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Math 247A (Topics in Analysis, W 2009) Rough Path Theory

March 11, 2009 *File:rpaths.tex*

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Part I

Rough Path Analysis

Here are a few suggested references for this course, [12, 15, 1]. The latter two references are downloadable if you are logging into MathSci net through your UCSD account. For a proof that all $p-$ variation paths have some extension to a rough path see, [14] and also see [6, Theorem 9.12 and Remark 9.13]. For other perspectives on the the theory, see [3] and also see Gubinelli [7, 8] Also see, [9, 4, 7] look interesting. A recent paper which deals with global existence issues for non-bounded vector fields is Lejay [11].

From Feynman Heuristics to Brownian Motion

In the physics literature one often finds the following informal expression,

$$d\mu_T(\omega) = \text{``}\frac{1}{Z(T)} e^{-\frac{1}{2}\int_0^T |\omega'(\tau)|^2 d\tau} \mathcal{D}_T\omega\text{''} \text{ for } \omega \in W_T, \tag{1.1}$$

where W_T is the set of continuous paths, $\omega : [0,T] \to \mathbb{R}$ (or $\mathbb{R}^d$), such that $\omega(0) = 0$,

$$\mathcal{D}_T\omega = \text{``}\prod_{0<t\le T} m(d\omega(t))\text{''} \; (m \text{ is Lebesgue measure here})$$

and $Z(T)$ is a normalization constant such that $\mu_T(W_T) = 1$.

We begin by giving meaning to this expression. For $0 \le s \le t \le T$, let

$$E_{[s,t]}(\omega) := \int_s^t |\omega'(\tau)|^2 d\tau.$$

If we decompose $\omega(\tau)$ as $\sigma(\tau) + \gamma(\tau)$ where

$$\sigma(\tau) := \omega(s) + \frac{\tau - s}{t - s}(\omega(t) - \omega(s)) \text{ and } \gamma(\tau) := \omega(\tau) - \sigma(\tau),$$

then we have, $\sigma\prime(\tau) = \frac{\omega(t)-\omega(s)}{t-s}$, $\gamma(s) = \gamma(t) = 0$, and hence

$$\begin{aligned}\int_s^t \sigma'(\tau)\cdot\gamma'(\tau)\,d\tau &= \int_s^t \sigma'(\tau)\cdot\gamma'(\tau)\,d\tau \\ &= \frac{\omega(t)-\omega(s)}{t-s}\cdot(\gamma(t)-\gamma(s)) = 0.\end{aligned}$$

Thus it follows that

$$\begin{aligned}E_{[s,t]}(\omega) &= E_{[s,t]}(\sigma) + E_{[s,t]}(\gamma) = \left|\frac{\omega(t)-\omega(s)}{t-s}\right|^2 (t-s) + E_{[s,t]}(\gamma) \\ &= \frac{|\omega(t)-\omega(s)|^2}{t-s} + E_{[s,t]}(\gamma).\end{aligned} \tag{1.2}$$

Thus if $f(\omega) = F\left(\omega|_{[0,s]}, \omega(t)\right)$, we will have,

$$\begin{aligned}&\frac{1}{Z_t}\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}E_t(\omega)}\mathcal{D}_t\omega \\ &\quad = \frac{1}{Z_t}\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}\left[E_s(\omega)+E_{[s,t]}(\omega)\right]}\mathcal{D}_t\omega\end{aligned}$$

and now fixing $\omega|_{[0,s]}$ and $\omega(t)$ and then doing the integral over $\omega|_{(s,t)}$ implies,

$$\begin{aligned}&\int F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}\left[E_s(\omega)+E_{[s,t]}(\omega)\right]}\mathcal{D}_{(s,t)}\omega \\ &= \int F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}\left[E_s(\omega)+\frac{|\omega(t)-\omega(s)|^2}{t-s}+E_{[s,t]}(\gamma)\right]}\mathcal{D}_{(s,t)}\gamma \\ &= C(s,t)\int F\left(\omega|_{[0,s]}, \omega(t)\right)\frac{e^{-\frac{1}{2}E_s(\omega)}}{Z(s)} e^{-\frac{1}{2}\frac{|\omega(t)-\omega(s)|^2}{t-s}}.\end{aligned}$$

Multiplying this equation by $\frac{1}{Z_t}\mathcal{D}\omega_{[0,s]}\cdot d\omega(t)$ and integrating the result then implies,

$$\begin{aligned}&\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) d\mu_t(\omega) \\ &\quad = \frac{C(s,t)}{Z_t}\int\left[\int_{\mathbb{R}^d} F\left(\omega|_{[0,s]}, y\right) e^{-\frac{1}{2}\frac{|y-\omega(s)|^2}{t-s}}dy\right]\frac{e^{-\frac{1}{2}E_s(\omega)}}{Z(s)}\mathcal{D}\omega_{[0,s]} \\ &\quad = \frac{C(s,t)}{Z_t}\int_{W_s}\left[\int_{\mathbb{R}^d} F(\omega, y)\, e^{-\frac{1}{2}\frac{|y-\omega(s)|^2}{t-s}}dy\right]d\mu_s(\omega).\end{aligned}$$

