t(k)\, dm(k) < \infty$$

which justifies differentiating under the integral sign. Thus

$$\begin{aligned}\hat{X}_i P_t f(0) &= \int_G \frac{d}{ds}\Big|_{s=0} f((se_i, 0) \star k) p_t(k)\, dm(k) \\ &= \int_G \hat{X}_i f(k) p_t(k)\, dm(k) \\ &= P_t \hat{X}_i f(0).\end{aligned}$$

This completes the proof. □

Thus Theorem 5.1.2 reduces to showing

$$\left|(P_1 \hat{\nabla} f)(0)\right| \leq K P_1(|\nabla f|)(0) \tag{5.3.2}$$

or in other words

$$\left|\int_G (\hat{\nabla} f) p_1\, dm\right| \leq K \int_G |\nabla f|\, p_1\, dm \tag{5.3.3}$$

for which it suffices to show

$$\left|\int_G ((\nabla - \hat{\nabla})f)p_1\, dm\right| \le K \int_G |\nabla f|\, p_1\, dm. \tag{5.3.4}$$

A similar argument can be used to verify the following integration by parts formula.

Proposition 5.3.2. *If $f \in C$, then*

$$\begin{aligned} \int_G (\nabla f)p_1\, dm &= -\int_G (\nabla p_1) f\, dm \\ \int_G (\hat{\nabla} f)p_1\, dm &= -\int_G (\hat{\nabla} p_1) f\, dm \end{aligned} \tag{5.3.5}$$

Proof. As in Theorem 2.4.7, take $\psi_n \in C_c^\infty(G)$ to be a sequence of compactly supported smooth functions such that $\psi_n \to 1$ and $\nabla\psi_n \to 0$ boundedly. By item 2 of Proposition 2.3.7, we have

$$\begin{aligned} \int_G (\nabla p_1)(f\psi_n)\, dm &= -\int_G p_1 \nabla(f\psi_n)\, dm \\ &= -\int_G p_1 f \nabla\psi_n\, dm - \int_G p_1 \psi_n \nabla f\, dm \end{aligned}$$

Since $p_1 f, p_1\nabla f, (\nabla p_1) f \in L^1(G)$ by definition of C and the heat kernel bounds, two applications of the dominated convergence theorem complete the proof. □

We now introduce an alternate coordinate system on G, similar but not exactly analogous to the so-called "polar coordinate" system used in [5]. As shown in Section 3.1, there is a unique (up to reparametrization) shortest horizontal path from the identity 0 to each point $(x, z) \in G$ with x, z nonzero; it has as its projection onto $\mathbb{R}^{2n} \times 0$ an arc of a circle lying in the plane spanned by x and $J_z x$, with the origin as one endpoint, and x as the other. The region in this plane bounded by the arc and the straight line from 0 to x has area equal to $|z|$. The projection onto $0 \times \mathbb{R}^m$ is a straight line from 0 to z.

Our new coordinate system will identify a point (x, z) with the point $u \in \mathbb{R}^{2n}$ which is the center of the arc, and a vector $\eta \in \mathbb{R}^m$ which is parallel to z and whose magnitude equals the angle subtended by the arc. The change of coordinates $(u, \eta) \mapsto (x, z)$ will be denoted by

$$\Phi : \{(u, \eta) \in \mathbb{R}^{2n+m} : 0 < |\eta| < 2\pi\} \to \{(x, z) \in G : x \neq 0, z \neq 0\}$$

where

$$\Phi(u,\eta) := \left(\left(I - e^{J_\eta}\right)u, \frac{|u|^2}{2}\left(1 - \frac{\sin|\eta|}{|\eta|}\right)\eta\right)$$
$$= \left((1-\cos|\eta|)u + \frac{\sin|\eta|}{|\eta|}J_\eta u, \frac{|u|^2}{2}\left(1 - \frac{\sin|\eta|}{|\eta|}\right)\eta\right)$$

by Proposition 2.1.7, items 2 and 3. Φ has the property that for each (u,η), the path $s \mapsto \Phi(u, s\eta)$ traces the shortest horizontal path between any two of its points, and has constant speed $|u|\,|\eta|$. In particular,

$$d(0, \Phi(u,\eta)) = |u|\,|\eta|\,. \tag{5.3.6}$$

Also, for any $f \in C^1(G)$,

$$\left|\frac{d}{ds} f(\Phi(u, s\eta))\right| \le |u|\,|\eta|\,|\nabla f(\Phi(u, s\eta))|\,. \tag{5.3.7}$$

Note that if $(x,z) = \Phi(u,\eta)$, we have

$$|x|^2 = |u|^2\,(2 - 2\cos|\eta|)$$
$$|z| = \frac{|u|^2}{2}(|\eta| - \sin|\eta|).$$

To compare this with the "polar coordinates" (u,s) used in [5], take $u = u$ and $s = |u|\,\eta$.

In (u,η) coordinates, the heat kernel estimate (4.1.2) reads

$$p_1(\Phi(u,\eta)) \asymp \frac{1 + (|u|\,|\eta|)^{2n-m-1}}{1 + (|u|^2\,|\eta|\,\sqrt{2 - 2\cos|\eta|})^{n-\frac{1}{2}}} e^{-\frac{1}{4}(|u||\eta|)^2} \tag{5.3.8}$$
$$\asymp \frac{1 + (|u|\,|\eta|)^{2n-m-1}}{1 + \left(|u|^2\,|\eta|^2\,(2\pi - |\eta|)\right)^{n-\frac{1}{2}}} e^{-\frac{1}{4}(|u||\eta|)^2} \tag{5.3.9}$$

since $1 - \cos\theta \asymp \theta^2(2\pi - \theta)^2$ for $\theta \in [0, 2\pi]$. We will often abuse notation and write $p_1(u,\eta)$ for $p_1(\Phi(u,\eta))$, when no confusion will result.

We now begin the proof of Theorem 5.1.2, which occupies the rest of this section.

We begin by computing the Jacobian determinant of the change of coordinates Φ, so that we can use (u,η) coordinates in explicit computations.

Lemma 5.3.3. *Let $A(u,\eta)$ denote the Jacobian determinant of Φ, so that $dm = A(u,\eta)\,du\,d\eta$. Then*

$$A(u,\eta) = |u|^{2m} \left(\frac{1}{2} - \frac{\sin|\eta|}{2|\eta|}\right)^{m-1} (2 - 2\cos|\eta|)^{n-1} (2 - 2\cos|\eta| - |\eta|\sin|\eta|). \tag{5.3.10}$$

Note that $A(u,\eta)$ depends on u,η only through their absolute values $|u|, |\eta|$. By an abuse of notation we may occasionally use A with u or η replaced by scalars, so that $A(r,\rho)$ means $A(r\hat{u},\rho\hat{\eta})$ for arbitrary unit vectors $\hat{u},\hat{\eta}$.

For the Heisenberg group with $n = m = 1$, this reduces to

$$A(u,\eta) = |u|^2 (2 - 2\cos|\eta| - |\eta|\sin|\eta|).$$

The analogous expression appearing in [5] is slightly incorrect. However, it does have the same asymptotics as the correct expression (see Corollary 5.3.4), which is sufficient for the rest of the argument in [5], so that its overall correctness is not affected.

