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The Taylor Map on Complex Path Groups

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by

Matthew Steven Cecil

Committee in charge:

Professor Bruce Driver, Chair
Professor Peter Ebenfelt
Professor George Fuller
Professor Ken Intrilligator
Professor Ruth Williams

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The dissertation of Matthew Steven Cecil is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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To Helen Atterberry,
who knew the value of education
and passed it on to her entire family.

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VITA

1978	Born, Indianapolis, Indiana.
2001	B. S. Mathematics, Indiana University. B. A. Physics, Indiana University. B. A. French, Indiana University.
2003	M. A. Mathematics, University of California, San Diego.
2006	Ph. D. Mathematics, University of California, San Diego.

ABSTRACT OF THE DISSERTATION

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by

Matthew Steven Cecil
Doctor of Philosophy in Mathematics

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Professor Bruce Driver, Chair

The heat kernel measure ν_t is constructed on $\mathcal{W}(G)$, the group of paths based at the identity on a simply connected complex Lie group G. An isometric map, the Taylor map, is established from the space of $L^2(\nu_t)-$holomorphic functions on $\mathcal{W}(G)$ to a subspace of the dual of the universal enveloping algebra of Lie$(H(G))$, where $H(G)$ is the Lie subgroup of finite energy paths. Surjectivity of this Taylor map can be shown in the case where G is stratified nilpotent.

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Introduction

1.1 Background

A holomorphic function $u : \mathbb{C} \to \mathbb{C}$ is determined by its derivatives at the origin. One can recover values of u by its everywhere convergent Taylor expansion

$$u(z) = \sum_{k=0}^{\infty} \frac{u^{(k)}(0)z^k}{k!}. \tag{1.1}$$

Let μ_t denote the Gaussian $\mu_t(z) = \frac{1}{\pi t} e^{\frac{-|z|^2}{t}}$. The following equation is easy to verify by switching to polar coordinates.

$$\int_{\mathbb{C}} z^k \overline{z}^l \mu_t(z) dx dy = \delta_{kl} t^k k!. \tag{1.2}$$

Our goal is to use this orthogonality of powers of z along with our Taylor expansion of u to relate the $L^2(\mu_t)$ norm of u to its derivatives at the origin.

Consider

$$u_n(z) = \sum_{k=0}^{n} \frac{u^{(k)}(0)z^k}{k!}. \tag{1.3}$$

Then Eq. (1.1) indicates that $u_n \to u$ pointwise and therefore uniformly on any compact set. Furthermore, for any $f \in L^2(\mu_t)$ and $R > 0$,

$$||f||^2_{L^2(1_{|z|\leq R}\mu_t)} \leq ||f||^2_{L^2(\mu_t)},$$

and $||f||^2_{L^2(1_{|z|\le R}\mu_t)}$ is increasing as a function of R. So by the MCT,

$$\lim_{R\to\infty} ||f||^2_{L^2(1_{|z|\le R}\mu_t)} = ||f||^2_{L^2(\mu_t)}.$$

Combining these results with Eq. (1.2) and Eq. (1.3) yields

$$\begin{aligned}
||u||^2_{L^2(\mu_t)} &= \overline{\lim_{R\to\infty}} ||u||^2_{L^2(1_{|z|\le R}\mu_t)} \\
&= \overline{\lim_{R\to\infty}} \left(\overline{\lim_{n\to\infty}} ||u_n||^2_{L^2(1_{|z|\le R}\mu_t)} \right) \\
&= \overline{\lim_{n\to\infty} \lim_{R\to\infty}} ||u_n||^2_{L^2(1_{|z|\le R}\mu_t)} \\
&= \overline{\lim_{n\to\infty}} ||u_n||^2_{L^2(\mu_t)} \\
&= \overline{\lim_{n\to\infty}} \int_{\mathbb{C}} |u_n(z)|^2 \mu_t(z) dxdy \\
&= \overline{\lim_{n\to\infty}} \int_{\mathbb{C}} \sum_{k,l=0}^{n} \left(\frac{u^{(k)}(0) z^k}{k!} \frac{\overline{u^{(l)}(0) z^l}}{l!} \right) \mu_t(z) dxdy \\
&= \overline{\lim_{n\to\infty}} \sum_{k,l=0}^{n} \frac{u^{(k)}(0)\overline{u^{(l)}(0)}}{k!l!} \int_{\mathbb{C}} z^k \overline{z}^l \mu_t(z) dxdy \\
&= \overline{\lim_{n\to\infty}} \sum_{k=0}^{n} \frac{t^k}{k!} |u^{(k)}(0)|^2 \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} |u^{(k)}(0)|^2.
\end{aligned}$$

More generally, if $u : \mathbb{C}^d \to \mathbb{C}$ is holomorphic and $\mu_t(z) = \left(\frac{1}{\pi t}\right)^d e^{\frac{-|z|^2}{t}}$, then

$$||u||^2_{L^2(\mu_t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1,\ldots,i_k=1}^{d} |(\partial_{e_{i_1}} \partial_{e_{i_2}} \cdots \partial_{e_{i_k}} u)(0)|^2, \tag{1.4}$$

where $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{C}^d$. The proof of Eq. (1.4) is exactly analagous to the above one dimensional case.

Let $T(\mathbb{C}^d)$ denote the tensor algebra over $\mathbb{C}^d$, that is $T(\mathbb{C}^d) \equiv \oplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k}$. To every holomorphic $u : \mathbb{C}^d \to \mathbb{C}$ we can associate an element $\alpha_u = \oplus_{k=0}^{\infty} \alpha_k \in T(\mathbb{C}^d)$, where $\alpha_k \in (\mathbb{C}^d)^{\otimes k}$ is the symmetric tensor defined by

$$(\alpha_k, z_1 \otimes z_2 \otimes \cdots \otimes z_k)_{(\mathbb{C}^d)^{\otimes k}} = (\partial_{z_1} \partial_{z_2} \cdots \partial_{z_k} u)(0)$$

for every $z_1, z_2, \cdots, z_k \in \mathbb{C}^d$. Here $(\ ,\ )_{(\mathbb{C}^d)^{\otimes k}}$ denotes the inner product on $(\mathbb{C}^d)^{\otimes k}$ arising from the standard one on $\mathbb{C}^d$. If we define a norm $||\cdot||_t$ on $T(\mathbb{C}^d)$ by

$$||\beta||_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} ||\beta_k||^2_{(\mathbb{C}^d)^{\otimes k}}$$

for $\beta = \oplus_{k=0}^{\infty} \beta_k$ with $\beta_k \in \left(\mathbb{C}^d\right)^{\otimes k}$, then Eq. (1.4) indicates that the map $u \to \alpha_u$ is unitary.

The physicist V. A. Fock introduced this isomorphism in 1932 in [7], and the work was later clarified by Segal and Bargmann in the '50's and '60's (see [1, 21, 22]). The correspondence proves useful in understanding the structure of quantum fields. In the above classical case, if one considers $(\mathbb{C}^d)^{\otimes k}$ as the k-particle state space, then the map $u \in \mathcal{H}L^2(\mu_t) \to \alpha_u \in T(\mathbb{C}^d)$ exhibits the wave-particle duality of a bosonic system. It is also closely related to the characterization theorem for generalized function in white noise analysis (see, for example, [12, 16, 15]).

In [4], Driver and Gross proved a generalization of the above result on a complex connected Lie group G with given Hermitian inner product $(\ ,\ )$ on the Lie algebra $\mathfrak{g} \equiv T_e G$. In this context, μ_t denotes heat kernel measure on G with respect to a right invariant Haar measure dx. Let $T(\mathfrak{g})$ denote the tensor algebra over $\mathfrak{g}$, and for each $t > 0$, define a norm $||\cdot||_t$ on $T(\mathfrak{g})$ by

$$||\beta||_t^2 := \sum_{k=0}^{\infty} \frac{k!}{t^k} ||\beta_k||^2_{\mathfrak{g}^{\otimes k}} \tag{1.5}$$

for $\beta = \oplus_{k=0}^{\infty} \beta_k$ with $\beta_k \in \mathfrak{g}^{\otimes k}$, where $||\cdot||_{(\mathfrak{g})^{\otimes k}}$ denotes the cross norm on $\mathfrak{g}^{\otimes k}$ arising from the inner product on $\mathfrak{g}^{\otimes k}$ determined by the given inner product on $\mathfrak{g}$. If we let $T(\mathfrak{g})_t$ denote the completion of $T(\mathfrak{g})$ with respect to this norm, then $T(\mathfrak{g})_t$ is a complex Hilbert space with respect to the Hermitian inner product given by polarizing the norm in Eq. (1.5) above.

Let $T(\mathfrak{g})'$ denote the algebraic dual of $T(\mathfrak{g})$. Then we can identify the topological dual space of $T(\mathfrak{g})_t$ with the subspace of $T(\mathfrak{g})'$ consisting of those $\alpha \in T(\mathfrak{g})'$ such that

$$||\alpha||_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} ||\alpha_k||^2_{(\mathfrak{g}^*)^{\otimes k}} < \infty,$$

where $\alpha = \oplus_{k=0}^{\infty}\alpha_k$ with $\alpha_k \in (\mathfrak{g}^*)^{\otimes k}$, where $||\cdot||_{(\mathfrak{g}^*)^{\otimes k}}$ denotes the cross norm on $(\mathfrak{g}^*)^{\otimes k}$ arising from the inner product on $\mathfrak{g}^*$ dual to the given inner product on $\mathfrak{g}$. Denote this space $T(\mathfrak{g})_t^*$.

Let J denote the ideal in $T(\mathfrak{g})$ generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] : \xi, \eta \in \mathfrak{g}\}$, and $J_t^0 = \{\alpha \in T(\mathfrak{g})_t^* : \langle \alpha, v\rangle = 0 \text{ for all } v \in J\}$. To any holomorphic function u on G, we can associate an element α_u of J_t^0 given by

$$\langle \alpha_u, \xi_1 \otimes \cdots \otimes \xi_k\rangle = (\tilde{\xi}_1 \cdots \tilde{\xi}_k u)(e).$$

Then the main theorm of [4] states that if G is simply connected, then the map $u \in \mathcal{H}L^2(G, \mu_t(x)dx) \to \alpha_u \in J_t^0$ is unitary.

Infinite dimensional analogues have been proven by Gordina in [9] and [10] on $GL(H)$, the group of invertible operators on a complex Hilbert space H, and groups associated with a $II_1 - factor$. The goal of this work is to establish yet another infininte dimensional Taylor map, this one on $\mathcal{W}(G)$, the groups of paths based at the identity on a simply connected complex Lie group G.

1.2 Statement of Results

Let G be an arbitrary complex simply connected Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. Assume there is a given Hermitian inner product $(\ ,\ )_{\mathfrak{g}}$ on $\mathfrak{g}$. Let $\langle\ ,\ \rangle$ denote the real left invariant Riemannian metric on G determined by

$$\langle \tilde{A}, \tilde{B}\rangle = \operatorname{Re}(A, B)_{\mathfrak{g}} \qquad \forall\ A, B \in \mathfrak{g}$$

where $\tilde{A}$ denotes the unique left invariant vector field satisfying $\tilde{A}(e) = A \in \mathfrak{g}$. We will use $\langle\ ,\ \rangle_{\mathfrak{g}}$ to denote this inner product on $\mathfrak{g}$.

Choose $\mathfrak{X}_{\mathbb{C}}$ to be an orthonormal basis for the complex inner product space $(\mathfrak{g}, (\ ,\ )_{\mathfrak{g}})$. If we denote the complex structure on $\mathfrak{g}$ by $\mathcal{J}$, then $\mathfrak{X}_{\mathbb{R}} = \{\mathfrak{X}_{\mathbb{C}}, \mathcal{J}\mathfrak{X}_{\mathbb{C}}\}$ is an orthonormal basis of the real inner product space $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}})$. Define the Laplacian on G by

$$\Delta_G = \sum_{A\in\mathfrak{X}_{\mathbb{C}}} \tilde{A}^2 + \widetilde{\mathcal{J}A}^2 = \sum_{A\in\mathfrak{X}_{\mathbb{R}}} \tilde{A}^2. \tag{1.6}$$

Then Δ_G is a stongly elliptic operator and in the case where G is unimodular, it is the Laplace-Beltrami operator (see Remark 2.2 in [4]). Let $\mathcal{H}(G)$ denote the space of complex valued holomorphic functions on G. Let dx denote a fixed right invariant Haar measure.

Define $\mathcal{W}(G)$ to be the based path group on G, i.e. the continuous paths $\sigma : [0,1] \to G$ such that $\sigma(0) = e$. Similarly, we'll let $\mathcal{W}(\mathfrak{g})$ denote the continuous paths $h : [0,1] \to \mathfrak{g}$ such that $h(0) = 0$. Define the energy of a path $\sigma \in \mathcal{W}(G)$ by

$$E(\sigma) := \begin{cases} \int_0^1 |L_{\sigma(s)^{-1}*}\sigma'(s)|_{\mathfrak{g}}^2 ds, & \text{if } \sigma \text{ is absolutely continuous} \\ \infty, & \text{otherwise} \end{cases}$$

The *finite energy subgroup of* $\mathcal{W}(G)$ is then given by

$$H(G) = \{\sigma \in \mathcal{W}(G) |\ E(\sigma) < \infty\}.$$

Similarly, for a $h \in \mathcal{W}(\mathfrak{g})$, let

$$(h,h)_{H(\mathfrak{g})} := \begin{cases} \int_0^1 |h'(s)|_{\mathfrak{g}}^2 ds, & \text{if } h \text{ is absolutely continuous} \\ \infty, & \text{otherwise} \end{cases}$$

We define the *Cameron-Martin subspace of* $\mathcal{W}(\mathfrak{g})$ as

$$H(\mathfrak{g}) = \{h \in \mathcal{W}(\mathfrak{g}) |\ (h,h)_{H(\mathfrak{g})} < \infty\}.$$

Given $h, k \in H(\mathfrak{g})$, we can define a Hermitian inner product on $H(\mathfrak{g})$ by

$$(h,k)_{H(\mathfrak{g})} = \int_0^1 (h'(s), k'(s))_{\mathfrak{g}} ds.$$

With this inner product, $H(\mathfrak{g})$ is a Hilbert space. As above, we let $\langle h,k\rangle_{H(\mathfrak{g})} = \mathrm{Re}(h,k)_{H(\mathfrak{g})}$. It is often convenient to think of $H(\mathfrak{g})$ as the "Lie algebra" of $\mathcal{W}(G)$.

Let $S_{\mathbb{C}} \subset H(\mathfrak{g})$ be an orthonormal basis for the complex inner product space $(H(\mathfrak{g}), (\ ,\ )_{H(\mathfrak{g})})$. The complex structure $\mathcal{J}$ on $H(\mathfrak{g})$ is that on $\mathfrak{g}$ defined pointwise. That is, for $h \in H(\mathfrak{g})$, $\mathcal{J}h \in H(\mathfrak{g})$ is given by $(\mathcal{J}h)(t) = \mathcal{J}(h(t))$ for all $t \in [0,1]$. Then $S_{\mathbb{R}} = \{S_{\mathbb{C}}, \mathcal{J}S_{\mathbb{C}}\}$ is an orthonormal basis for the real inner product space $(H(\mathfrak{g}), \langle\ ,\ \rangle_{H(\mathfrak{g})})$.

Our goal is to extend the results of [4, 9, 10, 22, 21, 1] to holomorphic functions on $\mathcal{W}(G)$. In order to do so, we will need a notion of heat kernel measure on $\mathcal{W}(G)$.

We construct a $\mathcal{W}(G)$-valued Brownian motion, and define ν_t, our heat kernel measure, to be the endpoint distribution of this process. Specifically, let $\{\beta(t,s)\}_{0\leq s\leq 1, 0\leq t<\infty}$ be a $\mathfrak{g}$-valued Brownian sheet with half the usual covariance defined on some probability space $(\Omega, \mathcal{F}, P)$ (For more details, see section 3.2). The following theorem is the main result of chapter 3.

Theorem 1.1 (Theorem 3.8)**.** *Suppose G is a Lie group with left invariant Riemannian metric $\langle\ ,\ \rangle$ and $g_0 \in \mathcal{W}(G)$. Then there exists a continuous adapted $\mathcal{W}(G)$-valued process $\{\Sigma(t)\}_{t\geq 0}$ on a filtered probability space $(\mathcal{W}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$ such that for each $s \in [0,1]$, $\Sigma(\cdot, s)$ solves the stochastic differntial equation:*

$$\Sigma(\delta t, s) = L_{\Sigma(t,s)*}\beta(\delta t, s) \qquad \text{with } \Sigma(0,s) = g_0(s).$$

More precisely,

$$\Sigma(\delta t, s) = \sum_{A\in\mathfrak{X}_{\mathbb{R}}} \tilde{A}(\Sigma(t,s))\beta^A(\delta t, s) \qquad \text{with } \Sigma(0,s) = g_0(s).$$

where $\beta^A(t,s) = \langle A, \beta(t,s)\rangle_{\mathfrak{g}}$. Here $\beta^A(\delta t, s)$ denotes the Stratonovich differential of the process $t \to \beta^A(t,s)$. We will use "δ" for the Stratonovich differential and "d" for the Itô differential of a semimartingale.

Definition 1.2. Let $\nu_t := Law(\Sigma(t,\cdot))$.

Given a partition of $[0,1]$, $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$, and $g \in \mathcal{W}(G)$, define $\pi_{\mathcal{P}} : \mathcal{W}(G) \to G^n$ by

$$\pi_{\mathcal{P}}(g) = (g(s_1), g(s_2), ..., g(s_n)).$$

Definition 1.3. A function f is a *holomorphic cylinder function* on $\mathcal{W}(G)$ if there exists a partition $\mathcal{P}$ and a holomorphic function $F : G^n \to \mathbb{C}$ such that $f = F \circ \pi_{\mathcal{P}}$.

Definition 1.4. Let $\mathcal{H}_t$ denote the $L^2(\nu_t)$-closure of the holomorphic cylinder functions on $\mathcal{W}(G)$.

$\mathcal{H}_t$ will serve as our Hilbert space of holomorphic functions. In order to state our version of the Taylor map, we must establish a suitable notion of "derivatives at the origin" for a function $f \in \mathcal{H}_t$. The following theorem is motivated by the results of Sugita and others ([23, 24]) in the setting of an abstract Wiener space and can be found in Chapter 4.

Notation 1.5. Let $\mathcal{H}(H(G))$ denote the functions on $H(G)$ which are holomorphic in the sense of Notation 4.5.

Theorem 1.6 (Theorem 4.7)**.** *There exists an injective linear map $R : \mathcal{H}_t \to \mathcal{H}(H(G))$ with the following properties:*

1. *For f a holomorphic cylinder function, $Rf = f|_{H(G)}$.*
2. *For $g \in H(G)$, $|(Rf)(g)|^2 \leq ||f||^2_{L^2(\nu_t)} e^{\frac{|g|^2_{H(G)}}{t}}$, where $|g|_{H(G)}$ denotes the Riemannian distance between g and the identity path in $H(G)$.*

Denote by $T(H(\mathfrak{g}))$ the tensor algebra over the complex vector space $H(\mathfrak{g})$. For each $t > 0$, define a norm on $T(H(\mathfrak{g}))$ by

$$||\beta||_t^2 = \sum_{k=0}^{\infty} \frac{k!}{t^k} |\beta_k|^2 \quad \text{where } \beta = \oplus_{k=0}^{\infty} \beta_k$$

with $\beta_k \in H(\mathfrak{g})^{\otimes k}$ for $k = 0, 1, 2, ...$, where $|\beta_k|$ denotes the cross norm on $H(\mathfrak{g})^{\otimes k}$ arising from the inner product on $H(\mathfrak{g})^{\otimes k}$ determined by the given inner product on $H(\mathfrak{g})$. We'll denote the completion of $T(H(\mathfrak{g}))$ with respect to this norm by $T(H(\mathfrak{g}))_t$. Then the Hermitian inner product on $T(H(\mathfrak{g}))_t$ given by polarizing the above turns $T(H(\mathfrak{g}))_t$ into complex Hilbert space.

The topological dual space of $T(H(\mathfrak{g}))_t$ may be identified with the subspace $T(H(\mathfrak{g}))_t^*$ of the algebraic dual $T(H(\mathfrak{g}))'$ of $T(H(\mathfrak{g}))$ consisting of those $\alpha \in T(H(\mathfrak{g}))'$ such that

$$||\alpha||_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2_{(H(\mathfrak{g})^*)^{\otimes k}} < \infty,$$

where $\alpha_k \in (H(\mathfrak{g})^*)^{\otimes k}$ and $|\alpha_k|_{(H(\mathfrak{g})^*)^{\otimes k}}$ denotes the cross norm on $(H(\mathfrak{g})^*)^{\otimes k}$ determined by the Hermitian inner product on $H(\mathfrak{g})^*$ dual to the given Hermitian inner product on $H(\mathfrak{g})$.

For $u \in \mathcal{H}(H(G))$, let $\alpha_u \in T(H(\mathfrak{g}))'$ be defined by

$$\langle \alpha_u, h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle = (\tilde{h}_1 \tilde{h}_2 \cdots \tilde{h}_n u)(\underline{e})$$

for $h_j \in H(\mathfrak{g})$ for $j = 1, ..., n$, where $\underline{e}$ represents the identity path in $\mathcal{W}(G)$ and

$$(\tilde{h}u)(g) \equiv \frac{d}{dt}|_{t=0} u(g \cdot e^{th})$$

for $g \in H(G)$ and $h \in H(\mathfrak{g})$. We will sometimes write $\alpha_u = (1-D)_{\underline{e}}^{-1} u$. Then by definition of the Lie bracket on $H(\mathfrak{g})$, α_u annihilates the two sided ideal

$$J(H(\mathfrak{g})) := \langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] | \xi, \eta \in H(\mathfrak{g}) \rangle.$$

Let $J^0(H(\mathfrak{g}))$ denote the annihilator of $J(H(\mathfrak{g}))$, that is

$$J^0(H(\mathfrak{g})) := \{\alpha \in T(H(\mathfrak{g}))' | \alpha|_{J(H(\mathfrak{g}))} \equiv 0\},$$

and let

$$J_t^0(H(\mathfrak{g})) := J^0(H(\mathfrak{g})) \cap T(H(\mathfrak{g}))_t^*.$$

We are now able to define the Taylor map on $\mathcal{H}_t$. Using the above notation, we send $f \in \mathcal{H}_t \to \alpha_{Rf} \in J_t^0(H(\mathfrak{g}))$. Then we are able to show the following in Chapter 4.

Theorem 1.7 (Corollary 4.13)**.** *For any complex Lie group G, the Taylor map, $(1 - D)_{\underline{e}}^{-1} R : \mathcal{H}_t \to J_t^0(H(\mathfrak{g}))$, is an isometry.*

In the previous cases [4, 9, 10, 22, 21, 1], the analagous Taylor map was also surjective. Chapter 5 is devoted to proving the the following special case.

Theorem 1.8 (Theorem 5.12)**.** *Suppose G is a stratified nilpotent Lie group. Then the Taylor map, $f \in \mathcal{H}_t \to \alpha_{Rf} \in J_t^0(H(\mathfrak{g}))$, is unitary.*

The appendix contains a section on reproducing kernels, a section containing example calculations, as well as a section devoted to stating and proving a theorem orignially found in [6] which is essential to proving surjectivity of the Taylor map when G is a stratified nilpotent Lie group.

2

Finite Dimensional Approximations

The primary purpose of this chapter is to summarize relations between the infinite group $\mathcal{W}(G)$ and finite products of G based on a partition of $[0,1]$. The relations will be used often throughout the sequel.

2.1 Approximations to $\mathcal{W}(G)$

For the entirety of this chapter, we'll let $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ denote a partition of $[0,1]$. We will also use the notation $\#(\mathcal{P}) = n$, the number of partition points of $\mathcal{P}$.

A partition $\mathcal{P}$ gives rise to a cannonical map on $\mathcal{W}(G)$, $\pi_{\mathcal{P}} : \mathcal{W}(G) \to G^{\#(\mathcal{P})}$ defined by

$$\pi_{\mathcal{P}}(g) = (g(s_1), g(s_2), ..., g(s_n)). \tag{2.1}$$

Notation 2.1. Let $\underline{e}$ denote the identity path. That is $\underline{e}(t) = e \in G$ for all $t \in [0,1]$.

Notice that for $h \in H(\mathfrak{g})$,

$$\pi_{\mathcal{P}*\underline{e}}(h) := \frac{d}{dt}|_{t=0}\pi_{\mathcal{P}}(e^{th}) \tag{2.2}$$

$$= \frac{d}{dt}|_{t=0}(e^{th(s_1)}, ..., e^{th(s_n)})$$

$$= (h(s_1), ..., h(s_n)). \tag{2.3}$$

Furthermore, for any $g \in H(G)$, $L_{g*}h \in T_g H(G)$, and we have the relationship

$$\pi_{\mathcal{P}*g}(L_{g*}h) = \frac{d}{dt}|_{t=0}\pi_{\mathcal{P}}(ge^{th})$$

$$= \frac{d}{dt}|_{t=0}(g(s_1)e^{th(s_1)}, ..., g(s_n)e^{th(s_n)})$$

$$= L_{\pi_{\mathcal{P}}(g)*}\left(\pi_{\mathcal{P}*\underline{e}}(h)\right). \tag{2.4}$$

We will revisit Eq. (2.3) and Eq. (2.4) in the next section.

Functions on $G^{\#(\mathcal{P})}$ determine a natural class of functions on $\mathcal{W}(G)$ via the map $\pi_{\mathcal{P}}$.

Definition 2.2. A function $f : \mathcal{W}(G) \to \mathbb{C}$ is a *smooth cylinder function* if there exists a partition $\mathcal{P} = \{0 = s_0 < s_1 < ... < s_n \leq 1\}$ of $[0, 1]$ and a $F \in C^\infty(G^{\#(\mathcal{P})})$ such that $f(g) = F(g(s_1), ..., g(s_n))$ for all $g \in \mathcal{W}(G)$. That is, $f = F \circ \pi_{\mathcal{P}}$. The collection of smooth cylinder functions is denoted $\mathcal{F}C^\infty(\mathcal{W})$.

Notation 2.3. We write $f \in \mathcal{F}C_c^\infty(\mathcal{W})$ if $f = F \circ \pi_{\mathcal{P}}$ for an $F \in C_c^\infty(G^{\#(\mathcal{P})})$.

Definition 2.4. A function $f \in \mathcal{F}C^\infty(\mathcal{W})$ is a *holomorphic cylinder function* if there exists an $F \in \mathcal{H}(G^{\#(\mathcal{P})})$ such that $f = F \circ \pi_{\mathcal{P}}$. The collection of holomorphic cylinder functions is denoted $\mathcal{H}\mathcal{F}C^\infty(\mathcal{W})$.

Expressions involving cylinder functions often reduce to related finite dimensional expressions. For example Remark 2.7 below indicates that differentiation of a cylinder function $f = F \circ \pi_{\mathcal{P}}$ is equivalent to a differentiation of F. In addition, the set of cylinder functions is closed under the operation of differentiation.

Definition 2.5. Given $h \in H(\mathfrak{g})$ and $f \in \mathcal{F}C^\infty(\mathcal{W})$, define

$$(\tilde{h}f)(g) := \frac{d}{dt}|_0 f(g \cdot e^{th}) \quad \forall g \in \mathcal{W}(G)$$

where $g \cdot e^{th} \in \mathcal{W}(G)$ is defined by $(g \cdot e^{th})(s) = g(s) \cdot e^{th(s)}$ for all $s \in [0, 1]$.

Notation 2.6. Suppose $f = F \circ \pi_{\mathcal{P}}$ where $F \in C^{\infty}(G^{|\mathcal{P}|})$. Then for $A \in \mathfrak{g}$ and $i \in \{1, 2, ..., n\}$ let

$$\tilde{A}^{(i)} F(x_1, x_2, ..., x_n) := \frac{d}{dt}|_0 F(x_1, ..., x_i \cdot e^{tA}, x_{i+1}, ..., x_n). \tag{2.5}$$

Remark 2.7. Notice that for $h \in H(\mathfrak{g})$,

$$\tilde{h} f = \sum_{i=1}^{n} (\widetilde{h(s_i)}^{(i)} F) \circ \pi_{\mathcal{P}}. \tag{2.6}$$

In particular, note that $\tilde{h} f$ is still a smooth cylinder function based on the same partition $\mathcal{P}$.

2.2 Approximations to $H(\mathfrak{g})$

The differential of the map $\pi_{\mathcal{P}} : \mathcal{W}(G) \to G^{\#(\mathcal{P})}$ maps $H(\mathfrak{g})$ to $\mathfrak{g}^{\#(\mathcal{P})}$ as seen in Eq. (2.3). Proposition 2.15 shows that there is an isometric Lie algebra isomorphism between a subspace of $H(\mathfrak{g})$ and $\mathfrak{g}^{\#(\mathcal{P})}$, where the metric on $\mathfrak{g}^{\#(\mathcal{P})}$ is described below. $K : [0, 1]^2 \to \mathbb{R}$ will be used to denote the reproducing kernel for $H(\mathbb{R})$ and $H(\mathbb{C})$, i.e. $K(s, t) = s \wedge t$ as in Notation 6.4. See section 1 of the appendix for more details.

Definition 2.8. Define $(\, , \,)_{\mathcal{P}}$ to be the unique left invariant Hermitian inner product on the fibers of $TG^{\#(\mathcal{P})}$ such that for $1 \leq i, j \leq n$,

$$(A^{(i)}, B^{(j)})_{\mathcal{P}} = (A, B)_{\mathfrak{g}} Q_{ij} \quad \text{for all } A, B \in \mathfrak{g},$$

where Q is the inverse of the matrix $\{K(s_i, s_j)\}_{i,j=1}^{n}$ and $A^{(i)}$ and $B^{(j)}$ are defined as in Remark 2.7.

Remark 2.9. Staying consistent with earlier notation, we'll let $\langle \, , \, \rangle_{\mathcal{P}} \equiv \mathrm{Re}(\, , \,)_{\mathcal{P}}$ denote the corresponding real left invariant Riemannian metric on the fibers of $TG^{\#(\mathcal{P})}$.

Definition 2.10. Let $H_{\mathcal{P}}(\mathfrak{g})$ denote the subspace of $H(\mathfrak{g})$ given by

$$H_{\mathcal{P}}(\mathfrak{g}) \equiv \{h \in H(\mathfrak{g}) \cap C^2\left((0, 1) \backslash \mathcal{P}\right) | h'' = 0 \text{ on } [0, 1] \backslash \mathcal{P}\}.$$

Remark 2.11. Notice that $H_{\mathcal{P}}(\mathfrak{g})$ is a closed subspace of $H(\mathfrak{g})$, but not a Lie subalgebra with the inherited pointwise commutator.

Proposition 2.12. *Let $\pi_{\mathcal{P}*\underline{e}} : H(\mathfrak{g}) \to \mathfrak{g}^{\#(\mathcal{P})}$ be given by Eq. (2.2), that is*

$$\pi_{\mathcal{P}*\underline{e}} h = (h(s_1), ..., h(s_n)).$$

Then $Nul(\pi_{\mathcal{P}*\underline{e}}) = H_{\mathcal{P}}(\mathfrak{g})^{\perp}$.

Proof. First suppose that $h \in Nul(\pi_{\mathcal{P}*\underline{e}})$, that is $h(s_i) = 0$ for all $i = 0, 1, ..., n$. Let $k \in H_{\mathcal{P}}(\mathfrak{g})$. Then there exist $A_0, ..., A_{n-1} \in \mathfrak{g}$ such that

$$k(t) = \sum_{i=0}^{n-1} A_i(t \wedge s_{i+1} - t \wedge s_i).$$

Notice that

$$k'(t) = \sum_{i=0}^{n-1} A_i 1_{s_{i-1}<t<s_i}.$$

Then

$$\begin{aligned} (h, k)_{H(\mathfrak{g})} &= \int_0^1 \left(h'(t), k'(t)\right)_{\mathfrak{g}} dt \\ &= \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \left(h'(t), A_i\right)_{\mathfrak{g}} dt \\ &= \sum_{i=0}^{n-1} \left(h(s_{i+1}) - h(s_i), A_i\right)_{\mathfrak{g}} \\ &= \sum_{i=0}^{n-1} (0, A_i)_{\mathfrak{g}} \\ &= 0. \end{aligned}$$

Therefore, $Nul(\pi_{\mathcal{P}*\underline{e}}) \subseteq H_{\mathcal{P}}(\mathfrak{g})^{\perp}$.

Now suppose that $h \in H_{\mathcal{P}}(\mathfrak{g})^{\perp}$. Let $A_i = h(s_{i+1}) - h(s_i) \in \mathfrak{g}$ for $i = 0, 1, ..., n-1$. Again, set

$$k(t) = \sum_{i=0}^{n-1} A_i(t \wedge s_{i+1} - t \wedge s_i).$$

Then $k \in H_{\mathcal{P}}(\mathfrak{g})$ and, as above,

$$k'(t) = \sum_{i=0}^{n-1} A_i 1_{s_{i-1}<t<s_i}.$$

We necessarily have

$$\begin{aligned} 0 &= (h,k)_{H(\mathfrak{g})} \\ &= \sum_{i=0}^{n-1} \left(h\left(s_{i+1}\right) - h\left(s_i\right), A_i\right)_{\mathfrak{g}} \\ &= \sum_{i=0}^{n-1} \|h(s_{i+1}) - h\left(s_i\right)\|_{\mathfrak{g}}^2, \end{aligned}$$

which clearly implies that $h(s_{i+1}) - h\left(s_i\right) = 0$ for $i = 0, 1, ..., n-1$. But since $h(0) = 0$, we have that $h(s_i) = 0$ for all $i = 0, 1, ..., n$. Therefore, $h \in Nul(\pi_{\mathcal{P}*\underline{e}})$, and $H_{\mathcal{P}}(\mathfrak{g})^{\perp} \subseteq Nul(\pi_{\mathcal{P}*\underline{e}})$. ☐

Remark 2.13. Proposition 2.12 indicates that

$$H(\mathfrak{g}) = H_{\mathcal{P}}(\mathfrak{g}) \overset{\perp}{\oplus} Nul(\pi_{\mathcal{P}*\underline{e}}).$$

In particular, if $P_{\mathcal{P}} : H(\mathfrak{g}) \to H_{\mathcal{P}}(\mathfrak{g})$ is orthogonal projection, then $P_{\mathcal{P}}h$ is the element of $H_{\mathcal{P}}(\mathfrak{g})$ that agrees with h at all partition points. This projection will be important in Chapter 5.

As indicated in Remark 2.11, $H_{\mathcal{P}}(\mathfrak{g})$ is not a Lie algebra with the inherited pointwise commutator. We can, however, define a new bracket on $H_{\mathcal{P}}(\mathfrak{g})$ using the above projection map.

Proposition 2.14. *Define* $[\ ,\ ]_{\mathcal{P}}$ *on* $H_{\mathcal{P}}(\mathfrak{g})$ *by* $[h,k]_{\mathcal{P}} = P_{\mathcal{P}}[h,k]$. *Then* $(H_{\mathcal{P}}(\mathfrak{g}), [\ ,\ ]_{\mathcal{P}})$ *is a Lie algebra.*

Proof. One simply needs to verify the Jacobi identity. For any $h, k \in H_{\mathcal{P}}(\mathfrak{g})$, $[h,k]_{\mathcal{P}}$ is piecewise linear and therefore determined by its values on the partition points. Since for any $s_i \in \mathcal{P}$, $[h,k]_{\mathcal{P}}(s_i) = [h\left(s_i\right), k\left(s_i\right)]$, the Jacobi identity follows from that for $[\ ,\ ]$ on $\mathfrak{g}$. ☐

Proposition 2.15. *Consider* $H_{\mathcal{P}}(\mathfrak{g})$ *as described in Definition 2.10 with inner product* $(\ ,\ )_{H(\mathfrak{g})}$ *and commutator* $[\ ,\ ]_{\mathcal{P}}$, *and* $\mathfrak{g}^{\#(\mathcal{P})}$ *with inner product* $(\ ,\ )_{\mathcal{P}}$ *and commutator* $[\ ,\ ]$. *Then the map* $\pi_{\mathcal{P}*\underline{e}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\#(\mathcal{P})}$, *the map described in Proposition 2.12 restricted to* $H_{\mathcal{P}}(\mathfrak{g})$, *is an isometric Lie algebra isomorphism.*

Proof. To see that $\pi_{\mathcal{P}*\underline{e}}$ is an isometry, associate to $A = (A_1, ..., A_n) \in \mathfrak{g}^{\#(\mathcal{P})}$ a path $h_A(t) \equiv \sum_{i=1}^n K(s_i, t)A_i \in H_{\mathcal{P}}(\mathfrak{g})$. Then if $B = (B_1, ..., B_n) \in \mathfrak{g}^{\#(\mathcal{P})}$,

$$\begin{aligned}(h_A, h_B)_{H(\mathfrak{g})} &= \sum_{i,j=1}^n (K(s_i, \cdot), K(s_j, \cdot))_{H(\mathbb{C})} (A_i, B_j)_{\mathfrak{g}} \\ &= \sum_{i,j=1}^n K(s_i, s_j)(A_i, B_j)_{\mathfrak{g}}, \end{aligned} \tag{2.7}$$

where we have used Remark 6.5 of the appendix.

$\{K(s_i, s_j)\}_{i,j=1}^n$ is a postive definite matrix, so setting $B = A$ in Eq. (2.7) shows that $A \to h_A$ is injective and hence surjective by the rank nullity theorem. By Definition 2.8,

$$\begin{aligned}\left(\pi_{\mathcal{P}*\underline{e}}(h_A), \pi_{\mathcal{P}*\underline{e}}(h_B)\right)_{\mathcal{P}} &= \sum_{k,l=1}^n \left(h_A(s_k)^{(k)}, h_B(s_l)^{(l)}\right)_{\mathcal{P}} \\ &= \sum_{k,l=1}^n Q_{kl} (h_A(s_k), h_B(s_l))_{\mathfrak{g}} \\ &= \sum_{i,j,k,l=1}^n Q_{kl} K(s_i, s_k) K(s_j, s_l)(A_i, B_j)_{\mathfrak{g}} \\ &= \sum_{i,j=1}^n K(s_i, s_j)(A_i, B_j)_{\mathfrak{g}} \\ &= (h_A, h_B)_{H(\mathfrak{g})}, \end{aligned}$$

where Eq. (2.7) was used in the last equality. Therefore, $\pi_{\mathcal{P}*\underline{e}}$ is an isometry. □

We end this section by showing how the above results on tangent spaces allow us to relate distances on our Lie groups $H(G)$ and $G^{\#(\mathcal{P})}$. We first prove the result in the case of a general Riemannian manifold.

Definition 2.16. Define the distance function on a Riemannian manifold, $d : M \times M \to \mathbb{R}$, by

$$d(m, n) = \inf \int_0^1 |\sigma'(s)| ds,$$

where the infimum is taken over all C^1-paths σ such that $\sigma(0) = m$ and $\sigma(1) = n$. Notice that $d(m, n) = d(n, m)$.

