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Analysis Tools with Examples

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This is the table of contents for the whole file. I am just giving you this so you know that I still have a lot of choices to make as to what should be included in the book.

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