Proof. Fix u,η. Form an orthonormal basis for $T_{(u,\eta)}\Phi^{-1}(G) \cong \mathbb{R}^{2n+m}$ as follows. Let $\hat{u}$ be a unit vector in the direction of $(u,0)$, $\hat{v}$ a unit vector in the direction of $(J_\eta u, 0)$. For $i = 1,\dots,n-1$ let $\hat{w}_i, \hat{y}_i \in \mathbb{R}^{2n} \times 0$ be unit vectors such that $\hat{w}_i$ is orthogonal to $\hat{u},\hat{v},\hat{w}_j,\hat{y}_j, 1 \le j < i$, and let $\hat{y}_i$ be in the direction of $J_\eta \hat{w}_i$ so that $\hat{y}_i$ is orthogonal to $\hat{u},\hat{v},\hat{w}_j,\hat{y}_j, 1 \le j < i$ as well as to $\hat{w}_i$. (To see this, note that if $\langle x,y\rangle = 0$ and $\langle x, J_z y\rangle = 0$, then $\langle J_z x, y\rangle = 0$ and $\langle J_z x, J_z y\rangle = -|z|^2 \langle x,y\rangle = 0$.) Let $\hat{\eta}$ be a unit vector in the direction of $(0,\eta)$, and let $\hat{\zeta}_k, k = 1,\dots,m-1$ be orthonormal vectors in $0 \times \mathbb{R}^m$ which are orthogonal to $\hat{\eta}$. Then $\{\hat{u},\hat{v},\hat{w}_i,\hat{y}_i,\hat{\eta},\hat{\zeta}_k\}$ form an orthonormal basis for $\mathbb{R}^{2n+m}$. Note $J_\eta\hat{u} = |\eta|\,\hat{v}$, $J_\eta\hat{v} = -|\eta|\,\hat{u}$,$J_\eta\hat{w}_i = |\eta|\,\hat{y}_i$, $J_\eta\hat{y}_i = -|\eta|\,\hat{w}_i$. Then

$$\begin{aligned}
\partial_{\hat{u}}\Phi(u,\eta) &= (1-\cos|\eta|)\hat{u} + \sin|\eta|\,\hat{v} + |u|\,(|\eta| - \sin|\eta|)\,\hat{\eta}\\
\partial_{\hat{v}}\Phi(u,\eta) &= (1-\cos|\eta|)\hat{v} - \sin|\eta|\,\hat{u}\\
\partial_{\hat{w}_i}\Phi(u,\eta) &= (1-\cos|\eta|)\hat{w}_i + \sin|\eta|\,\hat{y}_i\\
\partial_{\hat{y}_i}\Phi(u,\eta) &= (1-\cos|\eta|)\hat{y}_i - \sin|\eta|\,\hat{w}_i\\
\partial_{\hat{\eta}}\Phi(u,\eta) &= |u|\,(\sin|\eta|)\hat{u} + |u|\,(\cos|\eta|)\hat{v} + \frac{|u|^2}{2}(1-\cos|\eta|)\,\hat{\eta}\\
\partial_{\hat{\zeta}_k}\Phi(u,\eta) &= \frac{\sin|\eta|}{|\eta|} J_{\hat{\zeta}_k} u + \frac{|u|^2}{2}\left(1 - \frac{\sin|\eta|}{|\eta|}\right)\hat{\zeta}_k.
\end{aligned}$$

In this basis, the Jacobian matrix has the form

$$J = \begin{pmatrix} 1-\cos|\eta| & -\sin|\eta| & 0 & |u|\sin|\eta| & 0 \\ \sin|\eta| & 1-\cos|\eta| & 0 & |u|\cos|\eta| & 0 \\ 0 & 0 & B & 0 & * \\ |u|\left(|\eta|-\sin|\eta|\right) & 0 & 0 & \frac{|u|^2}{2}(1-\cos|\eta|) & 0 \\ 0 & 0 & 0 & 0 & D \end{pmatrix}_{(2n+m)\times(2n+m)} \tag{5.3.11}$$

where

$$B := \begin{pmatrix} 1-\cos|\eta| & -\sin|\eta| & & & \\ \sin|\eta| & 1-\cos|\eta| & & & \\ & & \ddots & & \\ & & & 1-\cos|\eta| & -\sin|\eta| \\ & & & \sin|\eta| & 1-\cos|\eta| \end{pmatrix}_{2(n-1)\times 2(n-1)} \tag{5.3.12}$$

is a block-diagonal matrix of 2×2 blocks, and

$$D := \begin{pmatrix} \frac{|u|^2}{2}\left(1-\frac{\sin|\eta|}{|\eta|}\right) & & \\ & \ddots & \\ & & \frac{|u|^2}{2}\left(1-\frac{\sin|\eta|}{|\eta|}\right) \end{pmatrix}_{(m-1)\times(m-1)} \tag{5.3.13}$$

is diagonal. Note $|B| = (2-2\cos|\eta|)^{n-1}$ and $|D| = \left(\frac{|u|^2}{2}\left(1-\frac{\sin|\eta|}{|\eta|}\right)\right)^{m-1}$.

So factoring out $|D|$ and expanding about the $\hat{\eta}$ row, we have

$$
\begin{aligned}
|J| &= |D| \left(|u| \left(|\eta| - \sin|\eta|\right) \begin{vmatrix} -\sin|\eta| & 0 & |u|\sin|\eta| \\ 1-\cos|\eta| & 0 & |u|\cos|\eta| \\ 0 & B & 0 \end{vmatrix} \right. \\
&\quad \left. + \frac{|u|^2}{2}(1-\cos|\eta|) \begin{vmatrix} 1-\cos|\eta| & -\sin|\eta| & 0 \\ \sin|\eta| & 1-\cos|\eta| & 0 \\ 0 & 0 & B \end{vmatrix} \right) \\
&= \left(\frac{|u|^2}{2} \left(1 - \frac{\sin|\eta|}{|\eta|} \right) \right)^{m-1} \\
&\quad \times \left(|u| \left(|\eta| - \sin|\eta|\right)(-|u|\sin|\eta|)(2-2\cos|\eta|)^{n-1} + \frac{|u|^2}{2}(1-\cos|\eta|)(2-2\cos|\eta|)^n \right) \\
&= \left(\frac{|u|^2}{2} \left(1 - \frac{\sin|\eta|}{|\eta|} \right) \right)^{m-1} |u|^2 (2-2\cos|\eta|)^{n-1} \left((|\eta| - \sin|\eta|)(-\sin|\eta|) + (1-\cos|\eta|)^2 \right) \\
&= |u|^{2m} \left(\frac{1}{2} - \frac{\sin|\eta|}{2|\eta|} \right)^{m-1} (2-2\cos|\eta|)^{n-1} (2 - 2\cos|\eta| - |\eta|\sin|\eta|)
\end{aligned}
$$

□

Corollary 5.3.4.

$$A(u,\eta) \asymp |u|^{2m} |\eta|^{2(m+n)} (2\pi - |\eta|)^{2n-1} \tag{5.3.14}$$

Proof. The asymptotic equivalence near $|\eta| = 0$ and $|\eta| = 2\pi$ follows from a routine Taylor series computation.

It then suffices to show that $A(u,\eta) > 0$ for all $0 < |\eta| < 2\pi$. We have $\frac{1}{2} - \frac{\sin|\eta|}{2|\eta|} > 0$ for all $|\eta| > 0$, since $x > \sin x$ for all $x > 0$. We also have $2 - 2\cos|\eta| > 0$ for all $0 < |\eta| < 2\pi$.

Finally, to show $f(|\eta|) := 2 - 2\cos|\eta| - |\eta|\sin|\eta| > 0$, let $\theta = \frac{1}{2}|\eta|$. Using double-angle identities, we have $f(2\theta) = 4\sin\theta(\sin\theta - \theta\cos\theta)$. For $0 < \theta < \pi$ we have $\sin\theta > 0$ so it suffices to show $g(\theta) := \sin\theta - \theta\cos\theta > 0$. But we have $g(0) = 0$ and $g'(\theta) = \theta\sin\theta > 0$ for $0 < \theta < \pi$. □

The heat kernel estimates will be used to prove a technical lemma regarding integrating the heat kernel along a geodesic. The proof requires the following simple fact from calculus, a close relative of Lemma 4.5.3.

Lemma 5.3.5. *For any $q \in \mathbb{R}$, $a_0 > 0$ there exists a constant $C = C_{q,a_0}$ such that for any $a \geq a_0$ we have*

$$\int_{t=1}^{t=\infty} t^q e^{-(at)^2}\, dt \leq C\frac{1}{a^2}e^{-a^2}. \tag{5.3.15}$$

Proof. Apply Lemma 4.5.3, taking $w_0 = a^2$, $\alpha = (q-1)/2$, and making the change of variables $w = (at)^2$. □

Let $B := \{g : d(0, g) \leq 1\}$ be the Carnot-Carathéodory unit ball.

Lemma 5.3.6. *For each $q \in \mathbb{R}$ there exists a constant C_q such that for all u, η with $\Phi(u, \eta) \in B^C$, i.e. $|u|\,|\eta| \geq 1$, we have*

$$\int_{t=1}^{t=\frac{2\pi}{|\eta|}} p_1(u, t\eta)A(u, t\eta)t^q\, dt \leq \frac{C_q}{(|u|\,|\eta|)^2} p_1(u, \eta)A(u, \eta) \tag{5.3.16}$$

$$\leq C_q p_1(u, \eta)A(u, \eta). \tag{5.3.17}$$

Note that (5.3.17) follows immediately from the stronger statement (5.3.16), since by assumption $|u|\,|\eta| \geq 1$. In fact, we shall only use (5.3.17) in the sequel.