Remark 2.17. In the case where the manifold is a Lie group G with a left invariant metric, it follows that for all $x, y, z \in G$,

$$d(zx, zy) = d(x, y).$$

Notation 2.18. For $x \in G$, we will sometimes use the notation

$$|x| := d(e, x).$$

Notice by Remark 2.17,

$$|x| = |x^{-1}|$$

and

$$d(x, y) = |x^{-1}y| = |y^{-1}x|.$$

Proposition 2.19. *Suppose (M, g) and (N, h) are Riemannian manifolds with $\pi : M \to N$ a surjective map such that $\pi_{*m} : Nul(\pi_{*m})^{\perp} \to T_{\pi(m)}N$ is an isometric isomorphism for all $m \in M$. If d^M and d^N denotes the distance on M and N respectively, then for all $m_1, m_2 \in M$,*

$$d^N(\pi(m_1), \pi(m_2)) \leq d^M(m_1, m_2).$$

Proof. For all $m \in M$ and all $v_m \in T_mM$, we can write $v_m = w_m + w_m^{\perp}$, where $w_m \in Nul(\pi_{*m})$ and $w_m^{\perp} \in Nul(\pi_{*m})^{\perp}$. Since w_m and $w_m^{\perp}$ are orthogonal, $|w_m^{\perp}|_g \leq |v_m|_g$. Finally, since $\pi_{*m}v_m = \pi_{*m}w_m^{\perp}$, we have

$$|\pi_{*m}v_m|_h = |\pi_{*m}w_m^{\perp}|_h = |w_m^{\perp}|_g \leq |v_m|_g.$$

Now let $\sigma : [0, 1] \to M$ be a C^1-path such that $\sigma(0) = m_1$ and $\sigma(1) = m_2$. Then $\pi \circ \sigma : [0, 1] \to N$ is a path connecting $\pi(m_1)$ to $\pi(m_2)$, and

$$d^N(\pi(m_1), \pi(m_2)) \leq l(\pi \circ \sigma) = \int_0^1 |\pi_{*\sigma(s)}\sigma'(s)|_h ds \leq \int_0^1 |\sigma'(s)|_g ds = l(\sigma).$$

Taking the infimum over all paths σ gives the desired result. ∎

Corollary 2.20. *For any partition $\mathcal{P}$ and any $g \in H(G)$,*

$$|\pi_{\mathcal{P}}g|_{\mathcal{P}} \leq |g|_{H(G)}.$$

Proof. We apply Proposition 2.19 with $(M, g) = (H(G), \langle\ ,\ \rangle_{H(G)})$, $(N, h) = (G^{\#(\mathcal{P})}, \langle\ ,\ \rangle_{\mathcal{P}})$, and $\pi = \pi_{\mathcal{P}}$. Notice by Eq. (2.4), for any $g \in H(G)$ and $h \in H(\mathfrak{g})$, $\pi_{\mathcal{P}*g}(L_{g*}h) = L_{\pi_{\mathcal{P}}g*}\pi_{\mathcal{P}*\underline{e}}h$. Since all metrics are left invariant, Proposition 2.15 indicates that $\pi_{\mathcal{P}*g} : L_{g*}H_{\mathcal{P}}(\mathfrak{g}) \to L_{\pi_{\mathcal{P}}g*}\mathfrak{g}^{\#(\mathcal{P})}$ is an isometric isomorphism. Therefore,

$$|\pi_{\mathcal{P}}g|_{\mathcal{P}} = d^{G^{\#(\mathcal{P})}}(\pi_{\mathcal{P}}\underline{e}, \pi_{\mathcal{P}}g) \leq d^{H(G)}(\underline{e}, g) = |g|_{H(G)}.$$

□

2.3 Associated Laplacians

Remark 2.7 indicates that taking derivatives is a well defined operation on cylinder functions. So too is the following natural Laplacian.

Definition 2.21. For $f \in \mathcal{F}C_c^\infty(\mathcal{W})$, define the Laplacian $\Delta_{H(G)}$ by

$$\Delta_{H(G)}f = \sum_{h \in S_{\mathbb{R}}} \tilde{h}^2 f.$$

If $f = F \circ \pi_{\mathcal{P}}$, then by Eq. (2.6) and Proposition 6.12 of the appendix,

$$\begin{aligned} \Delta_{H(G)}f &= \sum_{h \in S_{\mathbb{R}}} \sum_{i,j=1}^{n} (\widetilde{h(s_j)}^{(j)} \widetilde{h(s_i)}^{(i)} F) \circ \pi_{\mathcal{P}} \qquad (2.8) \\ &= \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \sum_{i,j=1}^{n} K(s_j, s_i)(\widetilde{A}^{(j)}\widetilde{A}^{(i)}F) \circ \pi_{\mathcal{P}}. \end{aligned}$$

So if we define an operator $\Delta_{\mathcal{P}}$ on $C_c^\infty(G^{\#(\mathcal{P})})$ by

$$\Delta_{\mathcal{P}}F \equiv \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \sum_{i,j=1}^{n} K(s_j, s_i)(\widetilde{A}^{(j)}\widetilde{A}^{(i)}F),$$

then we have the relationship

$$\Delta_{H(G)}(F \circ \pi_{\mathcal{P}}) = (\Delta_{\mathcal{P}}F) \circ \pi_{\mathcal{P}}.$$

Remark 2.22. Given the map $\pi_{\mathcal{P}*\underline{e}} : H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\#(\mathcal{P})}$ as described in Proposition 2.12 , for $f = F \circ \pi_{\mathcal{P}}$,

$$\begin{aligned} \Delta_{H(G)}f &= \sum_{h \in S_{\mathbb{R}}} \sum_{i,j=1}^{n} (\widetilde{h(s_j)}^{(j)} \widetilde{h(s_i)}^{(i)} F) \circ \pi_{\mathcal{P}} \\ &= \sum_{h \in S_{\mathbb{R}}} (\widetilde{\pi_{\mathcal{P}*\underline{e}}h}^2 F) \circ \pi_{\mathcal{P}}, \end{aligned}$$

or in other words

$$\Delta_{\mathcal{P}} F = \sum_{h \in S_{\mathbb{R}}} \widetilde{\pi_{\mathcal{P}*\underline{e}} h}^2 F.$$

In particular, since $H(\mathfrak{g}) = H_{\mathcal{P}}(\mathfrak{g}) \overset{\perp}{\oplus} Nul(\Pi_{\mathcal{P}})$, if $S_{\mathbb{R}}^{\mathcal{P}}$ is an orthonormal basis for the real inner product space $(H_{\mathcal{P}}(\mathfrak{g}), \langle \ , \ \rangle_{H(\mathfrak{g})})$, then

$$\begin{aligned}\Delta_{H(G)} f &= \sum_{h \in S_{\mathbb{R}}^{\mathcal{P}}} \tilde{h}^2 f \\ &= \sum_{h \in S_{\mathbb{R}}^{\mathcal{P}}} (\widetilde{\pi_{\mathcal{P}*\underline{e}} h}^2 F) \circ \pi_{\mathcal{P}},\end{aligned}$$

and

$$\Delta_{\mathcal{P}} F = \sum_{h \in S_{\mathbb{R}}^{\mathcal{P}}} \widetilde{\pi_{\mathcal{P}*\underline{e}} h}^2 F. \tag{2.9}$$

Suppose $f \in \mathcal{F}C^{\infty}(\mathcal{W})$ and $f = F \circ \pi_{\mathcal{P}}$ for some partition $\mathcal{P}$. Then for any partition $\tilde{\mathcal{P}} \supset \mathcal{P}$, we can also write $f = \tilde{F} \circ \pi_{\tilde{\mathcal{P}}}$ for an appropriate $\tilde{F} \in C^{\infty}(G^{|\tilde{\mathcal{P}}|})$. Regardless of the choice of representation of f as a cylinder function, $\Delta_{H(G)} f$ is well defined.

Proposition 2.23. *Suppose $\tilde{\mathcal{P}} \supset \mathcal{P}$ are two partitions of $[0,1]$ and $f \in \mathcal{F}C^{\infty}(\mathcal{W})$ has the property that $f = F \circ \pi_{\mathcal{P}} = \tilde{F} \circ \pi_{\tilde{\mathcal{P}}}$ for appropriate $F \in C^{\infty}(G^{\#(\mathcal{P})})$ and $\tilde{F} \in C^{\infty}(G^{|\tilde{\mathcal{P}}|})$. Then $(\Delta_{\mathcal{P}} F) \circ \pi_{\mathcal{P}} \equiv (\Delta_{\tilde{\mathcal{P}}} \tilde{F}) \circ \pi_{\tilde{\mathcal{P}}}$.*

Proof. For convenience, we'll consider the case where $\#(\mathcal{P}) = n$ and $\tilde{\mathcal{P}} = \mathcal{P} \cup \{s_{n+1}\}$ for some $s_n < s_{n+1} \leq 1$. The general case will follow by analagous compuations and iteration. For any $A \in \mathfrak{g}$, $g \in \mathcal{W}(G)$, and $i = 1, 2, ..., n$,

$$\begin{aligned}(\tilde{A}^{(i)} \tilde{F})(\pi_{\tilde{\mathcal{P}}}(g)) &= \frac{d}{dt}|_{t=0} \tilde{F}(g(s_1), ..., g(s_{i-1}), g(s_i) e^{tA}, g(s_{i+1}), ..., g(s_{n+1})) \\ &= \frac{d}{dt}|_{t=0} F(g(s_1), ..., g(s_{i-1}), g(s_i) e^{tA}, g(s_{i+1}), ..., g(s_n)) \\ &= (\tilde{A}^{(i)} F)(\pi_{\mathcal{P}}(g)).\end{aligned}$$

Also,

$$
\begin{aligned}
(\tilde{A}^{(n+1)}\tilde{F})(\pi_{\tilde{\mathcal{P}}}(g)) &= \frac{d}{dt}|_{t=0}\tilde{F}(g(s_1),...,g(s_n),g(s_{n+1})e^{tA}) \\
&= \frac{d}{dt}|_{t=0}F(g(s_1),...,g(s_n)) \\
&= 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\Delta_{\tilde{\mathcal{P}}}\tilde{F})\circ\pi_{\tilde{\mathcal{P}}} &= \sum_{A\in\mathfrak{X}_{\mathbb{R}}}\sum_{i,j=1}^{n+1}K(s_j,s_i)(\widetilde{A}^{(j)}\widetilde{A}^{(i)}\tilde{F})\circ\pi_{\tilde{\mathcal{P}}} \\
&= \sum_{A\in\mathfrak{X}_{\mathbb{R}}}\sum_{i,j=1}^{n}K(s_j,s_i)(\widetilde{A}^{(j)}\widetilde{A}^{(i)}F)\circ\pi_{\mathcal{P}} \\
&= (\Delta_{\mathcal{P}}F)\circ\pi_{\mathcal{P}}.
\end{aligned}
$$

□

The following is a summary of definitions and basic properties of strongly continuous semigroups of operators on a Banach space X. A more detailed exposition can be found in a variety of sources, specifically [18] and [8].

Definition 2.24. Let X be a Banach space. Then a collection of bounded linear operators S_t for $t \geq 0$ is a strongly continuous semigroup on X if

1. $S_0 = I$.
2. $S_{s+t} = S_s S_t$.
3. $S_{(\cdot)}f \in C([0,\infty), X)$ for all $f \in X$.

Definition 2.25. The generator of a strongly continuous semigroup S_t is the linear operator L given by

$$
Lf = \lim_{t\to 0}\frac{S_t f - f}{t},
$$

for all f such that the limit exists.

Remark 2.26. Any generator of a strongly continuous semigroup is closed and densely defined. See, for example, the proposition on page 237 of [18].

Proposition 2.27. *Suppose S_t is a strongly continuous semigroup on X with generator L. Then $u_t := S_t f$ satisfies*

$$\frac{\partial}{\partial t} u_t = L u_t \quad \textit{with } u_0 = f.$$

Proof. The proof follows readily from the above definitions. Certainly $u_0 = S_0 f = If = f$ by property (1) of Definition 2.24. Furthermore, by property (2) of Definition 2.24 and Definition 2.25,

$$\begin{aligned}
\frac{\partial}{\partial t} u_t &= \frac{d}{ds}|_{s=0} u_{s+t} \\
&= \frac{d}{ds}|_{s=0} S_{s+t} f \\
&= \frac{d}{ds}|_{s=0} S_s(S_t f) \\
&= \lim_{s \to 0} \frac{S_s(S_t f) - (S_t f)}{s} \\
&= L(S_t f) \\
&= L u_t.
\end{aligned}$$

□

Notation 2.28. In light of the above Proposition, if S_t is a strongly continuous semigroup on X with generator L, we will write

$$S_t = e^{tL}.$$

Given a linear operator, it is natural to ask if it generates some semigroup. This question is answered in generality by the Hille-Yoshida Theorem (pg. 238 of [18]). We are primarily concerned with operators on Hilbert spaces, in which case the following proposition will be sufficient.

Proposition 2.29. *Suppose L is a self-adjoint operator defined on a dense subset of a Hilbert space H. Then the closure of L generates a strongly continous semigroup on H.*

Notation 2.30. We will abuse notation and use the same symbol to denote the operator and its closure.

The left invariant Laplacians Δ_G and $\Delta_{\mathcal{P}}$ are essentially self-adjoint with respect to our right invariant Haar measure on their domains of definition, the complactly supported smooth functions. Hence their closures generate strongly contiuous semigroups on $L^2(G, dx)$ and $L^2(G^{\#(\mathcal{P})}, dx)$ respectively, where dx denotes the appropriate right invariant Haar measure.

Definition 2.31. If e^{tL} is a strongly continuous semigroup such that for all $f \in L^2(G, dx)$

$$\left(e^{tL} f\right)(y) = \int_G f(yx^{-1}) p_t(x) dx$$

for some $p_t \in L^2(G, dx)$, then we call p_t the *convolution semigroup kernel of* e^{tL}.

Definition 2.32. Let G be a Lie group with $\{A_i\}_{i=1}^d$ an orthonormal basis for $\mathfrak{g}$ with respect to a real left invariant Riemannian metric $\langle\ ,\ \rangle$. A left invariant second order differential operator L is *strongly elliptic* if for any $f \in C^2(G)$,

$$(Lf)(g) = \sum_{i,j=1}^{d} a_{ij} \left(\tilde{A}_i \tilde{A}_j f\right)(g) + \sum_{i=1}^{d} b_i \left(\tilde{A}_i f\right)(g) + cf(g),$$

where

$$\sum_{i,j=1}^{d} a_{ij} \xi^i \xi^j \geq C_0 |\xi|^2$$

for some $C_0 > 0$ and for all $\xi \in \mathbb{R}^d$.

Remark 2.33. That $\Delta_{\mathcal{P}}$ is a strongly elliptic operator is evident from Eq. (2.9). In this case, $\{a_{ij}\}_{i,j=1}^d$ is the identity matrix and hence

$$\sum_{i,j=1}^{d} a_{ij} \xi^i \xi^j = |\xi|^2.$$

The following theorem summarizes some important properties found in Robinson [19] of strongly continuous semigroups generated by strongly elliptic operators and their corresponding convolution kernels. [19] treats the case where dx represents left invariant Haar measure, though the case of right invariant Haar measure is similar. For the reader's convenience, we will cite each property separately.

Theorem 2.34. *Let L be a strongly elliptic second-order operator with no zeroth order coefficient ($c = 0$ in Definition 2.32) on a Lie group G of dimension d. Let dx denote right invariant Haar measure. Then there exists a strictly positive convolution semigroup kernel $p_t \in C^\infty((0,\infty) \times G)$ satisfying:*

1. *$\int_G p_t(x)dx = 1$. (pg. 253 of [19])*

2. *p_t satisfies the following "heat" equation*

$$\frac{\partial}{\partial t} p_t(x) = L p_t(x)$$

with the initial condition

$$\lim_{t \to 0} p_t(x) = \delta(x),$$

with the limit interpreted in a weak sense. (pg. 253 of [19])

3. *There exist constants $a, b > 0$ and $\omega \geq 0$ such that for all $t > 0$ and $g \in G$,*

$$|p_t(g)| \leq a t^{\frac{-d}{2}} e^{\frac{-b|g|^2}{t}} e^{\omega t}.$$

(Theorem 4.1 of [19]).

Notation 2.35. Let $p_t^{\mathcal{P}}$ denote the smooth semigroup kernel for the operator $\frac{t}{4}\Delta_{\mathcal{P}}$, and let p_t^G denote the smooth semigroup kernel for the operator $\frac{t}{4}\Delta_G$.

Remark 2.36. The fact that $\Delta_{\mathcal{P}}$ and Δ_G are essentially self-adjoint implies that $p_t^{\mathcal{P}}$ and p_t^G are invariant under $x \to x^{-1}$, that is,

$$p_t^{\mathcal{P}}(x) = p_t^{\mathcal{P}}(x^{-1})$$

and

$$p_t^G(x) = p_t^G(x^{-1}).$$

3

Heat Kernel Measure

In this chapter, we construct the heat kernel measure on $\mathcal{W}(G)$. The measure is constructed as the law of a continuous $\mathcal{W}(G)$ valued process. To prove the existence of such a process, we first require some geometric estimates.

3.1 Geometric Preliminaries

The following theorem is well known and can be found in a variety of sources (for example, see [2, 20]).

Theorem 3.1 (Bishop's Comparision Theorem)**.** *Let (M, g) be an N-dimensional complete Riemannian manifold, $\kappa \geq 0$, and assume that*

$$Ric\langle \xi, \xi \rangle \geq -(N-1)\kappa g\langle \xi, \xi \rangle \quad \forall \xi \in TM$$

Let $o \in M$ and $V(r)$ denote the Riemannian volume of the ball of radius r centered at $o \in M$. Then

$$V(r) \leq \omega_{N-1} \int_0^r \left(\frac{\sinh \sqrt{\kappa}\rho}{\sqrt{\kappa}} \right)^{N-1} d\rho,$$

where ω_{N-1} is the surface area of the unit N-1 sphere in $\mathbb{R}^N$. Also,

$$V(r) \leq \omega_{N-1} r^N e^{\sqrt{\kappa} r}.$$

Proposition 3.2. *Let G be a finite dimensional Lie group with left invariant metric $\langle\ ,\ \rangle$. Then $(G, \langle\ ,\ \rangle)$ satisifies the hypotheses of Bishop's Comparison Theorem (Theorem 3.1). That is, there exists a κ such that*

$$Ric\langle \xi, \xi \rangle \geq \kappa \langle \xi, \xi \rangle \quad \forall \xi \in TG. \tag{3.1}$$

Proof. Since both $R\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle$ are bilinear, it suffices to show Eq. (3.1) for vectors $\xi \in TM$ with $|\xi| = 1$. The set $\{(e, \xi) \in TG | \xi \in T_eG \text{ with } |\xi| = 1\}$ is compact and hence has Ricci curature bounded below by some constant κ. Then since $\langle\ ,\ \rangle$ is left invariant, $Ric\langle \xi, \xi \rangle \geq \kappa$ for any $g \in G$ and for all $\xi \in T_gG$ with $|\xi| = 1$. □

Remark 3.3. In particular, Proposition 3.2 indicates that $(G^{\#(\mathcal{P})}, \langle\ ,\ \rangle_{\mathcal{P}})$ satisfies the hypotheses of Bishop's Comparision Theorem.

The following proposition can be found in [5]. We include the proof for completeness.

Proposition 3.4. *Let G be a Lie group. Then there exists a constant $c < \infty$ such that for all $x \in G$, $||Ad_x|| \leq e^{c|x|}$, where $|| \cdot ||$ denotes the operator norm.*

Proof. Let $\sigma : [0, 1] \to G$ be a C^1-path such that $\sigma(0) = e$ and $\sigma(1) = x$. Then

$$\frac{d}{dt} Ad_{\sigma(t)} = \frac{d}{d\varepsilon}|_{\varepsilon=0} Ad_{\sigma(t)} Ad_{\sigma(t)^{-1}\sigma(t+\varepsilon)} = Ad_{\sigma(t)} ad_{\theta\langle\sigma'(t)\rangle},$$

where $\theta\langle\sigma'(t)\rangle = L_{\sigma(t)^{-1}*}\sigma'(t)$. Hence,

$$\begin{aligned} ||Ad_{\sigma(t)}|| &= ||I + \int_0^t Ad_{\sigma(\tau)} ad_{\theta\langle\sigma'(\tau)\rangle} d\tau|| \\ &\leq 1 + c \int_0^t Ad_{\sigma(\tau)} ad_{\theta\langle\sigma'(\tau)\rangle} d\tau, \end{aligned}$$

where $c = \max\{||ad_\alpha|| : \alpha \in \mathfrak{g} \text{ and } ||\alpha|| = 1\}$ and $||ad_\alpha||$ denotes the operator norm of ad_α. Therefore by Gronwall's inequality,

$$||Ad_x|| = ||Ad_{\sigma(1)}|| \leq \exp\left(c \int_0^1 |\theta\langle\sigma'(t)\rangle| dt \right) = e^{cl(\sigma)} \leq e^{c|x|}.$$

□

In the proof of Theorem 3.8 to follow, we will need to estimate distances on $(G^{\#(\mathcal{P})}, d_{\mathcal{P}})$ in terms of distances on G. The following notation and Proposition 3.6 will be used in the proof of Theorem 3.8.

Notation 3.5. Let $\{x, y\}$ denote a point in $G \times G$. Hence, for a two point partition $\mathcal{P}$, $|\{x, y\}|_{\mathcal{P}} = d_{\mathcal{P}}(\{x, y\}, \{e, e\})$, where $d_{\mathcal{P}}$ is the distace function on $G \times G$ relative to the metric $\langle\ ,\ \rangle_{\mathcal{P}}$.

For the next two propositions, suppose $0 < u < v < 1$ and let $\mathcal{P} = \{0 < u < v < 1\}$. We'll let $|\cdot|^2_{\mathcal{P}} = \langle\cdot,\cdot\rangle_{\mathcal{P}}$ and $|\cdot|^2_{\mathfrak{g}} = \langle\cdot,\cdot\rangle_{\mathfrak{g}}$, so for $\{A, B\} \in \mathfrak{g} \times \mathfrak{g}$,

$$\begin{aligned}
|\{A, B\}|^2_{\mathcal{P}} &= \langle\{A, B\}, \{A, B\}\rangle_{\mathcal{P}} \\
&= \langle\{A, 0\} + \{0, B\}, \{A, 0\} + \{0, B\}\rangle_{\mathcal{P}} \\
&= \langle\{A, 0\}, \{A, 0\}\rangle_{\mathcal{P}} + \langle\{A, 0\}, \{0, B\}\rangle_{\mathcal{P}} \\
&+ \langle\{0, B\}, \{A, 0\}\rangle_{\mathcal{P}} + \langle\{0, B\}, \{0, B\}\rangle_{\mathcal{P}} \\
&= a|A|^2_{\mathfrak{g}} - 2b\langle A, B\rangle_{\mathfrak{g}} + c|B|^2_{\mathfrak{g}},
\end{aligned} \tag{3.2}$$

where $a, b, c \in \mathbb{R}$ are determined by the following special case of Definition 2.8,

$$\begin{aligned}
\begin{bmatrix} a & -b \\ -b & c \end{bmatrix} &= \begin{bmatrix} K(u,u) & K(u,v) \\ K(u,v) & K(v,v) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} u & u \\ u & v \end{bmatrix}^{-1} \\
&= \frac{1}{u(v-u)} \begin{bmatrix} v & -u \\ -u & u \end{bmatrix}.
\end{aligned}$$

That is, $a = \frac{v}{u(v-u)}$, and $b = c = \frac{1}{v-u}$.

Proposition 3.6. *For all $A, B \in \mathfrak{g}$,*

$$|A - \frac{u}{v}B|_{\mathfrak{g}} \le \sqrt{(u/v)(v-u)}|\{A, B\}|_{\mathcal{P}}, \tag{3.3}$$

and

$$|B|_{\mathfrak{g}} \le \sqrt{v}|\{A, B\}|_{\mathcal{P}} \tag{3.4}$$

Proof. By completing the squares in Eq. (3.2) we have

$$\begin{aligned}|\{A,B\}|_{\mathcal{P}}^2 &= a\left(|A-\frac{b}{a}B|_{\mathfrak{g}}^2+(\frac{c}{a}-\frac{b^2}{a^2})|B|_{\mathfrak{g}}^2\right)\\ &= a|A-\frac{b}{a}B|_{\mathfrak{g}}^2+\frac{ac-b^2}{a}|B|_{\mathfrak{g}}^2\\ &= \frac{v}{u(v-u)}|A-\frac{u}{v}B|_{\mathfrak{g}}^2+\frac{1}{v}|B|_{\mathfrak{g}}^2. \end{aligned} \tag{3.5}$$

Then since $\frac{1}{v}|B|_{\mathfrak{g}}^2 \geq 0$,

$$\frac{v}{u(v-u)}|A-\frac{u}{v}B|_{\mathfrak{g}}^2 \leq |\{A,B\}|_{\mathcal{P}}^2,$$

which implies that

$$|A-\frac{u}{v}B|_{\mathfrak{g}} \leq \sqrt{(u/v)(v-u)}|\{A,B\}|_{\mathcal{P}}.$$

Similarly, since $\frac{v}{u(v-u)}|A-\frac{u}{v}B|_{\mathfrak{g}}^2 \geq 0$, Eq. (3.5) also yields

$$|B|_{\mathfrak{g}} \leq \sqrt{v}|\{A,B\}|_{\mathcal{P}}.$$

□

Lemma 3.7. *For any $x, y \in G$, we have that*

$$d(x,y) = |x^{-1}y| \leq 2\sqrt{v-u}e^{c|\{x,y\}|_{\mathcal{P}}}|\{x,y\}|_{\mathcal{P}}, \tag{3.6}$$

where c is the same constant as in Proposition 3.4.

Proof. Let $x, y \in G$, $\sigma : [0,1] \to G$ and $\tau : [0,1] \to G$ be two smooth paths such that $\sigma(0) = \tau(0) = e$, $\sigma(1) = x$, and $\tau(1) = y$. Since $\sigma\tau^{-1} : [0,1] \to G$ is a path joining e to xy^{-1}, it follows that $|xy^{-1}| \leq \int_0^1 |\theta\langle(\sigma\tau^{-1})'(s)\rangle|ds$, where θ is the Maurer-Cartan form. Furthermore, $\{\sigma,\tau\} : [0,1] \to G \times G$ is a smooth path with $\{\sigma,\tau\}(0) = \{e,e\}$ and $\{\sigma,\tau\}(1) = \{x,y\}$. Define $A \equiv \theta\langle\sigma'(s)\rangle$ and $B \equiv \theta\langle\tau'(s)\rangle$. Then

$$\ell(\sigma) = \int_0^1 |A(s)|_{\mathfrak{g}}ds,$$

$$\ell(\tau) = \int_0^1 |B(s)|_{\mathfrak{g}}ds,$$

and

$$\ell_{\mathcal{P}}(\{\sigma,\tau\}) = \int_0^1 |\{A(s),B(s)\}|_{\mathcal{P}}ds.$$

Notice that by Eq. (3.4),

$$\begin{aligned}\ell(\tau) &= \int_0^1 |B(s)|_{\mathfrak{g}} ds \\ &\leq \sqrt{v} \int_0^1 |\{A(s), B(s)\}|_{\mathcal{P}} ds \\ &= \sqrt{v}\ell_{\mathcal{P}}(\{\sigma,\tau\}) \\ &\leq \ell_{\mathcal{P}}(\{\sigma,\tau\}). \end{aligned} \tag{3.7}$$

Set $\iota(g) \equiv g^{-1}$. Then

$$(\tau^{-1})' = \iota_* \tau' = \iota_* L_{\tau *} B = \frac{d}{dt}|_0 \iota(\tau e^{tB}) = \frac{d}{dt}|_0 e^{-tB}\tau^{-1} = -R_{-1_*} B,$$

Therefore,

$$\begin{aligned}\theta\langle(\sigma\tau^{-1})'(s)\rangle &= L_{\tau\sigma^{-1}(s)*}\{R_{\tau^{-1}(s)*}\sigma'(s) + L_{\sigma(s)*}(\tau^{-1})'(s)\} \\ &= L_{\tau\sigma^{-1}(s)*}\{R_{\tau^{-1}(s)*}L_{\sigma(s)*}A(s) - L_{\sigma(s)*}R_{\tau^{-1}(s)*}B(s)\} \\ &= Ad_{\tau(s)}(A(s) - B(s)).\end{aligned}$$

So using Eqs.(3.3) , (3.4), and Proposition 3.4,

$$\begin{aligned}|xy^{-1}| &\leq \int_0^1 |Ad_{\tau(s)}(A(s) - B(s))|_{\mathfrak{g}} ds \\ &\leq \int_0^1 \|Ad_{\tau(s)}\| [|A(s) - \frac{u}{v}B(s)|_{\mathfrak{g}} + (1 - \frac{u}{v})|B(s)|_{\mathfrak{g}}] ds \\ &\leq \int_0^1 e^{c|\tau(s)|}[|A(s) - \frac{u}{v}B(s)|_{\mathfrak{g}} + (1 - \frac{u}{v})|B(s)|_{\mathfrak{g}}] ds \\ &\leq e^{c\ell(\tau)} \int_0^1 (\sqrt{(u/v)(v-u)} + (1 - \frac{u}{v})\sqrt{v})|\{A(s), B(s)\}|_{\mathcal{P}} ds \\ &\leq e^{c\ell(\{\sigma,\tau\})}(\sqrt{(u/v)(v-u)} + \frac{(v-u)}{\sqrt{v}})\ell_{\mathcal{P}}(\{\sigma,\tau\}) \qquad (3.8) \\ &\leq e^{c\ell(\{\sigma,\tau\})}\sqrt{v-u}\left(\frac{\sqrt{u} + \sqrt{v-u}}{\sqrt{v}}\right)\ell_{\mathcal{P}}(\{\sigma,\tau\}) \\ &\leq 2e^{c\ell(\{\sigma,\tau\})}\sqrt{v-u}\ell_{\mathcal{P}}(\{\sigma,\tau\}).\end{aligned}$$

where in line (3.8) we have also used Eq. (3.7). Minimizing this last inequality over all σ joining e to x and all τ from e to y shows that

$$|xy^{-1}| \leq 2\sqrt{v-u}e^{c|\{x,y\}|_{\mathcal{P}}}|\{x,y\}|_{\mathcal{P}}.$$

Sending $x \to x^{-1}$ and $y \to y^{-1}$ in the above expression also gives

$$\begin{aligned} |x^{-1}y| &\leq 2\sqrt{v-u}e^{c|\{x^{-1},y^{-1}\}|_{\mathcal{P}}}|\{x^{-1},y^{-1}\}|_{\mathcal{P}} \\ &= 2\sqrt{v-u}e^{c|\{x,y\}|_{\mathcal{P}}}|\{x,y\}|_{\mathcal{P}}. \end{aligned}$$

□

3.2 Construction of ν_t

Let $\{\beta(t,s)\}_{0\leq s\leq 1, 0\leq t<\infty}$ be a $\mathfrak{g}$-valued Brownian sheet with half the usualy covariance. That is, $\{\beta(t,s)\}_{0\leq s\leq 1, 0\leq t<\infty}$ is a jointly continuous, mean zero gaussian $\mathfrak{g}$-valued process defined on a probability space $(\Omega, \mathcal{F}, P)$ such that, if $\beta^A(t,s) := \langle A, \beta(t,s)\rangle_{\mathfrak{g}}$ for $A \in \mathfrak{g}$, then

$$\begin{aligned} \mathbb{E}[\beta^A(t,s)\beta^B(\tau,\sigma)] &= \langle A,B\rangle_{\mathfrak{g}}(t\wedge\tau)\frac{1}{2}K(s,\sigma) \\ &= \frac{1}{2}\langle A,B\rangle_{\mathfrak{g}}(t\wedge\tau)(s\wedge\sigma) \end{aligned}$$

for all $s,\sigma \in [0,1]$, $t,\tau \in [0,\infty)$, and $A, B \in \mathfrak{g}$, where $\mathbb{E}$ denotes expectation relative to the measure P. In other words, for fixed s, $t \to \beta(t,s)$ is a $\mathfrak{g}$-valued Brownian motion with variance $\frac{1}{2}K(s,s)$, and for fixed t, $s \to \beta(t,s)$ is a $\mathfrak{g}$-valued Brownian motion with variance t.

We now are able to prove the existence of a Brownian motion on $\mathcal{W}(G)$, which gives us a heat kernel measure.

Theorem 3.8. *Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and left invariant Riemannian inner product $\langle\ ,\ \rangle$ and $g_0 \in \mathcal{W}(G)$. Then there exists a continuous adapted $\mathcal{W}(G)$-valued process $\{\Sigma(t)\}_{t\geq 0}$ on the filtered probability space $(\mathcal{W},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{F},P)$ such that for each $s \in [0,1]$, $\Sigma(\cdot,s)$ solves the stochastic differntial equation:*

$$\Sigma(\delta t,s) = L_{\Sigma(t,s)*}\beta(\delta t,s) \qquad \text{with } \Sigma(0,s) = g_0(s). \tag{3.9}$$

More precisely,

$$\Sigma(\delta t,s) = \sum_{A\in\mathfrak{X}_{\mathbb{R}}} \tilde{A}(\Sigma(t,s))\beta^A(\delta t,s) \qquad \text{with } \Sigma(0,s) = g_0(s), \tag{3.10}$$

where $\mathfrak{X}_{\mathbb{R}} \subset \mathfrak{g}$ is an orthonormal basis for the real inner product space, $\tilde{A}$ is the left invariant vector field on G satisfying $\tilde{A}(e) = A$, and $\beta^A(t,s) = \langle A, \beta(t,s)\rangle$. Here $\beta^A(\delta t, s)$ denotes the Stratonovich differential of the process $t \to \beta^A(t,s)$. We will use "δ" for the Stratonovich differential and "d" for the differential of a semimartingale.

Remark 3.9. For fixed s, the existence of a $G-$valued process $\Sigma(t,s)$ satisfying Eq. 3.9 follows from the existence of Brownian motion on a finite dimensional Lie group. See, for example, Theorem 4.8.7 in [14]. The challege in proving Theorem 3.8 is showing that there exists a jointly continuous version of Σ, that is $\Sigma(t,\cdot)$ is a $\mathcal{W}(G)$-valued process.

Before proving the existence of a continuous version of the process in Theorem 3.8, we first prove a couple of propositions regarding a related process.

Definition 3.10. Let $\{\Sigma^0(t)\}_{t\geq 0}$ denote the solution to Eq. 3.9 given by Remark 3.9 with initial condition $g_0(s) = e$ for all $s \in [0,1]$.

Notation 3.11. Given the processes $\beta(t,s)$ and $\Sigma^0(t)$ defined above, for $\mathcal{P}$ a partition of $[0,1]$, define a continuous $G^{\#(\mathcal{P})}$-valued process $\Sigma_{\mathcal{P}}$ by

$$\Sigma_{\mathcal{P}}(t) := \pi_{\mathcal{P}} \circ \Sigma^0(t,\cdot),$$

and

$$\beta_{\mathcal{P}}(t) := \ \pi_{\mathcal{P}*\underline{e}}\beta(t,\cdot) = (\beta(t,s_1), \beta(t,s_2), ..., \beta(t,s_n))$$

Proposition 3.12. *$\Sigma_{\mathcal{P}}$ solves the SDE*

$$\Sigma_{\mathcal{P}}(\delta t) = L_{\Sigma_{\mathcal{P}}(t)*}\beta_{\mathcal{P}}(\delta t)$$

with $\Sigma_{\mathcal{P}}(0) = (e,e,...,e) \in G^{\#(\mathcal{P})}$. Furthermore, $\Sigma_{\mathcal{P}}$ has generator $\frac{1}{4}\Delta_{\mathcal{P}}$.

Proof. Using Eq. (3.9) and Eq. (2.4) we see that $\Sigma_{\mathcal{P}}$ solves the SDE

$$\begin{aligned}
\Sigma_{\mathcal{P}}(\delta t) &= \pi_{\mathcal{P}*\Sigma^0(t,s)}\Sigma_0(\delta t, s) \\
&= \pi_{\mathcal{P}*\Sigma^0(t,s)}L_{\Sigma_0(t,s)*}\beta(\delta t, s) \\
&= L_{\Sigma_{\mathcal{P}}(t)*}\pi_{\mathcal{P}*\underline{e}}\beta(\delta t, s) \\
&= L_{\Sigma_{\mathcal{P}}(t)*}\beta_{\mathcal{P}}(\delta t, s), \qquad (3.11)
\end{aligned}$$

with initial condition $\Sigma_{\mathcal{P}}(0) = (e, e, ..., e) \in G^{\#(\mathcal{P})}$. Note that by Itô's lemma, for any function $F \in C^\infty(G^{\#(\mathcal{P})})$,

$$\begin{aligned}
dF(\Sigma_{\mathcal{P}}(t)) &= \sum_{i=1}^{n} \sum_{A\in\mathfrak{X}_{\mathbb{R}}} (A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\beta^A_{\mathcal{P}}(\delta t, s_i) \\
&= \sum_{i=1}^{n} \sum_{A\in\mathfrak{X}_{\mathbb{R}}} (A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\beta^A_{\mathcal{P}}(dt, s_i) \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} \sum_{A,B\in\mathfrak{X}_{\mathbb{R}}} (B^{(j)}A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\beta^A_{\mathcal{P}}(dt, s_i)\beta^B_{\mathcal{P}}(dt, s_j) \\
&= \sum_{i=1}^{n} \sum_{A\in\mathfrak{X}_{\mathbb{R}}} (A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\beta^A_{\mathcal{P}}(dt, s_i) \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} \sum_{A,B\in\mathfrak{X}_{\mathbb{R}}} (B^{(j)}A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\frac{1}{2}K(s_i, s_j)\langle A, B\rangle dt \\
&= \sum_{i=1}^{n} \sum_{A\in\mathfrak{X}_{\mathbb{R}}} (A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\beta^A_{\mathcal{P}}(dt, s_i) \\
&+ \frac{1}{4}(\Delta_{\mathcal{P}}F)(\Sigma_{\mathcal{P}}(t))dt,
\end{aligned}$$

which implies that $\Sigma_{\mathcal{P}}(t, \cdot)$ is a $G^{\#(\mathcal{P})}$-valued process with generator $\frac{1}{4}\Delta_{\mathcal{P}}$. □

Proposition 3.13. *Let $\mathcal{P}$ be a partition of $[0, 1]$. Then for any bounded measureable function $f : G^{\#(\mathcal{P})} \to \mathbb{C}$ and $T > 0$,*

$$\mathbb{E}[f(\Sigma_{\mathcal{P}}(T))] = \int_{G^{\#(\mathcal{P})}} f(x)p^{\mathcal{P}}_T(x)dx,$$

where $p^{\mathcal{P}}_T$ is the convolution semigroup kernel corresponding to the operator $\frac{1}{4}\Delta_{\mathcal{P}}$ (see Notation 2.35).

Proof. First assume that $f \in C^2_c(G^{\#(\mathcal{P})})$. For $0 \le t \le T$, define

$$F_t(x) := \int_{G^{\#(\mathcal{P})}} f(xy^{-1})p^{\mathcal{P}}_{T-t}(y)dy = \int_{G^{\#(\mathcal{P})}} f(x)p^{\mathcal{P}}_{T-t}(yx)dy.$$

Then F_t satisfies

$$\frac{\partial}{\partial t}F_t = -\frac{1}{4}\Delta_{\mathcal{P}}F_t,$$

for $0 \leq t \leq T$. By Itô's lemma and Proposition 3.12,

$$\begin{aligned} dF_t(\Sigma_{\mathcal{P}}(t)) &= \sum_{i=1}^n \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (A^{(i)} F_t)(\Sigma_{\mathcal{P}}(t)) \beta_{\mathcal{P}}^A(\delta t, s_i) + \left(\frac{\partial}{\partial t} F_t\right)(\Sigma_{\mathcal{P}}(t)) \\ &= \sum_{i=1}^n \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (A^{(i)} F_t)(\Sigma_{\mathcal{P}}(t)) \beta_{\mathcal{P}}^A(dt, s_i) \\ &+ \frac{1}{4}(\Delta_{\mathcal{P}} F_t)(\Sigma_{\mathcal{P}}(t))dt - \frac{1}{4}(\Delta_{\mathcal{P}} F_t)(\Sigma_{\mathcal{P}}(t))dt \\ &= \sum_{i=1}^n \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (A^{(i)} F_t)(\Sigma_{\mathcal{P}}(t)) \beta_{\mathcal{P}}^A(dt, s_i), \end{aligned}$$

which implies that $M_t := F_t(\Sigma_{\mathcal{P}}(t))$ is a local martingale. Our next goal is to show that M_t is square integrable. It suffices to show

$$\mathbb{E} \int_0^{T_0} \left| \left(A^{(i)} F_t\right)(\Sigma_{\mathcal{P}}(t)) \right|^2 dt < \infty, \tag{3.12}$$

for any $T_0 \in (0, T_0)$ and $i = 1, 2, ..., n$.