Taking $F \equiv 1$ in this equation then implies,

$$\begin{aligned}1 &= \frac{C(s,t)}{Z_t}\int_{W_s}\left[\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{|y-\omega(s)|^2}{t-s}}dy\right]d\mu_s(\omega) \\ &= \frac{C(s,t)}{Z_t}\int_{W_s}\left[(2\pi(t-s))^{d/2}\right]d\mu_s(\omega) = \frac{C(s,t)}{Z_t}(2\pi(t-s))^{d/2}.\end{aligned}$$

Thus the heuristic expression in Eq. (1.1) leads to the following **Markov property** for μ_t, namely.

Proposition 1.1 (Heuristic). *Suppose that $F : W_s \times \mathbb{R}^d \to \mathbb{R}$ is a reasonable function, then for any $t \ge s$ we have*

$$\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) d\mu_t(\omega)$$
$$= \int_{W_s} \left[\int_{\mathbb{R}^d} F(\omega, y)\, p_{t-s}(\omega(s), y)\, dy\right] d\mu_s(\omega),$$

where

$$p_s(x, y) := \left(\frac{1}{2\pi(t-s)}\right)^{d/2} e^{-\frac{1}{2}\frac{|y-x|^2}{t-s}}. \tag{1.3}$$

Corollary 1.2 (Heuristic). *If* $0 = s_0 < s_1 < s_2 < \cdots < s_n = T$ *and* $f : \left(\mathbb{R}^d\right)^n \to \mathbb{R}$ *is a reasonable function, then*

$$\int_{W_T} f(\omega(s_1), \ldots, \omega(s_n))\, d\mu_T(\omega) = \int_{(\mathbb{R}^d)^n} f(y_1, \ldots, y_n) \prod_{i=1}^n \left(p_{s_i - s_{i-1}}(y_{i-1}, y_i)\, dy_i\right) \tag{1.4}$$

where by convention, $y_0 = 0$.

Theorem 1.3 (Wiener 1923). *For all* $t > 0$ *there exists a unique probability measure,* μ_t, *on* W_t, *such that Eq. (1.4) holds for all* n *and all bounded measurable* $f : \left(\mathbb{R}^d\right)^n \to \mathbb{R}$.

Definition 1.4. *Let* $B_t(\omega) := \omega(t)$. *Then* $\{B_t\}_{0 \le t \le T}$ *as a process on* (W_T, μ_T) *is called* ***Brownian motion.*** *We further write* $\mathbb{E}f$ *for* $\int_{W_T} f(\omega)\, d\mu_T(\omega)$.

The following lemma is useful for computational purposes involving Brownian motion and follows readily form from Eq. (1.4).

Lemma 1.5. *Suppose that* $0 = s_0 < s_1 < s_2 < \cdots < s_n = t$ *and* $f_i : \mathbb{R}^d \to \mathbb{R}$ *are reasonable functions, then*

$$\mathbb{E}\left[\prod_{i=1}^n f_i\left(B_{s_i} - B_{s_{i-1}}\right)\right] = \prod_{i=1}^n \mathbb{E}\left[f_i\left(B_{s_i} - B_{s_{i-1}}\right)\right], \tag{1.5}$$

$$\mathbb{E}\left[f(B_t - B_s)\right] = \mathbb{E}\left[f(B_{t-s})\right], \tag{1.6}$$

and

$$\mathbb{E}\left[f(B_t)\right] = \mathbb{E}f\left(\sqrt{t}B_1\right). \tag{1.7}$$

As an example let us observe that

$$\mathbb{E}B_t = \int y p_t(y)\, dy = 0,$$
$$\mathbb{E}B_t^2 = t\mathbb{E}B_1^2 = t\int y^2 p_1(y)\, dy = t \cdot 1,$$

and for $s < t$,

$$\mathbb{E}\left[B_t B_s\right] = \mathbb{E}\left[(B_t - B_s) B_s\right] + \mathbb{E}B_s^2 = \mathbb{E}(B_t - B_s) \cdot \mathbb{E}B_s + s = s$$

and

$$\mathbb{E}\left[|B_t - B_s|^p\right] = |t-s|^{p/2}\, \mathbb{E}\left[|B_1|^p\right] = C_p |t-s|^{p/2}. \tag{1.8}$$

1.1 Construction and basic properties of Brownian motion

In this section we sketch one method of constructing Wiener measure ore equivalently Brownian motion. We begin with the existence of a measure ν_T on the $\tilde{W}_T := \prod_{0 \le s \le T} \bar{\mathbb{R}}$ which satisfies Eq. (1.4) where $\bar{\mathbb{R}}$ is a compactification of $\mathbb{R}$ – for example either one point compactificatoin so that $\bar{\mathbb{R}} \cong S^1$.