Proof. Assume throughout that $|u|\,|\eta| \geq 1$ and $0 < |\eta| < 2\pi$.

The proof involves the fact that a geodesic passes through (up to) three regions of G in which the estimates for p_1 and A simplify in different ways. We define these regions, which partition B^C, as follows. See Figure 5.1.

1. Region R_1 is the set of $\Phi(u, \eta)$ such that $0 < |\eta| \leq \pi$. (This corresponds to having $|x|^2 \lesssim |z|$.) In this region we have $|u| \geq \frac{1}{\pi}$ and $\pi \leq 2\pi - |\eta| < 2\pi$. Therefore (5.3.9) becomes

   $$p_1(u, \eta) \overset{R_1}{\asymp} (|u|\,|\eta|)^{-m} e^{-\frac{1}{4}(|u||\eta|)^2}$$

   and Corollary 5.3.4 yields

   $$A(u, \eta) \overset{R_1}{\asymp} |u|^{2m}\,|\eta|^{2(n+m)}$$

   so that

   $$p_1(u, \eta)A(u, \eta) \overset{R_1}{\asymp} |u|^m\,|\eta|^{2n+m}\, e^{-\frac{1}{4}(|u||\eta|)^2} =: F_1(u, \eta).$$

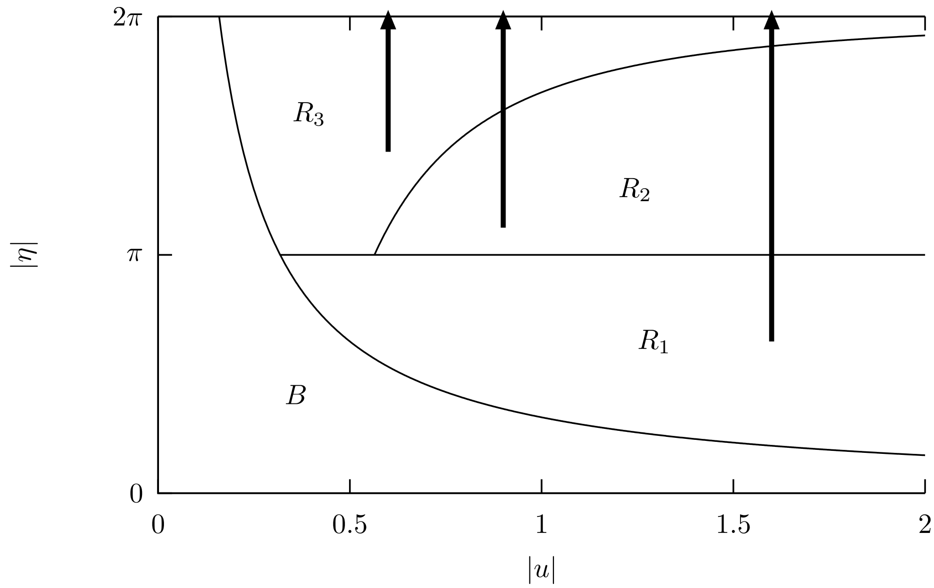

Figure 5.1: The regions R_1, R_2, R_3, seen in the $|u|$-$|\eta|$ plane. The dark lines indicate examples of the geodesic paths of integration used in (5.3.16).

2. Region R_2 is the set of $\Phi(u,\eta)$ such that $\pi < |\eta| \le 2\pi - \frac{1}{|u|^2}$. (This corresponds to having $|x|^2 \gtrsim |z|$ and $|x|^2\,|z| \gtrsim 1$.) In this region, we have $|u|^2\,|\eta|^2\,(2\pi - |\eta|) \ge \pi^2$, so that

$$\begin{aligned} p_1(u,\eta) &\stackrel{R_2}{\asymp} |u|^{-m}\,(2\pi - |\eta|)^{-n+\frac{1}{2}} e^{-\frac{1}{4}(|u||\eta|)^2} \\ A(u,\eta) &\stackrel{R_2}{\asymp} |u|^{2m}\,(2\pi - |\eta|)^{2n-1} \\ p_1(u,\eta)A(u,\eta) &\stackrel{R_2}{\asymp} |u|^{m}\,(2\pi - |\eta|)^{n-\frac{1}{2}} e^{-\frac{1}{4}(|u||\eta|)^2} =: F_2(u,\eta) \\ &\stackrel{R_2}{\asymp} |u|^{m}\,|\eta|^{2n+m}\,(2\pi - |\eta|)^{n-\frac{1}{2}} e^{-\frac{1}{4}(|u||\eta|)^2} =: \tilde{F}_2(u,\eta). \end{aligned}$$

We shall use the estimates $F_2, \tilde{F}_2$ at different times. Although $F_2 \stackrel{R_2}{\asymp} \tilde{F}_2$ (since $|\eta| \stackrel{R_2}{\asymp} 1$), they are not equivalent on R_1.

3. Region R_3 is the set of $\Phi(u,\eta)$ such that $|\eta| > \max\left(\pi, 2\pi - \frac{1}{|u|^2}\right)$. (This corresponds to having $|x|^2 \gtrsim |z|$ and $|x|^2\,|z| \lesssim 1$.) In this region, we have $|u|^2\,|\eta|^2\,(2\pi - |\eta|) < (2\pi)^2$, so that

$$\begin{aligned} p_1(u,\eta) &\stackrel{R_3}{\asymp} |u|^{2n-m-1}\, e^{-\frac{1}{4}(|u||\eta|)^2} \\ A(u,\eta) &\stackrel{R_3}{\asymp} |u|^{2m}\,(2\pi - |\eta|)^{2n-1} \\ p_1(u,\eta)A(u,\eta) &\stackrel{R_3}{\asymp} |u|^{2n+m-1}\,(2\pi - |\eta|)^{2n-1} e^{-\frac{1}{4}(|u||\eta|)^2} =: F_3(u,\eta) \end{aligned}$$

We observe that a geodesic starting from the origin (given by $t \mapsto \Phi(u, t\eta)$ for some fixed u, η) passes through these regions in order, except that it skips Region 2 if $|u| < \pi^{-1/2}$.

We now estimate the desired integral along a portion of a geodesic lying in a single region.

Claim 5.3.7. *Let $q \in \mathbb{R}$. Suppose that $F : G \to \mathbb{R}$ is given by*

$$F(u, \eta) = |u|^\alpha |\eta|^\beta (2\pi - |\eta|)^\gamma e^{-\frac{1}{4}(|u||\eta|)^2}$$

for some nonnegative powers α, β, γ, and that there is some region $R \subset G$ such that $F \overset{R}{\asymp} p_1 A$. Then there is a constant C depending on q, F, R such that for all $u, \eta, \tau_0, \tau_1, \tau_2$ satisfying

- $1 \le \tau_0 \le \tau_1 \le \tau_2 \le \frac{2\pi}{|\eta|}$*; and*
- $\Phi(u, t\eta) \in R$ *for all* $t \in [\tau_1, \tau_2]$

we have

$$\int_{t=\tau_1}^{t=\tau_2} p_1(u, t\eta) A(u, t\eta) t^q \, dt \le C \frac{\tau_0^{q-1}}{(|u|\,|\eta|)^2} F(u, \tau_0 \eta). \tag{5.3.18}$$