For any $A \in \mathfrak{g}^{\#(\mathcal{P})}$,

$$\frac{d}{ds} f(xe^{sA}y^{-1}) = \frac{d}{ds} f(xy^{-1}e^{sAd_y A}) = \langle Df(xy^{-1}), Ad_y A \rangle.$$

Recall from Proposition 3.4, we have $||Ad_y A||_{\mathfrak{g}} \leq ||A||_{\mathfrak{g}} e^{c|y|}$ for some $c > 0$. Therefore,

$$\left| \frac{d}{ds} f(xe^{sA}y^{-1}) \right| \leq ||Df||_\infty ||A||_{\mathfrak{g}} e^{c|y|}.$$

In addition, by part 3 of Theorem 2.34, it follows that

$$\int_{G^{\#(\mathcal{P})}} e^{c|y|} p_{T-t}^{\mathcal{P}}(y) dy = C(T-t),$$

where $\sup_{\varepsilon \leq \tau \leq T} C(\tau) < \infty$, for any $\varepsilon > 0$. We then have

$$\begin{aligned} \left(A^{(i)} F_t\right)(x) &= \frac{d}{ds}|_{s=0} \int_{G^{\#(\mathcal{P})}} f(xe^{sA^{(i)}} y^{-1}) p_{T-t}^{\mathcal{P}}(y) dy \\ &= \int_{G^{\#(\mathcal{P})}} \frac{d}{ds}|_{s=0} f(xe^{sA^{(i)}} y^{-1}) p_{T-t}^{\mathcal{P}}(y) dy \\ &= \int_{G^{\#(\mathcal{P})}} \langle Df(xy^{-1}), Ad_y A^{(i)} \rangle p_{T-t}^{\mathcal{P}}(y) dy. \end{aligned}$$

Moreover, for any $T_0 \in (0,T)$,

$$\sup_{0\le t\le T_0} |\left(A^{(i)}F_t\right)(x)| = \sup_{0\le t\le T_0}\left|\int_{G^{\#(\mathcal{P})}} \langle Df(xy^{-1}), Ad_y A^{(i)}\rangle p^{\mathcal{P}}_{T-t}(y)dy\right| \\ \le ||Df||_\infty |A^{(i)}| K(T_0), \tag{3.13}$$

where $K_0(T_0) = \sup_{T-T_0\le\tau\le T} C(\tau) < \infty$. From Eq. (3.13), it follows that

$$\mathbb{E}\int_0^{T_0} \left|\left(A^{(i)}F_t\right)(\Sigma_{\mathcal{P}}(t))\right|^2 dt \le ||Df||_\infty |A^{(i)}| K(T_0)T_0 < \infty,$$

which verifies Eq. (3.12). Therefore, $M_t = F_t(\Sigma_{\mathcal{P}}(t))$ is a Martingale for $0 \le t \le T$.

For any $t \in (0,T)$, $\mathbb{E}M_t = \mathbb{E}M_0$, that is

$$\begin{aligned}\mathbb{E}[F_t(\Sigma_{\mathcal{P}}(t))] &= \mathbb{E}[F_0(\Sigma_{\mathcal{P}}(0)] \\ &= F_0(e,e,...,e) \\ &= \int_{G^{\#(\mathcal{P})}} f(y^{-1}) p^{\mathcal{P}}_T(y)dy \\ &= \int_{G^{\#(\mathcal{P})}} f(y) p^{\mathcal{P}}_T(y)dy, \end{aligned}\tag{3.14}$$

where in line (3.14) we have used Remark 2.36.

Since $f \in C^2_c(G^{\#(\mathcal{P})})$, it is bounded, and therefore so is F. Furthermore, there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C|xy^{-1}| = C|y^{-1}x|.$$

Then for any $\omega \in \Omega$,

$$\begin{aligned}&\lim_{t\uparrow T} |F_t(\Sigma_{\mathcal{P}}(t)(\omega)) - f(\Sigma_{\mathcal{P}}(T)(\omega))| \\ &\le \lim_{t\uparrow T}\int_{G^{\#(\mathcal{P})}} |f(\Sigma_{\mathcal{P}}(t)(\omega)y^{-1}) - f(\Sigma_{\mathcal{P}}(T)(\omega))| p^{\mathcal{P}}_T(y)dy \\ &\le \lim_{t\uparrow T}\int_{G^{\#(\mathcal{P})}} C|\Sigma_{\mathcal{P}}(T)(\omega)^{-1}\Sigma_{\mathcal{P}}(t)(\omega)y^{-1}| p^{\mathcal{P}}_T(y)dy \\ &\le \lim_{t\uparrow T} C|\Sigma_{\mathcal{P}}(T)(\omega)^{-1}\Sigma_{\mathcal{P}}(t)(\omega)| \int_{G^{\#(\mathcal{P})}} |y^{-1}| p^{\mathcal{P}}_T(y)dy \\ &\le \lim_{t\uparrow T} \tilde{C}|\Sigma_{\mathcal{P}}(T)(\omega)^{-1}\Sigma_{\mathcal{P}}(t)(\omega)| \qquad (3.15) \\ &= 0,\end{aligned}$$

since $\Sigma_{\mathcal{P}}$ is continuous. The heat kernel growth bounds of Theorem 2.34 imply in line (3.15) that $\int_{G\#(\mathcal{P})} |y^{-1}| p_T^{\mathcal{P}}(y)dy < \infty$. Therefore, the DCT allows us to conclude

$$\mathbb{E}[f(\Sigma_{\mathcal{P}}(T))] = \lim_{t \uparrow T} \mathbb{E}[F_t(\Sigma_{\mathcal{P}}(t))] = \int_{G\#(\mathcal{P})} f(y) p_T^{\mathcal{P}}(y)dy.$$

□

Proof of Theorem 3.8. We first consider the process Σ^0 as given in Definition 3.10. Our immediate goal is to show that there exists a continuous version of this process.

Fix $\tau \in [0, \infty)$, and define a process $u(t) \equiv \Sigma^0(\tau, s)^{-1}\Sigma^0(t, s)$ for $t \geq \tau$. Then $u(\tau) = e \in G$ and $u(t)$ solves the stochastic differential equation:

$$\begin{aligned}
\delta u(t) &= L_{\Sigma^0(\tau,s)^{-1}*} \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \tilde{A}(\Sigma^0(t,s))\beta^A(\delta t, s) \\
&= \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \tilde{A}(\Sigma^0(\tau, s)^{-1}\Sigma^0(t,s))\beta^A(\delta t, s) \\
&= \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \tilde{A}(u(t))\beta^A(\delta t, s).
\end{aligned}$$

Therefore, if $f \in C^\infty(G)$ of polynomial growth with $f(e) = 0$, then using Itô's lemma, for all $t \geq \tau$, we get

$$\begin{aligned}
f(u(t)) &= \int_\tau^t \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (\tilde{A}f)(u(r))\beta^A(\delta r, s) \\
&= \int_\tau^t \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (\tilde{A}f)(u(r))\beta^A(dr, s) + \frac{1}{2}\int_\tau^t \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (\tilde{A}^2 f)(u(r))\frac{1}{2}K(s,s)dr \\
&= \int_\tau^t \sum_{A \in \mathfrak{X}_{\mathbb{R}}} (\tilde{A}f)(u(r))\beta^A(dr, s) + \frac{K(s,s)}{4}\int_\tau^t (\Delta_G f)(u(r))dr.
\end{aligned}$$

So we see that $u(t)$ is a process on G with generator $\frac{(t-\tau)G(s,s)}{4}\Delta_G$. It then follows that for any $f \in C(G)$,

$$\mathbb{E}(f(u(t)) = (e^{\frac{(t-\tau)K(s,s)}{4}\Delta_G} f)(e) = \int_G f(x) p^G_{(t-\tau)K(s,s)}(x)dx, \tag{3.16}$$

where $p_t^G(x)$ denotes the heat kernel on G (see Notation 2.35). We now set $f(x) = d(x,e)^p = |x|^p$. Let $V(r)$ denote the volume of the ball of radius r centered at $e \in G$, relative to the metric $\langle \, , \rangle$. Then by Bishop's Comparison theorem (Theorem 3.1), there is a constant $\gamma \in (0,\infty)$ such that $V(r) \le \gamma r^n e^{\gamma r}$. Then using our heat kernel estimate Theorem 2.34, and an integration by parts, we find that

$$\begin{aligned}
\mathbb{E}|u(t)|^p &= \int_G |x|^p p^G_{(t-\tau)K(s,s)}(x)dx \\
&\le a|t-\tau|^{\frac{n}{2}} \int_0^\infty r^p e^{\frac{-br^2}{(t-\tau)K(s,s)}} e^{\omega(t-\tau)K(s,s)} dV(r) \\
&\le -a|t-\tau|^{\frac{n}{2}} \int_0^\infty (pr^{p-1} - \frac{2br^{p+1}}{(t-\tau)K(s,s)}) e^{\frac{-br^2}{(t-\tau)K(s,s)}} e^{\omega(t-\tau)K(s,s)} V(r)dr \\
&\le K|t-\tau|^{\frac{n}{2}} \int_0^\infty \frac{r^{p+1}}{(t-\tau)K(s,s)} r^n e^{\frac{-br^2}{(t-\tau)K(s,s)}} e^{\gamma r} dr
\end{aligned}$$

where $K = 2a\gamma e^{\omega T}$. Using the following scaling argument

$$\int_0^\infty r^k e^{\frac{-r^2}{\alpha}} e^{\gamma r} dr = \alpha^{\frac{k+1}{2}} \int_0^\infty r^k e^{-r^2} e^{\sqrt{\alpha}\gamma r} dr,$$

we get

$$\mathbb{E}|u(t)|^p \le K|t-\tau|^{\frac{p}{2}} K(s,s)^{\frac{p}{2}} \int_0^\infty r^{p+1} e^{-r^2} e^{\sqrt{\frac{T}{b}}\gamma r} dr \le \tilde{K}^1_p |t-\tau|^{\frac{p}{2}} \tag{3.17}$$

since $K(s,s) \le 1$ for all s.

Consider the partition $\mathcal{P} = \{0 < s < \sigma < 1\}$. Then for any smooth cylinder function f of polynomial growth such that $f = F \circ \pi_{\mathcal{P}}$ for some smooth $F : G^2 \to \mathbb{C}$,

$$\mathbb{E}[f(\Sigma^0(t,\cdot)] = \mathbb{E}[F(\Sigma_{\mathcal{P}}(t,\cdot)] = (e^{\frac{t}{4}\Delta_{\mathcal{P}}} F)(e,e) = \int_{G^2} F(x,y) p^{\mathcal{P}}_t(x,y) dx dy. \tag{3.18}$$

Let $f(g) = d(g(s), g(\sigma))^p$ for all $g \in \mathcal{W}(G)$. Then $f = F \circ \pi_{\mathcal{P}}$ where $F(x,y) = d(x,y)^p$. Using our heat kernel estimates (Theorem 2.34) and Lemma 3.7, we get that

$$\begin{aligned}
\mathbb{E}[f(\Sigma^0(t,\cdot)] &= \int_{G^2} |x^{-1}y|^p p^{\mathcal{P}}_t(x,y) dx dy \\
&\le 2|s-\sigma|^{\frac{p}{2}} \int_{G^2} |\{x,y\}|^p e^{c|\{x,y\}|} p^{\mathcal{P}}_t(x,y) dx dy \\
&\le \tilde{K}^2_p |s-\sigma|^{\frac{p}{2}}
\end{aligned}$$

by a computation very similar to Eq.(3.17).

Consequently, for each $T \in (0,\infty)$, there is a constant $K_p(T)$ such that

$$\mathbb{E}[d(\Sigma^0(t,s), \Sigma^0(\tau,\sigma))^p] \le K_p(T)(|t-\tau|^{\frac{p}{2}} + |s-\sigma|^{\frac{p}{2}}),$$

for all $t, \tau \in [0, T]$ and $s, \sigma \in [0, 1]$. Therefore, by Kolmogorov's continuity criteria (see Theorem 1.4.1 of [14], or Theorem 53 of [17]) there is a continuous version $\Sigma(t, s)$ of $\Sigma^0(t, s)$ such that for all $\beta \in (0, \frac{1}{2})$ there exists a positive random variable K_β on $\mathcal{W}$ such that

$$d(\Sigma(t, s), \Sigma(\tau, \sigma)) \leq K_\beta(|t - \tau|^{\frac{p}{2}} + |s - \sigma|^{\frac{p}{2}}) \quad \text{a.s.}$$

Furthermore, $\mathbb{E}[K_\beta^p] < \infty$ for all $p \in (1, \infty)$. Since for each $s \in [0, 1]$, $\Sigma(\cdot, s)$ is a version of $\Sigma^0(\cdot, s)$, it follows that Σ satisfies the hypothesis of the theorem with $g_0(s) = e$.

For the general case, define $\hat{\Sigma}(t, s) = g_0(s)\Sigma(t, s)$. Then $\{\hat{\Sigma}(t)\}_{t \geq 0}$ is a continuous adapted $\mathcal{W}(G)$-valued process which satisfies (3.9). □

Definition 3.14. The measure $\nu_t = Law(\Sigma(t, \cdot))$ is called the heat kernel measure on $\mathcal{W}(G)$.

Definition 3.15. Let $\nu_t^{\mathcal{P}} = Law(\Sigma_{\mathcal{P}}(t))$.

Proposition 3.16. *$\nu_t^{\mathcal{P}}$ and ν_t satisfy the heat equations on $\mathcal{W}(G)$ and $G^{\#(\mathcal{P})}$ in the following weak sense. If $f = F \circ \pi_{\mathcal{P}}$ is a cylinder function, then*

$$\frac{\partial}{\partial t}\nu_t^{\mathcal{P}}(F) = \nu_t^{\mathcal{P}}(\frac{1}{4}\Delta_{\mathcal{P}}F) \tag{3.19}$$

and

$$\frac{\partial}{\partial t}\nu_t(f) = \nu_t(\frac{1}{4}\Delta_{H(G)}f). \tag{3.20}$$

Proof. Eq. (3.19) follows from the martingale decomposition in Proposition 3.12. That is

$$\begin{aligned}
\frac{\partial}{\partial t}\nu_t^{\mathcal{P}}(F) &= \frac{\partial}{\partial t}\mathbb{E}[F(\Sigma_{\mathcal{P}}(t))] \\
&= \frac{\partial}{\partial t}\mathbb{E}[M_t + \int_0^t \frac{1}{4}(\Delta_{\mathcal{P}}F)(\Sigma_{\mathcal{P}}(t))dt] \\
&= \mathbb{E}[\frac{1}{4}(\Delta_{\mathcal{P}}F)(\Sigma_{\mathcal{P}}(t))] \\
&= \nu_t^{\mathcal{P}}(\frac{1}{4}\Delta_{\mathcal{P}}F),
\end{aligned}$$

where M_t is the martingale

$$M_t = \sum_{i=1}^{n}\sum_{A \in \mathfrak{X}_{\mathbb{R}}}(A^{(i)}F)(\Sigma_{\mathcal{P}}(t))\beta_{\mathcal{P}}^A(dt, s_i).$$

Eq. (3.20) follows readily from the above, since

$$\frac{\partial}{\partial t}\nu_t(f) = \frac{\partial}{\partial t}\mathbb{E}[F \circ \pi_{\mathcal{P}}(\Sigma(t,\cdot)] = \frac{\partial}{\partial t}\mathbb{E}[F(\Sigma_{\mathcal{P}}(t)] = \nu_t^{\mathcal{P}}(\frac{1}{4}\Delta_{\mathcal{P}}F) = \nu_t(\frac{1}{4}\Delta_{H(G)}f).$$

□

Proposition 3.17. *Suppose* $f \in$ *such that* $f = F \circ \pi_{\mathcal{P}}$. *Then* $||f||_{L^2(\nu_t)} = ||F||_{L^2(\nu_t^{\mathcal{P}})}$.

Proof.

$$||f||^2_{L^2(\nu_t)} = \mathbb{E}[|f(\Sigma(t,\cdot))|^2] = \mathbb{E}[|F \circ \pi_{\mathcal{P}}(\Sigma(t,\cdot))|^2] = \mathbb{E}[|F(\Sigma_{\mathcal{P}}(t))|^2] = ||F||^2_{L^2(\nu_t^{\mathcal{P}})}.$$

□

4

The Taylor Map

4.1 Skeleton Theorem

For this chapter, we consider $T > 0$ to be fixed. In the previous chapter, we constructed ν_T, the heat kernel measure on $\mathcal{W}(G)$. Recall that $\mathcal{HFC}^\infty(\mathcal{W})$ is used to denote the holomorphic cylinder functions on $\mathcal{W}(G)$.

Definition 4.1. Let $\mathcal{H}_T$ denote the $L^2(\nu_T)$-closure of $\mathcal{HFC}^\infty(\mathcal{W})$.

We wish to establish our Taylor map on this space $\mathcal{H}_T$. In order to do so, we need a suitable notion of "derivatives at the origin" for a function $f \in \mathcal{H}_T$.

For $g \in H(G)$, define a function $R_g : \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T) \to \mathbb{C}$ by

$$R_g(f) = f(g).$$

Then R_g is clearly linear and is defined on a dense subset of $\mathcal{H}_T$. The following proposition indicates that R_g is bounded and has a continuous extension.

Proposition 4.2. *For all $g \in H(G)$, R_g can be extended uniquely to a continous linear functional on all of $\mathcal{H}_T$.*

Proof. Pick $g \in H(G)$, and let $f \in \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T)$ with $f = F \circ \pi_\mathcal{P}$ for some partition $\mathcal{P}$ of $[0, 1]$. Recall that by Definition 3.15, $\nu_T^\mathcal{P}$ is the heat kernel measure with respect to right invariant Haar measure on $G^{\#(\mathcal{P})}$ associated to the Laplacian $\frac{1}{4}\Delta_\mathcal{P}$.

Applying the finite dimensional results of Driver and Gross, specifically Remark 5.5 in [4], we find that

$$|R_g(f)|^2 = |F(\pi_{\mathcal{P}}(g))|^2 \leq ||F||^2_{L^2(\nu_T^{\mathcal{P}})} e^{\frac{|\pi_{\mathcal{P}}(g)|^2_{\mathcal{P}}}{T}}.$$

By Corollary 2.20, $|\pi_{\mathcal{P}}(g)|^2_{\mathcal{P}} \leq |g|^2_{H(G)}$, and so using Proposition 3.17,

$$|R_g(f)|^2 \leq ||f||^2_{L^2(\nu_T)} e^{\frac{|g|^2_{H(G)}}{T}}. \tag{4.1}$$

So $||R_g||^2 \leq e^{\frac{|g|^2_{H(G)}}{T}}$, and R_g is therefore continuous. For $f \in \mathcal{H}_T$, pick $\{f_n\}_{n=1}^{\infty} \subset \mathcal{HFC}^{\infty}(\mathcal{W}) \cap L^2(\nu_T)$ such that $f_n \to f$. We can then define $R_g(f) = \lim_{n\to\infty} R_g(f_n)$. □

Notation 4.3. In the sequel, R_g will refer to this extension.

Remark 4.4. Clearly, Proposition 4.2 imples that if $f_n \to f$ in $\mathcal{H}_T$, then for any $g \in H(G)$, $R_g f_n \to R_g f$. More precisely, Eq. (4.1) indicates that the convergence is locally uniform.

We will show that a function $f \in \mathcal{H}_T$ has a holomorphic "skeleton". That is, despite the fact that f is an $L^2(\nu_T)$ equivalence class, its values on $H(G)$ are determined and "$\left(f|_{H(G)}\right)(g)$" $:= R_g(f)$ is holomorphic. We prove this result in Theorem 4.7. We first need an appropriate notion of holomorphic functions on $H(G)$.

Notation 4.5. We will refer to a function $u : H(G) \to \mathbb{C}$ as holomorphic if it is holomorphic in the sense of Gross and Malliavin [11]. Specifically, we require that for every $g \in H(G)$, the map $h \in H(\mathfrak{g}) \to u(g \cdot e^h)$ is Frechét differentiable at $h = 0$ and that this Frechét derivative is complex linear and continuous in $H(\mathfrak{g})^*$ as a function of g.

Proposition 4.6. *Let G be a Lie group and suppose $F \in C^{\infty}(G)$. For every $g \in G$, define $dF_g \in \mathfrak{g}^*$ by*

$$(dF_g)(A) = \frac{d}{dt}|_{t=0} F(g \cdot e^{tA}).$$

Then dF_g is the Frechét derivative of F at g, and furthermore, dF_g is continuous in $\mathfrak{g}^*$ as a function of g.

Proof. That dF_g is continuous in $\mathfrak{g}^*$ as a function of g follows from the fact that $F \in C^{\infty}(G)$. To see that dF_g is the Frechét derivative of F at g, we need to show that

$$\lim_{A\to 0} \frac{|F(g \cdot e^A) - F(g) - (dF_g)(A)|}{||A||_{\mathfrak{g}}} = 0.$$

Notice that

$$\begin{aligned} F(g\cdot e^A) - F(g) - (dF_g)(A) &= \int_0^1 \left(\frac{d}{dt}F(g\cdot e^{tA}) - (dF_g)(A)\right) dt \\ &= \int_0^1 \left(\frac{d}{ds}|_{s=0}F(g\cdot e^{(s+t)A}) - (dF_g)(A)\right) dt \\ &= \int_0^1 \left(\frac{d}{ds}|_{s=0}F((g\cdot e^{tA})\cdot e^{sA}) - (dF_g)(A)\right) dt \\ &= \int_0^1 \left((dF_{g\cdot e^{tA}})(A) - (dF_g)(A)\right) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{A\to 0} \frac{|F(g\cdot e^A) - F(g) - (dF_g)(A)|}{||A||_{\mathfrak{g}}} &\le \lim_{A\to 0} \frac{\int_0^1 |(dF_{g\cdot e^{tA}})(A) - (dF_g)(A)|dt}{||A||_{\mathfrak{g}}} \\ &\le \lim_{A\to 0} \frac{\int_0^1 ||dF_{g\cdot e^{tA}} - dF_g||_{\mathfrak{g}^*}||A||_{\mathfrak{g}}dt}{||A||_{\mathfrak{g}}} \\ &= \lim_{A\to 0} \int_0^1 ||dF_{g\cdot e^{tA}} - dF_g||_{\mathfrak{g}^*}dt \\ &= 0, \end{aligned}$$

by the continuity in g of dF_g. □

Theorem 4.7 (Skeleton Theorem)**.** *There exists a linear map $R : \mathcal{H}_t \to \mathcal{H}(H(G))$ with the following properties:*

1. *For f a holomorphic cylinder function, $Rf = f|_{H(G)}$.*
2. *For $g \in H(G)$, $|(Rf)(g)|^2 \le ||f||^2_{L^2(\nu_T)} e^{\frac{|g|^2_{H(G)}}{t}}$.*

Proof. Given $f \in \mathcal{H}_t$, define Rf by

$$(Rf)(g) = R_g f$$

for all $g \in H(G)$. By the definition of R_g, if $f \in \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T)$, then

$$(Rf)(g) = f(g)$$

for all $g \in H(G)$. So (1) is satisfied. (2) follows from Eq. (4.1). It remains to show that $Rf \in \mathcal{H}(H(G))$.

We first suppose that $f \in \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T)$. Then $f = F \circ \pi_{\mathcal{P}}$ for some $F \in \mathcal{H}(G^{\#(\mathcal{P})})$ and some partition $\mathcal{P}$ of $[0,1]$. Let dF denote the Frechét derivative of F. Define for $g \in H(G)$,

$$(df)_g = (dF)_{\pi_{\mathcal{P}} g} \circ \pi_{\mathcal{P}*\underline{e}}.$$

Then

$$\begin{aligned}
&\lim_{h\to 0} \frac{|f(g\cdot e^h) - f(g) - (df)_g h|}{||h||_{H(\mathfrak{g})}} \\
&= \lim_{h\to 0} \frac{|F(\pi_{\mathcal{P}} g \cdot e^{\pi_{\mathcal{P}*\underline{e}} h}) - F(\pi_{\mathcal{P}} g) - (dF)_{\pi_{\mathcal{P}} g} \pi_{\mathcal{P}*\underline{e}} h|}{||h||_{H(\mathfrak{g})}} \\
&\leq \lim_{h\to 0} \frac{|F(\pi_{\mathcal{P}} g \cdot e^{\pi_{\mathcal{P}*\underline{e}} h}) - F(\pi_{\mathcal{P}} g) - (dF)_{\pi_{\mathcal{P}} g} \pi_{\mathcal{P}*\underline{e}} h|}{||\pi_{\mathcal{P}*\underline{e}} h||_{\mathcal{P}}} \qquad (4.2) \\
&= \lim_{\pi_{\mathcal{P}*\underline{e}} h\to 0} \frac{|F(\pi_{\mathcal{P}} g \cdot e^{\pi_{\mathcal{P}*\underline{e}} h}) - F(\pi_{\mathcal{P}} g) - (dF)_{\pi_{\mathcal{P}} g} \pi_{\mathcal{P}*\underline{e}} h|}{||\pi_{\mathcal{P}*\underline{e}} h||_{\mathcal{P}}} \\
&= 0,
\end{aligned}$$

by the fact that dF is the Frechét derivative of F and where in line (4.2) we used the fact that $||h||_{H(\mathfrak{g})} \geq ||\pi_{\mathcal{P}*\underline{e}} h||_{\mathcal{P}}$, which follows from Proposition 2.15. So f is Frechét differentiable at any $g \in H(G)$. df is continuous in g since both $dF_{\pi_{\mathcal{P}}\cdot}$ and $\pi_{\mathcal{P}}$ are. F holomorphic implies that dF is complex linear, and since $\pi_{\mathcal{P}*\underline{e}}$ is as well, df is complex linear. Therefore, $f|_{H(G)} = Rf \in \mathcal{H}(H(G))$.

For a general $f \in \mathcal{H}_t$, we fix $g \in H(G)$ and choose $\{f_n\}_{n=1}^\infty \subset \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T)$ such that $f_n \to f$. Remark 4.4 indicates that $(Rf_n)(g \cdot e^h) \to (Rf)(g\cdot e^h)$ uniformly for h in some neighborhood of 0. Therefore, by Theorem 3.18.1 of [13], $h \to (Rf)(g \cdot e^h)$ is holomorphic and jointly continuous in g. □

Remark 4.8. We will show later in Corollary 4.14 that R is injective.

4.2 The Taylor Isometry

Given the results of the previous section, we are able to define the Taylor map on $\mathcal{H}_T$.

Definition 4.9. Given $f \in \mathcal{H}_T$, define $\alpha_{Rf} \in J_T^0(H(\mathfrak{g}))$ by

$$\langle \alpha_{Rf}, h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle = (\tilde{h}_1 \tilde{h}_2 \cdots \tilde{h}_n Rf)(\underline{e}), \tag{4.3}$$

where $h_1, ..., h_n \in H(\mathfrak{g})$, and $\underline{e}$ denotes the identity path in $\mathcal{W}(G)$. Notice that by Proposition 4.6, $Rf \in \mathcal{H}(H(G))$, so the right hand side is well defined. The map $f \to \alpha_{Rf}$ will be called the Taylor map.

Notation 4.10. We will often use $(1-D)_{\underline{e}}^{-1} R$ to denote the above Taylor map.

Theorem 4.11. *Let $f \in \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T)$ and $\alpha_{Rf} \in T(H(\mathfrak{g}))'$ as given in the above. Then $||f||^2_{L^2(\nu_T)} = ||\alpha_{Rf}||^2_{J_T^0(H(\mathfrak{g}))}$.*

Proof. Suppose $f = F \circ \pi_{\mathcal{P}}$ where $\mathcal{P}$ is a partition of $[0,1]$. Let $S_{\mathbb{C}}^{\mathcal{P}}$ be an orthonormal basis for $(H_{\mathcal{P}}(\mathfrak{g}), (\ ,\ )_{H(\mathfrak{g})})$. Extend this to an orthonormal basis for $H(\mathfrak{g}) = H_{\mathcal{P}}(\mathfrak{g}) \overset{\perp}{\oplus} Nul(\pi_{\mathcal{P}*\underline{e}})$, which we will denote $S_{\mathbb{C}}$. Recall that by Proposition 2.12, $\mathfrak{X}_{\mathbb{C}}^{\mathcal{P}} := \{\pi_{\mathcal{P}*\underline{e}} h | h \in S_{\mathbb{C}}^{\mathcal{P}}\}$ is an orthonormal basis for $(\mathfrak{g}^{\#(\mathcal{P})}, (\ ,\ )_{\mathcal{P}})$. Note that for all $h \in H_{\mathcal{P}}(\mathfrak{g})^\perp$,

$$\partial_h f(\underline{e}) = \sum_{i=1}^n ((h(s_i)^{(i)} F) \circ \pi_{\mathcal{P}})(\underline{e}) = 0$$

since $h|_{\mathcal{P}} \equiv 0$. Then

$$\begin{aligned}
||\alpha_F||^2_{J_T^0(\mathfrak{g}^{\#(\mathcal{P})})} &= \sum_{k=0}^\infty \frac{t^k}{k!} \left(\sum_{A_1,...,A_k \in \mathfrak{X}_{\mathbb{C}}^{\mathcal{P}}} |\langle \alpha_F, A_1 \otimes \cdots \otimes A_k \rangle|^2 \right) \\
&= \sum_{k=0}^\infty \frac{t^k}{k!} \left(\sum_{A_1,...,A_k \in \mathfrak{X}_{\mathbb{C}}^{\mathcal{P}}} |(\tilde{A}_1 \cdots \tilde{A}_k F)(e, e, .., e)|^2 \right) \\
&= \sum_{k=0}^\infty \frac{t^k}{k!} \left(\sum_{h_1,...,h_k \in S_{\mathbb{C}}^{\mathcal{P}}} |(\tilde{h}_1 \cdots \tilde{h}_k f)(\underline{e})|^2 \right) \\
&= \sum_{k=0}^\infty \frac{t^k}{k!} \left(\sum_{h_1,...,h_k \in S_{\mathbb{C}}} |(\tilde{h}_1 \cdots \tilde{h}_k f)(\underline{e})|^2 \right) \\
&= ||\alpha_f||^2_{J_T^0(H(\mathfrak{g}))}.
\end{aligned}$$

Then using the finite dimensional results found in [4] and the fact that $Rf = f|_{H_0(G)}$,

$$||f||^2_{L^2(\nu_t)} = ||F||^2_{L^2(\nu_t)} = ||\alpha_F||^2_{J_T^0(\mathfrak{g}^{\#(\mathcal{P})})} = ||\alpha_f||^2_{J_T^0(H(\mathfrak{g}))} = ||\alpha_{Rf}||^2_{J_T^0(H(\mathfrak{g}))}.$$

□

Before proving Corollary 4.13, which extends this result to any $f \in \mathcal{H}_T$, we need the following theorem.

Proposition 4.12. *Suppose* $\{f_n\}_{n=1}^{\infty} \in \mathcal{HFC}^{\infty}(\mathcal{W}) \cap L^2(\nu_T)$ *with* $f_n \to f \in \mathcal{H}_T$. *Then for all* $h_1, h_2, ..., h_k \in H(\mathfrak{g})$ *and* $g \in H(G)$,

$$\left(\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k Rf_n\right)(g) \to \left(\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k Rf\right)(g).$$

Proof. Pick $g \in H(G)$, and $h_1, h_2, ..., h_k \in H(\mathfrak{g})$. Define a sequence of functions $F_n : \mathbb{C}^k \to \mathbb{C}$ by

$$F_n(\xi_1, \xi_2, ..., \xi_k) = Rf_n(g \cdot e^{\xi_1 h_1} \cdot e^{\xi_2 h_2} \cdots e^{\xi_k h_k}).$$

Similarly, define $F : \mathbb{C}^k \to \mathbb{C}$ by

$$F(\xi_1, \xi_2, ..., \xi_k) = Rf(g \cdot e^{\xi_1 h_1} \cdot e^{\xi_2 h_2} \cdots e^{\xi_k h_k}).$$

Then by Remark 4.4, $\forall \xi_1, \xi_2, ..., \xi_k \in \mathbb{C}$,

$$F_n(\xi_1, \xi_2, ..., \xi_k) \to F(\xi_1, \xi_2, ..., \xi_k)$$

uniformly in a neighborhood of the origin. Furthermore, F and F_n are holomorphic. Notice that

$$\left(\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k Rf_n\right)(g) = (\frac{d^k}{\xi_1\xi_2 \cdots \xi_k} F_n)(0, 0, ..., 0),$$

and

$$\left(\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k Rf\right)(g) = (\frac{d^k}{\xi_1\xi_2 \cdots \xi_k} F)(0, 0, ..., 0).$$

Repeated use of Cauchy's integral formula gives

$$\left(\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k Rf_n\right)(g) = \left(\frac{1}{2\pi i}\right)^k \oint_{|\xi_1|=R} \cdots \oint_{|\xi_k|=R} \frac{F_n(\xi_1, \xi_2, ..., \xi_k)}{\xi_1\xi_2 \cdots \xi_k} d\xi_k d\xi_{k-1} \cdots d\xi_1,$$

for $R>0$. The locally uniform convergence allows us to use the DCT to conclude

$$\begin{aligned}
&\lim_{n\to\infty}\left(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_kRf_n\right)(g)\\
&=\lim_{n\to\infty}\left(\frac{1}{2\pi i}\right)^k\oint_{|\xi_1|=R}\cdots\oint_{|\xi_k|=R}\frac{F_n(\xi_1,\xi_2,...,\xi_k)}{\xi_1\xi_2\cdots\xi_k}d\xi_kd\xi_{k-1}\cdots d\xi_1\\
&=\left(\frac{1}{2\pi i}\right)^k\oint_{|\xi_1|=R}\cdots\oint_{|\xi_k|=R}\lim_{n\to\infty}\frac{F_n(\xi_1,\xi_2,...,\xi_k)}{\xi_1\xi_2\cdots\xi_k}d\xi_kd\xi_{k-1}\cdots d\xi_1\\
&=\left(\frac{1}{2\pi i}\right)^k\oint_{|\xi_1|=R}\cdots\oint_{|\xi_k|=R}\frac{F(\xi_1,\xi_2,...,\xi_k)}{\xi_1\xi_2\cdots\xi_k}d\xi_kd\xi_{k-1}\cdots d\xi_1\\
&=\left(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_kRf\right)(g).
\end{aligned}$$

□

Corollary 4.13. *The Taylor map described in Definition 4.9 is an isometry, i.e. for all* $f\in\mathcal{H}_T$,

$$||f||^2_{L^2(\nu_T)}=||\alpha_{Rf}||^2_{J^0_T(H(\mathfrak{g}))}.$$

Proof. Let $\{f_n\}_{n=1}^\infty\subset\mathcal{HFC}^\infty(\mathcal{W})\cap L^2(\nu_T)$ such that $f_n\to f$. By Theorem 4.11, $\{\alpha_{Rf_n}\}_{n=1}^\infty\subset J^0_T(H(\mathfrak{g}))$ is Cauchy, and hence converges to some $\widehat{\alpha}\in J^0_T(H(\mathfrak{g}))$. It remains to show that $\widehat{\alpha}=\alpha_{Rf}$. That is, for any $h_1,h_2,...,h_k\in H(\mathfrak{g})$,

$$\langle\widehat{\alpha},h_1\otimes h_2\otimes\cdots\otimes h_k\rangle=\langle\alpha_{Rf},h_1\otimes h_2\otimes\cdots\otimes h_k\rangle.$$

By Proposition 4.12,

$$\begin{aligned}
\langle\widehat{\alpha},h_1\otimes h_2\otimes\cdots\otimes h_k\rangle&=\lim_{n\to\infty}\langle\alpha_{Rf_n},h_1\otimes h_2\otimes\cdots\otimes h_k\rangle\\
&=\lim_{n\to\infty}\left(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_kRf_n\right)(\underline{e})\\
&=\left(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_kRf\right)(\underline{e})\\
&=\langle\alpha_{Rf},h_1\otimes h_2\otimes\cdots\otimes h_k\rangle.
\end{aligned}$$

□

Corollary 4.14. *Since the Taylor map,* $(1-D)_e^{-1}R:\mathcal{H}_T\to J^0_T(H(\mathfrak{g}))$ *is injective, it necessarily follows that* $R:\mathcal{H}_T\to\mathcal{H}(H(G))$ *is injective.*

5

Surjectivity

In this chapter, we prove the surjectivity of the Taylor map when G is a stratified nilpotent Lie group. We also present a motivating example where G is the complex Heisenberg group.

5.1 Introduction

Let $\mathfrak{g}$ is a d-dimensional step r complex stratified nilpotent Lie algebra. This means that there is a sequence of nonzero subspaces V_i for $i = 1, .., r$ such that

$$\mathfrak{g} = \oplus_{i=1}^{r} V_i,$$

with $[V_1, V_j] \subset V_{j+1}$ for $j = 1, ..., r-1$ and $[V_1, V_r] = \{0\}$. It follows that $[V_i, V_j] \subset V_{i+j}$, with the convention that $V_s = \{0\}$ for $s > r$. This gives a decomposition of $H(\mathfrak{g})$,

$$H(\mathfrak{g}) = \oplus_{i=1}^{r} H(V_i),$$

with $[H(V_1), H(V_j)] \subset H(V_{j+1})$ for $j = 1, ..., r-1$ and $[H(V_1), H(V_r)] = \{0\}$. Therefore, $H(\mathfrak{g})$ is a step r complex stratified nilpotent Lie algebra as well. We will furthermore assume that the subspaces $\{V_i\}_{i=1}^{r}$ are orthogonal with respect to our inner product $(\, , \,)_{\mathfrak{g}}$.

Our goal is to show that given $\alpha \in J_T^0(H(\mathfrak{g}))$, there exists a function $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $(1-D)_{\underline{e}}^{-1} R \tilde{u}_\alpha = \alpha$. Of primary importance will be that finite rank tensors are dense in $J_T^0(H(\mathfrak{g}))$ when $H(\mathfrak{g})$ is stratified nilpotent. This was orignially proven in [6], and is included in section 6.2 of the appendix for completeness. The Taylor map will be show to be onto $\alpha \in J_T^0(H(\mathfrak{g}))$ of finite rank, and the following theorem states that this is sufficient.

Theorem 5.1. *Let $J \subset J_T^0(H(\mathfrak{g}))$ be a dense subset. If for every $\alpha \in J$ there exists a function $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $(1-D)_{\underline{e}}^{-1} R\tilde{u}_\alpha = \alpha$, then the result holds for all $\alpha \in J_T^0(H(\mathfrak{g}))$.*

Proof. Let $\alpha \in J_T^0(H(\mathfrak{g}))$, and pick a sequence $\{\alpha_n\}_{n=1}^\infty \subset J$ such that $\alpha_n \to \alpha$ as $n \to \infty$. For each α_n, there exists a $\tilde{u}_{\alpha_n} \in \mathcal{H}_T$ such that $(1-D)_{\underline{e}}^{-1} R\tilde{u}_{\alpha_n} = \alpha_n$. Recall by Corollary 4.13 that the Taylor map $(1-D)_{\underline{e}}^{-1} R : \mathcal{H}_T \to J_T^0(H(\mathfrak{g}))$ is an isometry. Since $\alpha_n \to \alpha$ in $J_T^0(H(\mathfrak{g}))$, $\{\alpha_n\}_{n=1}^\infty$ is Cauchy, and therefore so is $\{\tilde{u}_{\alpha_n}\}_{n=1}^\infty$ in $\mathcal{H}_T$. $\mathcal{H}_T$ is closed and hence there exists a $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $\tilde{u}_{\alpha_n} \to \tilde{u}_\alpha$. Finally, since the Taylor map is continuous,

$$\alpha = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (1-D)_{\underline{e}}^{-1} R\tilde{u}_{\alpha_n} = (1-D)_{\underline{e}}^{-1} R\tilde{u}_\alpha.$$

□

Remark 5.2. For the remainder of this chapter, it will be assumed that $\alpha \in J_T^0(H(\mathfrak{g}))$ is of finite rank.

Given $\alpha \in J_T^0(H(\mathfrak{g}))$, we wish to construct a converging sequence of approximating cylinder function. We first construct a holomorphic function with α as its set of derivatives at the identity. The following theorem is motivated by results in [3] and is the subject of section 5.3.

Theorem 5.3. *Let $\mathfrak{g}$ be a stratified Lie algebra and $\alpha \in J_T^0(H(\mathfrak{g}))$ be of finite rank. For every $g \in H(G)$ define*

$$u_\alpha(g) := \sum_{n=0}^\infty \langle \alpha, g^{\otimes n} \rangle / n!$$

Then u_α is a holomorphic function on $H(G)$ satisfying $(1-D)_{\underline{e}}^{-1} u_\alpha = \alpha$.

Let $\mathcal{P} = \{0 = s_0 < \cdots < s_n < s_{n+1} = 1\}$ be a partition of $[0,1]$.

Definition 5.4. Let $H_\mathcal{P}(G) = \{h \in H(G) | h'' \equiv 0 \text{ on } [0,1] \backslash \mathcal{P}\}$. Similarly, define $H_\mathcal{P}(\mathfrak{g}) = \{h \in H(\mathfrak{g}) | h'' \equiv 0 \text{ on } [0,1] \backslash \mathcal{P}\}$.

Remark 5.5. Note in the above definitions, $H_{\mathcal{P}}(\mathfrak{g})$ is a subspace of $H(\mathfrak{g})$, but not a Lie subalgebra. Furthermore, $H_{\mathcal{P}}(G)$ is not a subgroup of $H(G)$.

Definition 5.6. Let $\tilde{P}_{\mathcal{P}} : \mathcal{W}(G) \to H_{\mathcal{P}}(G)$ be defined so that $\tilde{P}_{\mathcal{P}}(g)$ and g agree at all partition points of $\mathcal{P}$ and $g'' \equiv 0$ on $[0,1]\backslash\mathcal{P}$. In a similar manner, define $P_{\mathcal{P}} : \mathcal{W}(\mathfrak{g}) \to H_{\mathcal{P}}(\mathfrak{g})$.

Remark 5.7. $P_{\mathcal{P}}$ defined above restricted to $H(\mathfrak{g})$ is orthogonal projection onto the subspace $H_{\mathcal{P}}(\mathfrak{g})$.

In order to constuct $\tilde{u}_\alpha$, we will construct a converging sequence of cylinder functions which we will show are Cauchy in $L^2(\nu_T)$. Our candidate cylinder functions are defined below.

Remark 5.8. Given a partition $\mathcal{P}$, $F_{\mathcal{P}} \equiv u_\alpha \circ \tilde{P}_{\mathcal{P}}$ defines a cylinder function.

Our first goal will be to estimate $||F_{\mathcal{P}}||^2_{L^2(\nu_t)}$ when α is of finite rank. Let $S_{\mathbb{C}}$ be an orthonormal basis for $H(\mathfrak{g})$. Then

$$||F_{\mathcal{P}}||^2_{L^2(\nu_t)} = \sum_{n=0}^{\infty} \frac{T^n}{n!} \left(\sum_{h_1,\ldots h_n \in S_{\mathbb{C}}} |(\tilde{h}_1 \cdots \tilde{h}_n F_{\mathcal{P}})(\underline{e})|^2 \right), \tag{5.1}$$

where $\underline{e}$ is the zero path, i.e. the path $\underline{e} \in H(G)$ such that $\underline{e}(t) = 0$ for all $t \in [0,1]$.

Definition 5.9. Given a partition $\mathcal{P}$, let $\alpha(\mathcal{P}) \equiv (1-D)^{-1}_{\underline{e}}(F_{\mathcal{P}})$.

Notation 5.10. For a function $u \in C^\infty(\mathcal{W}(G))$, $g \in \mathcal{W}(G)$, and $h_1, h_2, \ldots h_n \in H(\mathfrak{g})$, denote

$$\langle D^n u(g), h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle := (\tilde{h}_1 \tilde{h}_2 \cdots \tilde{h}_n u)(g). \tag{5.2}$$

Remark 5.11. Using the above notation,

$$||F_{\mathcal{P}}||^2_{L^2(\nu_t)} = \sum_{n=0}^{\infty} \frac{T^n}{n!} \left(\sum_{h_1,\ldots h_n \in S_{\mathbb{C}}} |\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle|^2 \right).$$

The following theorem summarizes the main result of this chapter.

Theorem 5.12. *Let $\{\mathcal{P}_k\}_{k=1}^\infty$ be a sequence of refining partitions and $\alpha \in J^0_T(H(\mathfrak{g}))$ of finite rank. Then for every $n > 0$, there exists a function $R^{\mathcal{P}_k}_n : H(\mathfrak{g})^n \to T(H(\mathfrak{g}))$ such that*

$$\langle \alpha(\mathcal{P}_k), h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle = \langle \alpha, P_{\mathcal{P}_k} h_1 \otimes P_{\mathcal{P}_k} h_2 \otimes \cdots \otimes P_{\mathcal{P}_k} h_n + R^{\mathcal{P}_k}_n(h_1, \ldots, h_n) \rangle$$

with the property that

$$\sum_{h_1,...h_n\in S_{\mathbb{C}}} |\langle \alpha, R_n^{\mathcal{P}_k}(h_1,...,h_n)\rangle|^2 \to 0 \quad as\ k\to\infty.$$

In particular,

$$||\alpha - \alpha(\mathcal{P}_k)||_{J_T^0(H(\mathfrak{g}))} \to 0 \quad as\ k\to\infty.$$

5.2 An example: the complex Heisenberg group

We present a proof of the surjectivity of the Taylor map in the simplest non-trivial case. The methods of this section will resemble, and perhaps motivate, those used in the general case to come. In the event that a proposition is more clearly stated or proved in the general case, we will refer the reader to the corresponding proposition in the sequel.

Let G denote the complex Heisenberg Lie group. We may realize G as $\mathbb{C}^3$ with the following group multiplication:

$$(a,b,c)\cdot(a',b',c') = (a+a', b+b', c+c'+\frac{1}{2}(ab'-a'b)).$$

Note that the origin acts as the identity, and $(a,b,c)^{-1} = (-a,-b,-c)$. Then $\mathfrak{g} \equiv T_eG$ is the complex Heisenberg Lie algebra, which again can be thought of as $\mathbb{C}^3$ with a bracket operation. If we let $X = (1,0,0)$, $Y = (0,1,0)$, and $Z = (0,0,1)$, then $\mathfrak{g} = span\{X,Y,Z\}$, where $[X,Y] = Z$ and Z is in the center of $\mathfrak{g}$. Throughout this paper, we will use coordinate notation for both elements of the group and elements of the Lie algebra. The standard inner product on $\mathbb{C}^3$,

$$((z_1,z_2,z_3),(w_1,w_2,w_3)) = z_1\overline{w}_1 + z_2\overline{w}_2 + z_3\overline{w}_3,$$

will be our given Hermitian inner product.

Remark 5.13. If we set $V_1 = span\{X,Y\}$ and $V_2 = span\{Z\}$, then $\mathfrak{g} = V_1 \overset{\perp}{\oplus} V_2$. Since $[V_1,V_1] = V_2$ and $[V_1,V_2] = [V_2,V_2] = 0$, we see that $\mathfrak{g}$ is a step 2 stratified nilpotent Lie algebra.

We intend to show that for fixed $T>0$ and $\alpha \in J_T^0(H(\mathfrak{g}))$, there exists a $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $R\tilde{u}_\alpha = u_\alpha$ where $u_\alpha \in \mathcal{H}(H(G))$ with $(1-D)_e^{-1}u_\alpha = \alpha$. We assume that α is of finite rank, i.e. there exists an $N>0$ such that $\sum_{n=0}^{\infty}\langle \alpha_n, v(n)\rangle = \sum_{n=0}^{N}\langle \alpha_n, v(n)\rangle$ for all $v(n) \in H(\mathfrak{g})^{\otimes n}$. By Theorem 5.1, the result will follow for all $\alpha \in J_T^0(H(\mathfrak{g}))$.

5.2.1 Construction of u_α

There is an obvious identification between G and $\mathfrak{g}$ in this example, and an easy calculation shows that for all $t \geq 0$,

$$e^{t(a,b,c)} = t(a,b,c).$$

This relationship holds pointwise for elements of the path group, and so for $g \in H(\mathfrak{g})$, $e^g = g \in H(G)$. Using the BCH formula, we have for $A, B \in \mathfrak{g}$,

$$\begin{aligned} A \cdot B &\equiv e^A \cdot e^B \\ &= A + B + \frac{1}{2}[A,B], \end{aligned}$$

where we are identifying $\mathfrak{g}$ and G as elements of $\mathbb{C}^3$.

The following Proposition show that elements of the Lie algebra and the path group can be thought of interchangably.

Lemma 5.14. *Suppose $g \in \mathcal{W}(\mathbb{C}^3)$. Then $g \in H(\mathfrak{g})$ iff $g \in H(G)$.*

Proof. Suppose $g = (g_1, g_2, g_3) \in H(\mathfrak{g})$, i.e. $||g'||_{L^2([0,1])} < \infty$. Note that

$$||g'||^2_{L^2([0,1])} = ||g_1'||^2_{L^2([0,1])} + ||g_2'||^2_{L^2([0,1])} + ||g_3'||^2_{L^2([0,1])},$$

so in particular, $||g_i'||^2_{L^2([0,1])} < \infty$ for $i = 1,2,3$. Also, for all $s \in [0,1]$,

$$\begin{aligned} |g_i(s)| &= |\int_0^s g_i'(r)dr| \\ &\leq \int_0^1 \left|g_i'(r)\right| dr \\ &= ||g_i'||_{L^1([0,1])} \\ &\leq ||g_i'||_{L^2([0,1])}. \end{aligned} \tag{5.3}$$

This implies that

$$\begin{aligned}||g_i||_{L^2([0,1])} &= \int_0^1 |g_i(s)|ds \\ &\leq \int_0^1 ||g_i'||_{L^2([0,1])}ds \\ &= ||g_i'||_{L^2([0,1])}. \end{aligned} \tag{5.4}$$

Observe the relationship

$$\begin{aligned} L_{g^{-1}(s)*}g'(s) &= \frac{d}{d\varepsilon}|_{\varepsilon=0}(-g(s))\cdot g(s+\varepsilon) \\ &= (g_1'(s), g_2'(s), g_3'(s) - \frac{1}{2}g_1(s)g_2'(s) + \frac{1}{2}g_2(s)g_1'(s)). \end{aligned} \tag{5.5}$$

Then considering g as an element of $\mathcal{W}(G)$ we calculate the energy:

$$\begin{aligned} E(g) &= \int_0^1 |L_{g^{-1}(s)*}g'(s)|^2 ds \\ &= \int_0^1 |(g_1'(s), g_2'(s), g_3'(s) - \frac{1}{2}(g_1(s)g_2'(s) - g_2(s)g_1'(s)))|^2 ds \\ &\leq 9\int_0^1 \left(|g'(s)|^2 + \frac{1}{4}|g_1(s)|^2|g_2'(s)|^2 + \frac{1}{4}|g_2(s)|^2|g_1'(s)|^2\right) ds \\ &\leq 9||g'||^2_{L^2([0,1])} + \frac{9}{2}||g_1'||^2_{L^2([0,1])}||g_2'||^2_{L^2([0,1])} \\ &< \infty. \end{aligned}$$

So $g \in H(G)$.

Now suppose $g \in H(G)$, i.e. $E(g) < \infty$. Using Eq. (5.5), we see that since

$$E(g)^2 = ||g_1'||^2_{L^2([0,1])} + ||g_1'||^2_{L^2([0,1])} + ||g_3' - \frac{1}{2}(g_1g_2' - g_2g_1')||^2_{L^2([0,1])},$$

we have that $||g_1'||_{L^2([0,1])}$, $||g_2'||_{L^2([0,1])}$, and $||g_3' - \frac{1}{2}(g_1g_2' - g_2g_1')||^2_{L^2([0,1])}$ are finite. So it is sufficient to show that $||g_3'||_{L^2([0,1])} < \infty$. Note that

$$\begin{aligned} ||g_3'||_{L^2([0,1])} &\leq ||g_3' - \frac{1}{2}(g_1g_2' - g_2g_1')||_{L^2([0,1])} \\ &\quad + \frac{1}{2}\left|\left|g_1g_2' - g_2g_1'\right|\right|_{L^2([0,1])}. \end{aligned}$$

By Eq. (5.4),

$$\begin{aligned} ||g_1 g_2' - g_2 g_1'||_{L^2([0,1])} &\leq ||g_1 g_2'||_{L^2([0,1])} + ||g_2 g_1'||_{L^2([0,1])} \\ &\leq 2||g_1'||_{L^2([0,1])}||g_2'||_{L^2([0,1])} \\ &< \infty. \end{aligned}$$

Therefore, $||g_3'||_{L^2([0,1])} < \infty$. □

Throughout the paper, we will make use of the above relationship, often without comment.

Theorem 5.15. *For every $\alpha \in J_T^0$ of finite rank, define*

$$u_\alpha(g) := \sum_{n=0}^{\infty} \langle \alpha, g^{\otimes n} \rangle / n!$$

Then $u_\alpha \in \mathcal{H}(H(G))$ such that $(1-D)_e^{-1} u_\alpha = \alpha$.

This theorem is restated and proved in the general case in Theorem 5.43.

5.2.2 Cylinder Function Approximations

Let $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ be a partition of $[0,1]$. In order to constuct $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $(1-D)_e^{-1} R\tilde{u}_\alpha = \alpha$, we will construct a sequence of cylinder functions which we will show are Cauchy in $L^2(\nu_T)$. Our candidate cylinder functions are defined below.

Remark 5.16. Given a partition $\mathcal{P}$, $F_\mathcal{P} := u_\alpha \circ P_\mathcal{P}$ defines a cylinder function, where $P_\mathcal{P}$ is defined as in Definition 5.6.

Our first goal will be to estimate $||F_\mathcal{P}||^2_{L^2(\nu_T)}$ when α is of finite rank. Let $S_\mathbb{C}$ be an orthonormal basis for $H(\mathfrak{g})$. Then

$$||F_\mathcal{P}||^2_{L^2(\nu_T)} = \sum_{n=0}^{\infty} \frac{T^n}{n!} \left(\sum_{h_1, \ldots h_n \in S_\mathbb{C}} |(\tilde{h}_1 \cdots \tilde{h}_n F_\mathcal{P})(\underline{e})|^2 \right), \tag{5.6}$$

What follows is a rewriting of the above in terms of derivatives of u_α at the zero path.

Remark 5.17. It is not difficult to check that sums like (5.6) above are basis independent. In the sequel, our calculations will be rewritings of such sums, and hence be basis independent as well.

Definition 5.18. Given a partition $\mathcal{P}$, let $\alpha(\mathcal{P}) \equiv (1-D)_{\underline{e}}^{-1}(F_{\mathcal{P}})$.

Notation 5.19. For a function $u \in C^\infty(\mathcal{W}(G))$, $g \in \mathcal{W}(G)$, and $h_1, h_2, ...h_n \in H(\mathfrak{g})$, denote

$$(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_n u)(g) \equiv \langle D^n u(g), h_1 \otimes h_2 \otimes \cdots \otimes h_n\rangle. \tag{5.7}$$

Lets first consider single derivatives of $F_{\mathcal{P}}$. A quick calculation yields

$$(\tilde{h}F_{\mathcal{P}})(g) = \frac{d}{dt}|_{t=0}F_{\mathcal{P}}(g\cdot th) = \frac{d}{dt}|_{t=0}u_\alpha(P_{\mathcal{P}}(g\cdot th)),$$

so if we set

$$h_{\mathcal{P}}(g) = L_{P_{\mathcal{P}}(g)^{-1}*}\frac{d}{dt}|_{t=0}P_{\mathcal{P}}(g\cdot th),$$

then we have

$$(\tilde{h}F_{\mathcal{P}})(g) = \langle Du_\alpha(P_{\mathcal{P}}(g)), h_{\mathcal{P}}(g)\rangle. \tag{5.8}$$

Notation 5.20. For the remainder of this section, upper indices will be used to denote coordinate functions.

Let $g = (g^1, g^2, g^3)$ and $h = (h^1, h^2, h^3)$. Then

$$\begin{aligned}
&h_{\mathcal{P}}(g)\\
&= \frac{d}{dt}|_0 P_{\mathcal{P}}(-g)\cdot P_{\mathcal{P}}(g\cdot th)\\
&= \frac{d}{dt}|_0(-P_{\mathcal{P}}g)\cdot(P_{\mathcal{P}}g + tP_{\mathcal{P}}h + (0,0,\frac{t}{2}P_{\mathcal{P}}(g^1h^2 - g^2h^1))\\
&= \frac{d}{dt}|_0(tP_{\mathcal{P}}h + \frac{t}{2}(0,0,P_{\mathcal{P}}(g^1h^2 - g^2h^1) - (P_{\mathcal{P}}g^1)(P_{\mathcal{P}}h^2) + (P_{\mathcal{P}}g^2)(P_{\mathcal{P}}h^1)))\\
&= P_{\mathcal{P}}h + \frac{1}{2}(P_{\mathcal{P}}[g,h] - [P_{\mathcal{P}}g, P_{\mathcal{P}}h]).
\end{aligned}$$

Notation 5.21. Let $R_{\mathcal{P}}(g,h) = \frac{1}{2}(P_{\mathcal{P}}[g,h] - [P_{\mathcal{P}}g, P_{\mathcal{P}}h])$. Therefore,

$$h_{\mathcal{P}}(g) = P_{\mathcal{P}}h + R_{\mathcal{P}}(g,h).$$

Remark 5.22. If we let $\delta_i = s_{i+1} - s_i$ and $\delta_i g^j = g^j(s_{i+1}) - g^j(s_i)$ for $i = 0, 1, ..., n-1$ and $j = 1, 2, 3$, then a calculation reveals that

$$R_{\mathcal{P}}(g,h)(s) = \sum_{i=0}^{n-1} 1_{(s_i,s_{i+1}]}(s)(0, 0, \frac{1}{2}(\delta_i g^1 \delta_i h^2 - \delta_i g^2 \delta_i h^1)(\frac{(s-s_i)}{\delta_i} - \frac{(s-s_i)^2}{\delta_i^2})). \quad (5.9)$$

Proposition 5.57 gives the equivalent statement in the general case.

Remark 5.23. Note that $R_{\mathcal{P}}(e,h) \equiv 0$, and since $R_{\mathcal{P}}(g,h)$ is zero except in the third component, we have $[R_{\mathcal{P}}(g,h), k] \equiv 0$ for all $k \in H(\mathfrak{g})$.

The above allows us to characterize the first derivatives of $F_{\mathcal{P}}$ at the zero path in terms of α,

$$\begin{aligned}(\tilde{h}F_{\mathcal{P}})(\underline{e}) &= \langle Du_\alpha(P_{\mathcal{P}}(\underline{e})), h_{\mathcal{P}}(\underline{e})\rangle \\ &= \langle Du_\alpha(P_{\mathcal{P}}(\underline{e})), P_{\mathcal{P}}h + R_{\mathcal{P}}(\underline{e}, h)\rangle \\ &= \langle Du_\alpha(\underline{e}), P_{\mathcal{P}}h\rangle \\ &= \langle \alpha, P_{\mathcal{P}}h\rangle,\end{aligned}$$

or in other words,

$$\langle \alpha(\mathcal{P}), h\rangle = \langle \alpha, P_{\mathcal{P}}h\rangle. \quad (5.10)$$

We wish to obtain the analagous result for higher order derivatives. We'll do the computation for second order derivatives, and the result for higher order derivatives will follow by similar computations. First, a claim with a simple proof:

Claim 5.24. $\frac{d}{dt}|_{t=0} R_{\mathcal{P}}(g \cdot tk, h) := R_{\mathcal{P}}(\tilde{k}(g), h) = R_{\mathcal{P}}(k, h)$.

For all $g \in \mathcal{W}(G)$ and $h, k \in H(\mathfrak{g})$,

$$\begin{aligned}(\tilde{k}\tilde{h}F_{\mathcal{P}})(g) &= \frac{d}{dt}|_{t=0}\left((P_{\mathcal{P}}h + \widetilde{R_{\mathcal{P}}(g \cdot tk, h)}u_\alpha\right)(P_{\mathcal{P}}(g \cdot tk)) \\ &= \left((P_{\mathcal{P}}k + \widetilde{R_{\mathcal{P}}}(g,k)(P_{\mathcal{P}}h + \widetilde{R_{\mathcal{P}}}(g,h)u_\alpha\right)(P_{\mathcal{P}}(g)) \\ &+ \left(\widetilde{R_{\mathcal{P}}(k,h)}u_\alpha\right)(P_{\mathcal{P}}(g)) \\ &= \langle D^2u_\alpha(P_{\mathcal{P}}(g)), (P_{\mathcal{P}}k + R_{\mathcal{P}}(g,k) \otimes (P_{\mathcal{P}}h + R_{\mathcal{P}}(g,h)\rangle \\ &+ \langle Du_\alpha(P_{\mathcal{P}}(g)), R_{\mathcal{P}}(k,h)\rangle.\end{aligned}$$

Therefore,

$$(\tilde{k}\tilde{h}F_{\mathcal{P}})(\underline{e}) := \langle \alpha_2(\mathcal{P}), k \otimes h\rangle = \langle \alpha_2, P_{\mathcal{P}}k \otimes P_{\mathcal{P}}h\rangle + \langle \alpha_1, R_{\mathcal{P}}(k,h)\rangle.$$

The general expression is best expressed using the following notation. Given $h_1, h_2, ..., h_k \in H(\mathfrak{g})$, let $\Omega_{h_1...h_k} := \{(h_i, h_j)|i < j\}$. Also, let $\theta_{h_1h_2...h_k} \equiv P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes P_{\mathcal{P}}h_k$, and if $x \in \Omega_{h_1...h_k}$ with $x = (h_i, h_j)$ then let $\theta_{h_1h_2...h_k}/\{x\} = P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes \widehat{P_{\mathcal{P}}h_i} \otimes \cdots \otimes \widehat{P_{\mathcal{P}}h_j} \otimes \cdots \otimes P_{\mathcal{P}}h_k$, where the hat denotes omission. Then we have that

$$\begin{aligned}
&(\tilde{h}_1 \cdots \tilde{h}_k F_{\mathcal{P}})(\underline{e}) \\
&= \langle \alpha(\mathcal{P}), h_1 \otimes \cdots \otimes h_k\rangle \\
&= \langle \alpha, \theta_{h_1...h_k}\rangle + \sum_{x_1 \in \Omega_{h_1...h_k}} \langle \alpha, (\theta_{h_1h_2...h_k}/\{x_1\}) \otimes R_{\mathcal{P}}(x_1)\rangle \\
&+ \sum_{x_1,x_2 \in \Omega_{h_1...h_k}} \langle \alpha, (\theta_{h_1...h_k}/\{x_1, x_2\}) \otimes R_{\mathcal{P}}(x_1) \otimes R_{\mathcal{P}}(x_2)\rangle + ... \\
&+ \sum_{x_1,...,x_{\lfloor \frac{k}{2} \rfloor} \in \Omega_{h_1...h_k}} \langle \alpha, (\theta_{h_1...h_k}/\{x_1, ..., x_{\lfloor \frac{k}{2} \rfloor}\}) \otimes_{i=1}^{\lfloor \frac{k}{2} \rfloor} R_{\mathcal{P}}(x_i))\rangle. \qquad (5.11)
\end{aligned}$$

Remark 5.25. In the above, we have pushed all the $R_{\mathcal{P}}$ terms to the right. This allowed by Remark 5.23.

For example, given $h_1, h_2, h_3, h_4 \in H(\mathfrak{g})$, we have that

$$\begin{aligned}
(\tilde{h}_1\tilde{h}_2\tilde{h}_3\tilde{h}_4 F_{\mathcal{P}})(e) &= \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes P_{\mathcal{P}}h_3 \otimes P_{\mathcal{P}}h_4\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}h_3 \otimes P_{\mathcal{P}}h_4 \otimes R_{\mathcal{P}}(h_1, h_2)\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}h_2 \otimes P_{\mathcal{P}}h_4 \otimes R_{\mathcal{P}}(h_1, h_3)\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}h_2 \otimes P_{\mathcal{P}}h_3 \otimes R_{\mathcal{P}}(h_1, h_4)\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_4 \otimes R_{\mathcal{P}}(h_2, h_3)\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_3 \otimes R_{\mathcal{P}}(h_2, h_4)\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes R_{\mathcal{P}}(h_3, h_4)\rangle \\
&+ \langle \alpha, R_{\mathcal{P}}(h_1, h_2) \otimes R_{\mathcal{P}}(h_3, h_4)\rangle \\
&+ \langle \alpha, R_{\mathcal{P}}(h_1, h_3) \otimes R_{\mathcal{P}}(h_2, h_4)\rangle \\
&+ \langle \alpha, R_{\mathcal{P}}(h_1, h_4) \otimes R_{\mathcal{P}}(h_2, h_3)\rangle.
\end{aligned}$$

5.2.3 $L^2(\nu_T)$ Estimates

The goal of this section is to show that for any partition $\mathcal{P}$, the cylinder function $F_{\mathcal{P}}$ is in $L^2(\nu_T)$. The following proposition is proven in the general case in Proposition 5.80 of the sequel.

Proposition 5.26. *Given $\alpha \in T(H(\mathfrak{g}))^*_T$, and $h_1, h_2, ...h_k \in S_0$, $\beta_{h_1 h_2 ... h_k} \in (H(\mathfrak{g})^*)^{\otimes n}$ given by $\beta_{h_1 h_2 ... h_k} = \alpha \circ L_{h_1 \otimes \cdots \otimes h_k}$ satisfies*

$$\langle \alpha, h_1 \otimes \cdots \otimes h_k \otimes \eta \rangle = \langle \beta_{h_1 h_2 .. h_k}, \eta \rangle, \tag{5.12}$$

for any $\eta \in H(\mathfrak{g})^{\otimes n}$, and furthermore,

$$\sum_{h_1, h_2, ..., h_k \in S_0} ||\beta_{h_1 h_2 .. h_k}||^2_{(H(\mathfrak{g})^*)^{\otimes n}} = ||\alpha_{n+k}||^2_{(H(\mathfrak{g})^*)^{\otimes k}} < \infty. \tag{5.13}$$

Section 1 of the appendix concerns the reproducing kernel for path spaces. For calculations in this specific case, the follow proposition is sufficient. A proof can be found in Proposition 6.3.

Proposition 5.27. *For $i, j = 1, 2, ... \dim(G)$, we have $\sum_{h \in S_{\mathbb{C}}} h^i(s)\, \overline{h^j(t)} = \delta_{ij}(s \wedge t)$.*

Proposition 5.28. *For every partition $\mathcal{P}$, $\sum_{h,k \in S_{\mathbb{C}}} ||R_{\mathcal{P}}(h,k)||^2 = \frac{1}{6}$.*

Proof. A quick calculation using Remark 5.22 reveals that

$$||R_{\mathcal{P}}(h,k)||^2 = \tilde{C} \sum_{i=0}^{n-1} \frac{|\delta_i h^1 \delta_i k^2 - \delta_i h^2 \delta_i k^1|^2}{\delta_i},$$

where

$$\tilde{C} = \frac{1}{4} \int_0^1 (1-2t)^2 dt = \frac{1}{12}.$$

Then

$$\sum_{h,k \in S_{\mathbb{C}}} ||R_{\mathcal{P}}(h,k)||^2 = \frac{1}{12} \sum_{i=0}^{n-1} \sum_{h,k \in S_{\mathbb{C}}} \frac{|\delta_i h^1 \delta_i k^2 - \delta_i h^2 \delta_i k^1|^2}{\delta_i}.$$

Multiplying out the expression on the right gives

$$\begin{aligned}\sum_{h,k\in S_{\mathbb{C}}} \frac{|\delta_i h^1 \delta_i k^2 - \delta_i h^2 \delta_i k^1|^2}{\delta_i} &= \sum_{h,k\in S_{\mathbb{C}}} \cdot \frac{|\delta_i h^1|^2 |\delta_i k^2|^2}{\delta_i} \\ &- \sum_{h,k\in S_{\mathbb{C}}} \frac{\delta_i h^1 \delta_i k^2 \overline{\delta_i h^2 \delta_i k^1}}{\delta_i} \\ &- \sum_{h,k\in S_{\mathbb{C}}} \frac{\delta_i h^2 \delta_i k^1 \overline{\delta_i h^1 \delta_i k^2}}{\delta_i} \\ &+ \sum_{h,k\in S_{\mathbb{C}}} \frac{|\delta_i h^2|^2 |\delta_i k^1|^2}{\delta_i}.\end{aligned}$$

Using Proposition 5.27, we get that the first and last terms in the above are each δ_i, while the middle terms are zero. Hence,

$$\sum_{h,k\in S_{\mathbb{C}}} ||R_{\mathcal{P}}(h,k)||^2 = \frac{1}{12}\sum_{i=0}^{n-1} 2\delta_i = \frac{1}{6}.$$

□

We are now able to show that our cylinder functions $F_{\mathcal{P}}$ are square integrable for every partition $\mathcal{P}$. Note that using our notation from before,

$$||F_{\mathcal{P}}||^2_{L^2(\nu_T)} = \sum_{n=0}^{\infty} \frac{T^n}{n!} \left(\sum_{h_1,\dots h_n \in S_{\mathbb{C}}} |\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle|^2 \right). \tag{5.14}$$

Our assumption that α is of finite rank, along with (5.11) allow us to rewrite the expression as

$$\begin{aligned}&||F_{\mathcal{P}}||^2_{L^2(\nu_T)} \\ &= \sum_{k=0}^{N} \frac{T^k}{k!} \sum_{h_1,\dots h_k \in S_{\mathbb{C}}} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{x_1,\dots,x_j \in \Omega_{h_1,\dots,h_k}} |\langle \alpha, (\theta_{h_1\dots,h_k}/\{x_1,\dots,x_j\}) \otimes_{i=1}^{j} R_{\mathcal{P}}(x_i) \rangle|^2.\end{aligned} \tag{5.15}$$

The important point is that now the right hand side only involves a finite number of sums over our basis elements, each of which can be controlled. So in order to show that $||F_{\mathcal{P}}||^2_{L^2(\nu_T)}$ is finite, it suffices to pick a "typical" term from the right hand side and show that it is finite. The following should be enough to convince the reader that a typical term in line (5.15) is finite without any of the cumbersome notation the general case would require. Consider

$$\sum_{h_1,\dots h_k\in S_{\mathbb{C}}} |\langle \alpha, P_{\mathcal{P}}h_1 \otimes \cdots \otimes P_{\mathcal{P}}h_{k-4} \otimes R_{\mathcal{P}}(h_{k-3},h_{k-2}) \otimes R_{\mathcal{P}}(h_{k-1},h_k)\rangle|^2. \tag{5.16}$$

Recall that $P_{\mathcal{P}}$ is orthogonal projection onto the subspace $H_{\mathcal{P}}(\mathfrak{g})$, so if we first choose an orthonormal basis $S_{\mathbb{C}}^{\mathcal{P}}$ for $H_{\mathcal{P}}(\mathfrak{g})$, then extend this to an orthonormal basis for $S_{\mathbb{C}}$, we get that the above (5.16) is equal to

$$\begin{aligned}
&\sum_{h_2,\dots h_k\in S_{\mathbb{C}}} \sum_{h_1\in S_{\mathbb{C}}^{\mathcal{P}}} |\langle \alpha, h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes P_{\mathcal{P}}h_{k-4} \otimes R_{\mathcal{P}}(h_{k-3},h_{k-2}) \otimes R_{\mathcal{P}}(h_{k-1},h_k)\rangle|^2 \\
&\le \sum_{h_1,\dots h_k\in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes P_{\mathcal{P}}h_{k-4} \otimes R_{\mathcal{P}}(h_{k-3},h_{k-2}) \otimes R_{\mathcal{P}}(h_{k-1},h_k)\rangle|^2.
\end{aligned}$$

We can repeat this procedure and in this manner change all of the $P_{\mathcal{P}}h_i$ terms into h_i. Finally, we use Proposition 5.26 in combination with Proposition 5.28 to bound the whole term.

$$\begin{aligned}
&\sum_{h_1,\dots h_k\in S_{\mathbb{C}}} |\langle \alpha, P_{\mathcal{P}}h_1 \cdots \otimes P_{\mathcal{P}}h_{k-4} \otimes R_{\mathcal{P}}(h_{k-3},h_{k-2}) \otimes R_{\mathcal{P}}(h_{k-1},h_k)\rangle|^2 \\
&\le \sum_{h_1,\dots h_k\in S_{\mathbb{C}}} |\langle \alpha, h_1 \cdots \otimes h_{k-4} \otimes R_{\mathcal{P}}(h_{k-3},h_{k-2}) \otimes R_{\mathcal{P}}(h_{k-1},h_k)\rangle|^2 \\
&= \sum_{h_1,\dots h_k\in S_{\mathbb{C}}} |\langle \beta_{h_1\dots h_{k-4}}, R_{\mathcal{P}}(h_{k-3},h_{k-2}) \otimes R_{\mathcal{P}}(h_{k-1},h_k)\rangle|^2 \\
&\le \sum_{h_1,\dots h_k\in S_{\mathbb{C}}} ||\beta_{h_1\dots h_{k-4}}||^2_{(H(\mathfrak{g})^*)^{\otimes k-4}} ||R_{\mathcal{P}}(h_{k-3},h_{k-2})||^2_{H(\mathfrak{g})} ||R_{\mathcal{P}}(h_{k-1},h_k)||^2_{H(\mathfrak{g})} \\
&= \left(\sum_{h_1,\dots h_{k-4}\in S_{\mathbb{C}}} ||\beta_{h_1\dots h_{k-4}}||^2_{(H(\mathfrak{g})^*)^{\otimes k-4}} \right) \left(\sum_{h_{k-3},h_{k-2}\in S_{\mathbb{C}}} ||R_{\mathcal{P}}(h_{k-3},h_{k-2})||^2_{H(\mathfrak{g})} \right) \\
&\times \left(\sum_{h_{k-1},h_k\in S_{\mathbb{C}}} ||R_{\mathcal{P}}(h_{k-1},h_k)||^2_{H(\mathfrak{g})} \right) \\
&= ||\alpha_{k-4}||^2_{(H(\mathfrak{g})^*)^{\otimes k-4}} \left(\frac{1}{6} \right)^2.
\end{aligned} \tag{5.17}$$

Remark 5.29. Note that the bound in (5.17) of the above is independent of our partition $\mathcal{P}$.

Since Eq. (5.15) involves only a finite number of sums like those in Eq. (5.16), each of which is bounded independent of partition by calculations similar to Eq. (5.17), we have now shown that $F_{\mathcal{P}} \in L^2(\nu_T)$, and $||F_{\mathcal{P}}||_{L^2(\nu_T)}$ is bounded independent of partition.

5.2.4 Convergence of $F_{\mathcal{P}}$

In this section, we'll show $||\alpha - \alpha(\mathcal{P}_n)||^2_{J^0_T(H(\mathfrak{g})} \to 0$ as $n \to 0$, where $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of refining partitions. This will imply that $F_{\mathcal{P}_n}$ is $L^2(\nu_T)-$Cauchy, hence the limiting function $\tilde{u}_\alpha$ is in $\mathcal{H}_T$. We first require the following proposition, which is true for any sequence of partitions with $|\mathcal{P}| \to 0$.

Proposition 5.30. *Given* $\alpha \in H(\mathfrak{g})^*$,

$$\sum_{h,k \in S_0} |\langle \alpha, R_{\mathcal{P}}(h,k)\rangle|^2 \to 0 \quad \text{as } |\mathcal{P}| \to 0.$$

Proof. For all $v \in H(\mathfrak{g})$, there exists an $\tilde{\alpha} \in H(\mathfrak{g})$ such that $\langle \alpha, v\rangle = (v, \tilde{\alpha})_{H(\mathfrak{g})}$. Suppose $\tilde{\alpha}(s) = (x\,(s)\,, y(s), z(s))$, and $R_{\mathcal{P}}(h_1, h_2)_3$ represents the third component of $R_{\mathcal{P}}(h_1, h_2)$. Then

$$\langle \alpha, R_{\mathcal{P}}(h,k)\rangle = (R_{\mathcal{P}}(h,k), \tilde{\alpha})_{H(\mathfrak{g})} = \int_0^1 R_{\mathcal{P}}(h,k)'_3(s)\overline{z'(s)}ds.$$

Summing over our orthonormal basis we get

$$\begin{aligned} \sum_{h,k \in S_{\mathbb{C}}} |\langle \alpha, R_{\mathcal{P}}(h,k)\rangle|^2 &= \sum_{h,k \in S_{\mathbb{C}}} \int_0^1 R_{\mathcal{P}}(h,k)'_3(s)\overline{z'(s)}ds \int_0^1 \overline{R_{\mathcal{P}}(h,k)'_3(t)}z'(t)dt \\ &= \int_{[0,1]^2} \overline{z'(s)}z'(t) \sum_{h,k \in S_{\mathbb{C}}} R_{\mathcal{P}}(h,k)'_3(s)\overline{R_{\mathcal{P}}(h,k)'_3(t)}dsdt, \end{aligned} \tag{5.18}$$

where the second equality is justified by Fubini's theorem and Proposition 5.28.

Define

$$G_{\mathcal{P}}(s,t) := \sum_{h,k \in S_0} \left(R_{\mathcal{P}}(h,k)'_3(s)\overline{R_{\mathcal{P}}(h,k)'_3(t)} \right). \tag{5.19}$$

It will be shown that $||G_{\mathcal{P}}||_{L^2([0,1]^2)} \to 0$ as $|\mathcal{P}| \to 0$. Note that by Eq. (5.9),

$$|G_{\mathcal{P}}(s,t)| = \sum_{h,k \in S_{\mathbb{C}}} \sum_{i,j=0}^{n-1} \frac{1}{\delta_i \delta_j} (\delta_i h^1 \delta_i k^2 - \delta_i h^2 \delta_i k^1)\overline{(\delta_j h^1 \delta_j k^2 - \delta_j h^2 \delta_j k^1)} K^{ij}_{\mathcal{P}}(s,t), \tag{5.20}$$

Where

$$K_{\mathcal{P}}^{ij}(s,t) := 1_{(s_i,s_{i+1}]}(s)1_{(s_j,s_{j+1}]}(t)\frac{1}{4}(1-\frac{2(s-s_i)}{\delta_i})(1-\frac{2(t-s_j)}{\delta_j}).$$

By computations similar to those in Proposition 5.28,

$$\sum_{h,k\in S_{\mathbb{C}}}(\delta_i h^1\delta_i k^2-\delta_i h^2\delta_i k^1)\overline{(\delta_j h^1\delta_j k^2-\delta_j h^2\delta_j k^1)}=2\delta_{ij}\delta_i\delta_j,$$

where δ_{ij} is the Kronecker delta. Therefore, Eq. (5.20) reduces to

$$|G_{\mathcal{P}}(s,t)|=\sum_{i=0}^{n-1}2K_{\mathcal{P}}^{ii}(s,t).$$

Hence the function $G_{\mathcal{P}}$ has support only on the set $\{(s,t)|s_i\leq s,t\leq s_{i+1}$ for some $i=0,..,n-1\}$, and since $G_{\mathcal{P}}$ is bounded independent of partition, we have that $||G_{\mathcal{P}}||_{L^2([0,1]^2)}\to 0$ as $|\mathcal{P}|\to 0$. Also,

$$\begin{aligned}||\overline{z'}\otimes z'||_{L^2([0,1]^2)} &= \left(\int_{[0,1]^2}|\overline{z'(s)}z'(t)|^2dsdt\right)^{1/2}\\ &= \left(\int_0^1|z'(s)|^2ds\int_0^1|z'(t)|^2dt\right)^{1/2}\\ &\leq ||\tilde{\alpha}||_{H(\mathfrak{g})}^2<\infty. \end{aligned}\tag{5.21}$$

In summary,

$$\begin{aligned}\sum_{h,k\in S_{\mathbb{C}}}|\langle\alpha,R_{\mathcal{P}}(h,k)\rangle|^2 &= \sum_{h,k\in S_{\mathbb{C}}}|(R_{\mathcal{P}}(h,k),\tilde{\alpha})_{H(\mathfrak{g})}|^2\\ &= \int_{[0,1]^2}\overline{z'(s)}z'(t)G_{\mathcal{P}}(s,t)dsdt\\ &= (\overline{z'}\otimes z',G_{\mathcal{P}})_{L^2([0,1]^2)}\\ &\leq ||\overline{z'}\otimes z'||_{L^2([0,1]^2)}||G_{\mathcal{P}}||_{L^2([0,1]^2)}. \end{aligned}\tag{5.22}$$

The result follows from taking the limit $|\mathcal{P}|\to 0$. □

The above proposition forms the basis of the more general result:

Proposition 5.31. *Given* $\alpha\in(H(\mathfrak{g})^{\otimes m})^*$,

$$\sum_{h_1,h_2,\ldots h_{2m}\in S_0}|\langle\alpha,R_{\mathcal{P}}(h_1,h_2)\otimes\cdots\otimes R_{\mathcal{P}}(h_{2m-1},h_{2m})\rangle|^2\to 0$$

as $|\mathcal{P}|\to 0$.