Proof of Claim 5.3.7. We have

$$\begin{aligned}
\int_{t=\tau_1}^{t=\tau_2} p_1(u, t\eta) A(u, t\eta) t^q \, dt &\le C \int_{t=\tau_1}^{t=\tau_2} F(u, t\eta) t^q \, dt \\
&\le C \int_{t=\tau_0}^{t=\tau_2} F(u, t\eta) t^q \, dt \\
&= C |u|^\alpha |\eta|^\beta \int_{t=\tau_0}^{t=\tau_2} t^{q+\beta_i} (2\pi - t|\eta|)^\gamma e^{-\frac{1}{4}(t|u||\eta|)^2} \, dt \\
&\le C |u|^\alpha |\eta|^\beta (2\pi - \tau_0 |\eta|)^\gamma \int_{t=\tau_0}^{t=\tau_2} t^{q+\beta} e^{-\frac{1}{4}(t|u||\eta|)^2} \, dt
\end{aligned}$$

since $t \ge \tau_0$. We now make the change of variables $t = t'\tau_0$:

$$\begin{aligned}
&\le C |u|^\alpha |\eta|^\beta (2\pi - \tau_0 |\eta|)^\gamma \tau_0^{q+\beta+1} \int_{t'=1}^{t'=\infty} t'^{q+\beta} e^{-\frac{1}{4}(t'\tau_0|u||\eta|)^2} \, dt' \\
&\le C' |u|^\alpha |\eta|^\beta (2\pi - \tau_0 |\eta|)^\gamma \tau_0^{q+\beta+1} \frac{1}{(\tau_0 |u|\,|\eta|)^2} e^{-\frac{1}{4}(\tau_0|u||\eta|)^2} \\
&= C' \frac{\tau_0^{q-1}}{(|u|\,|\eta|)^2} F_i(u, \tau_0 \eta)
\end{aligned}$$

where in the second-to-last line we applied Lemma 5.3.5 with $a = \frac{1}{2}\tau_0 |u|\,|\eta|$, $a_0 = \frac{1}{2}$. □

Now for fixed u, η, let

$$t_2 := \max\left(1, \frac{\pi}{|\eta|}\right)$$
$$t_3 := \max\left(t_2, \frac{1}{|\eta|}\left(2\pi - \frac{1}{|u|^2}\right)\right)$$

so that

$$\Phi(u, t\eta) \in R_1 \text{ for } 1 < t \leq t_2$$
$$\Phi(u, t\eta) \in R_2 \text{ for } t_2 < t < t_3$$
$$\Phi(u, t\eta) \in R_3 \text{ for } t_3 \leq t < \frac{2\pi}{|\eta|}.$$

We divide the remainder of the proof into cases, depending on the region where $\Phi(u, \eta)$ resides.

Case 1. Suppose that $\Phi(u, \eta) \in R_1$. We have

$$\int_{t=1}^{t=\frac{2\pi}{|\eta|}} p_1(u, t\eta) A(u, t\eta) t^q \, dt = \int_{t=1}^{t=t_2} + \int_{t=t_2}^{t=t_3} + \int_{t=t_3}^{t=\frac{2\pi}{|\eta|}}.$$

For the first integral, where $\Phi(u, t\eta) \in R_1$, we have by Claim 5.3.7 (taking $\tau_0 = \tau_1 = 1$, $\tau_2 = t_2$, $R = R_1$, $F = F_1$) that

$$\int_{t=1}^{t=t_2} p_1(u, t\eta) A(u, t\eta) t^q \, dt \leq \frac{C}{(|u|\,|\eta|)^2} F_1(u, \eta) \leq \frac{C'}{(|u|\,|\eta|)^2} p_1(u, \eta) A(u, \eta)$$

since $F_1 \overset{R_1}{\asymp} p_1 A$.

For the second integral, where $\Phi(u, t\eta) \in R_2$, we take $\tau_0 = 1$, $\tau_1 = t_2$, $\tau_2 = t_3$, $R = R_2$, $F = \tilde{F}_2$ in Claim 5.3.7 to obtain

$$\int_{t=t_2}^{t=t_3} p_1(u, t\eta) A(u, t\eta) t^q \, dt \leq \frac{C}{(|u|\,|\eta|)^2} \tilde{F}_2(u, \eta).$$

However, for $\Phi(u, \eta) \in R_1$ we have

$$\frac{\tilde{F}_2(u, \eta)}{F_1(u, \eta)} = (2\pi - |\eta|)^{n-\frac{1}{2}} \leq (2\pi)^{n-\frac{1}{2}}.$$

Thus

$$\begin{aligned}\int_{t=t_2}^{t=t_3} p_1(u, t\eta) A(u, t\eta) t^q \, dt &\leq \frac{C'}{(|u|\,|\eta|)^2} F_1(u, \eta) \\ &\leq \frac{C''}{(|u|\,|\eta|)^2} p_1(u, \eta) A(u, \eta)\end{aligned}$$

The third integral is more subtle. We apply Claim 5.3.7 with $\tau_0 = \tau_1 = t_3$, $\tau_3 = \frac{2\pi}{|\eta|}$, $R = R_3$, $F = F_3$:

$$\int_{t=t_3}^{t=\frac{2\pi}{|\eta|}} p_1(u,t\eta)A(u,t\eta)t^q\,dt \le C\frac{t_3^{q-1}}{(|u|\,|\eta|)^2}F_3(u,t_3\eta)$$

Then

$$\frac{t_3^{q-1}F_3(u,t_3\eta)}{F_1(u,\eta)} = t_3^{q-1}\,|u|^{2n-1}\,|\eta|^{-2n-m}\,(2\pi - t_3\,|\eta|)^{2n-1}e^{-\frac{1}{4}(|u||\eta|)^2(t_3^2-1)}. \tag{5.3.19}$$

We must show that this ratio is bounded. Fix some $\epsilon > 0$. If $|u| \ge (\pi-\epsilon)^{-1/2} > \pi^{-1/2}$, we have $2\pi - \frac{1}{|u|^2} > \pi + \epsilon$ and thus $t_3 = \frac{1}{|\eta|}\left(2\pi - \frac{1}{|u|^2}\right)$. Then

$$\begin{aligned}|\eta|^2\,(t_3^2-1) &= \left(2\pi - \frac{1}{|u|^2}\right)^2 - |\eta|^2\\ &\ge (\pi+\epsilon)^2 - \pi^2 = 2\pi\epsilon + \epsilon^2.\end{aligned}$$

So in this case (5.3.19) becomes

$$\begin{aligned}\frac{t_3^{q-1}F_3(u,t_3\eta)}{F_1(u,\eta)} &= \left(\frac{1}{|\eta|}\left(2\pi - \frac{1}{|u|^2}\right)\right)^{q-1}|u|^{2n-1}\,|\eta|^{-2n-m}\left(\frac{1}{|u|^2}\right)^{2n-1}e^{-\frac{1}{4}(|u||\eta|)^2(t_3^2-1)}\\ &= \left(2\pi - \frac{1}{|u|^2}\right)^{q-1}|u|^{-2n+1}\,|\eta|^{-2n-m-q+1}\,e^{-\frac{1}{4}(|u||\eta|)^2(t_3^2-1)}\\ &\le (2\pi)^{q-1}\,|u|^{m+q}\,e^{-\frac{1}{4}(2\pi\epsilon+\epsilon^2)|u|^2}\end{aligned}$$

since $|\eta| \le \frac{1}{|u|}$. This is certainly bounded by some constant. On the other hand, if $|u| \le (\pi-\epsilon)^{-1/2}$, then $|\eta| \ge (\pi-\epsilon)^{1/2}$ and $1 \le t_3 \le \left(\frac{\pi+\epsilon}{\pi-\epsilon}\right)^{1/2}$, so that the right side of (5.3.19) is clearly bounded.

Thus we have

$$\begin{aligned}\int_{t=t_3}^{t=\frac{2\pi}{|\eta|}} p_1(u,t\eta)A(u,t\eta)t^q\,dt &\le \frac{C'}{(|u|\,|\eta|)^2}F_1(u,\eta)\\ &\le \frac{C''}{(|u|\,|\eta|)^2}p_1(u,\eta)A(u,\eta)\end{aligned}$$

This completes the proof of this case.