Proof. For $\alpha \in (H(\mathfrak{g})^{\otimes m})^*$, define

$$\phi_{\mathcal{P}}(\alpha) := \sqrt{\sum_{h_1,h_2,\ldots h_{2m} \in S_{\mathbb{C}}} |\langle \alpha, R_{\mathcal{P}}(h_1,h_2) \otimes \cdots \otimes R_{\mathcal{P}}(h_{2m-1},h_{2m})\rangle|^2}.$$

Then $\phi_{\mathcal{P}}$ is a seminorm on $(H(\mathfrak{g})^{\otimes m})^*$. Using Proposition 5.28, we can see that

$$\begin{aligned}\phi_{\mathcal{P}}(\alpha)^2 &\leq ||\alpha||^2_{(H(\mathfrak{g})^{\otimes m})^*} \sum_{h_1,h_2,\ldots h_{2m} \in S_{\mathbb{C}}} ||R_{\mathcal{P}}(h_1,h_2) \otimes \cdots \otimes R_{\mathcal{P}}(h_{2m-1},h_{2m})||^2_{H(\mathfrak{g})^{\otimes m}} \\ &\leq ||\alpha||^2_{(H(\mathfrak{g})^{\otimes m})^*} \left(\frac{1}{6}\right)^m,\end{aligned}$$

or equivalently that

$$\phi_{\mathcal{P}}(\alpha) \leq ||\alpha||_{(H(\mathfrak{g})^{\otimes m})^*} \left(\frac{1}{6}\right)^{\frac{m}{2}}. \tag{5.23}$$

Suppose $\alpha = (\cdot, k_1 \otimes \cdots \otimes k_m)_{H(\mathfrak{g})^{\otimes m}}$. Then

$$\begin{aligned}\phi_{\mathcal{P}}(\alpha)^2 &= \sum_{h_1,h_2,\ldots h_{2m} \in S_{\mathbb{C}}} |(R_{\mathcal{P}}(h_1,h_2) \otimes \cdots \otimes R_{\mathcal{P}}(h_{2m-1},h_{2m}), k_1 \otimes \cdots \otimes k_m)_{H(\mathfrak{g})^{\otimes m}}|^2 \\ &\leq \sum_{h_1,h_2,\ldots h_{2m} \in S_{\mathbb{C}}} |(R_{\mathcal{P}}(h_1,h_2), k_1)|^2_{H(\mathfrak{g})} \times \cdots \times |(R_{\mathcal{P}}(h_{2m-1},h_{2m}), k_m)|^2_{H(\mathfrak{g})} \\ &= \left(||k_1||^2_{H(\mathfrak{g})} ||G_{\mathcal{P}}||_{L^2([0,1]^2)}\right) \times \cdots \times \left(||k_m||^2_{H(\mathfrak{g})} ||G_{\mathcal{P}}||_{L^2([0,1]^2)}\right) \\ &= ||G_{\mathcal{P}}||^m_{L^2([0,1]^2)} \prod_{i=1}^m ||k_i||^2_{H(\mathfrak{g})},\end{aligned}$$

where $G_{\mathcal{P}}$ is as given in Eq. (5.19). The proof of Proposition 5.30 tells us that $||G_{\mathcal{P}}||_{L^2([0,1]^2)} \to 0$ as $|\mathcal{P}| \to 0$. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$. Furthermore, the same is true for all finite linear combinations of such indecomposable α. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$ for all α of finite rank.

For $\alpha, \beta \in (H(\mathfrak{g})^{\otimes m})^*$,

$$\begin{aligned}\phi_{\mathcal{P}}(\alpha) &\leq \phi_{\mathcal{P}}(\alpha - \beta) + \phi_{\mathcal{P}}(\beta) \\ &\leq \left(\frac{1}{6}\right)^{\frac{m}{2}} ||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes m})^*} + \phi_{\mathcal{P}}(\beta).\end{aligned}$$

Let $\varepsilon > 0$ and choose $\beta \in (H(\mathfrak{g})^{\otimes m})^*$ finite rank such that $||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes m})^*} < \varepsilon$. Then

$$\begin{aligned}\lim_{|\mathcal{P}|\to 0} \phi_{\mathcal{P}}(\alpha) &\leq \lim_{|\mathcal{P}|\to 0} \left(\frac{1}{6}\right)^{\frac{m}{2}} ||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes m})^*} + \lim_{|\mathcal{P}|\to 0} \phi_{\mathcal{P}}(\beta) \\ &= \left(\frac{1}{6}\right)^{\frac{m}{2}} ||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes m})^*} \\ &< \left(\frac{1}{6}\right)^{\frac{m}{2}} \varepsilon.\end{aligned}$$

Our choice of ε was arbitrary. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$. ☐

Proposition 5.32.

$$\sum_{h_1,\ldots,h_{k+n}\in S_{\mathbb{C}}} |\langle \alpha, P_{\mathcal{P}}h_1 \otimes \cdots \otimes P_{\mathcal{P}}h_k \otimes R_{\mathcal{P}}(h_{k+1}, h_{k+2}) \otimes \cdots \otimes R_{\mathcal{P}}(h_{k+n-1}, h_{k+n})\rangle|^2 \to 0$$

as $|\mathcal{P}| \to 0$, for any integer k and even integer $n > 0$.

Proof. By choosing a basis for $H_{\mathcal{P}}(\mathfrak{g})$, then extending it to a basis $S_{\mathbb{C}}$ for $H(\mathfrak{g})$, we get

$$\begin{aligned}&\lim_{|\mathcal{P}|\to 0} \sum_{h_1,\ldots,h_{k+n}\in S_{\mathbb{C}}} |\langle \alpha, P_{\mathcal{P}}h_1 \otimes \cdots \otimes P_{\mathcal{P}}h_k \otimes R_{\mathcal{P}}(h_{k+1}, h_{k+2}) \otimes \cdots \otimes R_{\mathcal{P}}(h_{k+n-1}, h_{k+n})\rangle|^2 \\ &\leq \lim_{|\mathcal{P}|\to 0} \sum_{h_1,\ldots,h_{k+n}\in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k \otimes R_{\mathcal{P}}(h_{k+1}, h_{k+2}) \otimes \cdots \otimes R_{\mathcal{P}}(h_{k+n-1}, h_{k+n})\rangle|^2 \\ &= \lim_{|\mathcal{P}|\to 0} \sum_{h_1,\ldots,h_{k+n}\in S_{\mathbb{C}}} |\langle \beta_{h_1\ldots h_k}, R_{\mathcal{P}}(h_{k+1}, h_{k+2}) \otimes \cdots \otimes R_{\mathcal{P}}(h_{k+n-1}, h_{k+n})\rangle|^2,\end{aligned}$$

Where $\beta_{h_1\ldots h_k}$ comes from Proposition 5.26. Let

$$J_{\mathcal{P}} := \sum_{h_1,\ldots,h_{k+n}\in S_{\mathbb{C}}} |\langle \beta_{h_1\ldots h_k}, R_{\mathcal{P}}(h_{k+1}, h_{k+2}) \otimes \cdots \otimes R_{\mathcal{P}}(h_{k+n-1}, h_{k+n})\rangle|^2.$$

Notice that by Eq. (5.13) and Proposition 5.28,

$$\begin{aligned}|J_{\mathcal{P}}| &\leq \sum_{h_1,\ldots,h_{k+n}\in S_{\mathbb{C}}} ||\beta_{h_1\ldots h_k}||^2_{(H(\mathfrak{g})^*)^{\otimes n}} ||R_{\mathcal{P}}(h_{k+1}, h_{k+2})||^2_{H(\mathfrak{g})} \cdots ||R_{\mathcal{P}}(h_{k+n-1}, h_{k+n})||^2_{H(\mathfrak{g})} \\ &\leq ||\alpha_{n+k}||^2_{(H(\mathfrak{g})^*)^{\otimes k}} \left(\frac{1}{6}\right)^n < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{|\mathcal{P}|\to 0} J_{\mathcal{P}} \\
&= \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \lim_{|\mathcal{P}|\to 0} \sum_{h_{k+1},\ldots,h_{k+n}\in S_{\mathbb{C}}} |\langle \beta_{h_1\ldots h_k}, R_{\mathcal{P}}(h_{k+1},h_{k+2})\otimes\cdots\otimes R_{\mathcal{P}}(h_{k+n-1},h_{k+n})\rangle|^2 \\
&= 0,
\end{aligned}$$

by Proposition 5.31. □

For the remainder of this section, we restrict ourselves to refining partitions. That is, a sequence of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$ such that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n > 0$. The following two results are shown in the general case in Lemma 5.85 and Proposition 5.88.

Lemma 5.33. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions. Then $\cup_{n=1}^{\infty} H_{\mathcal{P}_n}(\mathfrak{g})$ is dense in $H(\mathfrak{g})$.*

Remark 5.34. Since $P_{\mathcal{P}_n}$ is orthogonal projection onto the subspace $H_{\mathcal{P}_n}(\mathfrak{g})$ and $H_{\mathcal{P}_n}(\mathfrak{g}) \subset H_{\mathcal{P}_{n+1}}(\mathfrak{g})$, the above lemma implies that for $h \in H(\mathfrak{g})$,

$$\lim_{n\to\infty} ||h - P_{\mathcal{P}_n} h||_{H(\mathfrak{g})} = 0.$$

Proposition 5.35. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions. Then for $h_1, ..., h_k \in S_{\mathbb{C}}$,*

$$||h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k||^2_{H(\mathfrak{g})^{\otimes k}} \le k^2 \sum_{j=1}^{k} ||h_j - P_{\mathcal{P}_n} h_j||^2_{H(\mathfrak{g})}. \tag{5.24}$$

In addition, for $\alpha \in T(H(\mathfrak{g}))^*_T$,

$$\sum_{h_1,\ldots,h_k\in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k\rangle|^2 < k^3 ||\alpha_k||^2_{J^0_T(H(\mathfrak{g}))}. \tag{5.25}$$

We are now set to prove

Proposition 5.36. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions and $\alpha \in J^0_T(H(\mathfrak{g}))$ of finite rank. Then*

$$\lim_{n\to\infty} ||\alpha - \alpha(\mathcal{P}_n)||^2_{J^0_T(H(\mathfrak{g}))} = 0 \quad .$$

Proof. Again, the fact that α is of finite degree is essential to this proposition. Assume α has rank N. Let $S_{\mathbb{C}}$ be a basis for $H(\mathfrak{g})$ adapted to our sequence of partitions in the following sense. First construct an orthonormal basis for $H_{\mathcal{P}_1}(\mathfrak{g})$, then extend inductively, i.e. given an orthonormal basis for $H_{\mathcal{P}_i}(\mathfrak{g})$, extend to an orthonormal basis for $H_{\mathcal{P}_{i+1}}(\mathfrak{g})$ for all $i > 0$. Lemma 5.33 guarantees we can construct a basis $S_{\mathbb{C}}$ for $H(\mathfrak{g})$ in this manner. Using this basis, we calculate

$$\begin{aligned}
&||\alpha - \alpha(\mathcal{P}_n)||^2_{J^0_T(H(\mathfrak{g})}\\
&= \sum_{k=0}^{\infty} \frac{T^k}{k!} \sum_{h_1,h_2,\dots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle - \langle \alpha(\mathcal{P}_n), h_1 \otimes \cdots \otimes h_k\rangle|^2\\
&= \sum_{k=0}^{N} \frac{T^k}{k!} \sum_{h_1,\dots,h_k \in S_{\mathbb{C}}} \left| \begin{array}{c} \langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle - \\ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{x_1,\dots,x_j \in \Omega_{h_1,\dots,h_k}} \langle \alpha, (\theta_{h_1,\dots,h_k}/\{x_1,\dots,x_j\}) \otimes_{i=1}^{j} R_{\mathcal{P}_n}(x_i)\rangle \end{array} \right|^2
\end{aligned}$$

Now for any $1 \le k \le N$,

$$\begin{aligned}
&\sum_{h_1,\dots,h_k \in S_{\mathbb{C}}} \left| \begin{array}{c} \langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle - \\ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{x_1,\dots,x_j \in \Omega_{h_1,\dots,h_k}} \langle \alpha, (\theta_{h_1,\dots,h_k}/\{x_1,\dots,x_j\}) \otimes_{i=1}^{j} R_{\mathcal{P}_n}(x_i)\rangle \end{array} \right|^2\\
&\le C_k \sum_{h_1,\dots,h_k \in S_{\mathbb{C}}} \left(\begin{array}{c} |\langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle - \langle \alpha, P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k\rangle|^2 + \\ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{x_1,\dots,x_j \in \Omega_{h_1,\dots,h_k}} |\langle \alpha, (\theta_{h_1,\dots,h_k}/\{x_1,\dots,x_j\}) \otimes_{i=1}^{j} R_{\mathcal{P}_n}(x_i)\rangle|^2 \end{array} \right)
\end{aligned} \tag{5.26}$$

for a constant C_k which only depends on k. Calculations like Eq. (5.17) show that 5.26 is bounded independent of partition. Since every term in 5.26 involving $R_{\mathcal{P}_n}$ goes to zero as $n \to \infty$ by Proposition 5.32, we get

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{N} \sum_{h_1,\dots,h_k \in S_{\mathbb{C}}} \left| \begin{array}{c} \langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle - \\ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{x_1,\dots,x_j \in \Omega_{h_1,\dots,h_k}} \langle \alpha, (\theta_{h_1,\dots,h_k}/\{x_1,\dots,x_j\} \otimes_{i=1}^{j} R_{\mathcal{P}_n}(x_i)\rangle \end{array} \right|^2\\
&\le C(N) \sum_{k=1}^{N} \lim_{n\to\infty} \sum_{h_1,\dots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle - \langle \alpha, P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k\rangle|^2,
\end{aligned} \tag{5.27}$$

where $C(N) = \sum_{k=1}^{N} C_k < \infty$. Eq. (5.25) of Proposition 5.35 allows us to use the Dominated Convergence Theorem with dominating function $4|\langle \alpha, h_1 \otimes \cdots \otimes h_k\rangle|^2$. That is

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{h_1,\ldots,h_k\in S_{\mathbb{C}}}|\langle\alpha,h_1\otimes\cdots\otimes h_k\rangle-\langle\alpha,P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle|^2\\
&=\sum_{h_1,\ldots,h_k\in S_{\mathbb{C}}}\lim_{n\to\infty}|\langle\alpha,h_1\otimes\cdots\otimes h_k\rangle-\langle\alpha,P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle|^2\\
&\le k^2\sum_{h_1,\ldots,h_k\in S_{\mathbb{C}}}\lim_{n\to\infty}\left(\sum_{j=1}^{k}||h_j-P_{\mathcal{P}_n}h_j||^2_{H(\mathfrak{g})}\right)\\
&=0,
\end{aligned}$$

where we have used Eq. (5.24) in conjunction with Remark 5.34.

Therefore,

$$\lim_{n\to 0}||\alpha-\alpha(\mathcal{P}_n)||^2_{J_T^0(H(\mathfrak{g}))}=0.$$

□

Definition 5.37. Define $\tilde{u}_\alpha\in\mathcal{H}_T$ as $\tilde{u}_\alpha=L^2(\nu_T)-\lim_{n\to 0}F_{\mathcal{P}_n}$.

It remains to be seen that $R\tilde{u}_\alpha=u_\alpha$. This, however, is a short and straightforward calculation given results already obtained in the general case. For $g\in H(G)$,

$$\begin{aligned}
R\tilde{u}_\alpha(g)&=\lim_{n\to 0}RF_{\mathcal{P}_n}(g)\\
&=\lim_{n\to 0}F_{\mathcal{P}_n}(g)\\
&=\lim_{n\to 0}u_\alpha(\pi_{\mathcal{P}_n}g)\\
&=u_\alpha(g).
\end{aligned}$$

5.3 Construction of u_α

We now return to the case where G is a general simply connected step r nilpotent complex Lie group. In choosing a basis $\{X_i\}_{i=1}^d$ for $\mathfrak{g}$, we are able to realize G as $\mathbb{C}^d$ under exponential coordinates with a multiplication law given by the Baker-Campbell-

Hausdorff formula. Furthermore, since $e^A = A$ as vectors in $\mathbb{C}^d$,

$$\begin{aligned} A \cdot B &= e^A \cdot e^B \\ &= e^{\Gamma(A,B)} \\ &= \Gamma(A,B), \end{aligned} \tag{5.28}$$

where $\Gamma(A,B)$ is given by the BCH formula. That is if we define

$$\mathcal{I}_k \equiv \left\{ (m,n) \in \mathbb{Z}_+^{2k} | m+n > 0 \right\},$$

where $m+n>0$ means that $m_i + n_i > 0$ for all $i = 1, ..., k$, and set $m! = m_1! \cdots m_k!$ and $|m| = m_1 + \cdots + m_k$, then

$$\begin{aligned} \Gamma(A,B) &= A + B \\ &+ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \sum_{(m,n)\in\mathcal{I}_k} \frac{1}{m!n!(|n|+1)} ad_A^{m_k} ad_B^{n_k} \cdots ad_A^{m_1} ad_B^{n_1} B. \end{aligned} \tag{5.29}$$

Though $ad_B^{n_1} B = 0$ if $n_1 > 0$, it will be notationally convenient to include the term. In our case, Eq. (5.29) only contains a finite number of terms, since if $|m| + |n| \geq r$, $ad_A^{m_k} ad_B^{n_k} \cdots ad_A^{m_1} ad_B^{n_1} B = 0$ for any $A, B \in \mathfrak{g}$. Also notice that $(m,n) \in \mathcal{I}_k$ implies that $|m| + |n| \geq k$. Therefore, a more efficient writing of Eq. (5.29) for our purposes is

$$\begin{aligned} \Gamma(A,B) &= A + B \\ &+ \sum_{k=1}^{r-1} \frac{(-1)^{k+1}}{k(k+1)} \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} \frac{1}{m!n!(|n|+1)} ad_A^{m_k} ad_B^{n_k} \cdots ad_A^{m_1} ad_B^{n_1} B. \end{aligned} \tag{5.30}$$

The gist of Eq. (5.28) is that we can consider $\mathfrak{g}$ and G interchangably as the vector space $\mathbb{C}^d$ with two operations, $[\ ,\ ]$, the Lie bracket which comes from $\mathfrak{g}$, and group multiplication $\cdot$, which is a linear combination and composition of $[\ ,\ ]$ operations.

Remark 5.38. Using exponential coordinates, the identity and inverses are given by 0 and $-g$ respectively, i.e.

$$0 \cdot g = g \cdot 0 = g$$

and

$$g \cdot (-g) = (-g) \cdot g = 0$$

for all $g \in G$.

The pointwise application of the above gives us the same relations on the path space, namely $\mathcal{W}(\mathfrak{g})$ and $\mathcal{W}(G)$ are considered interchangeably as the vector space $\mathcal{W}\left(\mathbb{C}^d\right)$ with operations $[\ ,\ ]$ and $\cdot$.

Proposition 5.39. *Suppose $g \in \mathcal{W}(\mathbb{C}^d)$. Then $g \in H(G)$ iff $g \in H(\mathfrak{g})$.*

Proof. First observe two important facts. For all $s \in [0,1]$, we have

$$\begin{aligned} ||g(s)||_{\mathfrak{g}} &= \left\| \int_0^s g'(r)dr \right\|_{\mathfrak{g}} \\ &\leq \int_0^1 ||g'(r)||_{\mathfrak{g}} dr \\ &= ||g'||_{L^1([0,1])} \\ &\leq ||g'||_{L^2([0,1])} \\ &= ||g||_{H(\mathfrak{g})}. \end{aligned} \tag{5.31}$$

We also calculate the Maurer-Cartan form,

$$\begin{aligned} L_{g^{-1}(s)*}g'(s) &= \frac{d}{d\varepsilon}|_{\varepsilon=0}(-g(s)) \cdot g(s+\varepsilon) \\ &= \frac{d}{d\varepsilon}|_{\varepsilon=0}\Gamma(-g(s), g(s+\epsilon)) \\ &= g'(s) + \sum_{i=1}^{r-1} \Lambda^i(g(s), g'(s)), \end{aligned} \tag{5.32}$$

where

$$\Lambda^i(g(s), g'(s)) \equiv \sum_{k=1}^{i} \frac{(-1)^{k+1}}{k(k+1)} \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|=i}} \frac{(-1)^{|m|}}{m!n!(|n|+1)} ad^i_{g(s)} g'(s).$$

Notice that $\Lambda^i(g(s), g'(s))$ is a constant times $ad^i_{g(s)}(g'(s))$. Also notice that since $||ad_A B||_{\mathfrak{g}} \leq C||A||_{\mathfrak{g}}||B||_{\mathfrak{g}}$ for some constant C, $||\Lambda^i(g(s), g'(s))||_{\mathfrak{g}} \leq C_i||g(s)||^i||g'(s)||$ for some constant C_i.

First suppose $g \in H(\mathfrak{g})$. Then $||g||_{H(\mathfrak{g})} = ||g'||_{L^2([0,1])} < \infty$. Considering g as an element of $\mathcal{W}(G)$, we calculate

$$\begin{aligned}
E\left(g\right) &:= \int_0^1 ||L_{g^{-1}(s)*}g'\left(s\right)||_{\mathfrak{g}}^2 ds \\
&= \int_0^1 ||g'(s) + \sum_{i=1}^{r-1} \Lambda^i(g\left(s\right), g'\left(s\right))||_{\mathfrak{g}}^2 ds \\
&\leq r^2 \left(\int_0^1 \left|\left|g'(s)\right|\right|_{\mathfrak{g}}^2 ds + \sum_{i-1}^{r-1} \int_0^1 \left|\left|\Lambda^i(g\left(s\right), g'\left(s\right)\right|\right|_{\mathfrak{g}}^2 ds \right) \\
&\leq r^2 \left(\left|\left|g'\right|\right|_{L^2([0,1])}^2 + \sum_{i=1}^{r-1} C_i^2 \int_0^1 ||g(s)||_{\mathfrak{g}}^{2i} \left|\left|g'(s)\right|\right|_{\mathfrak{g}}^2 ds \right) \\
&\leq r^2 \left(\left|\left|g'\right|\right|_{L^2([0,1])}^2 + \sum_{i=1}^{r-1} C_i^2 \left|\left|g'\right|\right|_{L^2([0,1])}^{2i+2} \right) \qquad (5.33) \\
&= poly(||g||_{H(\mathfrak{g})}) \qquad (5.34) \\
&< \infty,
\end{aligned}$$

where in line (5.33) we have used Eq. (5.31).

Now suppose $g \in H(G)$. Considering $g \in \mathcal{W}(\mathfrak{g})$, we write $g = (g_1, g_2, ..., g_r)$ where $g_i \in \mathcal{W}(V_i)$. The fact that $g \in H(G)$, tells us that $E(g) < \infty$, or in other words

$$\sum_{i=1}^{r} || \left(L_{g^{-1}(\cdot)*}g'\left(\cdot\right)\right)_i ||_{L^2([0,1])}^2 < \infty.$$

In particular, $\left|\left|\left(L_{g^{-1}(\cdot)*}g'\left(\cdot\right)\right)_i\right|\right|_{L^2([0,1])}^2 < \infty$ for all $i = 1, ..., r$. We wish to show that for all $i = 1, ..., r$, $||g_i'||_{L^2([0,1])} < \infty$. First note that $\Lambda^i(g(s), g'(s)) \in \oplus_{j=i+1}^r \mathcal{W}(V_j)$, i.e. it is identically zero in the first i coordinates. This being the case, then (5.32) tells us that

$$\left|\left|g_1'\right|\right|_{L^2([0,1])}^2 = || \left(L_{g^{-1}(\cdot)*}g'\left(\cdot\right)\right)_1 ||_{L^2([0,1])}^2 < \infty.$$

Now for the second coordinate, we have

$$\left(L_{g^{-1}(s)*}g'\left(s\right)\right)_2 = g_2'(s) + \Lambda^1(g(s), g'(s))_2.$$

Therefore,

$$\begin{aligned}||g_2'||^2_{L^2([0,1])} &= \int_0^1 ||\left(L_{g^{-1}(s)*}g'(s)\right)_2 + \Lambda^1(g(s), g'(s))_2||^2_{\mathfrak{g}} ds \\ &\leq 4\left(||\left(L_{g^{-1}(\cdot)*}g'(\cdot)\right)_2||^2_{L^2([0,1])} + C_1^2 \int_0^1 ||g_1(s)||^2_{\mathfrak{g}} \left|\left|g_1'(s)\right|\right|^2_{\mathfrak{g}} ds\right) \\ &\leq 4\left(||\left(L_{g^{-1}(\cdot)*}g'(\cdot)\right)_2||^2_{L^2([0,1])} + C_1^2 \left|\left|g_1'\right|\right|^4_{L^2([0,1])}\right) \\ &< \infty,\end{aligned}$$

where we have used Eq. (5.31) restricted to the first coordinate. In this manner, we can inductively get $||g_i'||_{L^2([0,1])} < \infty$ for all $i = 1, ..., r$. □

We now have an expression for the product on $H(G)$. That is, using the above proposition and Eq. (5.30), we have for all $g, h \in H(G)$,

$$\begin{aligned}g \cdot h &= \Gamma(g,h) \\ &= g + h \\ &+ \sum_{k=1}^{r-1} \frac{(-1)^{k+1}}{k(k+1)} \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} \frac{1}{m!n!(|n|+1)} ad_g^{m_k} ad_h^{n_k} \cdots ad_g^{m_1} ad_h^{n_1} h,\end{aligned} \tag{5.35}$$

where $(g \cdot h)(s) = g(s) \cdot h(s)$ and $(ad_g h)(s) = ad_{g(s)} h(s)$ for all $s \in [0,1]$. The following characterizations of Eq. (5.35) will be useful in the sequel.

Remark 5.40. Notice that

$$g \cdot h = g + h + \sum_{l=0}^{r-1} C_l ad_g^l h + \sum_{l=2}^{r-1} Q_l(g,h), \tag{5.36}$$

for some constants $C_1, ..., C_{r-1}$ and functions $Q_l(g,h)$ which satisfy $Q_l(g, zh) = z^l Q(g,h)$ for $z \in \mathbb{C}$, and for fixed g, $||Q_l(g,h)||_{\mathfrak{g}} \leq C_l(g)||h||^l_{\mathfrak{g}}$.

Definition 5.41. For $h \in H(\mathfrak{g})$ and $g \in H(G)$, we'll denote

$$\begin{aligned}
\tilde{h}(g) &:= \frac{d}{dt}|_{t=0}\left(g \cdot e^{th}\right) \\
&= \frac{d}{dt}|_{t=0}\left(g \cdot th\right) \\
&= \frac{d}{dt}|_{t=0}\left(g + th + t\sum_{l=0}^{r-1} C_l ad_g^l h + \sum_{l=2}^{r-1} t^l Q(g,h)\right) \\
&= h + \sum_{l=0}^{r-1} C_l ad_g^l h,
\end{aligned}$$

where we are using the notation as in Remark 5.40.

Remark 5.42. Observe that

$$g \cdot h = g + \tilde{h}(g) + \sum_{l=2}^{r-1} Q_l(g,h).$$

Given $\alpha \in J_T^0$, we would like to construct a holomorphic function u_α on $H(G)$ such that $(1-D)_{\underline{e}}^{-1} u_\alpha = \alpha$. Recall from Chapter 4 that we require that for every $g \in H(G)$, the map $h \in H(\mathfrak{g}) \to u_\alpha(g \cdot e^h)$ is Frechét differentiable at $h = 0$ and that this Frechét derivative is complex linear and continuous in $H(\mathfrak{g})^*$ as a function of g. The following theorem is motivated by results from [3], specifically Remark 5.6 and Proposition 6.2.

Theorem 5.43. *Given $\alpha \in J_T^0(H(\mathfrak{g}))$ of rank N, for every $g \in H(G)$ define*

$$u_\alpha(g) := \sum_{n=0}^{N} \langle \alpha, g^{\otimes n} \rangle / n!$$

In defining u_α, we are using the identification between $H(G)$ and $H(\mathfrak{g})$ exhibited in Proposition 5.39. Then u_α is a holomorphic function on $H(G)$ satisfying $(1-D)_{\underline{e}}^{-1} u_\alpha = \alpha$.

Proof. For $0 < n \leq N$, define $f_n : H(G) \to \mathbb{C}$ by

$$f_n(g) = \frac{1}{n!}\langle \alpha, g^{\otimes n} \rangle$$

Since finite sums of holomorphic functions are holomorphic, showing that f_n is holomorphic is sufficient to prove that u_α is holomorphic.

For $h \in H(\mathfrak{g})$ and $g \in H(G)$, define

$$(df_n)_g\, h := \frac{1}{n!}\langle \alpha, \sum_{k=0}^{n-1} g^{\otimes k} \otimes \tilde{h}(g) \otimes g^{\otimes n-k-1}\rangle,$$

where $\tilde{h}(g)$ is given by Definition 5.41. That is

$$\tilde{h}(g) = h + \sum_{l=0}^{r-1} C_l ad_g^l h.$$

Notice that $\tilde{h}(g)$ is complex linear in h and continuous in g.

To see that $(df_n)_g$ is the Frechét derivative of f_n at $g \in H(G)$, we first observe that, using the same notation as Remark 5.40,

$$\begin{aligned}(g \cdot h)^{\otimes n} &= (g + \tilde{h}(g) + \sum_{l=2}^{r-1} Q_l(g,h))^{\otimes n} \\ &= g^{\otimes n} + \sum_{k=0}^{n-1} g^{\otimes k} \otimes \tilde{h}(g) \otimes g^{\otimes n-k-1} + R^n(g,h),\end{aligned}$$

where $R^n(g,h)$ is a sum of tensors each containing at least one $Q_l(g,h)$, for some $l \geq 2$. Therefore, $||R^n(g,h)||_{H(\mathfrak{g})^{\otimes n}} \leq C_g ||h||^2_{H(\mathfrak{g})}$ for an appropriate constant C_g. Therefore

$$\begin{aligned}&\lim_{h\to 0} \frac{|f_n(g\cdot h) - f_n(g) - (df_n)_g\, h|}{||h||_{H(\mathfrak{g})}} \\ &\leq \lim_{h\to 0} \frac{||\alpha||_{(H(\mathfrak{g})^*)^{\otimes n}} ||(g\cdot h)^{\otimes n} - g^{\otimes n} - \sum_{k=0}^{n-1} g^{\otimes k} \otimes \tilde{h}(g) \otimes g^{\otimes n-k-1}\rangle||}{n!||h||_{H(\mathfrak{g})}} \\ &= \lim_{h\to 0} \frac{||\alpha||_{(H(\mathfrak{g})^*)^{\otimes n}} ||R^n(g,h)||_{H(\mathfrak{g})^{\otimes n}}}{n!||h||_{H(\mathfrak{g})}} \\ &\leq \lim_{h\to 0} \frac{||\alpha||_{(H(\mathfrak{g})^*)^{\otimes n}} C_g ||h||^2_{H(\mathfrak{g})}}{n!||h||_{H(\mathfrak{g})}} \\ &= 0.\end{aligned}$$

This proves that f_n is holomorphic, and therefore so is u_α.

To see that $(1-D)^{-1}_{\underline{e}} u_\alpha = \alpha$, observe that for $h \in H(\mathfrak{g})$,

$$u_\alpha(e^{th}) = u_\alpha(th) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\langle \alpha, h^{\otimes n}\rangle.$$

Then

$$\overbrace{\tilde{h}\tilde{h}\cdots\tilde{h}}^{\text{k times}}u_\alpha(\underline{e}) = \frac{d}{dt_1}|_{t_1=0}\cdots\frac{d}{dt_k}|_{t_k=0}u_\alpha(e^{t_k h}\cdot e^{t_{k-1}h}\cdots e^{t_1 h})$$
$$= \frac{d}{dt_1}|_{t_1=0}\cdots\frac{d}{dt_k}|_{t_k=0}u_\alpha(e^{(t_k+t_{k-1}+\cdots+t_1)h})$$
$$= \frac{d}{dt_1}|_{t_1=0}\cdots\frac{d}{dt_k}|_{t_k=0}\sum_{n=0}^{\infty}\frac{(t_k+\cdots+t_1)^n}{n!}\langle\alpha, h^{\otimes n}\rangle$$
$$= \langle\alpha, h^{\otimes k}\rangle.$$

Polarization then gives the result for symmetric tensors. The fact that $\alpha \in J_T^0$ and the Burkhoff-Witt theorem gives that for $h_1, h_2, ..., h_k \in H(\mathfrak{g})$,

$$\tilde{h}_1\tilde{h}_2...\tilde{h}_k u_\alpha(\underline{e}) = \langle\alpha, h_1\otimes h_2\otimes\cdots\otimes h_k\rangle.$$

□

5.4 Derivatives of $F_{\mathcal{P}}$

Our goal in this section is to characterize the derivatives of $F_{\mathcal{P}}$ in terms of our given α. Notice that since $F_{\mathcal{P}} = u_\alpha \circ \tilde{P}_{\mathcal{P}}$, given $h \in H(\mathfrak{g})$ and $g \in \mathcal{W}(G)$,

$$(\tilde{h}F_{\mathcal{P}})(g) = \frac{d}{dt}|_{t=0}F_{\mathcal{P}}(g\cdot e^{th})$$
$$= \frac{d}{dt}|_{t=0}u_\alpha(\tilde{P}_{\mathcal{P}}(g\cdot e^{th})).$$

Setting

$$h_{\mathcal{P}}(g) \equiv L_{\tilde{P}_{\mathcal{P}}(g)^{-1}*}\frac{d}{dt}|_{t=0}\tilde{P}_{\mathcal{P}}(g\cdot e^{th})$$
$$= \frac{d}{dt}|_{t=0}(-\tilde{P}_{\mathcal{P}}(g)\cdot\tilde{P}_{\mathcal{P}}(g\cdot e^{th}))$$

then

$$(\tilde{h}F_{\mathcal{P}})(g) = \langle Du_\alpha(\tilde{P}_{\mathcal{P}}(g)), h_{\mathcal{P}}(g)\rangle,$$

and in particular

$$(\tilde{h}F_{\mathcal{P}})(\underline{e}) = \langle\alpha, h_{\mathcal{P}}(\underline{e})\rangle$$

since $\tilde{P}_{\mathcal{P}}(\underline{e}) = \underline{e}$.

In our case, the identification of $\mathcal{W}(G)$ and $\mathcal{W}(\mathfrak{g})$ via the exponential map greatly simplifies the calculation. Using the fact that as functions on $\mathcal{W}(\mathbb{C}^d)$, $\tilde{P}_{\mathcal{P}} \equiv P_{\mathcal{P}}$, we have that

$$\begin{aligned} h_{\mathcal{P}}(g) &= \frac{d}{dt}|_{t=0}(-\tilde{P}_{\mathcal{P}}(g) \cdot \tilde{P}_{\mathcal{P}}(g \cdot e^{th})) \\ &= \frac{d}{dt}|_{t=0}(-P_{\mathcal{P}}(g) \cdot P_{\mathcal{P}}(g \cdot th)). \end{aligned}$$

$P_{\mathcal{P}}$ is a linear function from $H(\mathfrak{g})$ to $H(\mathfrak{g})$. We now wish to work out $h_{\mathcal{P}}(g)$ in more detail. We first perform the calculations with $P_{\mathcal{P}}$ replaced by $P : H(\mathfrak{g}) \to H(\mathfrak{g})$ a general bounded linear function. In particular, we are interested in the quantity

$$\frac{d}{dt}|_{t=0}(-P(g)) \cdot P(g \cdot th),$$

since replacing P with $P_{\mathcal{P}}$ gives us our desired $h_{\mathcal{P}}(g)$.

Notice that by calculations similar to those in Remark 5.40, we have that

$$g \cdot th = g + t\sum_{l=0}^{r-1} C_l ad_g^l h + O(t^2), \tag{5.37}$$

for constants C_l. Therefore,

$$P(g \cdot th) = P(g) + t\sum_{l=0}^{r-1} C_l P(ad_g^l h) + O(t^2). \tag{5.38}$$

Proposition 5.44. *For any bounded linear function $P : H(\mathfrak{g}) \to H(\mathfrak{g})$, there exist constants $C_{l,m}$ independent of P, for $0 < l \leq m \leq r$, such that*

$$\frac{d}{dt}|_{t=0}(-P(g)) \cdot P(g \cdot th) = \sum_{m=0}^{r-1}\sum_{l=0}^{m} C_{l,m} ad^l_{P(g)} P(ad_g^{l-m} h),$$

with the property that $C_{0,0} = 1$ and for all $m \geq 1$, $\sum_{l=0}^m C_{l,m} = 0$.

Proof. For notational convenience, we'll set

$$\Lambda(g,h) = \sum_{l=0}^{r-1} C_l P(ad_g^l h),$$

so that Eq. (5.38) becomes

$$P(g \cdot th) = P(g) + t\Lambda(g,h) + O(t^2).$$

In addition, let

$$N_k = \frac{(-1)^{k+1}}{k(k+1)},$$

and for $m, n \in \mathbb{Z}_+^k$,

$$M_{m,n} = \frac{1}{m!n!(|n|+1)}.$$

With this notation, the simplified BCH formula, Eq. (5.30), becomes

$$\Gamma(A,B) = A + B + \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} M_{m,n} ad_A^{m_k} ad_B^{n_k} \cdots ad_A^{m_1} ad_B^{n_1} B.$$

Also, in the following computation we will often throw out terms of order greater than t from one equality to the next with the knowledge that we will be evaluating the derivative at zero, and hence they will not contribute. Using the BCH formula and the notation defined before,

$$
\begin{aligned}
&\frac{d}{dt}|_{t=0}(-P(g)) \cdot P(g \cdot th) \\
&= \frac{d}{dt}|_{t=0}(-P(g)) \cdot (P(g) + t\Lambda(g,h) + O(t^2)) \\
&= \frac{d}{dt}|_{t=0} t\Lambda(g,h) \\
&+ \frac{d}{dt}|_{t=0} \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} M_{m,n} ad^{m_k}_{-P(g)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{P(g)+t\Lambda(g,h)} \left(P(g) + t\Lambda\left(g,h\right)\right) \\
&= \Lambda(g,h) \\
&+ \frac{d}{dt}|_{t=0} \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} M_{m,n} ad^{m_k}_{-P(g)} ad^{n_k}_{P(g)+t\Lambda(g,h)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{P(g)+t\Lambda(g,h)} P(g) \\
&+ \frac{d}{dt}|_{t=0} t \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} M_{m,n} ad^{m_k}_{-P(g)} ad^{n_k}_{P(g)+t\Lambda(g,h)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{P(g)+t\Lambda(g,h)} \Lambda\left(g,h\right) \\
&= \Lambda(g,h) + \frac{d}{dt}|_{t=0} t \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r \\ n_1=1}} M_{m,n} ad^{m_k}_{-P(g)} ad^{n_k}_{P(g)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{\Lambda(g,h)} P(g) \\
&+ \frac{d}{dt}|_{t=0} t \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} M_{m,n} ad^{m_k}_{-P(g)} ad^{n_k}_{P(g)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{P(g)} \Lambda\left(g,h\right) \\
&= \Lambda(g,h) - \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r \\ n_1=1}} M_{m,n} ad^{m_k}_{-P(g)} ad^{n_k}_{P(g)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{P(g)} \Lambda\left(g,h\right) \\
&+ \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r}} M_{m,n} ad^{m_k}_{-P(g)} ad^{n_k}_{P(g)} \cdots ad^{m_1}_{-P(g)} ad^{n_1}_{P(g)} \Lambda\left(g,h\right)
\end{aligned}
$$

$$= \Lambda(g,h) + \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r \\ n_1\neq 1}} M_{m,n} ad_{-P(g)}^{m_k} ad_{P(g)}^{n_k} \cdots ad_{-P(g)}^{m_1} ad_{P(g)}^{n_1} \Lambda(g,h)$$

$$= \Lambda(g,h) + \sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ |m|+|n|<r \\ n_1\neq 1}} (-1)^{|m|} M_{m,n} ad_{P(g)}^{|m|+|n|} \Lambda(g,h)$$

$$= \Lambda(g,h) + \sum_{q=1}^{r-1} \left(\sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ n_1\neq 1 \\ |m|+|n|=q}} (-1)^{|m|} M_{m,n} \right) ad_{P(g)}^{q} \Lambda(g,h) \,.$$

Setting

$$K_q = \left(\sum_{k=1}^{r-1} N_k \sum_{\substack{(m,n)\in\mathcal{I}_k \\ n_1\neq 1 \\ |m|+|n|=q}} (-1)^{|m|} M_{m,n} \right),$$

we get

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}(-P(g)) \cdot P(g \cdot th) \\
&= \Lambda(g,h) + \sum_{q=1}^{r-1} K_q ad_{P(g)}^{q} \left(\Lambda(g,h)\right) \\
&= \sum_{l=0}^{r-1} C_l P(ad_g^l h) + \sum_{q=1}^{r-1} K_q ad_{P(g)}^{q} \left(\sum_{l=0}^{r-1} C_l P(ad_g^l h) \right) \\
&= \sum_{l=0}^{r-1} C_l P(ad_g^l h) + \sum_{q=1}^{r-1} \sum_{l=0}^{r-1} K_q C_l ad_{P(g)}^{q} P(ad_g^l h) \\
&= \sum_{l=0}^{r-1} C_l P(ad_g^l h) + \sum_{\substack{q+l=1 \\ q>0}}^{r-1} K_q C_l ad_{P(g)}^{q} P(ad_g^l h) \\
&\equiv \sum_{m=0}^{r-1} \sum_{l=0}^{m} C_{l,m} ad_{P(g)}^{l} P(ad_g^{l-m} h).
\end{aligned}$$

Now to see the relation among the constants, note that we have made no assumptions as to the nature of our linear function P. In particular, we could set P to be the indentity function and we would get

$$\begin{aligned} h &= \frac{d}{dt}|_{t=0}(-g)\cdot(g\cdot th) \\ &= \sum_{m=0}^{r-1}\sum_{l=0}^{m} C_{l,m} ad_g^l ad_g^{l=m} h \\ &= C_{0,0}h + \sum_{m=1}^{r-1}\left(\sum_{l=0}^{m} C_{l,m}\right) ad_g^m h. \end{aligned}$$

We also have made no additional assumptions on our Lie algebra, so if there is a step r stratified nilpotent Lie algebra in which there exists a g and h such that $\{ad_g^m h\}_{m=0}^{r-1}$ are linearly independent, then we necessarily have that $C_{0,0} = 1$ and $\sum_{l=0}^{m} C_{l,m} = 0$ for $m > 0$.