Case 2. Suppose that $\Phi(u,\eta) \in R_2$. We have

$$\int_{t=1}^{t=\frac{2\pi}{|\eta|}} p_1(u,t\eta)A(u,t\eta)t^q\,dt = \int_{t=1}^{t=t_3} + \int_{t=t_3}^{t=\frac{2\pi}{|\eta|}}.$$

Note that in this region we have $1 \le t_3 \le 2$. Again by Claim 5.3.7, with $\tau_0 = \tau_1 = 1$ and $\tau_2 = t_3$, we have

$$\int_{t=1}^{t=t_3} p_1(u, t\eta)A(u, t\eta)t^q \, dt \le \frac{C}{(|u|\,|\eta|)^2} F_2(u, \eta) \le \frac{C'}{(|u|\,|\eta|)^2} p_1(u, \eta)A(u, \eta).$$

For the second integral, we apply Claim 5.3.7 with $\tau_0 = 1$, $\tau_1 = t_3$, $\tau_2 = \frac{2\pi}{|\eta|}$ to get

$$\int_{t=t_3}^{t=\frac{2\pi}{|\eta|}} p_1(u, t\eta)A(u, t\eta)t^q \, dt \le \frac{C}{(|u|\,|\eta|)^2} F_3(u, \eta).$$

But $|\eta| \ge 2\pi - \frac{1}{|u|^2}$ on R_3, so we have

$$\begin{aligned} \frac{F_3(u, \eta)}{F_2(u, \eta)} &= |u|^{2n-1} (2\pi - |\eta|)^{n-\frac{1}{2}} \\ &\le |u|^{2n-1} \left(\frac{1}{|u|^2} \right)^{n-\frac{1}{2}} = 1. \end{aligned}$$

Thus

$$\begin{aligned} \int_{t=t_3}^{t=\frac{2\pi}{|\eta|}} p_1(u, t\eta)A(u, t\eta)t^q \, dt &\le \frac{C'}{(|u|\,|\eta|)^2} F_2(u, \eta) \\ &\le \frac{C''}{(|u|\,|\eta|)^2} p_1(u, \eta)A(u, \eta). \end{aligned}$$

Case 3. Suppose $\Phi(u, \eta) \in R_3$; we apply Claim 5.3.7 with $\tau_0 = \tau_1 = 1$, $\tau_2 = \frac{2\pi}{|\eta|}$ to get

$$\int_{t=1}^{t=\frac{2\pi}{|\eta|}} p_1(u, t\eta)A(u, t\eta)t^q \, dt \le C \frac{1}{(|u|\,|\eta|)^2} F_3(u, \eta) \le C' p_1(u, \eta)A(u, \eta).$$

The three cases together complete the proof of Lemma 5.3.6. □

Notation 5.3.8. For $f \in C^1(G)$, let $m_f := \frac{\int_B f \, dm}{\int_B dm}$, where B is the Carnot-Carathéodory unit ball.

To continue to follow the line of [5], we need the following Poincaré inequality. This theorem can be found in [22], and is a special case of a more general theorem appearing in [33].

Theorem 5.3.9. *There exists a constant C such that for any $f \in C^\infty(G)$,*

$$\int_B \left| f - m_f \right| \, dm \le C \int_B |\nabla f| \, dm. \tag{5.3.20}$$

Corollary 5.3.10. *There exists a constant C such that for any $f \in C^\infty(G)$,*

$$\int_B \left|f - m_f\right| p_1 \, dm \leq C \int_B |\nabla f| \, p_1 \, dm. \tag{5.3.21}$$

Proof. p_1 is bounded and bounded below away from 0 on B. ◻

Lemma 5.3.11 (akin to Lemma 5.2 of [5])**.** *There exists a constant C such that for all $f \in \mathcal{C}$,*

$$\int_{B^C} \left|f - m_f\right| p_1 \, dm \leq C \int_G |\nabla f| \, p_1 \, dm. \tag{5.3.22}$$

Proof. Changing to (u, η) coordinates, we wish to show

$$\int_{|u| \geq \frac{1}{2\pi}} \int_{\frac{1}{|u|} \leq |\eta| < 2\pi} \left|f(\Phi(u,\eta)) - m_f\right| p_1(\Phi(u,\eta)) A(u,\eta) \, d\eta \, du \leq C \int_G |\nabla f| \, p_1 \, dm. \tag{5.3.23}$$

By an abuse of notation we shall write $f(u,\eta)$ for $f(\Phi(u,\eta))$, $p_1(u,\eta)$ for $p_1(\Phi(u,\eta))$, $\nabla f(u,\eta)$ for $(\nabla f)(\Phi(u,\eta))$, et cetera.

Let $g(u,\eta) := f\left(u, \min\left(|\eta|, \frac{1}{|u|}\right) \frac{\eta}{|\eta|}\right)$. Then $g = f$ on B (in particular $m_g = m_f$), g is bounded, the function $s \mapsto g(u, s\eta)$ is absolutely continuous, and $\frac{d}{ds} g(u, s\eta) = 0$ for $s > \frac{1}{|u||\eta|}$.

Now $\left|f - m_f\right| \leq |f - g| + \left|g - m_f\right|$. We first observe that for $|u| \, |\eta| \geq 1$ we have

$$\begin{aligned} |f(u,\eta) - g(u,\eta)| &= \left| \int_{s=\frac{1}{|u||\eta|}}^{s=1} \left(\frac{d}{ds} f(u, s\eta) - \cancel{\frac{d}{ds} g(u, s\eta)} \right) ds \right| \\ &\leq \int_{s=\frac{1}{|u||\eta|}}^{s=1} |\nabla f(u, s\eta)| \, |u| \, |\eta| \, ds \end{aligned}$$

by (5.3.7). Thus

$$\int_{B^C} |f - g| \, p_1 \, dm = \int_{|u| \geq \frac{1}{2\pi}} \int_{|\eta| \geq \frac{1}{|u|}} |f(u,\eta) - g(u,\eta)| \, p_1(u,\eta) A(u,\eta) \, d\eta \, du$$

where the limits of integration come from the conditions $|u| \, |\eta| \geq 1$, $|\eta| < 2\pi$;

$$\begin{aligned} &\leq \int_{|u| \geq \frac{1}{2\pi}} \int_{|\eta| \geq \frac{1}{|u|}} \int_{s=\frac{1}{|u||\eta|}}^{s=1} |\nabla f(u, s\eta)| \, |u| \, |\eta| \, p_1(u,\eta) A(u,\eta) \, ds \, d\eta \, du \\ &= \int_{|u| \geq \frac{1}{2\pi}} \int_{s=0}^{s=1} \int_{\frac{1}{s|u|} \leq |\eta| \leq 2\pi} |\nabla f(u, s\eta)| \, |u| \, |\eta| \, p_1(u,\eta) A(u,\eta) \, d\eta \, ds \, du \end{aligned}$$

by Tonelli's theorem. We now make the change of variables $\eta' = s\eta$ to obtain

$$= \int_{|u|\geq\frac{1}{2\pi}} \int_{s=0}^{s=1} \int_{\frac{1}{|u|}\leq|\eta'|\leq 2\pi s} |\nabla f(u,\eta')|\,|u|\,\frac{1}{s}\,|\eta'|$$
$$\times\, p_1\left(u, \frac{1}{s}\eta'\right) A\left(u, \frac{1}{s}\eta'\right) \frac{1}{s^m}\, d\eta'\, ds\, du$$
$$= \int_{|u|\geq\frac{1}{2\pi}} \int_{\frac{1}{|u|}\leq|\eta'|\leq 2\pi} |\nabla f(u,\eta')|\,|u|\,|\eta'|$$
$$\times \left(\int_{s=\frac{|\eta'|}{2\pi}}^{s=1} p_1\left(u, \frac{1}{s}\eta'\right) A\left(u, \frac{1}{s}\eta'\right) \frac{1}{s^{m+1}}\, ds \right) d\eta'\, du$$

Make the further change of variables $t = \frac{1}{s}$ to get

$$= \int_{|u|\geq\frac{1}{2\pi}} \int_{\frac{1}{|u|}\leq|\eta'|\leq 2\pi} |\nabla f(u,\eta')|\,|u|\,|\eta'|$$
$$\times \left(\int_{t=1}^{t=\frac{2\pi}{|\eta'|}} p_1(u,t\eta')A(u,t\eta')t^{m-1}\, dt \right) d\eta'\, du.$$

Applying Lemma 5.3.6 to the bracketed term gives

$$\leq C \int_{|u|\geq\frac{1}{2\pi}} \int_{\frac{1}{|u|}\leq|\eta'|\leq 2\pi} \frac{1}{|u|\,|\eta'|} |\nabla f(u,\eta')|\, p_1(u,\eta')A(u,\eta')\, d\eta'\, du$$
$$\leq C' \int_{B^C} |\nabla f|\, p_1\, dm$$

converting back from geodesic coordinates and using the fact that $|u|\,|\eta'| \geq 1$.

To complete the proof, we must show that $\int_{B^C} \left|g - m_f\right| p_1\, dm \leq \int_G |\nabla f|\, p_1\, dm$. Note that for $\Phi(u,\eta) \in B^C$, i.e. $|u|\,|\eta| \geq 1$, we have $g(u,\eta) = f\left(u, \frac{1}{|u||\eta|}\eta\right)$, so

$$\int_{B^C} \left|g - m_f\right| p_1\, dm = \int_{|u|\geq\frac{1}{2\pi}} \int_{\frac{1}{|u|}\leq|\eta|\leq 2\pi} \left| f\left(u, \frac{1}{|u|\,|\eta|}\eta\right) - m_f \right| p_1(u,\eta)A(u,\eta)\, d\eta\, du. \tag{5.3.24}$$

Change the η integral to polar coordinates by writing $\eta = \rho\hat{\eta}$, where $\rho \geq 0$ and $|\hat{\eta}| = 1$. Note that $p_1(u,\eta), A(u,\eta)$ depend on η only through ρ and not $\hat{\eta}$.

$$= C \int_{|u|\geq\frac{1}{2\pi}} \int_{\hat{\eta}\in S^{m-1}} \left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - m_f \right| \tag{5.3.25}$$
$$\times \int_{\rho=\frac{1}{|u|}}^{\rho=2\pi} p_1(u,\rho)A(u,\rho)\rho^{m-1}\, d\rho\, d\hat{\eta}\, du \tag{5.3.26}$$

Now, for any $s \in [0,1]$ we have

$$\left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - m_f \right| \leq \left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - f\left(u, \frac{s}{|u|}\hat{\eta}\right) \right| + \left| f\left(u, \frac{s}{|u|}\hat{\eta}\right) - m_f \right|. \tag{5.3.27}$$

Let

$$D(u) := \int_{s=0}^{s=1} \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) ds. \tag{5.3.28}$$

By multiplying both sides of (5.3.27) by $\frac{1}{D(u)} \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right)$ and integrating we obtain

$$\begin{aligned} \left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - m_f \right| \leq \frac{1}{D(u)} \int_{s=0}^{s=1} & \left(\left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - f\left(u, \frac{s}{|u|}\hat{\eta}\right) \right| + \left| f\left(u, \frac{s}{|u|}\hat{\eta}\right) - m_f \right| \right) \\ & \times \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) ds. \end{aligned} \tag{5.3.29}$$

Let

$$R(u) := \frac{1}{D(u)} \int_{\rho=\frac{1}{|u|}}^{\rho=2\pi} p_1(u,\rho) A(u,\rho) \rho^{m-1}\, d\rho. \tag{5.3.30}$$

Then substituting (5.3.29) into (5.3.26) and using (5.3.30) we have

$$\int_{B^C} \left| g - m_f \right| p_1 \, dm \leq I_1 + I_2 \tag{5.3.31}$$

where

$$I_1 := \int_{|u| \geq \frac{1}{2\pi}} \int_{\hat{\eta} \in S^{m-1}} \int_{s=0}^{s=1} \left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - f\left(u, \frac{s}{|u|}\hat{\eta}\right) \right| \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) ds\, R(u)\, d\hat{\eta}\, du \tag{5.3.32}$$

$$I_2 := \int_{|u| \geq \frac{1}{2\pi}} \int_{\hat{\eta} \in S^{m-1}} \int_{s=0}^{s=1} \left| f\left(u, \frac{s}{|u|}\hat{\eta}\right) - m_f \right| \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) ds\, R(u)\, d\hat{\eta}\, du. \tag{5.3.33}$$

We now show that I_1, I_2 can each be bounded by a constant times $\int_G |\nabla f|\, p_1\, dm$, using the following claim.

Claim 5.3.12. *There exists a constant C such that for all $|u| \geq \frac{1}{2\pi}$ we have*

$$R(u) \leq C \left(2\pi - \frac{1}{|u|}\right)^{2n-1} \leq (2\pi)^{2n-1} C. \tag{5.3.34}$$

Proof of Claim. First, by Corollary 5.3.4 we have

$$\begin{aligned}
D(u) &:= \int_{s=0}^{s=1} \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) ds \\
&\geq C \int_{s=0}^{s=1} \frac{s^{m-1}}{|u|^m} |u|^{2m} \left(\frac{s}{|u|}\right)^{2(m+n)} \left(2\pi - \frac{s}{|u|}\right)^{2n-1} ds \\
&= C\,|u|^{-2n-m} \int_{s=0}^{s=1} s^{3m+2n-1} \left(2\pi - \frac{s}{|u|}\right)^{2n-1} ds \\
&\geq C\,|u|^{-2n-m} \int_{s=0}^{s=1} s^{3m+2n-1} \left(2\pi(1-s)\right)^{2n-1} ds \qquad \text{since } u \geq \frac{1}{2\pi} \\
&= C'\,|u|^{-2n-m}
\end{aligned}$$

since the s integral is a positive constant independent of u.

On the other hand, making the change of variables $\rho = \frac{t}{|u|}$ shows

$$\begin{aligned}
\int_{\rho=\frac{1}{|u|}}^{\rho=2\pi} p_1(u,\rho)A(u,\rho)\rho^{m-1}\,d\rho &= |u|^{-m} \int_{t=1}^{t=2\pi|u|} p_1\left(u, \frac{t}{|u|}\right) A\left(u, \frac{t}{|u|}\right) t^{m-1}\,dt \\
&\leq C\,|u|^{-m}\, p_1\left(u, \frac{1}{|u|}\right) A\left(u, \frac{1}{|u|}\right)
\end{aligned}$$

by taking $|\eta| = \frac{1}{|u|}$ in Lemma 5.3.6. Now $p_1\left(u, \frac{1}{|u|}\right)$ is the heat kernel evaluated at a point on the unit sphere of G, so this is bounded by a constant independent of u. Thus by Corollary 5.3.4 we have

$$\begin{aligned}
\int_{\rho=\frac{1}{|u|}}^{\rho=2\pi} p_1(u,\rho)A(u,\rho)\rho^{m-1}\,d\rho &\leq C\,|u|^{-m}\,|u|^{2m} \left(\frac{1}{|u|}\right)^{2(m+n)} \left(2\pi - \frac{1}{|u|}\right)^{2n-1} \\
&\leq C\left(2\pi - \frac{1}{|u|}\right)^{2n-1} |u|^{-2n-m}\,.
\end{aligned}$$

Combining this with the estimate on $D(u)$ proves the claim. □

To estimate I_1 (see (5.3.32)), we observe that

$$\begin{aligned}
\left| f\left(u, \frac{1}{|u|}\hat{\eta}\right) - f\left(u, \frac{s}{|u|}\hat{\eta}\right) \right| &= \left| \int_{t=s}^{t=1} \frac{d}{dt} f\left(u, \frac{t}{|u|}\hat{\eta}\right) dt \right| \\
&\leq \int_{t=s}^{t=1} \left| \frac{d}{dt} f\left(u, \frac{t}{|u|}\hat{\eta}\right) \right| dt \\
&\leq \int_{t=s}^{t=1} \left| \nabla f\left(u, \frac{t}{|u|}\hat{\eta}\right) \right| dt
\end{aligned}$$

by (5.3.7). Thus

$$I_1 \leq \int_{|u|\geq\frac{1}{2\pi}} \int_{\hat{\eta}\in S^{m-1}} \int_{s=0}^{s=1} \int_{t=s}^{t=1} \left|\nabla f\left(u, \frac{t}{|u|}\hat{\eta}\right)\right| \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) dt\, ds\, R(u)\, d\hat{\eta}\, du \tag{5.3.35}$$

$$= \int_{|u|\geq\frac{1}{2\pi}} \int_{\hat{\eta}\in S^{m-1}} \int_{t=0}^{t=1} \left|\nabla f\left(u, \frac{t}{|u|}\hat{\eta}\right)\right| \frac{1}{|u|^m} \left(R(u) \int_{s=0}^{s=t} s^{m-1} A\left(u, \frac{s}{|u|}\right) ds\right) dt\, d\hat{\eta}\, du. \tag{5.3.36}$$

Now by Claim 5.3.12 and Corollary 5.3.4, we have for all $t \in [0, 1]$:

$$\begin{aligned}
R(u) \int_{s=0}^{s=t} s^{m-1} A\left(u, \frac{s}{|u|}\right) ds &\leq C\left(2\pi - \frac{1}{|u|}\right)^{2n-1} \\
&\quad \times \int_{s=0}^{s=t} s^{m-1} |u|^{2m} \left(\frac{s}{|u|}\right)^{2(m+n)} \left(2\pi - \frac{s}{|u|}\right)^{2n-1} ds \\
&\leq C\left(2\pi - \frac{t}{|u|}\right)^{2n-1} (2\pi)^{2n-1} |u|^{-2n} \int_{s=0}^{s=t} s^{3m+2n-1}\, ds \\
&= C'\left(2\pi - \frac{t}{|u|}\right)^{2n-1} |u|^{-2n}\, t^{3m+2n} \\
&= C'\left(2\pi - \frac{t}{|u|}\right)^{2n-1} |u|^{2m} \left(\frac{t}{|u|}\right)^{2(m+n)} t^m \\
&\leq C'' A\left(u, \frac{t}{|u|}\right) t^m \\
&\leq C'' A\left(u, \frac{t}{|u|}\right) t^{m-1}.
\end{aligned}$$