Let V be a finite dimensional vector space, and consider the truncated tensor algebra $T^{(r)} \equiv \oplus_{i=1}^{r} V^{\otimes i}$. For $v, w \in T^{(r)}$, the product $[v,w] = v\otimes w - w\otimes v$, along with the convention that we elminate products of length greater than r, defines a Lie bracket on $T^{(r)}$. It then follows that for any $v, w \in V$ with $v, w \neq 0$, $ad_v^m w \in V^{\otimes(m+1)}$ and hence the set $\{ad_v^m w\}_{m=0}^{r-1}$ are linearly independent. $\square$

Remark 5.45. Since for $m > 0$, $\sum_{l=0}^{m} C_{l,m} = 0$ in the above Proposition, we can rewrite

$$\begin{aligned} \frac{d}{dt}|_{t=0}(-P(g))\cdot P(g\cdot th) &= \sum_{m=0}^{r-1}\sum_{l=0}^{m} C_{l,m} ad_{P(g)}^l P(ad_g^{l-m}h) \\ &= \sum_{m=0}^{r-1}\sum_{l=0}^{m-1} \tilde{C}_{l,m}\left(ad_{P(g)}^l P(ad_g^{l-m}h) - ad_{P(g)}^{l+1} P(ad_g^{l-m-1}h)\right), \end{aligned}$$

where $\tilde{C}_{l,m} = \sum_{j=0}^{l} C_{j,m}$.

Proof. Let $m > 0$. Then for sequences $\{a_i\}_{i=0}^{l}$ and $\{b_i\}_{i=0}^{l}$, if we set $A_{-1} = 0$ and

$$A_l := \sum_{k=0}^{l} a_k,$$

for $0 < l \leq m$, then

$$\begin{aligned}\sum_{l=0}^{m} a_l b_l &= \sum_{l=0}^{m} (A_l - A_{l-1}) b_l \\ &= \sum_{l=0}^{m} A_l b_l - \sum_{l=0}^{m-1} A_l b_{l+1} \\ &= \sum_{l=0}^{m-1} A_l (b_l - b_{l+1}) + A_m b_m.\end{aligned}$$

Our result follows from the above by setting $a_l = C_{l,m}$ and $b_l = ad^l_{P(g)} P(ad^{l-m}_g h)$. It then follows that $A_l = \tilde{C}_{l,m}$. Notice that since $\sum_{l=0}^{m} C_{l,m} = 0$,

$$A_m = \sum_{l=0}^{m} a_l = \sum_{l=0}^{m} C_{l,m} = 0.$$

□

Substituting $P_{\mathcal{P}}$ for P in the previous calculation gives

$$h_{\mathcal{P}}(g) = \sum_{m=0}^{r-1} \sum_{l=0}^{m-1} \tilde{C}_{l,m} \left(ad^l_{P_{\mathcal{P}}(g)} P_{\mathcal{P}}(ad^{l-m}_g h) - ad^{l+1}_{P_{\mathcal{P}}(g)} P_{\mathcal{P}}(ad^{l-m-1}_g h) \right) \tag{5.39}$$

$$= P_{\mathcal{P}} h + \sum_{m=1}^{r-1} \sum_{l=0}^{m-1} \tilde{C}_{l,m} \left(ad^l_{P_{\mathcal{P}}(g)} P_{\mathcal{P}}(ad^{l-m}_g h) - ad^{l+1}_{P_{\mathcal{P}}(g)} P_{\mathcal{P}}(ad^{l-m-1}_g h) \right). \tag{5.40}$$

Definition 5.46. For $1 \leq k < j \leq r$ and $h_1, ..., h_j \in H(\mathfrak{g})$, define $R^{\mathcal{P}}_{j,k} : H(\mathfrak{g})^j \to H(\mathfrak{g})$ by

$$\begin{aligned}R^{\mathcal{P}}_{j,k}(h_1, ..., h_j) \equiv\ & ad_{P_{\mathcal{P}}(h_1)} \cdots ad_{P_{\mathcal{P}}(h_{k-1})} \left(P_{\mathcal{P}}(ad_{h_k} \cdots ad_{h_{j-1}} h_j) \right) \\ & - ad_{P_{\mathcal{P}}(h_1)} \cdots ad_{P_{\mathcal{P}}(h_k)} \left(P_{\mathcal{P}}(ad_{h_{k+1}} \cdots ad_{h_{j-1}} h_j) \right).\end{aligned}$$

Observe that $R^{\mathcal{P}}_{j,k}$ is a multilinear function.

Using this definition, we can then write

$$h_{\mathcal{P}}(g) = P_{\mathcal{P}} h + \sum_{1 \leq k < j \leq r} \tilde{C}_{j,k} R^{\mathcal{P}}_{j,k}(\overbrace{g, ..., g}^{j-1 \text{ times}}, h), \tag{5.41}$$

where the constants $\tilde{C}_{j,k}$ are given by Remark 5.45.

Example 5.47. Suppose that $\mathfrak{g}$ is a step 3 stratified Lie algebra. Then

$$g \cdot h = g + h + \frac{1}{2}[g,h] + \frac{1}{12}([g,[g,h]] - [h,[g,h]])$$

and therefore

$$P_{\mathcal{P}}(g \cdot th) = P_{\mathcal{P}}g + tP_{\mathcal{P}}h + \frac{t}{2}P_{\mathcal{P}}[g,h] + \frac{t}{12}P_{\mathcal{P}}[g,[g,h]] - \frac{t^2}{12}P_{\mathcal{P}}[h,[g,h]].$$

Calculating $h_{\mathcal{P}}(g)$ yields

$$\begin{aligned}
h_{\mathcal{P}}(g) &= \frac{d}{dt}|_{t=0} P_{\mathcal{P}}(-g) \cdot P_{\mathcal{P}}(g \cdot th) \\
&= \frac{d}{dt}|_{t=0}(-P_{\mathcal{P}}g) \cdot (P_{\mathcal{P}}g + tP_{\mathcal{P}}h + \frac{t}{2}P_{\mathcal{P}}[g,h] + \frac{t}{12}P_{\mathcal{P}}[g,[g,h]] - \frac{t^2}{12}P_{\mathcal{P}}[h,[g,h]]) \\
&= \frac{d}{dt}|_{t=0} \begin{pmatrix} tP_{\mathcal{P}}h + \frac{t}{2}P_{\mathcal{P}}[g,h] + \frac{t}{12}P_{\mathcal{P}}[g,[g,h]] - \frac{t^2}{12}P_{\mathcal{P}}[h,[g,h]] \\ -\frac{1}{2}[P_{\mathcal{P}}g, P_{\mathcal{P}}g + tP_{\mathcal{P}}h + \frac{t}{2}P_{\mathcal{P}}[g,h]] + \frac{1}{12}[P_{\mathcal{P}}g,[P_{\mathcal{P}}g, P_{\mathcal{P}}g + tP_{\mathcal{P}}h]] \\ +\frac{1}{12}[P_{\mathcal{P}}g + tP_{\mathcal{P}}h, [P_{\mathcal{P}}g, P_{\mathcal{P}}g + tP_{\mathcal{P}}h]] \end{pmatrix} \\
&= P_{\mathcal{P}}h + \frac{1}{2}P_{\mathcal{P}}[g,h] + \frac{1}{12}P_{\mathcal{P}}[g,[g,h]] - \frac{1}{2}[P_{\mathcal{P}}g, P_{\mathcal{P}}h] \\
&- \frac{1}{4}[P_{\mathcal{P}}g, P_{\mathcal{P}}[g,h]] + \frac{1}{12}[P_{\mathcal{P}}g,[P_{\mathcal{P}}g,P_{\mathcal{P}}h]] + \frac{1}{12}[P_{\mathcal{P}}g,[P_{\mathcal{P}}g,P_{\mathcal{P}}h]] \\
&= P_{\mathcal{P}}h + \frac{1}{2}\left(P_{\mathcal{P}}[g,h] - [P_{\mathcal{P}}g,P_{\mathcal{P}}h]\right) + \frac{1}{12}\left(P_{\mathcal{P}}[g,[g,h]] - [P_{\mathcal{P}}g, P_{\mathcal{P}}[g,h]]\right) \\
&- \frac{1}{6}\left([P_{\mathcal{P}}g, P_{\mathcal{P}}[g,h]] - [P_{\mathcal{P}}g,[P_{\mathcal{P}}g,P_{\mathcal{P}}h]]\right) \\
&= P_{\mathcal{P}}h + \frac{1}{2}R^{\mathcal{P}}_{2,1}(g,h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(g,g,h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(g,g,h).
\end{aligned}$$

Notation 5.48. For notational convenience, we will continue using $\underline{e}$ to denote the identity path in $\mathcal{W}(G)$, while 0 will be used to denote the identity path in $\mathcal{W}(\mathfrak{g})$.

Remark 5.49. Notice that if any one of $h_1, ..., h_j$ is 0, then $R^{\mathcal{P}}_{j,k}(h_1, ..., h_j) \equiv 0$.

Recall that our goal is to analyze derivatives of $F_{\mathcal{P}}$ at the identity path. Then Remark 5.49 imples that

$$\begin{aligned}
(\tilde{h}F_{\mathcal{P}})(\underline{e}) &= \langle \alpha, h_{\mathcal{P}}(\underline{e}) \rangle \\
&= \langle \alpha, P_{\mathcal{P}}h + \sum_{1 \le k < j \le r} \tilde{C}_{j,k} R^{\mathcal{P}}_{j,k}(\underline{e}, ..., \underline{e}, h) \rangle \\
&= \langle \alpha, P_{\mathcal{P}}h \rangle. \qquad (5.42)
\end{aligned}$$

Now we consider the second order case.

Notation 5.50. Suppose R is a function defined on $H(\mathfrak{g})^j$. For $k \in H(\mathfrak{g})$, denote

$$(\tilde{k}^{(i)}R)(g_1, ..., g_j) \equiv \frac{d}{dt}|_{t=0}R(g_1, ..., g_{i-1}, g_i \cdot tk, g_{i+1}, ..., g_j).$$

Note that if R is multilinear, then

$$\begin{aligned}(\tilde{k}^{(i)}R)(g_1, ..., g_j) &= \frac{d}{dt}|_{t=0}R(g_1, ..., g_{i-1}, g_i + \sum_{l=0}^{r} C_l ad_{g_i}^l h + O(t^2), g_{i+1}, ..., g_j) \\ &= \sum_{l=0}^{r-1} C_l R(g_1, ..., g_{i-1}, ad_{g_i}^l h, g_{i+1}, ..., g_j). \qquad (5.43)\end{aligned}$$

The notation above allows us to write

$$\begin{aligned}\frac{d}{dt}|_{t=0}h_{\mathcal{P}}(g \cdot tk) &= \frac{d}{dt}|_{t=0}P_{\mathcal{P}}h + \sum_{1\le k<j\le r} \tilde{C}_{j,k}R_{j,k}^{\mathcal{P}}(g \cdot tk, ..., g \cdot tk, h) \\ &= \sum_{1\le k<j\le r}\sum_{i=1}^{j-1} \tilde{C}_{j,k}\tilde{k}^{(i)}R_{j,k}^{\mathcal{P}}(g, ..., g, h).\end{aligned}$$

Since $R_{j,k}^{\mathcal{P}}$ is multilinear, Eq. (5.43) gives

$$\begin{aligned}\tilde{k}^{(i)}R_{j,k}^{\mathcal{P}}(g, ..., g, h) &= \sum_{l=0}^{r-1} C_l R_{j,k}^{\mathcal{P}}(g_1, ..., g_{i-1}, ad_{g_i}^l k, g_{i+1}, ..., g_j) \\ &= \sum_{l=0}^{r-j-1} C_l R_{j,k}^{\mathcal{P}}(g_1, ..., g_{i-1}, ad_{g_i}^l k, g_{i+1}, ..., g_j).\end{aligned}$$

Therefore

$$\begin{aligned}(\tilde{k}\tilde{h}F_{\mathcal{P}})(g) &= \frac{d}{dt}|_{t=0}(\tilde{h}_{\mathcal{P}}(g \cdot tk)u_\alpha)(P_{\mathcal{P}}(g \cdot tk)) \\ &= (\tilde{k}_{\mathcal{P}}(g)\tilde{h}_{\mathcal{P}}(g)u_\alpha)(P_{\mathcal{P}}(g)) + (\frac{d}{dt}|_{t=0}\tilde{h}_{\mathcal{P}}(g \cdot tk)u_\alpha)(P_{\mathcal{P}}(g)) \\ &= (\tilde{k}_{\mathcal{P}}(g)\tilde{h}_{\mathcal{P}}(g)u_\alpha)(P_{\mathcal{P}}(g)) \\ &+ \left(\left(\sum_{1\le k<j\le r}\sum_{i=1}^{j-1} \tilde{C}_{j,k}\tilde{k}^{(i)}R_{j,k}^{\mathcal{P}}(g, ..., g, h)\right) u_\alpha\right)(P_{\mathcal{P}}(g)) \\ &= \langle D^2u_\alpha(P_{\mathcal{P}}(g))), k_{\mathcal{P}}(g) \otimes h_{\mathcal{P}}(g)\rangle \\ &+ \sum_{1\le k<j\le r}\sum_{i=1}^{j-1} \tilde{C}_{j,k}\langle Du_\alpha(P_{\mathcal{P}}(g))), \tilde{k}^{(i)}R_{j,k}^{\mathcal{P}}(g, ..., g, h)\rangle.\end{aligned}$$

In particular,

$$\begin{aligned}(\tilde{k}\tilde{h}F_{\mathcal{P}})(\underline{e}) &= \langle D^2 u_\alpha(\underline{e}), P_{\mathcal{P}}k \otimes P_{\mathcal{P}}h\rangle + \tilde{C}_{2,1}\langle Du_\alpha(\underline{e}), R^{\mathcal{P}}_{2,1}(k,h)\rangle \\ &= \langle \alpha, P_{\mathcal{P}}k \otimes P_{\mathcal{P}}h + \tilde{C}_{2,1}R^{\mathcal{P}}_{2,1}(k,h)\rangle.\end{aligned}$$

For higher order derivatives, it is not difficult to see that the result of evaluating derivatives of $F_{\mathcal{P}}$ at the identity yields α acting on a sum of tensor products of terms like $P_{\mathcal{P}}h$ and $R^{\mathcal{P}}_{j,k}$,with arguents possibly nested brackets. This is summarized in the following theorem.

Theorem 5.51. *Let $\mathcal{P}$ a partition of $[0,1]$, and $F_{\mathcal{P}}$ defined as in Remark 5.8. Then for any $h_1, h_2, ..., h_k \in H(\mathfrak{g})$,*

$$\begin{aligned}\left(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_kF_{\mathcal{P}}\right)(\underline{e}) &= \langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle \\ &= \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes P_{\mathcal{P}}h_k + R^{\mathcal{P}}_k(h_1, ..., h_k)\rangle,\end{aligned}$$

where $R^{\mathcal{P}}_k \in T(H(\mathfrak{g}))$ is a sum of tensor products determined by iteration of the product rule and differentiation as in Notation 5.50.

The above theorem is really a summary of the notation that we have built up over the past section. The truth of Theorem 5.51 can best be seen using examples. The following continuation of Example 5.47 should provide insight into the general case.

Example 5.52. Again, let $\mathfrak{g}$ be a step 3 stratified Lie algebra. Recall from Example 5.47 that

$$h_{\mathcal{P}}(g) = P_{\mathcal{P}}h + \frac{1}{2}R^{\mathcal{P}}_{2,1}(g,h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(g,g,h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(g,g,h).$$

Also note that since

$$g \cdot tk = g + tk + \frac{t}{2}[g,k] + \frac{t}{12}[g,[g,k]] + O(t^2),$$

the following compuations are justified:

$$\begin{aligned}\tilde{k}^{(1)}R^{\mathcal{P}}_{2,1}(g,h) &= R^{\mathcal{P}}_{2,1}(k,h) + R^{\mathcal{P}}_{2,1}([g,k],h) \\ \tilde{k}^{(1)}R^{\mathcal{P}}_{3,1}(g,g,h) &= R^{\mathcal{P}}_{3,1}(k,g,h) \\ \tilde{k}^{(2)}R^{\mathcal{P}}_{3,1}(g,g,h) &= R^{\mathcal{P}}_{3,1}(g,k,h) \\ \tilde{k}^{(1)}R^{\mathcal{P}}_{3,2}(g,g,h) &= R^{\mathcal{P}}_{3,2}(k,g,h) \\ \tilde{k}^{(2)}R^{\mathcal{P}}_{3,2}(g,g,h) &= R^{\mathcal{P}}_{3,2}(g,k,h)\end{aligned}$$

Now computing the first derivative, we see that

$$
\begin{aligned}
(\tilde{h}F_{\mathcal{P}})(g) &= \langle \alpha, h_{\mathcal{P}}(g)\rangle \\
&= \langle \alpha, P_{\mathcal{P}}h + \frac{1}{2}R^{\mathcal{P}}_{2,1}(g,h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(g,g,h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(g,g,h)\rangle,
\end{aligned}
$$

and so

$$
\begin{aligned}
(\tilde{h}F_{\mathcal{P}})(\underline{e}) &= \langle \alpha, P_{\mathcal{P}}h + \frac{1}{2}R^{\mathcal{P}}_{2,1}(\underline{e},h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(\underline{e},\underline{e},h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(\underline{e},\underline{e},h)\rangle \\
&= \langle \alpha, P_{\mathcal{P}}h\rangle.
\end{aligned}
$$

For the second derivatives,

$$
\begin{aligned}
(\tilde{k}\tilde{h}F_{\mathcal{P}})(g) &= \langle D^2 u_\alpha(P_{\mathcal{P}}(g))), k_{\mathcal{P}}(g)\otimes h_{\mathcal{P}}(g)\rangle \\
&+ \sum_{1\le k<j\le 3}\sum_{i=1}^{j-1}\tilde{C}_{j,k}\langle Du_\alpha(P_{\mathcal{P}}(g))), \tilde{k}^{(i)}R^{\mathcal{P}}_{j,k}(g,...,g,h)\rangle \\
&= \langle \alpha, k_{\mathcal{P}}(g)\otimes h_{\mathcal{P}}(g)\rangle + \frac{1}{2}\langle \alpha, \tilde{k}^{(1)}R^{\mathcal{P}}_{2,1}(g,h)\rangle + \frac{1}{12}\langle \alpha, \tilde{k}^{(1)}R^{\mathcal{P}}_{3,1}(g,g,h)\rangle \\
&+ \frac{1}{12}\langle \alpha, \tilde{k}^{(2)}R^{\mathcal{P}}_{3,1}(g,g,h)\rangle - \frac{1}{6}\langle \alpha, \tilde{k}^{(1)}R^{\mathcal{P}}_{3,2}(g,g,h)\rangle - \frac{1}{6}\langle \alpha, \tilde{k}^{(2)}R^{\mathcal{P}}_{3,2}(g,g,h)\rangle \\
&= \langle \alpha, k_{\mathcal{P}}(g)\otimes h_{\mathcal{P}}(g)\rangle + \frac{1}{2}\langle \alpha, R^{\mathcal{P}}_{2,1}(k,h) + R^{\mathcal{P}}_{2,1}([g,k],h)\rangle \\
&+ \frac{1}{12}\langle \alpha, R^{\mathcal{P}}_{3,1}(k,g,h) + R^{\mathcal{P}}_{3,1}(g,k,h)\rangle - \frac{1}{6}\langle \alpha, R^{\mathcal{P}}_{3,2}(k,g,h) + R^{\mathcal{P}}_{3,2}(g,k,h)\rangle.
\end{aligned}
$$

Therefore

$$
(\tilde{k}\tilde{h}F_{\mathcal{P}})(\underline{e}) = \langle \alpha, P_{\mathcal{P}}k\otimes P_{\mathcal{P}}h\rangle + \frac{1}{2}\langle \alpha, R^{\mathcal{P}}_{2,1}(k,h)\rangle.
$$

For the third derivatives, we get the following expression (the calculation is carried out in section 3 of the appendix).

$$
\begin{aligned}
(\tilde{l}\tilde{k}\tilde{h}F_{\mathcal{P}})(\underline{e}) &= \langle \alpha, P_{\mathcal{P}}l\otimes P_{\mathcal{P}}k\otimes P_{\mathcal{P}}h\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}k\otimes \frac{1}{2}R^{\mathcal{P}}_{2,1}(l,h)\rangle + \langle \alpha, \frac{1}{2}R^{\mathcal{P}}_{2,1}(l,k)\otimes P_{\mathcal{P}}h\rangle \\
&+ \langle \alpha, P_{\mathcal{P}}l\otimes \frac{1}{2}R^{\mathcal{P}}_{2,1}(k,h)\rangle + \langle \alpha, \frac{1}{2}R^{\mathcal{P}}_{2,1}([l,k],h)\rangle \\
&+ \langle \alpha, \frac{1}{12}R^{\mathcal{P}}_{3,1}(k,l,h)\rangle + \langle \alpha, \frac{1}{12}R^{\mathcal{P}}_{3,1}(l,k,h)\rangle \\
&- \langle \alpha, \frac{1}{6}R^{\mathcal{P}}_{3,2}(l,k,h)\rangle - \langle \alpha, \frac{1}{6}R^{\mathcal{P}}_{3,2}(k,l,h)\rangle.
\end{aligned}
$$

5.5 Increments and Multilinear Functions on $H(\mathfrak{g})$

The terms $R_{j,k}^{\mathcal{P}}$ as defined in Definition 5.46 are multilinear functions on $H(\mathfrak{g})$. While the structure of our remainder terms can get quite complicated, the fact that they are multilinear allows us to estimate sums over an orthonormal basis.

Notation 5.53. Given a partition $\mathcal{P} = \{0 = s_0 < s_1 < ... < s_n < s_{n+1} = 1\}$, we'll let δ_i denote the increment function for $i = 0, ..., n-1$. That is, if V is a vector space, and $T : [0,1] \to V$ is a path, then

$$\delta_i T := T(s_{i+1}) - T(s_i).$$

In the case that T is the identity function on $[0,1]$, we omit the T in the notation, i.e.

$$\delta_i := s_{i+1} - s_i.$$

Remark 5.54. The notation above will be used often in the following contexts. If $h \in H(\mathfrak{g})$, then

$$\delta_i h := h(s_{i+1}) - h(s_i).$$

If $h_1, ..., h_k \in H(\mathfrak{g})$, then $(h_1, ..., h_k) \in H(\mathfrak{g})^k$, and

$$\delta_i(h_1, ..., h_k) := (h_1(s_{i+1}), ..., h_k(s_{i+1})) - (h_1(s_i), ..., h_k(s_i)).$$

Finally, if $h_1, ..., h_k \in H(\mathfrak{g})$ and T is a multilinear function on $\mathfrak{g}^k$, then

$$\delta_i T(h_1, ..., h_k) := T(h_1(s_{i+1}), ..., h_k(s_{i+1})) - T(h_1(s_i), ..., h_k(s_i)).$$

Notation 5.55. For a given partition $\mathcal{P} = \{0 = s_0 < s_1 < ... < s_n < s_{n+1} = 1\}$, let

$$t_i(t) := 1_{(s_i, s_{i+1}]}(t)\left(\frac{t - s_i}{\delta_i}\right).$$

In the sequel, we will often omit the t in the notation, that is $t_i(t) = t_i$.

Remark 5.56. Notice that

$$\frac{d}{dt} t_i(t) = 1_{(s_i, s_{i+1}]}(t)\frac{1}{\delta_i}.$$

With this notation, for $h \in H(\mathfrak{g})$, we have

$$P_{\mathcal{P}} h = \sum_{i=0}^{n-1} \left(h\left(s_i\right) 1_{(s_i, s_{i+1}]} + \left(\delta_i h\right) t_i \right).$$

Proposition 5.57. *For any $g, h \in H(\mathfrak{g})$ and any partition $\mathcal{P}$,*

$$P_{\mathcal{P}}[g,h] - [P_{\mathcal{P}} g, P_{\mathcal{P}} h] = \sum_{i=0}^{n-1} [\delta_i g, \delta_i h](t_i - t_i^2),$$

or in other words,

$$P_{\mathcal{P}}\left(ad_g h\right) - ad_{P_{\mathcal{P}} g}\left(P_{\mathcal{P}} h\right) = \sum_{i=0}^{n-1} ad_{\delta_i g}\left(\delta_i h\right)(t_i - t_i^2).$$

Proof. It suffices to show the result on an individual partition increment. Consider the interval $(s_i, s_{i+1}]$. Then on $(s_i, s_{i+1}]$,

$$\begin{aligned} P_{\mathcal{P}}[g,h] - [P_{\mathcal{P}} g, P_{\mathcal{P}} h] &= [g\left(s_i\right), h(s_i)] + \delta_i [g,h] t_i - [g\left(s_i\right) + \delta_i g t_i, h\left(s_i\right) + \delta_i h t_i] \\ &= [g\left(s_i\right), h(s_i)] + \delta_i [g,h] t_i - [g\left(s_i\right), h(s_i)] \\ &- [\delta_i g, h\left(s_i\right)] t_i - [g\left(s_i\right), \delta_i h] t_i - [\delta_i g, \delta_i h] t_i^2 \\ &= \left(\delta_i [g,h] - [\delta_i g, h\left(s_i\right)] - [g\left(s_i\right), \delta_i h]\right) t_i - [\delta_i g, \delta_i h] t_i^2 \\ &= [\delta_i g, \delta_i h](t_i - t_i^2). \end{aligned}$$

□

In particular, Proposition 5.57 lets us rewrite our remainder terms.

$$\begin{aligned} R_{j,k}^{\mathcal{P}}(h_1, ..., h_j) &= ad_{P_{\mathcal{P}}(h_1)} \cdots ad_{P_{\mathcal{P}}(h_{k-1})} P_{\mathcal{P}}(ad_{h_k} \cdots ad_{h_{j-1}} h_j) \\ &- ad_{P_{\mathcal{P}}(h_1)} \cdots ad_{P_{\mathcal{P}}(h_k)} P_{\mathcal{P}}(ad_{h_{k+1}} \cdots ad_{h_{j-1}} h_j) \\ &= \sum_{i=0}^{n-1} ad_{P_{\mathcal{P}}(h_1)} \cdots ad_{P_{\mathcal{P}}(h_{k-1})} ad_{\delta_i h_k} \left(\delta_i (ad_{h_{k+1}} \cdots ad_{h_{j-1}} h_j)\right)(t_i - t_i^2), \end{aligned}$$

or in bracket notation

$$R_{j,k}^{\mathcal{P}}(h_1, ..., h_j) = \sum_{i=0}^{n-1} [P_{\mathcal{P}} h_1, [..., [P_{\mathcal{P}} h_{k-1}, [\delta_i h_k, \delta_i [h_{k+1}, [..., [h_{j-1}, h_j]] \cdots](t_i - t_i^2). \quad (5.44)$$

As we saw in the previous section, the more general remainder term is of the form

$$R^{\mathcal{P}}_{j,k}(B_1, B_2, ..., B_j)(h_1, ..., h_p), \tag{5.45}$$

where B_i are multilinear functions coming from nested brackets. It is to be understood in writing (5.45) that there exist $p_1, ..., p_j > 0$ such that $\sum_{i=1}^{j} p_i = p$ and $B_i : H(\mathfrak{g})^{p_i} \to H(\mathfrak{g})$, in which case

$$\begin{aligned} &R^{\mathcal{P}}_{j,k}(B_1, B_2, ..., B_j)(h_1, ..., h_p) \\ &\equiv R^{\mathcal{P}}_{j,k}(B_1(h_1, ..., h_{p_1}), B_2(h_{p_1+1}, ..., h_{p_1+p_2}), ..., B_j(h_{p-p_j+1}, ..., h_p)). \end{aligned} \tag{5.46}$$

We will see that the structure and domain of the B_i terms are not important for our calculations. Therefore Eq. (5.45) is useful shorthand. We will often further shorten the expression in the following manner:

$$R^{\mathcal{P}}_{j,k}(B_1, B_2, ..., B_j)(\overline{h}) := R^{\mathcal{P}}_{j,k}(B_1, B_2, ..., B_j)(h_1, ..., h_p). \tag{5.47}$$

The meaning of $\overline{h}$ should be clear from context.

From Eq. (5.44),

$$\begin{aligned} &R^{\mathcal{P}}_{j,k}(B_1, ..., B_j)(h_1, ..., h_p) \\ &:= \sum_{i=0}^{n-1} [P_{\mathcal{P}} B_1, [..., [P_{\mathcal{P}} B_{k-1}, [\delta_i B_k, \delta_i [B_{k+1}, [..., [B_{j-1}, B_j]] \cdots](h_1, ..., h_p)(t_i - t_i^2). \end{aligned}$$

We will see that a futher rewriting of the above is useful. First some more notation.

Notation 5.58. Let $S(\mathbb{C})$ denote a basis for $H(\mathbb{C})$, and $\mathfrak{X}_{\mathbb{C}}$ denote a basis for the complex inner product space $(\mathfrak{g}, (\ , \ )_{\mathfrak{g}})$.

Remark 5.59. If $S(\mathbb{C}) = \{u_i\}_{i=1}^{\infty}$, and $\mathfrak{X}_{\mathbb{C}} = \{A_j\}_{j=1}^{d}$, then $\{u_i A_j\}_{i,j=1}^{i=\infty, j=d}$ forms a basis for $(H(\mathfrak{g}), (\ , \ )_{H(\mathfrak{g})})$.

Notation 5.60. For integers $k \geq l \geq 0$, let Ω_k^l denote the set of subsets of size l of the integers $1, 2, ..., k$. That is

$$\Omega_k^l := \{\omega \in 2^{\{1,2,...,k\}} | \#(\omega) = l\}.$$

Notation 5.61. Suppose $u_1, ..., u_k \in H(\mathbb{C})$ and $\omega \in \Omega_k^l$. Then

$$\delta_i^\omega(u_1 \cdots u_k) := \prod_{j\in\omega} \delta_i u_{r_l} \prod_{j\notin\omega} u_{p_q}(s_i).$$

Similarly, if $h_1, ..., h_k \in H(\mathfrak{g})$, then

$$\delta_i^\omega(h_1, ..., h_k) \equiv (\widehat{h}_1, ..., \widehat{h}_k)$$

where

$$\widehat{h}_j = \begin{cases} \delta_i h_j & \text{if } j \in \omega \\ h_j(s_i) & \text{otherwise} \end{cases}.$$

Example 5.62. Suppose $\omega = \{1, 3, 4\} \in \Omega_5^3$. Then

$$\delta_i^\omega(u_1 \cdots u_5) = (\delta_i u_1)(\delta_i u_3)(\delta_i u_4)(u_2(s_i))(u_5(s_i))$$

and

$$\delta_i^\omega(h_1, ..., h_5) = (\delta_i h_1, h_2(s_i), \delta_i h_3, \delta_i h_4, h_5(s_i)).$$

Observe that if T is a bilinear function on $\mathfrak{g}^2$. Then

$$\begin{aligned}
\delta_i T(h_1, h_2) &= T(h_1(s_{i+1}), h_2(s_{i+1})) - T(h_1(s_i), h_2(s_i)) \\
&= T(h_1(s_{i+1}), h_2(s_i)) - T(h_1(s_i), h_2(s_i)) + T(h_1(s_i), h_2(s_{i+1})) \\
&- T(h_1(s_i), h_2(s_i)) + T(h_1(s_{i+1}), h_2(s_{i+1})) - T(h_1(s_{i+1}), h_2(s_i)) \\
&- T(h_1(s_i), h_2(s_{i+1})) + T(h_1(s_i), h_2(s_i)) \\
&= T(h_1(s_{i+1}), h_2(s_i)) - T(h_1(s_i), h_2(s_i)) + T(h_1(s_i), h_2(s_{i+1})) \\
&- T(h_1(s_i), h_2(s_i)) + T(h_1(s_{i+1}) - h_1(s_i), h_2(s_{i+1}) - h_2(s_i)) \\
&= T(\delta_i h_1, h_2(s_i)) + T(h_1(s_i), \delta_i h_2) + T(\delta_i h_1, \delta_i h_2) \\
&= \sum_{l=1}^{2} \sum_{\omega\in\Omega_2^l} T(\delta_i^\omega(h_1, h_2)).
\end{aligned}$$

This suggests the following Proposition.

Proposition 5.63 (Product Rule)**.** *Suppose T be a multilinear function on $\mathfrak{g}^k$. Then*

$$\delta_i T(h_1, ..., h_k) = \sum_{l=1}^{k} \sum_{\omega\in\Omega_k^l} T\left(\delta_i^\omega(h_1, ..., h_k)\right).$$

Proof. For $k = 1$, the result is trivial. For $k = 2$, the result follows by the above calculation. Now suppose it is true for all multilinear functions on $\mathfrak{g}^k$. Then if T is a multilinear function on $\mathfrak{g}^{k+1}$,

$$\begin{aligned}
&\delta_i T(h_1, ..., h_{k+1}) \\
&= T(h_1(s_{i+1}), ..., h_{k+1}(s_{i+1})) - T(h_1(s_i), ..., h_{k+1}(s_i)) \\
&= T(h_1(s_{i+1}), ..., h_k(s_{i+1}), h_{k+1}(s_{i+1})) - T(h_1(s_{i+1}), ..., h_k(s_{i+1}), h_{k+1}(s_i)) \\
&- T(h_1(s_i), ..., h_k(s_i), h_{k+1}(s_{i+1})) + T(h_1(s_i), ..., h_k(s_i), h_{k+1}(s_i)) \\
&+ T(h_1(s_{i+1}), ..., h_k(s_{i+1}), h_{k+1}(s_i)) - T(h_1(s_i), ..., h_k(s_i), h_{k+1}(s_i)) \\
&= T(\delta_i(h_1, ..., h_k), \delta_i h_{k+1}) + T(\delta_i(h_1, ..., h_k), h_{k+1}(s_i)) \\
&= T(\sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} (\delta_i^\omega(h_1, ..., h_k)), \delta_i h_{k+1}) + T(\sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} (\delta_i^\omega(h_1, ..., h_k)), h_{k+1}(s_i)) \\
&= \sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} T((\delta_i^\omega(h_1, ..., h_k)), \delta_i h_{k+1}) + \sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} T((\delta_i^\omega(h_1, ..., h_k)), h_{k+1}(s_i)) \\
&= \sum_{l=1}^{k+1} \sum_{\{\omega \in \Omega_{k+1}^l | k+1 \in \omega\}} T(\delta_i^\omega(h_1, ..., h_{k+1})) + \sum_{l=1}^{k+1} \sum_{\{\omega \in \Omega_{k+1}^l | k+1 \notin \omega\}} T(\delta_i^\omega(h_1, ..., h_{k+1})) \\
&= \sum_{l=1}^{k+1} \sum_{\omega \in \Omega_{k+1}^l} T(\delta_i^\omega(h_1, ..., h_{k+1})).
\end{aligned}$$

□

Remark 5.64. Letting T be the identity map on $\mathfrak{g}^k$, the above Proposition tells us that

$$\delta_i(h_1, ..., h_k) = \sum_{l=1}^{k} \sum_{\omega \in \Omega_n^l} \delta_i^\omega(h_1, ..., h_k).$$

Furthermore, if $T : \mathbb{C}^k \to \mathbb{C}$ by $T(z_1, ..., z_k) = z_1 \cdots z_k$, then we get

$$\delta_i(u_1 \cdots u_k) = \sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} \delta_i^\omega(u_1 \cdots u_k)$$

for $u_1, ..., u_k \in H(\mathbb{C})$.

The following proposition is a summary of results that can be found in section 1 of the appendix.

Proposition 5.65. *Suppose* $\mathcal{P}$ $=\{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ *is a partition of* $[0,1]$. *For any* $s,t \in [0,1]$ *and any* $1 \le i,j \le n$,

1. $\sum_{u \in S(\mathbb{C})} u(s)\overline{u(t)} = s \wedge t$
2. $\sum_{u \in S(\mathbb{C})} |\delta_i u|^2 = K(\delta_i,\delta_i) = \delta_i$
3. $\sum_{u \in S(\mathbb{C})} (\delta_i u)\overline{u(s_j)} = \delta_{j>i}\delta_i$
4. $\sum_{u \in S(\mathbb{C})} (\delta_i u)\overline{(\delta_j u)} = \delta_{ij}\delta_i$

   where δ_{ij} *denotes the Kronecker delta and*

$$\delta_{j>i} = \begin{cases} \delta_i & \text{if } j > i \\ 0 & \text{if } j \le i \end{cases}.$$

Proposition 5.66. *Suppose* $\omega \in \Omega_n^l$, *and* $\theta \in \Omega_n^m$, *with* $n \ge 2$ *and* $l,m \ge 1$. *Then*

$$\sum_{u_1,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) = \delta_{ij}\delta_{\omega\theta}\,(\delta_i)^l\,(s_i)^{n-l},$$

where $\delta_{\omega\theta} = 1$ *if* $\omega = \theta$ *and* 0 *otherwise.*

Proof. First suppose $\omega \ne \theta$. If $\omega \cap \theta \ne \emptyset$, then WLOG there exists elements $p \in \omega$ with $p \notin \theta$, and $q \in \omega \cap \theta$. WLOG, say $p = 1$ and $q = 2$. Then

$$\begin{aligned}
&\sum_{u_1,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) \\
&= \left(\sum_{u_1 \in S(\mathbb{C})} (\delta_i u_1)\left(\overline{u_1(s_j)}\right)\right)\left(\sum_{u_2 \in S(\mathbb{C})} (\delta_i u_2)\left(\overline{\delta_j u_2}\right)\right) \times \cdots \\
&\cdots \times \left(\sum_{u_3,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega\setminus\{1,2\}}(u_3 \cdots u_n)\delta_j^{\theta\setminus\{2\}}(\overline{u_3 \cdots u_n})\right) \\
&= (\delta_{j>i}\delta_i)(\delta_{ij}\delta_i)\left(\sum_{u_3,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega\setminus\{1,2\}}(u_3 \cdots u_n)\delta_j^{\theta\setminus\{2\}}(\overline{u_3 \cdots u_n})\right) \\
&= 0,
\end{aligned}$$

since $\delta_{j>i}\delta_{ij} = 0$. If $\omega \cap \theta = \emptyset$, then there exist elements p, q such that $p \in \omega, p \notin \theta, q \notin \omega$, and $q \in \theta$. WLOG, say $p = 1$ and $q = 2$. Then

$$\begin{aligned}
&\sum_{u_1,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) \\
&= \left(\sum_{u_1 \in S(\mathbb{C})} (\delta_i u_1) \left(\overline{u_1 (s_j)}\right) \right) \left(\sum_{u_2 \in S(\mathbb{C})} (u_2 (s_i)) \left(\overline{\delta_j u_2}\right) \right) \times \cdots \\
&\cdots \times \left(\sum_{u_3,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega \setminus \{1\}}(u_3 \cdots u_n)\delta_j^{\theta \setminus \{2\}}(\overline{u_3 \cdots u_n}) \right) \\
&= (\delta_{j>i}\delta_i)(\delta_{i>j}\delta_j) \left(\sum_{u_3,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega \setminus \{1\}}(u_3 \cdots u_n)\delta_j^{\theta \setminus \{2\}}(\overline{u_3 \cdots u_n}) \right) \\
&= 0,
\end{aligned}$$

since $\delta_{j>i}\delta_{i>j} = 0$. Now we assume that $\omega = \theta$. Then

$$\begin{aligned}
&\sum_{u_1,\ldots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) \\
&= \prod_{p \in \omega} \left(\sum_{u_p \in S(\mathbb{C})} (\delta_i u_p) \left(\overline{\delta_j u_p}\right) \right) \prod_{q \in \omega^c} \left(\sum_{u_q \in S(\mathbb{C})} (u_q (s_i)) \left(\overline{u_q (s_j)}\right) \right) \\
&= \delta_{ij} (\delta_i)^l (s_i)^{n-l}.
\end{aligned}$$

□

Corollary 5.67. $\sum_{u_1,\ldots,u_n \in S(\mathbb{C})} \delta_i(u_1 \cdots u_n)\delta_j(\overline{u_1 \cdots u_n}) = \delta_{ij} \sum_{l=1}^{n} \binom{n}{l} (\delta_i)^l (s_i)^{n-l}.$

Proof.

$$
\begin{aligned}
&\sum_{u_1,...,u_n\in S(\mathbb{C})} \delta_i(u_1\cdots u_n)\delta_j(\overline{u_1\cdots u_n})\\
&= \sum_{u_1,...,u_n\in S(\mathbb{C})} \left(\sum_{l=1}^{n}\sum_{\omega\in\Omega_n^l} \delta_i^{\omega}(u_1\cdots u_n)\right)\left(\sum_{m=1}^{n}\sum_{\theta\in\Omega_n^m} \delta_j^{\theta}(\overline{u_1\cdots u_n})\right)\\
&= \sum_{l,m=1}^{n}\sum_{\omega\in\Omega_n^l}\sum_{\theta\in\Omega_n^m}\left(\sum_{u_1,...,u_n\in S(\mathbb{C})} \delta_i^{\omega}(u_1\cdots u_n)\delta_j^{\theta}(\overline{u_1\cdots u_n})\right)\\
&= \sum_{l,m=1}^{n}\sum_{\omega\in\Omega_n^l}\sum_{\theta\in\Omega_n^m} \delta_{ij}\delta_{\omega\theta}\left(\delta_i\right)^l\left(s_i\right)^{n-l}\\
&= \delta_{ij}\sum_{l=1}^{n}\sum_{\omega\in\Omega_n^l}\left(\delta_i\right)^l\left(s_i\right)^{n-l}\\
&= \delta_{ij}\sum_{l=1}^{n}\binom{n}{l}\left(\delta_i\right)^l\left(s_i\right)^{n-l}\\
&= \delta_{ij}(s_{i+1}^n - s_i^n).
\end{aligned}
$$

□

Remark 5.68. Notice that if $h_j = u_jA_j$ for $u_j \in S(\mathbb{C})$ and $A_j \in \mathfrak{X}_{\mathbb{C}}$, then

$$\delta_i^{\omega}T(h_1, ..., h_n) = \delta_i^{\omega}(u_1\cdots u_n)T(A_1, ..., A_n)$$

and

$$\delta_iT(h_1, ..., h_n) = \delta_i(u_1\cdots u_n)T(A_1, ..., A_n).$$

Notation 5.69. If T_1 and T_2 are $\mathfrak{g}-$valued multilinear functions on $\mathfrak{g}^n$, then define $C_n(T_1, T_2) \in \mathfrak{g}\otimes\mathfrak{g}$ by

$$C_n(T_1, T_2) := \sum_{A_1,...,A_n\in\mathfrak{X}_{\mathbb{C}}} T_1(A_1, ..., A_n)\otimes\overline{T_2(A_1, ..., A_n)}.$$

Corollary 5.70. *Suppose $\omega \in \Omega_n^l$, and $\theta \in \Omega_n^m$, with $n \geq 2$ and $l, m \geq 1$. Let T_1 and T_2 be $\mathfrak{g}-$valued multilinear functions on $\mathfrak{g}^n$. Then*

$$
\begin{aligned}
&\sum_{h_1,...,h_n\in S_{\mathbb{C}}} T_1(\delta_i^{\omega}(h_1, ..., h_n))\otimes\overline{T_2(\delta_j^{\theta}(h_1, ..., h_n))}\\
&= \delta_{ij}\delta_{\omega\theta}\left(\delta_i\right)^l\left(s_i\right)^{n-l}C_n(T_1, T_2)
\end{aligned}
$$

Proof.

$$\begin{aligned}
&\sum_{h_1,\dots,h_n\in S_{\mathbb{C}}} T_1(\delta_i^{\omega}(h_1,...,h_n))\otimes\overline{T_2(\delta_j^{\theta}(h_1,...,h_n))}\\
&=\sum_{A_1,\dots,A_n\in\mathfrak{X}_{\mathbb{C}}}\sum_{u_1,\dots,u_n\in S(\mathbb{C})}\delta_i^{\omega}(u_1\cdots u_n)\delta_j^{\theta}(\overline{u_1\cdots u_n})T_1(A_1,...,A_n)\otimes\overline{T_2(A_1,...,A_n)}\\
&=\delta_{ij}\delta_{\omega\theta}\left(\delta_i\right)^l\left(s_i\right)^{n-l}\sum_{A_1,\dots,A_n\in\mathfrak{X}_{\mathbb{C}}}T_1(A_1,...,A_n)\otimes\overline{T_2(A_1,...,A_n)}\\
&=\delta_{ij}\delta_{\omega\theta}\left(\delta_i\right)^l\left(s_i\right)^{n-l}C_n(T_1,T_2).
\end{aligned}$$

□

Corollary 5.71. *Let T_1 and T_2 be $\mathfrak{g}-$valued multilinear functions on $\mathfrak{g}^n$. Then*

$$\begin{aligned}
&\sum_{h_1,\dots,h_n\in S_{\mathbb{C}}}\delta_iT_1(h_1,...,h_n)\otimes\overline{\delta_jT_2(h_1,...,h_n)}\\
&=\delta_{ij}\sum_{l=1}^{n}\binom{n}{l}\left(\delta_i\right)^l\left(s_i\right)^{n-l}C_n(T_1,T_2)\\
&=\delta_{ij}C_n(T_1,T_2)(s_{i+1}^n-s_i^n).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\sum_{h_1,\dots,h_n\in S_{\mathbb{C}}}\delta_iT_1(h_1,...,h_n)\otimes\overline{\delta_jT_2(h_1,...,h_n)}\\
&=\sum_{h_1,\dots,h_n\in S_{\mathbb{C}}}\left(\sum_{l=1}^{n}\sum_{\omega\in\Omega_n^l}T_1(\delta_i^{\omega}(h_1,...,h_n))\right)\otimes\left(\sum_{m=1}^{n}\sum_{\theta\in\Omega_n^m}\overline{T_2(\delta_j^{\theta}(h_1,...,h_n))}\right)\\
&=\sum_{l=1}^{n}\sum_{\omega\in\Omega_n^l}\sum_{m=1}^{n}\sum_{\theta\in\Omega_n^m}\left(\sum_{h_1,\dots,h_n\in S_{\mathbb{C}}}T_1(\delta_i^{\omega}(h_1,...,h_n))\otimes\overline{T_2(\delta_j^{\theta}(h_1,...,h_n))}\right)\\
&=\sum_{l=1}^{n}\sum_{\omega\in\Omega_n^l}\sum_{m=1}^{n}\sum_{\theta\in\Omega_n^m}\delta_{ij}\delta_{\omega\theta}\left(\delta_i\right)^l\left(s_i\right)^{n-l}\sum_{A_1,\dots,A_n\in\mathfrak{X}_{\mathbb{C}}}T_1(A_1,...,A_n)\otimes\overline{T_2(A_1,...,A_n)}\\
&=\delta_{ij}\sum_{l=1}^{n}\sum_{\omega\in\Omega_n^l}\left(\delta_i\right)^l\left(s_i\right)^{n-l}C_n(T_1,T_2)\\
&=\delta_{ij}\sum_{l=1}^{n}\binom{n}{l}\left(\delta_i\right)^l\left(s_i\right)^{n-l}C_n(T_1,T_2).
\end{aligned}$$

□

Remark 5.72. Using the bilinear map $h\otimes\overline{k}\in\mathfrak{g}\otimes\mathfrak{g}\to(h,k)_{\mathfrak{g}}$, it is clear that for $\omega\in\Omega_n^l$ and $\theta\in\Omega_n^m$

$$\begin{aligned}&\sum_{h_1,...,h_n\in S_{\mathbb{C}}}\Big(T_1(\delta_i^{\omega}(h_1,...,h_n)),T_2(\delta_j^{\theta}(h_1,...,h_n))\Big)_{\mathfrak{g}}\\&=\delta_{ij}\delta_{\omega\theta}\left(\delta_i\right)^l\left(s_i\right)^{n-l}\tilde{C}_n(T_1,T_2)\end{aligned}$$

where

$$\tilde{C}_n(T_1,T_2):=\sum_{A_1,...,A_n\in\mathfrak{X}_{\mathbb{C}}}\left(T_1(A_1,...,A_n),T_2(A_1,...,A_n)\right)_{\mathfrak{g}}.$$

In particular,

$$\begin{aligned}\sum_{h_1,...,h_n\in S_{\mathbb{C}}}||T(\delta_i^{\omega}(h_1,...,h_n))||_{\mathfrak{g}}^2&=\left(\delta_i\right)^l\left(s_i\right)^{n-l}\sum_{A_1,...,A_n\in\mathfrak{X}_{\mathbb{C}}}||T(A_1,...,A_n)||_{\mathfrak{g}}^2\\&=\tilde{C}_n(T,T)\left(\delta_i\right)^l\left(s_i\right)^{n-l}.\end{aligned}$$

5.6 Remainder Estimates

In general, we can write

$$R_{j,k}^{\mathcal{P}}(B_1,...,B_j)(h_1,...,h_p)=\sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\delta_i^{\omega}T(h_1,...,h_p)f_{\omega}(t_i),\tag{5.48}$$

where $T:\mathfrak{g}^p\to\mathfrak{g}$ is a mulitlinear function and $f_{\omega}(t_i)$ is a polynomial, possibly zero, in t_i depending on ω, where t_i is defined in Notation 5.55. Refer to Eq. (5.46) for the meaning of the left hand side, while Notation 5.61 indicates the meaning of the right hand side.

Example 5.73. For $r\geq 6$, the following remainder term appears in a sixth order derivative of $F_{\mathcal{P}}$ evaluated at the identity path.

$$\begin{aligned}
&R_{3,2}^{\mathcal{P}}([h_1,[h_2,h_3]],[h_4,h_5],h_6)\\
&=\sum_{i=0}^{n-1}[P_{\mathcal{P}}[h_1,[h_2,h_3]],[\delta_i[h_4,h_5],\delta_i h_6](t_i-t_i^2)\\
&=\sum_{i=0}^{n-1}\left(\begin{array}{c}[[h_1(s_i),[h_2(s_i),h_3(s_i)]],[\delta_i[h_4,h_5],\delta_i h_6](t_i-t_i^2)\\ +[\delta_i[h_1,[h_2,h_3]],[\delta_i[h_4,h_5],\delta_i h_6](t_i^2-t_i^3)\end{array}\right)\\
&=\sum_{i=0}^{n-1}\left(\begin{array}{c}[[h_1(s_i),[h_2(s_i),h_3(s_i)]],[\sum_{l=1}^{2}\sum_{\omega\in\Omega_2^l}\delta_i^{\omega}[h_4,h_5],\delta_i h_6](t_i-t_i^2)\\ +[\sum_{l=1}^{3}\sum_{\omega\in\Omega_3^l}\delta_i^{\omega}[h_1,[h_2,h_3]],[\sum_{l=1}^{2}\sum_{\omega\in\Omega_2^l}\delta_i^{\omega}[h_4,h_5],\delta_i h_6](t_i^2-t_i^3)\end{array}\right)\\
&=\sum_{i=0}^{n-1}\sum_{l=2}^{6}\sum_{\omega\in\Omega_6^l}\delta_i^{\omega}[[h_1,[h_2,h_3]],[h_4,h_5],h_6]f_{\omega}(t_i),
\end{aligned}$$

where

$$f_{\omega}(t_i)=\begin{cases}0 & \text{if} \quad \omega\cap\{6\}=\emptyset \text{ or } \omega\cap\{4,5\}=\emptyset\\ t_i-t_i^2 & \text{if} \quad \omega\cap\{1,2,3\}=\emptyset,\ \omega\cap\{6\}\neq\emptyset,\ \text{and } \omega\cap\{4,5\}\neq\emptyset\\ t_i^2-t_i^3 & \text{if} \quad \omega\cap\{1,2,3\}\neq\emptyset,\ \omega\cap\{6\}\neq\emptyset,\ \text{and } \omega\cap\{4,5\}\neq\emptyset\end{cases}$$

Proposition 5.74. *Suppose $f\in H(\mathbb{C})$ and $A\in\mathfrak{g}$. Then $fA\in H(\mathfrak{g})$ and*

$$||fA||_{H(\mathfrak{g})}^2=||f||_{H(\mathbb{C})}^2||A||_{\mathfrak{g}}^2.$$

Proof. The proof is a straightforward calculation.

$$\begin{aligned}
||fA||_{H(\mathfrak{g})}^2&=\int_0^1||f'(t)A||_{\mathfrak{g}}^2dt\\
&=\int_0^1|f'(t)|^2||A||_{\mathfrak{g}}^2dt\\
&=||A||_{\mathfrak{g}}^2\int_0^1|f'(t)|^2dt\\
&=||f||_{H(\mathbb{C})}^2||A||_{\mathfrak{g}}^2.
\end{aligned}$$

∎

Proposition 5.75. *Suppose $1\leq k<j\leq r$, and for $1\leq i\leq j$, $B_i:\mathfrak{g}^{p_i}\to\mathfrak{g}$ be multilinear functions such that $\sum_{i=1}^{j}p_i=p$. Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining*

partitions with $\#(\mathcal{P}_n) = n$. *Then*

$$\limsup_{n\to\infty} \sum_{\overline{h}\in(S_{\mathbb{C}})^p} ||R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})||^2_{H(\mathfrak{g})} < \infty.$$

Proof. By Eq. (5.48), we can write

$$R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h}) = \sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} \delta_i^\omega T(\overline{h}) f_\omega(t_i),$$

where $T : \mathfrak{g}^p \to \mathfrak{g}$ is a mulitlinear function and, for each $\omega \in \Omega_p^l$, $f_\omega(t_i)$ is a polynomial, possibly zero, in t_i. Notice that

$$\begin{aligned}
||R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})||^2_{H(\mathfrak{g})} &= ||\sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} \delta_i^\omega T(\overline{h}) f_\omega(t_i)||^2_{H(\mathfrak{g})} \\
&= \sum_{i=0}^{n-1}||\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} \delta_i^\omega T(\overline{h}) f_\omega(t_i)||^2_{H(\mathfrak{g})} \\
&\le C(p)\sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} ||\delta_i^\omega T(\overline{h}) f_\omega(t_i)||^2_{H(\mathfrak{g})} \\
&= C(p)\sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} ||\delta_i^\omega T(\overline{h})||^2_{\mathfrak{g}} ||f_\omega(t_i)||^2_{H(\mathbb{C})}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
||f_\omega(t_i)||^2_{H(\mathbb{C})} &= \int_{s_i}^{s_{i+1}} |\frac{d}{dt} f_\omega(t_i)|^2 dt \\
&= \int_{s_i}^{s_{i+1}} |\frac{d}{dt} f_\omega(\frac{t-s_i}{\delta_i})|^2 dt \\
&= \frac{1}{\delta_i^2}\int_{s_i}^{s_{i+1}} |f'_\omega(\frac{t-s_i}{\delta_i})|^2 dt \\
&= \frac{1}{\delta_i}\int_0^1 \left|f'_\omega(u)\right|^2 du \\
&= \frac{C(\omega)}{\delta_i},
\end{aligned}$$

where $C(\omega)$ is an appropriate constant. By Remark 5.72, $\sum_{\overline{h}\in(S_{\mathbb{C}})^p}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} ||\delta_i^\omega T(\overline{h})||^2_{\mathfrak{g}}$

is $O(\delta_i^2)$. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{\overline{h}\in(S_{\mathbb{C}})^p} ||R_{j,k}^{\mathcal{P}_n}(B_1,...,B_j)(\overline{h})||_{H(\mathfrak{g})}^2 \\
&\leq \limsup_{n\to\infty} \sum_{\overline{h}\in(S_{\mathbb{C}})^p} C(p) \sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} ||\delta_i^\omega T(\overline{h})||_{\mathfrak{g}}^2 ||f_\omega(t_i)||_{H(\mathbb{C})}^2 \\
&\leq \limsup_{n\to\infty} \tilde{C}(p) \sum_{i=0}^{n-1} \frac{1}{\delta_i} \left(\sum_{\overline{h}\in(S_{\mathbb{C}})^p} \sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} ||\delta_i^\omega T(\overline{h})||_{\mathfrak{g}}^2 \right) \\
&= \limsup_{n\to\infty} \tilde{C}(p) \sum_{i=0}^{n-1} \frac{1}{\delta_i} O(\delta_i^2) \\
&= \limsup_{n\to\infty} \tilde{C}(p) \sum_{i=0}^{n-1} O(\delta_i) \\
&< \infty.
\end{aligned}$$

□

Proposition 5.76. *Suppose $1 \leq k < j \leq r$, and for $1 \leq i \leq j$, $B_i : H(\mathfrak{g})^{p_i} \to H(\mathfrak{g})$ be bounded multilinear functions such that $\sum_{i=1}^{j} p_j = p$. Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions with $|\mathcal{P}_n| = n$. Define a function $G_{j,k}^{\mathcal{P}_n}(B_1,...,B_j) : [0,1]^2 \to \mathfrak{g}^{\otimes 2}$ by*

$$\begin{aligned}
&G_{j,k}^{\mathcal{P}_n}(B_1,...,B_j)(u,t) \\
&\equiv \sum_{\overline{h}\in(S_{\mathbb{C}})^p} \frac{d}{du} R_{j,k}^{\mathcal{P}_n}(B_1,...,B_j)(\overline{h})(u) \otimes \frac{d}{dt} \overline{R_{j,k}^{\mathcal{P}_n}(B_1,...,B_j)(\overline{h})}(t).
\end{aligned}$$

Then $||G_{j,k}^{\mathcal{P}_n}(B_1,...,B_j)||_{L^2([0,1]^2;\mathfrak{g}^{\otimes 2})} \to 0$ as $n \to 0$.

Proof. Again by Eq. (5.48), we can write

$$\begin{aligned}
\frac{d}{dt} R_{j,k}^{\mathcal{P}_n}(B_1,...,B_j)(\overline{h})(t) &= \frac{d}{dt} \sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} \delta_i^\omega T(\overline{h}) f_\omega(t_i(t)) \\
&= \sum_{i=0}^{n-1}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l} \delta_i^\omega T(\overline{h}) \frac{1_{(s_i,s_{i+1}]}(t)}{\delta_i} f_\omega'(t_i(t)).
\end{aligned}$$

So

$$G^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(u,t)$$
$$= \sum_{i,j=0}^{n-1} \sum_{\overline{h}\in(S_{\mathbb{C}})^p} \left(\begin{array}{c} \left(\sum_{l=2}^p \sum_{\omega\in\Omega_p^l} \delta_i^\omega T(\overline{h}) \frac{1_{(s_i,s_{i+1}]}(u)}{\delta_i} f'_\omega(t_i(u))\right) \\ \otimes \left(\sum_{l'=2}^p \sum_{\theta\in\Omega_p^{l'}} \overline{\delta_j^\theta T(\overline{h}) \frac{1_{(s_j,s_{j+1}]}(t)}{\delta_j} f'_\theta(t_j(t))}\right) \end{array} \right)$$
$$= \sum_{i,j=0}^{n-1} \sum_{l,l'=2}^{p} \sum_{\substack{\omega\in\Omega_p^l \\ \theta\in\Omega_p^{l'}}} 1_{(s_i,s_{i+1}]}(u) 1_{(s_j,s_{j+1}]}(t) \frac{f'_\omega(t_i(u))\overline{f'_\theta(t_j(t))}}{\delta_i\delta_j} \left(\sum_{\overline{h}\in(S_{\mathbb{C}})^p} \delta_i^\omega T(\overline{h}) \otimes \overline{\delta_j^\theta T(\overline{h})} \right)$$
$$= \left(\sum_{i,j=0}^{n-1} \sum_{l,l'=2}^{p} \sum_{\substack{\omega\in\Omega_p^l \\ \theta\in\Omega_p^{l'}}} 1_{(s_i,s_{i+1}]}(u) 1_{(s_j,s_{j+1}]}(t) \frac{f'_\omega(t_i(u))\overline{f'_\theta(t_j(t))}}{\delta_i\delta_j} \right)$$
$$\times \left(\delta_{ij}\delta_{\omega\theta} \left(\delta_i\right)^l \left(s_i\right)^{p-l} \sum_{A_1,...,A_p\in\mathfrak{X}_{\mathbb{C}}} T(A_1,...,A_p) \otimes \overline{T(A_1,...,A_p)} \right)$$
$$= \sum_{i=0}^{n-1} \sum_{l=2}^{p} \sum_{\omega\in\Omega_p^l} 1_{(s_i,s_{i+1}]}(u) 1_{(s_i,s_{i+1}]}(t) \frac{f'_\omega(t_i(u))\overline{f'_\omega(t_i(t))}}{\delta_i^2} \left(\delta_i\right)^l \left(s_i\right)^{p-l} C_p(T,T)$$
$$= \sum_{i=0}^{n-1} \sum_{l=2}^{p} \sum_{\omega\in\Omega_p^l} 1_{(s_i,s_{i+1}]}(u) 1_{(s_i,s_{i+1}]}(t) f'_\omega(t_i(u))\overline{f'_\omega(t_i(t))} \left(\delta_i\right)^{l-2} \left(s_i\right)^{p-l} C_p(T,T),$$

where $C_p(T,T) \in \mathfrak{g}\otimes\mathfrak{g}$ is defined as in Notation 5.69. So $G^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)$ has support concentrated near the diagonal, on the set $\{(u,t) \in [0,1]^2 | s_i \leq u,t \leq s_{i+1}$ for some $i = 0,...,n-1\}$, which is going to zero in measure as $n \to \infty$. Also note that our functions f_ω are polynomials, and hence are bounded on $[0,1]$ by some constant. For some $i = 0,...,n-1$ and $(u,t) \in [0,1]^2$ such that $s_i \leq u,t \leq s_{i+1}$, we have

$$\begin{aligned} ||G^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(u,t)||_{\mathfrak{g}\otimes\mathfrak{g}} &= ||\sum_{l=2}^{p} \sum_{\omega\in\Omega_p^l} f'_\omega(u_i)\overline{f'_\omega(t_i)} \left(\delta_i\right)^{l-2} \left(s_i\right)^{p-l} C_p(T,T)||_{\mathfrak{g}\otimes\mathfrak{g}} \\ &\leq C(p)\tilde{C}^2|| \left(\delta_i\right)^{l-2} \left(s_i\right)^{p-l} C_p(T,T)||_{\mathfrak{g}\otimes\mathfrak{g}} \\ &\leq C(p)\tilde{C}^2||C_p(T,T)||_{\mathfrak{g}\otimes\mathfrak{g}} \\ &< \infty. \end{aligned}$$

So $G^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(u,t)$ is pointwise bounded independent of partition with measure of the support going to zero as $n\to\infty$. Therefore, $||G^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)||_{L^2([0,1]^2)}\to 0$ as $n\to 0$. $\square$

Corollary 5.77. *Suppose $1\le k<j\le r$, and for $1\le i\le j$, $B_i: H(\mathfrak{g})^{p_i}\to H(\mathfrak{g})$ be linear functions such that $\sum_{i=1}^j p_i = p$. Let $\{\mathcal{P}_n\}_{n=1}^\infty$ be a sequence of refining partitions with $|\mathcal{P}_n| = n$. Given $\alpha\in T(H(\mathfrak{g}))^*_T$, then*

$$\sum_{\overline{h}\in(S_\mathbb{C})^p} |\langle\alpha, R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})\rangle|^2\to 0 \quad \textit{as } n\to\infty.$$

Proof. For all $\alpha\in H(\mathfrak{g})^*$, there exists an $\widetilde{\alpha}\in H(\mathfrak{g})$, such that $\langle\alpha, v\rangle = (v,\widetilde{\alpha})_{H(\mathfrak{g})}$ for all $v\in H(\mathfrak{g})$. Then

$$\begin{aligned}\langle\alpha, R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})\rangle &= \left(R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h}),\widetilde{\alpha}\right)_{H(\mathfrak{g})}\\ &= \int_0^1\left(R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h}),\widetilde{\alpha}'(t)\right)_{\mathfrak{g}} dt.\end{aligned}$$

In particular,

$$\begin{aligned}&\sum_{\overline{h}\in(S_\mathbb{C})^p} |\langle\alpha, R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})\rangle|^2\\ &=\sum_{\overline{h}\in(S_\mathbb{C})^p}\int_0^1\left(R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})'(t),\widetilde{\alpha}'(t)\right)_{\mathfrak{g}} dt\overline{\int_0^1\left(R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})'(u),\widetilde{\alpha}'(u)\right)_{\mathfrak{g}} du}\\ &=\int_{[0,1]^2}\sum_{\overline{h}\in(S_\mathbb{C})^p} R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})'(t)\otimes\overline{R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})'(u)},\widetilde{\alpha}'(u)\otimes\overline{\widetilde{\alpha}'(u)})_{\mathfrak{g}^{\otimes 2}}dt\otimes du \qquad (5.49)\\ &=(G^{\mathcal{P}_n}_{j,k}(B_1,...,B_j),\widetilde{\alpha}'\otimes\overline{\widetilde{\alpha}'})_{L^2([0,1]^2,\mathfrak{g}^{\otimes 2})},\end{aligned}$$

Where line (5.49) is justified by Fubini's theorem and Proposition 5.75. By Cauchy-Schwartz,

$$\begin{aligned}\sum_{\overline{h}\in(S_\mathbb{C})^p} |\langle\alpha, R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h})\rangle|^2 &= \sum_{\overline{h}\in(S_\mathbb{C})^p}|\left(R^{\mathcal{P}_n}_{j,k}(B_1,...,B_j)(\overline{h}),\widetilde{\alpha}\right)_{H(\mathfrak{g})}|^2\\ &\le ||\widetilde{\alpha}'\otimes\overline{\widetilde{\alpha}'}||_{L^2([0,1]^2,\mathfrak{g}^{\otimes 2})}\left|\left|G^{\mathcal{P}}_{j,k}(B_1,...,B_j)\right|\right|_{L^2([0,1]^2,\mathfrak{g}^{\otimes 2})}.\end{aligned}$$

We've shown in the above proposition that $||G_{j,k}^{\mathcal{P}_n}(B_1, ..., B_j)||_{L^2([0,1]^2,\mathfrak{g}^{\otimes 2})} \to 0$ as $n \to 0$. Our result will be proven if we can show $||\widetilde{\alpha}' \otimes \overline{\widetilde{\alpha}'}||_{L^2([0,1]^2,\mathfrak{g}^{\otimes 2})} < \infty$. An easy calculation yields

$$\begin{aligned}
||\widetilde{\alpha}' \otimes \overline{\widetilde{\alpha}'}||_{L^2([0,1]^2,\mathfrak{g}^{\otimes 2})} &= \left(\int_{[0,1]^2} ||\widetilde{\alpha}'(t) \otimes \overline{\widetilde{\alpha}'(u)}||^2_{\mathfrak{g}^{\otimes 2}} dt \otimes du \right)^{1/2} \\
&= \left(\int_{[0,1]^2} ||\widetilde{\alpha}'(t)||^2_{\mathfrak{g}} ||\widetilde{\alpha}'(u)||^2_{\mathfrak{g}} dt \otimes du \right)^{1/2} \\
&= \int_0^1 ||\widetilde{\alpha}'(t)||^2_{\mathfrak{g}} dt \\
&= ||\tilde{\alpha}||^2_{H(\mathfrak{g})} \\
&< \infty.
\end{aligned}$$

□

We wish to extend the result of Corollary 5.77 to arbitrary tensor products of remainder terms. The existing notation, unfortunately, is becoming cumbersome, so we introduce some more.

Notation 5.78. Suppose $1 \leq k_l < j_l \leq r$ for $l = 1, 2, ..., q$. Furthermore, for each l, suppose for $1 \leq i \leq j_l$, $B_i^l : H(\mathfrak{g})^{p_i^l} \to H(\mathfrak{g})$ be linear functions such that $\sum_{l=1}^q \sum_{i=1}^{j_l} p_j = p$. Let $\mathcal{P}$ be a partition of $[0, 1]$. Then we will use

$$R_{j_1,k_1}^{\mathcal{P}_n}(\overline{B}^1) \otimes \cdots \otimes R_{j_q,k_q}^{\mathcal{P}_n}(\overline{B}^q)(\overline{h})$$

to denote

$$R_{j_1,k_1}^{\mathcal{P}_n}(B_1^1, ..., B_{j_1}^1)(h_1, ..., h_{p_{j_1}^1}) \otimes \cdots \otimes R_{j_q,k_q}^{\mathcal{P}_n}(B_1^q, ..., B_{j_q}^q)(h_{p-p_{j_q}^q+1}, ..., h_p), \tag{5.50}$$

where in line (5.50) we are using the notation from Eq. (5.45).

Proposition 5.79. *Assume the same setup as in Notation 5.78 above. Let $\{\mathcal{P}_n\}_{n=1}^\infty$ be a sequence of refining partitions. Given $\alpha \in T(H(\mathfrak{g}))_T^*$, then*

$$\sum_{\overline{h} \in (S_{\mathbb{C}})^p} |\langle \alpha, R_{j_1,k_1}^{\mathcal{P}_n}(\overline{B}^1) \otimes \cdots \otimes R_{j_q,k_q}^{\mathcal{P}_n}(\overline{B}^q)(\overline{h}) \rangle|^2 \to 0 \quad \textit{as } n \to \infty.$$

Proof. For $\alpha \in (H(\mathfrak{g})^{\otimes q})^*$, define

$$\phi_{\mathcal{P}}(\alpha) := \sqrt{\sum_{\overline{h} \in (S_{\mathbb{C}})^p} |\langle \alpha, R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}) \rangle|^2}.$$

Then $\phi_{\mathcal{P}}$ is a seminorm on $(H(\mathfrak{g})^{\otimes q})^*$. Using Proposition 5.75, we can say that

$$\begin{aligned}
\phi_{\mathcal{P}}(\alpha)^2 &\leq ||\alpha||^2_{(H(\mathfrak{g})^{\otimes q})^*} \sum_{\overline{h} \in (S_{\mathbb{C}})^p} ||R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})||^2_{H(\mathfrak{g})^{\otimes q}} \\
&\leq ||\alpha||^2_{(H(\mathfrak{g})^{\otimes q})^*} \left(\sum_{\overline{h} \in (S_{\mathbb{C}})^{p^1_{j_1}}} ||R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1)(\overline{h})||^2_{H(\mathfrak{g})} \right) \times \cdots \\
&\cdots \times \left(\sum_{\overline{h} \in (S_{\mathbb{C}})^{p^q_{j_q}}} ||R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})||^2_{H(\mathfrak{g})} \right) \\
&\leq C^2 ||\alpha||^2_{(H(\mathfrak{g})^{\otimes q})^*},
\end{aligned}$$

for some $C^2 < \infty$. Equivalently,

$$\phi_{\mathcal{P}}(\alpha) \leq C||\alpha||_{(H(\mathfrak{g})^{\otimes q})^*}. \tag{5.51}$$

Suppose $\alpha = (\cdot, k_1 \otimes \cdots \otimes k_q)_{H(\mathfrak{g})^{\otimes q}}$. Then

$$\begin{aligned}
\phi_{\mathcal{P}}(\alpha)^2 &= \sum_{\overline{h} \in (S_{\mathbb{C}})^p} |(R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}), k_1 \otimes \cdots \otimes k_q)_{H(\mathfrak{g})^{\otimes q}}|^2 \\
&\leq \sum_{\overline{h} \in (S_{\mathbb{C}})^p} |(R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1), k_1)|^2_{H(\mathfrak{g})} \times \cdots \times |(R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q), k_m)|^2_{H(\mathfrak{g})} \\
&= \left(||k_1||^2_{H(\mathfrak{g})} ||G^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1)||_{L^2([0,1]^2)} \right) \times \cdots \times \left(||k_m||^2_{H(\mathfrak{g})} ||G^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)||_{L^2([0,1]^2)} \right),
\end{aligned}$$

where $G^{\mathcal{P}_n}_{j_i,k_i}(\overline{B}^i)$ is as given in Proposition 5.76. Proposition 5.76 also tells us that $||G^{\mathcal{P}_n}_{j_i,k_i}(\overline{B}^i)||_{L^2([0,1]^2)} \to 0$ as $|\mathcal{P}| \to 0$ for $i = 1, ..., q$. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$. Furthermore, the same is true for all finite linear combinations of such α. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$ for all α of finite rank.

For $\alpha, \beta \in (H(\mathfrak{g})^{\otimes q})^*$,

$$\begin{aligned}
\phi_{\mathcal{P}}(\alpha) &\leq \phi_{\mathcal{P}}(\alpha - \beta) + \phi_{\mathcal{P}}(\beta) \\
&\leq C||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes q})^*} + \phi_{\mathcal{P}}(\beta).
\end{aligned}$$

Let $\varepsilon > 0$ and choose $\beta \in (H(\mathfrak{g})^{\otimes q})^*$ finite rank such that $||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes q})^*} < \varepsilon$. Then

$$\begin{aligned}\lim_{|\mathcal{P}|\to 0} \phi_{\mathcal{P}}(\alpha) &\leq \lim_{|\mathcal{P}|\to 0} C||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes q})^*} + \lim_{|\mathcal{P}|\to 0} \phi_{\mathcal{P}}(\beta) \\ &= C||\alpha - \beta||_{(H(\mathfrak{g})^{\otimes q})^*} \\ &< C\varepsilon.\end{aligned}$$

Our choice of ε was arbitrary. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$. □

The typical remainder term is a tensor product of $P_{\mathcal{P}}h_i$ and $R^{\mathcal{P}}_{j,k}$ terms. Adding the $P_{\mathcal{P}}h_i$ terms doesnt not effect the results. First a proposition.

Proposition 5.80. *Given $\alpha \in T(H(\mathfrak{g}))^*_T$ and $h_1, h_2, ...h_k \in S_{\mathbb{C}}$, there exists $\beta_{h_1h_2...h_k} \in (H(\mathfrak{g})^*)^{\otimes n}$ which satisfies*

$$\langle \alpha, h_1 \otimes \cdots \otimes h_k \otimes \eta \rangle = \langle \beta_{h_1h_2..h_k}, \eta \rangle, \tag{5.52}$$

for any $\eta \in H(\mathfrak{g})^{\otimes n}$, and furthermore,

$$\sum_{h_1,h_2,...,h_k \in S_{\mathbb{C}}} ||\beta_{h_1h_2..h_k}||^2_{(H(\mathfrak{g})^*)^{\otimes n}} = ||\alpha_{n+k}||^2_{(H(\mathfrak{g})^*)^{\otimes(n+k)}} < \infty. \tag{5.53}$$

Proof. Since $\alpha \in T(H(\mathfrak{g}))^*_T$, we can write

$$\alpha_{n+k} = \sum_{h_1,h_2,...,h_{n+k} \in S_{\mathbb{C}}} a_{h_1h_2...h_{n+k}} (\cdot, h_1 \otimes \cdots \otimes h_{n+k})_{H(\mathfrak{g})^{\otimes n+k}},$$

for some square summable $a_{h_1h_2...h_{n+k}} \in \mathbb{C}$, i.e.

$$\sum_{h_1,h_2,...,h_{n+k} \in S_{\mathbb{C}}} |a_{h_1h_2...h_{n+k}}|^2 < \infty.$$

Set

$$\langle \beta_{h_1h_2...h_k}, \cdot \rangle = \sum_{h_{k+1},...,h_{n+k} \in S_{\mathbb{C}}} a_{h_1h_2...h_{n+k}} (\cdot, h_{k+1} \otimes \cdots \otimes h_{n+k})_{H(\mathfrak{g})^{\otimes n}}.$$

Then it is clear that (5.52) is satisfied. It then follows that

$$\begin{aligned}\sum_{h_1,h_2,...,h_k \in S_{\mathbb{C}}} ||\beta_{h_1h_2..h_k}||^2_{(H(\mathfrak{g})^*)^{\otimes n}} &= \sum_{h_1,h_2,...,h_{n+k} \in S_{\mathbb{C}}} |a_{h_1h_2...h_{n+k}}|^2 \\ &= ||\alpha_{n+k}||^2_{(H(\mathfrak{g})^*)^{\otimes(n+k)}} \\ &< \infty.\end{aligned}$$

□

Remark 5.81. The above proposition remains true if the $h_1, h_2, ...h_k$ terms of our tensor product are in any position, not just the beginning of our tensor product.

Proposition 5.82. *Again, we assume the same setup as Notation 5.78. Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions with $\#(\mathcal{P}_n) = n$. Given $\alpha \in T(H(\mathfrak{g}))_T^*$ and $q > 0$, then as $n \to \infty$*

$$\sum_{\substack{h_1,\ldots,h_q \in S_{\mathbb{C}} \\ \overline{h} \in (S_{\mathbb{C}})^p}} |\langle \alpha, P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_q \otimes R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})\rangle|^2 \to 0.$$

Proof. First notice that for any partition $\mathcal{P}$, we can first select a basis $S^{\mathcal{P}}_{\mathbb{C}}$ for $H_{\mathcal{P}}(\mathfrak{g})$, and then extend it to a basis $S_{\mathbb{C}}$ for $H(\mathfrak{g})$. Then it follows that

$$\begin{aligned} &\sum_{\substack{h_1,\ldots,h_q \in S_{\mathbb{C}} \\ \overline{h} \in (S_{\mathbb{C}})^p}} |\langle \alpha, P_{\mathcal{P}} h_1 \otimes \cdots \otimes P_{\mathcal{P}} h_q \otimes R^{\mathcal{P}}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}}_{j_q,k_q}(\overline{B}^q)(\overline{h})\rangle|^2 \\ &\le \sum_{\substack{h_1,\ldots,h_q \in S_{\mathbb{C}} \\ \overline{h} \in (S_{\mathbb{C}})^p}} |\langle \alpha, h_1 \otimes \cdots \otimes h_q \otimes R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})\rangle|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\lim_{n\to\infty} \sum_{\substack{h_1,\ldots,h_q \in S_{\mathbb{C}} \\ \overline{h} \in (S_{\mathbb{C}})^p}} |\langle \alpha, P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_q \otimes R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})\rangle|^2 \\ &\le \lim_{n\to\infty} \sum_{\substack{h_1,\ldots,h_q \in S_{\mathbb{C}} \\ \overline{h} \in (S_{\mathbb{C}})^p}} |\langle \alpha, h_1 \otimes \cdots \otimes h_q \otimes R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})\rangle|^2 \\ &= \lim_{n\to\infty} \sum_{\substack{h_1,\ldots,h_q \in S_{\mathbb{C}} \\ \overline{h} \in (S_{\mathbb{C}})^p}} |\langle \beta_{h_1 \cdots h_q}, R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h})\rangle|^2 \qquad (5.54) \\ &= 0. \end{aligned}$$

In line (5.54) we have used Proposition 5.80. We are able to move the limit inside by the DCT, which is justified by Eq. (5.53) and Proposition 5.75. □

Remark 5.83. Again, for the above proposition, the $P_{\mathcal{P}} h$ terms need not occur at the beginning of the tensor product.