Thus

$$I_1 \leq C \int_{|u|\geq\frac{1}{2\pi}} \int_{\hat{\eta}\in S^{m-1}} \int_{t=0}^{t=1} \left|\nabla f\left(u, \frac{t}{|u|}\hat{\eta}\right)\right| A\left(u, \frac{t}{|u|}\right) \frac{t^{m-1}}{|u|^m}\, dt\, d\hat{\eta}\, du \tag{5.3.37}$$

Make the change of variables $r = \frac{t}{|u|}$:

$$= C \int_{|u|\geq\frac{1}{2\pi}} \int_{\hat{\eta}\in S^{m-1}} \int_{r=0}^{r=\frac{1}{|u|}} |\nabla f(u, r\hat{\eta})|\, A(u, r)\, r^{m-1}\, dr\, d\hat{\eta}\, du \tag{5.3.38}$$

$$\leq C \int_{u\in\mathbb{R}^{2n}} \int_{\hat{\eta}\in S^{m-1}} \int_{r=0}^{r=\frac{1}{|u|}} |\nabla f(u, r\hat{\eta})|\, A(u, r)\, r^{m-1}\, dr\, d\hat{\eta}\, du \tag{5.3.39}$$

$$= C \int_B |\nabla f|\, dm \tag{5.3.40}$$

$$\leq \frac{C}{\inf_B p_1} \int_B |\nabla f|\, p_1\, dm \tag{5.3.41}$$

$$\leq C' \int_G |\nabla f|\, p_1\, dm. \tag{5.3.42}$$

where we have used the fact that p_1 is bounded away from 0 on B.

For I_2 (see (5.3.33)), we have by Claim 5.3.12 that

$$I_2 \le C \int_{|u|\ge \frac{1}{2\pi}} \int_{\hat\eta\in S^{m-1}} \int_{s=0}^{s=1} \left| f\left(u, \frac{s}{|u|}\hat\eta\right) - m_f \right| \frac{s^{m-1}}{|u|^m} A\left(u, \frac{s}{|u|}\right) ds\, d\hat\eta\, du. \tag{5.3.43}$$

Make the change of variables $r = \frac{s}{|u|}$:

$$= C \int_{|u|\ge \frac{1}{2\pi}} \int_{\hat\eta\in S^{m-1}} \int_{r=0}^{r=\frac{1}{|u|}} \left| f(u, r\hat\eta) - m_f \right| r^{m-1} A(u,r)\ dr\, d\hat\eta\, du \tag{5.3.44}$$

$$\le C \int_{u\in\mathbb{R}^{2n}} \int_{\hat\eta\in S^{m-1}} \int_{r=0}^{r=\frac{1}{|u|}} \left| f(u, r\hat\eta) - m_f \right| r^{m-1} A(u,r)\ dr\, d\hat\eta du \tag{5.3.45}$$

$$= C \int_B \left| f - m_f \right|\ dm \tag{5.3.46}$$

$$\le C \int_B |\nabla f|\ dm \tag{5.3.47}$$

by Theorem 5.3.9. The inequalities (5.3.40–5.3.42) now show that $I_2 \le C' \int_G |\nabla f|\, p_1\, dm$, as desired. □

Corollary 5.3.13. *There exists a constant C such that for all $f \in C$,*

$$\int_G \left| f - m_f \right| p_1\, dm \le C \int_G |\nabla f|\, p_1\, dm. \tag{5.3.48}$$

Proof. Add (5.3.21) and (5.3.22). □

We can now prove some cases of the desired gradient inequality (5.3.4).

Notation 5.3.14. Let $D(R) = \{(x,z) : |x| \le R\}$ denote the "cylinder about the z axis" of radius R.

Lemma 5.3.15. *For fixed $R > 0$, (5.3.4) holds, with a constant $C = C(R)$ depending on R, for all $f \in C$ which are supported on $D(R)$ and satisfy $m_f = 0$.*

Proof.

$$\begin{aligned}
\left| \int_G ((\nabla - \hat\nabla) f) p_1 \right| dm &= \left| \int_G f(\nabla - \hat\nabla) p_1 \right| dm && \text{by integration by parts (5.3.5)} \\
&\le \int_G |f| \left| (\nabla - \hat\nabla) p_1 \right| dm \\
&= \int_G |f|\, |x|\, |\nabla_z p_1|\ dm && \text{by (2.3.7)} \\
&\le CR \int_G |f|\, p_1\, dm && \text{by (4.1.9).}
\end{aligned}$$

(Note that $|x| \le R$ on the support of f.)

$$\le C'R \int_G |\nabla f|\, p_1 \, dm \qquad \text{by Corollary 5.3.13.}$$

□

Notation 5.3.16. If $T : G \to M_{2n\times 2n}$ is a matrix-valued function on G, with $k\ell$th entry $a_{k\ell}$, let $\nabla \cdot T : G \to \mathbb{R}^{2n}$ be defined as

$$\nabla \cdot T(g) := \sum_{k,\ell=1}^{2n} X_\ell a_{k\ell}(g) e_k. \tag{5.3.49}$$

Note that for $f : G \to \mathbb{R}$ we have the product formula

$$\nabla \cdot (fT) = T\nabla f + f\nabla \cdot T. \tag{5.3.50}$$

Lemma 5.3.17. *For fixed $R > 1$, (5.3.4) holds, with a constant $C = C(R)$ depending on R, for all $f \in C$ which are supported on the complement of $D(R)$.*

Proof. Applying (2.3.5) we have

$$\nabla p_1(x,z) = \nabla_x p_1(x,z) + \frac{1}{2} J_{\nabla_z p_1(x,z)} x.$$

Now p_1 is a "radial" function (that is, $p_1(x,z)$ depends only on $|x|$ and $|z|$). Thus we have that $\nabla_x p_1(x,z)$ is a scalar multiple of x, and also that $\nabla_z p_1(x,z)$ is a scalar multiple of z, so that $J_{\nabla_z p_1(x,z)} x$ is a scalar multiple of $J_z x$.

For nonzero $x \in \mathbb{R}^{2n}$, let $T(x) \in M_{2n\times 2n}$ be orthogonal projection onto the m-dimensional subspace of $\mathbb{R}^{2n}$ spanned by the orthogonal vectors $J_{u_1}x, \ldots, J_{u_m}x$. (Recall $\left\langle J_{u_i}x, J_{u_j}x \right\rangle = -\left\langle u_i, u_j \right\rangle \|x\|^2 = -\delta_{ij}\|x\|^2$.) Thus for any $z \in \mathbb{R}^m$, $T(x)J_z x = J_z x$, and $T(x)x = 0$; in particular,

$$T(x)\nabla p_1(x,z) = \frac{1}{2} J_{\nabla_z p_1(x,z)} x = \frac{1}{2}(\nabla - \hat{\nabla}) p_1(x,z). \tag{5.3.51}$$

Explicitly, we have

$$T(x) = \frac{1}{|x|^2} \sum_{j=1}^{m} J_{u_j} x (J_{u_j} x)^T.$$

Note that $|T(x)| = 1$ (in operator norm) for all $x \neq 0$, and a routine computation verifies that $|\nabla \cdot T(x)| = |\nabla_x \cdot T(x)| \leq \frac{C}{|x|}$. Indeed, the $k\ell$th entry of $T(x)$ is

$$a_{k\ell}(x) = \frac{1}{|x|^2} \sum_{j=1}^{m} \left\langle J_{u_j} x, e_k \right\rangle \left\langle J_{u_j} x, e_\ell \right\rangle$$

so that $|X_k a_{k\ell}(x)| = \left|\frac{\partial}{\partial x^k} a_{k\ell}(x)\right| \leq \frac{3m}{|x|}$; thus $|\nabla \cdot T(x)| \leq \frac{3m(2n)^2}{|x|}$.