Remark 5.84. For all $k > 0$ and any partition $\mathcal{P}$, $R_k^{\mathcal{P}}$ is defined by the following expression

$$\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_k \rangle = \langle \alpha, P_{\mathcal{P}} h_1 \otimes P_{\mathcal{P}} h_2 \otimes \cdots \otimes P_{\mathcal{P}} h_k + R_k^{\mathcal{P}}(h_1, ..., h_k) \rangle.$$

Since we have shown that $R_k^{\mathcal{P}}$ consists of a finite sum of terms like those in Proposition 5.82, we have shown that for a refining sequence of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$,

$$\sum_{h_1,...,h_k \in S_{\mathbb{C}}} |\langle \alpha, R_k^{\mathcal{P}_n}(h_1, ..., h_k) \rangle| \to 0 \quad \text{as } n \to \infty.$$

Lemma 5.85. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions. Then $\cup_{n=1}^{\infty} H_{\mathcal{P}_n}(\mathfrak{g})$ is dense in $H(\mathfrak{g})$.*

Proof. Recall from Proposition 2.12 that

$$H_{\mathcal{P}}(\mathfrak{g})^{\perp} = Nul(\pi_{\mathcal{P}*\underline{e}}) = \{h \in H(\mathfrak{g}) | h|_{\mathcal{P}} \equiv 0\}.$$

Now suppose that $h \in H(\mathfrak{g})$ with $h \perp \cup_{n=1}^{\infty} H_{\mathcal{P}_n}(\mathfrak{g})$. Then $h \in H_{\mathcal{P}_n}(\mathfrak{g})^{\perp}$ for all n, which imples that $h|_{\mathcal{P}_n} \equiv 0$ for all n. Therefore, $h|_{\cup_{n=1}^{\infty} \mathcal{P}_n} \equiv 0$, and since h is continuous, we necessarily have that $h \equiv 0$. ☐

Corollary 5.86. *Since $P_{\mathcal{P}_n}$ is orthogonal projection onto the subspace $H_{\mathcal{P}_n}(\mathfrak{g})$ and $H_{\mathcal{P}_n}(\mathfrak{g}) \subset H_{\mathcal{P}_{n+1}}(\mathfrak{g})$ for all $n = 1, 2., ...,$ the above lemma implies that for $h \in H(\mathfrak{g})$,*

$$\lim_{n \to \infty} ||h - P_{\mathcal{P}_n} h||_{H(\mathfrak{g})} = 0.$$

Remark 5.87. If $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of refining partitions, then Corollary 5.86 allows us to construct an orthonormal basis for $H(\mathfrak{g})$ adapted to our sequence of partitions in the following sense. After first constructing an orthonormal basis $S_{\mathbb{C}}^{\mathcal{P}_1}$ for $H_{\mathcal{P}_1}(\mathfrak{g})$, extend this basis inductively from $H_{\mathcal{P}_i}(\mathfrak{g})$ to $H_{\mathcal{P}_{i+1}}(\mathfrak{g})$ for $i = 1, 2, ...$ Then $S_{\mathbb{C}} = \cup_{n=1}^{\infty} S_{\mathbb{C}}^{\mathcal{P}_n}$ is an orthonormal basis for $H(\mathfrak{g})$.

Proposition 5.88. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions. Then for $h_1, ..., h_k \in S_{\mathbb{C}}$,*

$$||h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k||^2_{H(\mathfrak{g})^{\otimes k}} \leq k^2 \sum_{j=1}^{k} ||h_j - P_{\mathcal{P}_n} h_j||^2_{H(\mathfrak{g})}. \tag{5.55}$$

In addition, for $\alpha \in T(H(\mathfrak{g}))_T^*$,

$$\sum_{h_1,...,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 < k^3 ||\alpha_k||^2_{J_T^0(H(\mathfrak{g}))}. \tag{5.56}$$

Proof. To see Eq. (5.55), notice that we can rewrite

$$\begin{aligned}
&h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \\
&= (h_1 - P_{\mathcal{P}_n} h_1) \otimes h_2 \otimes \cdots \otimes h_k \\
&+ P_{\mathcal{P}_n} h_1 \otimes (h_2 - P_{\mathcal{P}_n} h_2) \otimes h_3 \otimes \cdots \otimes h_k + \cdots \\
&\cdots + P_{\mathcal{P}_n} h_1 \otimes P_{\mathcal{P}_n} h_2 \otimes \cdots \otimes (h_k - P_{\mathcal{P}_n} h_k),
\end{aligned} \tag{5.57}$$

where the RHS is a sum of k terms. Since $||P_{\mathcal{P}_n} h_j||^2_{H(\mathfrak{g})} \le ||h_j|||^2_{H(\mathfrak{g})} \le 1$, the result easily follows using the relation $|| \sum_{j=1}^k a_j||^2 \le k^2 \sum_{j=1}^k ||a_j||^2$.

For Eq. (5.56), we choose a basis $S_{\mathbb{C}}$ adapted to our sequence of partition as in Remark 5.87. Then for every $n > 0$ and $h \in S_{\mathbb{C}}$, $P_{\mathcal{P}_n} h = h$ if $h \in H_{\mathcal{P}_n}(\mathfrak{g})$ and 0 otherwise. Similarly, $h - P_{\mathcal{P}_n} h = 0$ if $h \in H_{\mathcal{P}_n}(\mathfrak{g})$ and h otherwise. Therefore, using Eq. (5.57),

$$\begin{aligned}
&\sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 \\
&= \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left| \begin{array}{c} \langle \alpha, (h_1 - P_{\mathcal{P}_n} h_1) \otimes h_2 \otimes \cdots \otimes h_k \rangle \\ + \langle \alpha, P_{\mathcal{P}_n} h_1 \otimes (h_2 - P_{\mathcal{P}_n} h_2) \otimes h_3 \otimes \cdots \otimes h_k \rangle + \cdots \\ \cdots + \langle \alpha, P_{\mathcal{P}_n} h_1 \otimes P_{\mathcal{P}_n} h_2 \otimes \cdots \otimes (h_k - P_{\mathcal{P}_n} h_k) \rangle \end{array} \right|^2 \\
&\le k^2 \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left(\begin{array}{c} |\langle \alpha, (h_1 - P_{\mathcal{P}_n} h_1) \otimes h_2 \otimes \cdots \otimes h_k \rangle|^2 \\ + |\langle \alpha, P_{\mathcal{P}_n} h_1 \otimes (h_2 - P_{\mathcal{P}_n} h_2) \otimes h_3 \otimes \cdots \otimes h_k \rangle|^2 + \cdots \\ \cdots + |\langle \alpha, P_{\mathcal{P}_n} h_1 \otimes P_{\mathcal{P}_n} h_2 \otimes \cdots \otimes (h_k - P_{\mathcal{P}_n} h_k) \rangle|^2 \end{array} \right) \\
&\le k^3 \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes h_2 \otimes \cdots \otimes h_k \rangle|^2 \\
&= k^3 ||\alpha_k||^2_{J_T^0(H(\mathfrak{g}))}.
\end{aligned}$$

□

Corollary 5.89. *Let $\{\mathcal{P}_n\}_{n=1}^\infty$ be a sequence of refining partitions and $\alpha \in T(H(\mathfrak{g}))^*_T$. Then for any $h_1, \ldots, h_k \in S_{\mathbb{C}}$,*

$$\lim_{n \to \infty} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 = 0.$$

Proof. This follows easily from Eq. (5.55) and Corollary 5.86. Specifically,

$$\begin{aligned}
&\lim_{n\to\infty} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 \\
&\leq \lim_{n\to\infty} ||\alpha_k||^2_{J^0_T(H(\mathfrak{g}))} ||h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k||^2_{H(\mathfrak{g})^{\otimes k}} \\
&\leq \lim_{n\to\infty} k^2 ||\alpha_k||^2_{J^0_T(H(\mathfrak{g}))} \sum_{j=1}^{k} ||h_j - P_{\mathcal{P}_n} h_j||^2_{H(\mathfrak{g})} \\
&= k^2 ||\alpha_k||^2_{J^0_T(H(\mathfrak{g}))} \sum_{j=1}^{k} \lim_{n\to\infty} ||h_j - P_{\mathcal{P}_n} h_j||^2_{H(\mathfrak{g})} \\
&= 0.
\end{aligned}$$

$\square$

We are now set to prove the main theorem of the chapter.

Theorem 5.90. *Let* $\{\mathcal{P}_n\}_{n=1}^{\infty}$ *be a sequence of refining partitions. Then*

$$\lim_{n\to\infty} ||\alpha - \alpha(\mathcal{P}_n)||_{J^0_T(H(\mathfrak{g}))} \to 0 \quad \text{as } n \to \infty.$$

Proof. The fact that α is of finite rank is essential to this result. We assume that α is of rank N. Choose $S_{\mathbb{C}}$ to be an orthonormal basis for $H(\mathfrak{g})$ adapted to $\{\mathcal{P}_n\}_{n=1}^{\infty}$ as in Remark 5.87. Then

$$\begin{aligned}
&\lim_{n\to\infty} ||\alpha - \alpha(\mathcal{P}_n)||^2_{J^0_T(H(\mathfrak{g}))} \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} \sum_{h_i,\ldots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k \rangle - \langle \alpha(\mathcal{P}_n), h_1 \otimes \cdots \otimes h_k \rangle|^2 \\
&= \lim_{n\to\infty} \sum_{k=0}^{N} \sum_{h_i,\ldots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle - \langle \alpha, R_k(h_i, \ldots, h_k) \rangle|^2 \\
&\leq \lim_{n\to\infty} 4 \sum_{k=0}^{N} \sum_{h_i,\ldots,h_k \in S_{\mathbb{C}}} \begin{pmatrix} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 \\ + |\langle \alpha, R_k^{\mathcal{P}_n}(h_i, \ldots, h_k) \rangle|^2 \end{pmatrix}.
\end{aligned}$$

We have shown already in Remark 5.84 that

$$\lim_{n\to\infty} \sum_{h_i,\ldots,h_k \in S_{\mathbb{C}}} |\langle \alpha, R_k^{\mathcal{P}_n}(h_i, \ldots, h_k) \rangle|^2 = 0.$$

Therefore, there is some finite constant C_N such that

$$\begin{aligned}
&\lim_{n\to\infty} ||\alpha - \alpha(\mathcal{P}_n)||^2_{J^0_T(H(\mathfrak{g}))} \\
&\leq \lim_{n\to\infty} C_N \sum_{k=0}^{N} \sum_{h_i,\ldots,h_k \in S_{\mathbb{C}}} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 \\
&= C_N \sum_{k=0}^{N} \sum_{h_i,\ldots,h_k \in S_{\mathbb{C}}} \lim_{n\to\infty} |\langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle|^2 \\
&= 0,
\end{aligned}$$

by Corollary 5.89. □

The argument is now complete. We present a brief summary to end.

Given $\alpha \in J^0_T(H(\mathfrak{g}))$, we wish to find an $f \in \mathcal{H}_T$ such that $(1-D)^{-1}_{\underline{e}} Rf = \alpha$. By Theorem 5.1 and Theorem 6.15 of the appendix, it suffices to assume that α is of finite rank. Under this assumption, we then constructed a function $u_\alpha \in \mathcal{H}(H(G))$ in Theorem 5.43 such that $(1-D)^{-1}_{\underline{e}} u_\alpha = \alpha$. Given a refining sequence of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$, we construct a sequence of cylinder functions $\{F_{\mathcal{P}_n}\}_{n=1}^{\infty} \subset \mathcal{H}\mathcal{F}C^{\infty}(\mathcal{W}) \cap L^2(\nu_T)$ such that

$$(1-D)^{-1}_{\underline{e}} RF_{\mathcal{P}_n} = (1-D)^{-1}_{\underline{e}} F_{\mathcal{P}_n} = \alpha(\mathcal{P}_n).$$

We have shown in Theorem 5.90 that $\alpha(\mathcal{P}_n) \to \alpha$. Since the Taylor map is an isometry, there is a function $f \in \mathcal{H}_T$ such that $F_{\mathcal{P}_n} \to f$. Finally, since the Taylor map is continuous,

$$(1-D)^{-1}_{\underline{e}} Rf = \lim_{n\to\infty} (1-D)^{-1}_{\underline{e}} RF_{\mathcal{P}_n} = \lim_{n\to\infty} \alpha(\mathcal{P}_n) = \alpha.$$

6

Appendix

6.1 Reproducing Kernels

The following is a summary of the properties of the reproducing kernels on path spaces, which form the basis of many of our calculations in the previous sections. Given a continuous function $h : [0, 1] \to \mathbb{R}$, define

$$\langle h, h \rangle_{H(\mathbb{R})} \equiv \begin{cases} \int_0^1 |h'(s)|^2 ds, & \text{if} \quad h \text{ is absolutely continuous} \\ \infty, & \text{if} \quad \text{otherwise} \end{cases}.$$

Define the *real Cameron-Martin space*,

$$H(\mathbb{R}) = \{h \in C([0, 1]; \mathbb{R}) | h(0) = 0 \text{ and } \langle h, h \rangle_{H(\mathbb{R})} < \infty\}.$$

Then equipped with the inner product

$$\langle h, k \rangle_{H(\mathbb{R})} \equiv \int_0^1 h'(s)k'(s)ds,$$

$H(\mathbb{R})$ is a Hilbert space.

Similarly, for a continuous function $h : [0, 1] \to \mathbb{C}$, we define

$$(h, h)_{H(\mathbb{C})} \equiv \begin{cases} \int_0^1 |h'(s)|^2 ds, & \text{if} \quad h \text{ is absolutely continuous} \\ \infty, & \text{if} \quad \text{otherwise} \end{cases},$$

and the *complex Cameron-Martin space*

$$H(\mathbb{C}) = \{h \in C([0,1];\mathbb{C}) | h(0) = 0 \text{ and } (h,h)_{H(\mathbb{C})} < \infty\},$$

where

$$(h,h)_{H(\mathbb{C})} \equiv \int_0^1 h'(s)\overline{k'(s)}ds.$$

Notation 6.1. We will use $S(\mathbb{R})$ and $S(\mathbb{C})$ to denote orthonormal bases of $H(\mathbb{R})$ and $H(\mathbb{C})$ respectively.

Remark 6.2. It is easy to check that if $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis for $H(\mathbb{R})$, then $\{\frac{1}{\sqrt{2}}(u_j + iu_k)\}_{j,k=1}^{\infty}$ is an orthonormal basis for $H(\mathbb{C})$.

Proposition 6.3. *For any* $s, t \in [0,1]$,

$$\sum_{u \in S(\mathbb{R})} u(s)\, u(t) = \sum_{u \in S(\mathbb{C})} u(s)\, \overline{u(t)} = s \wedge t. \tag{6.1}$$

Proof. By the Fundamental Theorem of Calculus,

$$u(s) = \int_0^1 1_{\tau \leq s} u'(\tau) d\tau. \tag{6.2}$$

Let $\kappa_s \in H(\mathbb{R})$ be given by $\kappa_s(\cdot) = s \wedge \cdot$. If $u \in H(\mathbb{R})$, then line (6.2) is equivalent to

$$u(s) = \langle u, \kappa_s \rangle_{H(\mathbb{R})},$$

and so

$$\begin{aligned} \sum_{u \in S(\mathbb{R})} u(s)\, u(t) &= \sum_{u \in S(\mathbb{R})} \langle u, \kappa_s \rangle_{H(\mathbb{R})} \langle \kappa_t, u \rangle_{H(\mathbb{R})} \\ &= \langle \kappa_t, \kappa_s \rangle_{H(\mathbb{R})} \\ &= \int_0^1 1_{[0,t]}(\tau) 1_{[0,s]}(\tau) d\tau \\ &= s \wedge t. \end{aligned} \tag{6.3}$$

To prove the results on the complex Cameron-Martin space, one simply repeats the above calculations after using Remark 6.2 to notice that

$$\begin{aligned}
\sum_{u\in S(\mathbb{C})} u\,(s)\,\overline{u(t)} &= \frac{1}{2}\sum_{u,v\in S(\mathbb{R})}(u(s)+iv(s))(u(t)-iv(t)) \\
&= \frac{1}{2}\left(\sum_{u\in S(\mathbb{R})}u(s)u(t)+\sum_{v\in S(\mathbb{R})}v(s)v(t)+i\sum_{u,v\in S(\mathbb{R})}(u(t)v(s)-u(s)v(t))\right) \\
&= \sum_{u\in S(\mathbb{R})}u(s)u(t).
\end{aligned}$$

□

Notation 6.4. We will let $K(s,t)\equiv s\wedge t$ denote the *reproducing kernel* of $H(\mathbb{R})$ and $H(\mathbb{C})$. That is

$$\sum_{u\in S(\mathbb{R})} u\,(s)\,u\,(t) = \sum_{u\in S(\mathbb{C})} u\,(s)\,\overline{u\,(t)} = K(s,t).$$

Remark 6.5. Since in the proof of Proposition 6.3 $\kappa_s(\cdot)=K(s,\cdot)$, by line (6.3) we have the relationship

$$\langle K(s,\cdot),K(t,\cdot)\rangle_{H(\mathbb{R})} = (K(s,\cdot),K(t,\cdot))_{H(\mathbb{C})} = K(s,t).$$

Corollary 6.6. *For all* $s\in[0,1]$,

$$\sum_{u\in S(\mathbb{R})}|u(s)|^2 = \sum_{u\in S(\mathbb{C})}|u(s)|^2 = K(s,s) = s.$$

Notation 6.7. Suppose $\mathcal{P}=\{0=s_0<s_1<\cdots<s_n<s_{n+1}=1\}$ is a partition of $[0,1]$. Then for $i=1,2,...,n$

$$\delta_i := s_{i+1}-s_i$$

and, for $u\in H(\mathbb{R})$ or $H(\mathbb{C})$, let

$$\delta_i u := u(s_{i+1})-u(s_i).$$

Corollary 6.8. *Suppose* $\mathcal{P}$ *is a partition of* $[0,1]$. *Then*

$$\sum_{u\in S(\mathbb{R})}|\delta_i u|^2 = \sum_{u\in S(\mathbb{C})}|\delta_i u|^2 = K(\delta_i,\delta_i) = \delta_i.$$

Proof. The proof is a straightforward use of Proposition 6.3. For example,

$$\begin{aligned}
\sum_{u\in S(\mathbb{R})} |\delta_i u|^2 &= \sum_{u\in S(\mathbb{R})} (u(s_{i+1}) - u(s_i))(u(s_{i+1}) - u(s_i)) \\
&= \sum_{u\in S(\mathbb{R})} |u(s_{i+1})|^2 - u(s_{i+1})u(s_i) - u(s_i)u(s_{i+1}) + |u(s_i)|^2 \\
&= K(s_{i+1}, s_{i+1}) - K(s_{i+1}, s_i) - K(s_i, s_{i+1}) + K(s_i, s_i) \\
&= s_{i+1} - s_i - s_i + s_i \\
&= \delta_i.
\end{aligned}$$

□

Corollary 6.9. *Suppose $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ is a partition of $[0,1]$. Then for $1 \le i, j \le n$,*

$$\sum_{u\in S(\mathbb{R})} (\delta_i u)\, u(s_j) = \sum_{u\in S(\mathbb{C})} (\delta_i u)\, \overline{u(s_j)} = \delta_{j>i}\delta_i$$

where .$\delta_{j>i} = 1$ if $j > i$ and 0 otherwise.

Proof. Again, we only show the proof in the real cases.

$$\begin{aligned}
\sum_{u\in S(\mathbb{R})} (\delta_i u)\, u(s_j) &= \sum_{u\in S(\mathbb{R})} u(s_{i+1})u(s_j) - u(s_i)u(s_j) \\
&= K(s_{i+1}, s_j) - K(s_i, s_j) \\
&= s_{i+1} \wedge s_j - s_i \wedge s_j \\
&= \begin{cases} \delta_i & \text{if } j > i \\ 0 & \text{if } j \le i \end{cases} \\
&= \delta_{j>i}\delta_i.
\end{aligned}$$

□

Corollary 6.10. *Suppose $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ is a partition of $[0,1]$. Then for $1 \le i, j \le n$,*

$$\sum_{u\in S(\mathbb{R})} (\delta_i u)\, (\delta_j u) = \sum_{u\in S(\mathbb{C})} (\delta_i u)\, \overline{(\delta_j u)} = \delta_{ij}\delta_i$$

where δ_{ij} denotes the Dirac delta function.

Proof. This is an application of Corollary 6.9.

$$\begin{aligned}\sum_{u\in S(\mathbb{R})}(\delta_i u)(\delta_j u) &= \sum_{u\in S(\mathbb{R})}(\delta_i u)(u(s_{j+1})-u(s_j))\\ &= \delta_{j+1>i}\delta_i - \delta_{j>i}\delta_i\\ &= \delta_{ij}\delta_i.\end{aligned}$$

□

We deal mostly with path spaces on a Lie algebra $\mathfrak{g}$. Refer to Chapter 1 for the definition of $H(\mathfrak{g})$.

Proposition 6.11. *Suppose $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H(\mathbb{R})$ and $\{A_i\}_{i=1}^{2d}$ is an orthonormal basis for $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}})$. Then $\{u_iA_j\}_{i,j=1}^{\infty,d}$ is an orthonormal basis for $(H(\mathfrak{g}),\langle\ ,\ \rangle_{H(\mathfrak{g})})$. Similarly, if $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H(\mathbb{C})$ and $\{A_i\}_{i=1}^{d}$ is an orthonormal basis for $(\mathfrak{g},(\ ,\ )_{\mathfrak{g}})$. Then $\{u_iA_j\}_{i,j=1}^{\infty,2d}$ is an orthonormal basis for $(H(\mathfrak{g}),(\ ,\ )_{H(\mathfrak{g})})$.*

Proof. The proof is a straightforward calculation. In the real case,

$$\begin{aligned}\langle u_iA_j,u_kA_l\rangle_{H(\mathfrak{g})} &= \int_0^1\langle u_i'(s)A_j,u_k'(s)A_l\rangle_{\mathfrak{g}}ds\\ &= \int_0^1 u_i'(s)u_k'(s)\langle A_j,A_l\rangle_{\mathfrak{g}}ds\\ &= \langle A_j,A_l\rangle_{\mathfrak{g}}\int_0^1 u_i'(s)u_k'(s)ds\\ &= \langle A_j,A_l\rangle_{\mathfrak{g}}\langle u_i,u_k\rangle_{H(\mathbb{R})}\\ &= \delta_{jl}\delta_{ik}.\end{aligned}$$

□

Proposition 6.12. *For any $s,t\in[0,1]$,*

$$\begin{aligned}\sum_{h\in S_{\mathbb{R}}} h(s)\otimes h(t) &= K(s,t)\sum_{A\in\mathfrak{X}_{\mathbb{R}}}A\otimes A,\\ \sum_{h\in S_{\mathbb{C}}} h(s)\otimes\overline{h(t)} &= K(s,t)\sum_{A\in\mathfrak{X}_{\mathbb{C}}}A\otimes A,\end{aligned}$$

where $\mathfrak{X}_{\mathbb{R}}$ ($\mathfrak{X}_{\mathbb{C}}$) is an orthonormal basis for the real (complex) inner product space $(\mathfrak{g},\langle\ ,\ \rangle_{\mathfrak{g}})$ $((\mathfrak{g},(\ ,\ )_{\mathfrak{g}}))$.

Proof. We use Proposition 6.11 to write $S_{\mathbb{R}} = \{u_i A_j\}_{i,j=1}^{\infty,d}$ where $\{u_i\}_{i=1}^{\infty} = S(\mathbb{R})$ is an orthonormal basis for $H(\mathbb{R})$ and $\{A_i\}_{i=1}^{d} = \mathfrak{X}_{\mathbb{R}}$ is an orthonormal basis for $(\mathfrak{g}, \langle\ ,\ \rangle_{\mathfrak{g}})$. Then

$$\begin{aligned}\sum_{h\in S_{\mathbb{R}}} h(s)\otimes h(t) &= \sum_{u\in S(\mathbb{R})}\sum_{A\in\mathfrak{X}_{\mathbb{R}}} u(s)A\otimes u(t)A \\ &= \sum_{u\in S(\mathbb{R})} u(s)u(t)\sum_{A\in\mathfrak{X}_{\mathbb{R}}} A\otimes A \\ &= K(s,t)\sum_{A\in\mathfrak{X}_{\mathbb{R}}} A\otimes A.\end{aligned}$$

□

6.2 Density of Finite Rank Tensors

The theorem below is reproduced from [6] in the context of path spaces and is essential to our proofs of surjectivity. We suppose that $\mathfrak{g}$ is a step r complex stratified nilpotent Lie algebra. Recall that this means that there is a sequence of nonzero subspaces V_i for $i = 1, .., r$ such that

$$\mathfrak{g} = \oplus_{i=1}^{r} V_i,$$

with $[V_1, V_j] \subset V_{j+1}$ for $j = 1, ..., r-1$ and $[V_1, V_r] = \{0\}$. It follows that $[V_i, V_j] \subset V_{i+j}$, with the convention that $V_s = \{0\}$ for $s > r$. In our case, we assume that these subspaces are orthogonal. This gives an orthogonal decomposition of $H(\mathfrak{g})$,

$$H(\mathfrak{g}) = \oplus_{i=1}^{r} H(V_i),$$

with $[H(V_1), H(V_j)] \subset H(V_{j+1})$ for $j = 1, ..., r-1$ and $[H(V_1), H(V_r)] = \{0\}$.

Definition 6.13. For $\lambda \in \mathbb{C}$, define $\delta_\lambda : H(\mathfrak{g}) \to H(\mathfrak{g})$ by

$$\delta_\lambda(h_1 + h_2 + \cdots + h_r) = \sum_{i=1}^{r} \lambda^i h_i$$

for $h_i \in H(V_i)$.

Proposition 6.14. *For $\lambda \neq 0$, δ_λ is an Lie algebra automorphism.*

Proof. Let $\lambda \neq 0$. δ_λ is certainly a bijective linear map, so we only need to show it is a Lie algebra homomorphism. Let $h = \sum_{i=1}^r h_i$ and $k = \sum_{i=1}^r k_i$ with $h_i, k_i \in H(V_i)$. Then

$$\begin{aligned}
[\delta_\lambda h, \delta_\lambda k] &= \left[\sum_{i=1}^r \lambda^i h_i, \sum_{j=1}^r \lambda^j k_j\right] \\
&= \sum_{i,j=1}^r \lambda^{i+j}[h_i, k_j] \\
&= \sum_{i=1}^r \lambda^i \left(\sum_{j=1}^{i-j} [h_j, k_{i-j}]\right) \\
&= \delta_\lambda \left(\sum_{j=1}^{i-j} [h_j, k_{i-j}]\right) \\
&= \delta_\lambda [h, k].
\end{aligned}$$

□

Theorem 6.15. *Suppose $\mathfrak{g}$ is complex stratified nilpotent Lie algebra. Then the finite rank tensors in $J_t^0(H(\mathfrak{g}))$ are dense in $J_t^0(H(\mathfrak{g}))$ for each $t > 0$.*

Proof. Let $\Gamma_\theta : T(H(\mathfrak{g})) \to T(H(\mathfrak{g}))$ be the automorphism induced by the automorphism $\delta_{e^{i\theta}}$ on $H(\mathfrak{g})$. Then for any $h, k \in H(\mathfrak{g})$, we have

$$\Gamma_\theta(h \otimes k - k \otimes h - [h, k]) = (\delta_{e^{i\theta}} h) \otimes (\delta_{e^{i\theta}} k) - (\delta_{e^{i\theta}} k) \otimes (\delta_{e^{i\theta}} h) - \delta_{e^{i\theta}}[h, k],$$

and so Γ_θ takes $J(H(\mathfrak{g}))$ into and onto $J(H(\mathfrak{g}))$. If we let Γ'_θ denote the transpose, then for any $\alpha \in J^0(H(\mathfrak{g}))$ and $v \in J(H(\mathfrak{g}))$,

$$0 = \langle \alpha, \Gamma_\theta v \rangle = \langle \Gamma'_\theta \alpha, v \rangle.$$

Therefore, Γ'_θ takes $J^0(H(\mathfrak{g}))$ into itself. To see that Γ'_θ is onto $J^0(H(\mathfrak{g}))$, note that for any $\alpha \in J^0(H(\mathfrak{g}))$, if we define $\beta \in J^0(H(\mathfrak{g}))$ by $\langle \beta, v \rangle = \langle \alpha, \Gamma_{-\theta} v \rangle$ for all $v \in T(H(\mathfrak{g}))$, then it is easy to check that $\alpha = \Gamma'_\theta \beta$. In addition, $\theta \to \delta_{e^{i\theta}} h$ is continuous for any norm on $H(\mathfrak{g})$ and therefore so is $\theta \to \Gamma_\theta$ and $\theta \to \Gamma'_\theta$.

For every $n \in \mathbb{Z}_+$, let

$$F_n(\theta) = \frac{1}{2\pi n} \sum_{k=0}^{n-1} \sum_{l=-k}^{k} e^{il\theta} = \frac{1}{2\pi n} \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}$$

denote Fejer's kernel. Then

$$\int_{-\pi}^{\pi} F_n(\theta) d\theta = 1 \tag{6.4}$$

for all n, and if ϕ is continuous on $[-\pi, \pi]$, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} F_n(\theta)\phi(\theta) d\theta = \phi(0). \tag{6.5}$$

In addition, if $m \in \mathbb{Z}_+$ with $m > n$, then one can show that

$$\int_{-\pi}^{\pi} F_n(\theta) e^{im\theta} d\theta = 0. \tag{6.6}$$

Consider $\beta = h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H(\mathfrak{g})^{\otimes k}$, where $h_p \in H(V_{j_p})$ for $p = 1, ..., k$, where $1 \le j_p \le r$. Then

$$\begin{aligned} \Gamma_\theta \beta &= (e^{ij_1\theta} h_1) \otimes (e^{ij_2\theta} h_2) \otimes \cdots \otimes (e^{ij_k\theta} h_k) \\ &= \left(e^{i\theta \sum_{p=1}^{k} j_p} \right) \beta. \end{aligned}$$

If $k > n$, then $\sum_{p=1}^{k} j_p > n$ as well, and by Eq. (6.6),

$$\int_{-\pi}^{\pi} F_n(\theta) \Gamma_\theta \beta d\theta = 0.$$

Any element of $H(\mathfrak{g})^{\otimes k}$ can be written as a sum of elements like β, and so in fact

$$\int_{-\pi}^{\pi} F_n(\theta) \Gamma_\theta \beta d\theta = 0 \text{ for all } \beta \in H(\mathfrak{g})^{\otimes k} \text{ with } k > n.$$

Consequently,

$$\int_{-\pi}^{\pi} F_n(\theta) \Gamma'_\theta \alpha d\theta = 0 \text{ for all } \alpha \in (H(\mathfrak{g})^*)^{\otimes k} \text{ with } k > n. \tag{6.7}$$

Since for all $\alpha_k \in (H(\mathfrak{g})^*)^{\otimes k}$, $|\Gamma'_\theta \alpha_k|_{(H(\mathfrak{g})^*)^{\otimes k}} = |\alpha_k|_{(H(\mathfrak{g})^*)^{\otimes k}}$, it follows that for each $\alpha \in J_T^0(H(\mathfrak{g}))$, $\Gamma'_\theta \alpha \in J_T^0(H(\mathfrak{g}))$ and $||\Gamma'_\theta \alpha||_t = ||\alpha||_t$. Hence $\theta \to \Gamma'_\theta$ is strongly continuous in $J_T^0(H(\mathfrak{g}))$. For $\alpha \in J_T^0(H(\mathfrak{g}))$. Define

$$\gamma_n := \int_{-\pi}^{\pi} F_n(\theta) \Gamma'_\theta \alpha d\theta.$$

Then $\gamma_n \in J_T^0(H(\mathfrak{g}))$ for all $n > 0$ and by Eq. (6.7), it is zero in all ranks greater than n. Therefore

$$\begin{aligned}\lim_{n\to\infty} ||\gamma_n - \alpha||_t &= \lim_{n\to\infty} \left|\left| \int_{-\pi}^{\pi} F_n(\theta)(\Gamma'_\theta \alpha - \alpha) d\theta \right|\right|_t \\ &\leq \lim_{n\to\infty} \int_{-\pi}^{\pi} F_n(\theta) ||\Gamma'_\theta \alpha - \alpha||_t d\theta \\ &= 0\end{aligned}$$

by Eq. (6.5). □

6.3 A Continuation of Example 5.52

The following is a continuation of Example 5.52.

$$
\begin{aligned}
(\tilde{l}\tilde{k}\tilde{h}F_{\mathcal{P}})(g) &= \frac{d}{dt}|_{t=0}(\tilde{k}\tilde{h}F_{\mathcal{P}})(g\cdot tl) \\
&= \frac{d}{dt}|_{t=0}\langle D^2u_\alpha(P_{\mathcal{P}}(g\cdot tl))), k_{\mathcal{P}}(g\cdot tl)\otimes h_{\mathcal{P}}(g\cdot tl)\rangle \\
&+ \frac{d}{dt}|_{t=0}\sum_{1\le k<j\le 3}\sum_{i=1}^{j-1}\tilde{C}_{j,k}\langle Du_\alpha(P_{\mathcal{P}}(g\cdot tl))), \tilde{k}^{(i)}R^{\mathcal{P}}_{j,k}(g\cdot tl, ..., g\cdot tl, h)\rangle \\
&= \frac{d}{dt}|_{t=0}\langle D^2u_\alpha(P_{\mathcal{P}}(g\cdot tl)), k_{\mathcal{P}}(g)\otimes h_{\mathcal{P}}(g)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle D^2u_\alpha(P_{\mathcal{P}}(g)), k_{\mathcal{P}}(g)\otimes h_{\mathcal{P}}(g\cdot tl)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle D^2u_\alpha(P_{\mathcal{P}}(g)), k_{\mathcal{P}}(g\cdot tl)\otimes h_{\mathcal{P}}(g)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g\cdot tl)), \frac{1}{2}R^{\mathcal{P}}_{2,1}(k,h) + \frac{1}{2}R^{\mathcal{P}}_{2,1}([g,k],h)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g\cdot tl)), \frac{1}{12}R^{\mathcal{P}}_{3,1}(k,g,h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(g,k,h)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g\cdot tl)), -\frac{1}{6}R^{\mathcal{P}}_{3,2}(g,k,h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(k,g,h)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g)), \frac{1}{2}R^{\mathcal{P}}_{2,1}([g\cdot tl,k],h)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g)), \frac{1}{12}R^{\mathcal{P}}_{3,1}(k,g\cdot tl,h)\rangle \\
&+ \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g)), \frac{1}{12}R^{\mathcal{P}}_{3,1}(g\cdot tl,k,h)\rangle \\
&- \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g)), \frac{1}{6}R^{\mathcal{P}}_{3,2}(g\cdot tl,k,h)\rangle \\
&- \frac{d}{dt}|_{t=0}\langle Du_\alpha(P_{\mathcal{P}}(g)), \frac{1}{6}R^{\mathcal{P}}_{3,2}(k,g\cdot tl,h)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle D^3 u_\alpha(P_{\mathcal{P}}(g)), l_{\mathcal{P}}(g) \otimes k_{\mathcal{P}}(g) \otimes h_{\mathcal{P}}(g)\rangle \\
&+ \sum_{1\le k<j\le 3} \sum_{i=1}^{j-1} \langle D^2 u_\alpha(P_{\mathcal{P}}(g)), k_{\mathcal{P}}(g) \otimes \tilde{C}_{j,k} \tilde{l}^{(i)} R^{\mathcal{P}}_{j,k}(g,...,g,h)\rangle \\
&+ \sum_{1\le k<j\le 3} \sum_{i=1}^{j-1} \langle D^2 u_\alpha(P_{\mathcal{P}}(g)), \tilde{C}_{j,k} \tilde{l}^{(i)} R^{\mathcal{P}}_{j,k}(g,...,g,k) \otimes h_{\mathcal{P}}(g)\rangle \\
&+ \langle D^2 u_\alpha(P_{\mathcal{P}}(g)), l_{\mathcal{P}}(g) \otimes \left(\frac{1}{2} R^{\mathcal{P}}_{2,1}(k,h) + \frac{1}{2} R^{\mathcal{P}}_{2,1}([g,k],h)\right)\rangle \\
&+ \langle D^2 u_\alpha(P_{\mathcal{P}}(g)), l_{\mathcal{P}}(g) \otimes \left(\frac{1}{12} R^{\mathcal{P}}_{3,1}(k,g,h) + \frac{1}{12} R^{\mathcal{P}}_{3,1}(g,k,h)\right)\rangle \\
&+ \langle D^2 u_\alpha(P_{\mathcal{P}}(g)), l_{\mathcal{P}}(g) \otimes \left(-\frac{1}{6} R^{\mathcal{P}}_{3,2}(g,k,h) - \frac{1}{6} R^{\mathcal{P}}_{3,2}(k,g,h)\right)\rangle \\
&+ \langle D u_\alpha(P_{\mathcal{P}}(g)), \frac{1}{2} R^{\mathcal{P}}_{2,1}([l,k],h)\rangle + \langle D u_\alpha(P_{\mathcal{P}}(g)), \frac{1}{12} R^{\mathcal{P}}_{3,1}(k,l,h)\rangle \\
&+ \langle D u_\alpha(P_{\mathcal{P}}(g)), \frac{1}{12} R^{\mathcal{P}}_{3,1}(l,k,h)\rangle - \langle D u_\alpha(P_{\mathcal{P}}(g)), \frac{1}{6} R^{\mathcal{P}}_{3,2}(l,k,h)\rangle \\
&- \langle D u_\alpha(P_{\mathcal{P}}(g)), \frac{1}{6} R^{\mathcal{P}}_{3,2}(k,l,h)\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\tilde{l}\tilde{k}\tilde{h}F_{\mathcal{P}})(\underline{e}) &= \langle \alpha, P_{\mathcal{P}} l \otimes P_{\mathcal{P}} k \otimes P_{\mathcal{P}} h\rangle \\
&+ \langle \alpha, P_{\mathcal{P}} k \otimes \frac{1}{2} R^{\mathcal{P}}_{2,1}(l,h)\rangle + \langle \alpha, \frac{1}{2} R^{\mathcal{P}}_{2,1}(l,k) \otimes P_{\mathcal{P}} h\rangle \\
&+ \langle \alpha, P_{\mathcal{P}} l \otimes \frac{1}{2} R^{\mathcal{P}}_{2,1}(k,h)\rangle + \langle \alpha, \frac{1}{2} R^{\mathcal{P}}_{2,1}([l,k],h)\rangle \\
&+ \langle \alpha, \frac{1}{12} R^{\mathcal{P}}_{3,1}(k,l,h)\rangle + \langle \alpha, \frac{1}{12} R^{\mathcal{P}}_{3,1}(l,k,h)\rangle \\
&- \langle \alpha, \frac{1}{6} R^{\mathcal{P}}_{3,2}(l,k,h)\rangle - \langle \alpha, \frac{1}{6} R^{\mathcal{P}}_{3,2}(k,l,h)\rangle.
\end{aligned}
$$

Bibliography

[1] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. Part II. A family of related function spaces. Application to distribution theory. *Comm. Pure Appl. Math.*, 20:1–101, 1967.

[2] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry.* North-Holland Publishing Co., Amsterdam, 1975. North-Holland Mathematical Library, Vol. 9.

[3] B. K. Driver. On the Kakutani-Itô-Segal-Gross and Segal-Bargmann-Hall isomorphisms. *J. Funct. Anal.*, 133(1):69–128, 1995.

[4] B. K. Driver and L. Gross. Hilbert spaces of holomorphic functions on complex Lie groups. In *New trends in stochastic analysis (Charingworth, 1994)*, pages 76–106. World Sci. Publishing, River Edge, NJ, 1997.

[5] B. K. Driver and T. Lohrenz. Logarithmic Sobolev inequalities for pinned loop groups. *J. Funct. Anal.*, 140(2):381–448, 1996.

[6] G. L. Driver, Bruce K and L. Saloff-Coste. Subelliptic heat kernel analysis over nilpotent lie groups. Preprint.

[7] V. Fock. Verallgemeinerung und lsung der diracschen statistischen gleichung. *Zeits. f. Phys.*, 49:339–357, 1928.

[8] A. Friedman. *Partial differential equations.* Holt, Rinehart and Winston, Inc., New York, 1969.

[9] M. Gordina. Heat kernel analysis and Cameron-Martin subgroup for infinite dimensional groups. *J. Funct. Anal.*, 171(1):192–232, 2000.

[10] M. Gordina. Taylor map on groups associated with a II_1-factor. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(1):93–111, 2002.

[11] L. Gross and P. Malliavin. Hall's transform and the Segal-Bargmann map. In *Itô's stochastic calculus and probability theory*, pages 73–116. Springer, Tokyo, 1996.

[12] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit. *White noise*, volume 253 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993. An infinite-dimensional calculus.

[13] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

[14] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[15] Y.-J. Lee. Generalized functions on infinite-dimensional spaces and its application to white noise calculus. *J. Funct. Anal.*, 82(2):429–464, 1989.

[16] N. Obata. *White noise calculus and Fock space*, volume 1577 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[17] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.

[18] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[19] D. W. Robinson. *Elliptic operators and Lie groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[20] R. Schoen and S.-T. Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso.

[21] I. E. Segal. Tensor algebras over Hilbert spaces. I. *Trans. Amer. Math. Soc.*, 81:106–134, 1956.

[22] I. E. Segal. Mathematical characterization of the physical vacuum for a linear Bose-Einstein field. (Foundations of the dynamics of infinite systems. III). *Illinois J. Math.*, 6:500–523, 1962.

[23] H. Sugita. Properties of holomorphic Wiener functions—skeleton, contraction, and local Taylor expansion. *Probab. Theory Related Fields*, 100(1):117–130, 1994.

[24] H. Sugita. Holomorphic Wiener function. In *New trends in stochastic analysis (Charingworth, 1994)*, pages 399–415. World Sci. Publishing, River Edge, NJ, 1997.