Since p_1 decays rapidly at infinity, we have the integration by parts formula

$$0 = \int_G \nabla \cdot (f p_1 T)\, dm = \int_G (f p_1 \nabla \cdot T + f T \nabla p_1 + p_1 T \nabla f)\, dm. \tag{5.3.52}$$

Thus

$$\begin{aligned}
\left| \int_G ((\nabla - \hat{\nabla}) f) p_1\, dm \right| &= \left| \int_G f (\nabla - \hat{\nabla}) p_1\, dm \right| \\
&= 2 \left| \int_G f T \nabla p_1\, dm \right| \\
&= 2 \left| \int_G f p_1 (\nabla \cdot T + T \nabla f)\, dm \right| \\
&\leq 2 \int_G |f|\, |\nabla \cdot T|\, p_1\, dm + 2 \int_G |T|\, |\nabla f|\, p_1\, dm \\
&\leq \frac{2C}{R} \int_G |f|\, p_1\, dm + 2 \int_G |\nabla f|\, p_1\, dm
\end{aligned}$$

since on the support of f, we have $|\nabla \cdot T| \leq \frac{C}{|x|} \leq \frac{C}{R}$, and $|T| = 1$. The second integral is the desired right side of (5.3.4). The first integral is bounded by the same by Corollary 5.3.13, where we note that $m_f = 0$ because f vanishes on $D(R) \supset B$. □

We can now complete the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. We prove (5.3.4) for general $f \in \mathcal{C}$. By replacing f by $f - m_f \in \mathcal{C}$, we can assume $m_f = 0$.

Let $\psi \in C^\infty(G)$ be a smooth function such that $\psi \equiv 1$ on $D(1)$ and ψ is supported in $D(2)$. Then $f = \psi f + (1 - \psi) f$.

ψf is supported on $D(2)$, so Lemma 5.3.15 applies to ψf. (Note that $m_{\psi f} = 0$

since $\psi \equiv 1$ on $D(1) \supset B$.) We have

$$\begin{aligned}\left|\int_G (\nabla - \hat{\nabla})(\psi f) p_1 \, dm\right| &\leq C \int_G |\nabla(\psi f)| \, p_1 \, dm \\ &\leq C \int_G |\nabla\psi| \, |f| \, p_1 \, dm + \int_G |\psi| \, |\nabla f| \, p_1 \, dm \\ &\leq C \sup_G |\nabla\psi| \int_G |f| \, p_1 \, dm + C \sup_G |\psi| \int_G |\nabla f| \, p_1 \, dm.\end{aligned}$$

The second integral is the right side of (5.3.4), and the first is bounded by the same by Corollary 5.3.13.

Precisely the same argument applies to $(1 - \psi)f$, which is supported on the complement of $D(1)$, by using Lemma 5.3.17 instead of Lemma 5.3.15. □

5.4 The optimal constant K

We observed previously that the constant K in (5.1.1) can be taken to be independent of t. We now show that the *optimal* constant is also independent of $t > 0$, and is discontinuous at $t = 0$. This distinguishes the current situation from the elliptic case, in which the constant is continuous at $t = 0$; see, for instance [4, Proposition 2.3]. This fact was initially noted for the Heisenberg group in [13], and the proof here is similar to the one found there.

Proposition 5.4.1. *For $t \geq 0$, let*

$$K_{\text{opt}}(t) := \sup\left\{\frac{|(\nabla P_t f)(g)|}{P_t(|\nabla f|)(g)} : f \in C, g \in G, P_t(|\nabla f|)(g) \neq 0\right\} \tag{5.4.1}$$

Then $K_{\text{opt}}(0) = 1$, and for all $t > 0$, $K_{\text{opt}}(t) \equiv K_{\text{opt}} > 1$ is independent of t, so that $K_{\text{opt}}(t)$ is discontinuous at $t = 0$. In particular, $K_{\text{opt}} \geq \sqrt{\frac{3n+5}{3n+1}}$.

Proof. It is obvious that $K_{\text{opt}}(0) = 1$.

As before, by the left invariance of P_t and ∇, it suffices to take $g = 0$ on the right side of (5.4.1). To show independence of $t > 0$, fix $t, s > 0$. If $f \in C$, then

$\tilde{f} := f \circ \varphi_{s^{1/2}}^{-1} \in C$ and $f = \tilde{f} \circ \varphi_{s^{1/2}}$. Then

$$\begin{aligned}
\frac{|(\nabla P_t f)(0)|}{P_t(|\nabla f|)(0)} &= \frac{\left|(\nabla P_t(\tilde{f} \circ \varphi_{s^{1/2}}))(0)\right|}{P_t\left(\left|\nabla(\tilde{f} \circ \varphi_{s^{1/2}})\right|\right)(0)} \\
&= \frac{\left|(\nabla(P_{st}\tilde{f}) \circ \varphi_{s^{1/2}})(0)\right|}{P_t\left(s^{1/2}\left|\nabla \tilde{f}\right| \circ \varphi_{s^{1/2}}\right)(0)} \\
&= \frac{s^{1/2}\left|(\nabla P_{st}\tilde{f})(\varphi_{s^{1/2}}(0))\right|}{s^{1/2}P_{st}\left(\left|\nabla \tilde{f}\right|\right)(\varphi_{s^{1/2}}(0))} \le K_{\mathrm{opt}}(st).
\end{aligned}$$

Taking the supremum over f shows that $K_{\mathrm{opt}}(t) \le K_{\mathrm{opt}}(st)$. s was arbitrary, so $K_{\mathrm{opt}}(t)$ is constant for $t > 0$.

In order to bound the constant, we explicitly compute a related ratio for a particular choice of function f. The function used is an obvious generalization of the example used in [13] for the Heisenberg group.

Fix a unit vector u_1 in the center of G, i.e. $u_1 \in 0 \times \mathbb{R}^m \subset \mathbb{R}^{2n+m}$. We note that the operator L and the norm of the gradient $|\nabla f|^2 = \frac{1}{2}(L(f^2) - 2fLf)$ are independent of the orthonormal basis $\{e_i\}$ chosen to define the vector fields $\{X_i\}$, so without loss of generality we suppose that $J_{u_1}e_1 = e_2$. Then take

$$\begin{aligned}
f(x,z) &:= \langle x, e_1\rangle + \langle z, u_1\rangle \langle x, e_2\rangle = x_1 + z_1 x_2 \\
k(t) &:= \frac{|(\nabla P_t f)(0)|}{P_t(|\nabla f|)(0)}.
\end{aligned}$$

Note that $k(t) \le K_{\mathrm{opt}}$ for all t. By the Cauchy-Schwarz inequality,

$$k(t)^2 \ge k_2(t) := \frac{|(\nabla P_t f)(0)|^2}{P_t\left(|\nabla f|^2\right)(0)}.$$

Since f is a polynomial, we can compute $P_t f$ by the formula $P_t f = f + \frac{t}{1!}Lf + \frac{t^2}{2!}L^2 f + \cdots$ since the sum terminates after a finite number of terms (specifically, two). The same is true of $|\nabla f|^2$, which is also a polynomial (three terms are needed). The formulas (2.3.2) are helpful in carrying out this tedious but straightforward computation. We find

$$k_2(t) = \frac{(1+t)^2}{1 - 2t + (3n+2)t^2}$$

which, by differentiation, is maximized at $t_{\max} = \frac{2}{3n+3}$, with $k_2(t_{\max}) = \frac{3n+5}{3n+1}$. Since $K_{\mathrm{opt}} \ge k(t_{\max}) \ge \sqrt{k_2(t_{\max})} = \sqrt{\frac{3n+5}{3n+1}}$, this is the desired bound. □

5.5 Consequences

Section 6 of [5] gives several important consequences of the gradient inequality (5.1.1). The proofs given there are generic (see their Remark 6.6); with Theorem 5.1.2 in hand, they go through without change in the case of H-type groups. These consequences include:

- Local Gross-Poincaré inequalities, or φ-Sobolev inequalities;
- Cheeger type inequalities; and
- Bobkov type isoperimetric inequalities.

We refer the reader to [5] for the statements and proofs of these theorems, and many references as well.

Chapter 5, in large part, is adapted from material awaiting publication as Eldredge, Nathaniel, "Gradient Estimates for the Subelliptic Heat Kernel on H-type Groups," submitted, *Journal of Functional Analysis*, 2009. The dissertation author was the sole author of this paper.

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