Theorem 1.6 (Kolmogorov's Existence Theorem). *There exists a probability measure,* ν_T, *on* $\tilde{W}_T$ *such that Eq. (1.4) holds.*

Proof. For a function $F(\omega) := f(\omega(s_1), \ldots, \omega(s_n))$ where $f \in C\left(\bar{\mathbb{R}}^n, \mathbb{R}\right)$, define

$$I(F) := \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) \prod_{i=1}^n \left(p_{s_i - s_{i-1}}(y_{i-1}, y_i)\, dy_i\right).$$

Using the semi-group property;

$$\int_{\mathbb{R}^d} p_t(x, y)\, p_s(y, z)\, dy = p_{s+t}(x, z)$$

along with the fact that $\int_{\mathbb{R}^d} p_t(x, y)\, dy = 1$ for all $t > 0$, one shows that $I(F)$ is well defined independently of how we represent F as a "finitely based" continuous function.

By Tychonoff's Theorem $\tilde{W}_T$ is a compact Hausdorff space. By the Stone Weierstrass Theorem, the finitely based continuous functions are dense inside of $C\left(\tilde{W}_T\right)$. Since $|I(F)| \le \|F\|_\infty$ for all finitely based continuous functions, we may extend I uniquely to a positive continuous linear functional on $C\left(\tilde{W}_T\right)$. An application of the Riesz Markov theorem now gives the existence of the desired measure, ν_T. ■

Theorem 1.7 (Kolmogorov's Continuity Criteria). *Suppose that* $(\Omega, \mathcal{F}, P)$ *is a probability space and* $\tilde{X}_t : \Omega \to S$ *is a process for* $t \in [0, T]$ *where* (S, ρ) *is a complete metric space. Assume there exists positive constants,* ε, β, *and* C, *such that*

$$\mathbb{E}[\rho(\tilde{X}_t, \tilde{X}_s)^\varepsilon] \le C|t-s|^{1+\beta} \tag{1.9}$$

for all $s,t\in[0,T]$. *Then for any* $\alpha\in(0,\beta/\varepsilon)$ *there is a modification,* X, *of* $\tilde{X}$ *(i.e.* $P\left(X_t=\tilde{X}_t\right)=1$ *for all* t*) which is* α*–Hölder continuous. Moreover, there is a random variable* K_α *such that,*

$$\rho(X_t,X_s)\le K_\alpha\left|t-s\right|^\alpha \text{ for all } s,t\in[0,T] \tag{1.10}$$

and $\mathbb{E}K_\alpha^p<\infty$ *for all* $p<\frac{\beta-\alpha\varepsilon}{1-\alpha}$.

Corollary 1.8. *Let* $\tilde{B}_t:\tilde{W}_T\to\mathbb{R}$ *be the projection map,* $\tilde{B}_t(\omega)=\omega(t)$. *Then there is a modifications,* $\{B_t\}$ *of* $\left\{\tilde{B}_t\right\}$ *for which* $t\to B_t$ *is* α *– Hölder continuous* ν_T *– almost surely for any* $\alpha\in(0,1/2)$.

Proof. Applying Theorem 1.7 with $\varepsilon:=p$ and $\beta:=p/2-1$ for any $p\in(2,\infty)$ shows there is a modification $\{B_t\}_{t\ge0}$ of $\left\{\tilde{B}_t\right\}$ which is almost surely α – Hölder continuous for any

$$\alpha\in(0,\beta/\varepsilon)=\left(0,\frac{p/2-1}{p}\right)=(0,1/2-1/p).$$

Letting $p\to\infty$ shows that $\{B_t\}_{t\ge0}$ is almost surely α – Hölder continuous for all $\alpha<1/2$. ■

We will see shortly that these Brownian paths are very rough. Before we do this we will pause to develop a quantitative measurement of roughness of a continuous path.

p – Variations and Controls

Let (E, d) be a metric space which will usually be assumed to be complete.

Definition 2.1. *Let $0 \leq a < b < \infty$. Given a **partition** $\Pi := \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ and a function $Z \in C([a, b], E)$, let $(t_i)_- := t_{i-1}$, $(t_i)_+ := t_{i+1}$, with the convention that $t_{-1} := t_0 = a$ and $t_{n+1} := t_n = T$. Furthermore for $1 \leq p < \infty$ let*

$$V_p(Z:\Pi) := \left(\sum_{j=1}^{n} d^p\left(Z_{t_j}, Z_{t_{j-1}}\right)\right)^{1/p} = \left(\sum_{t\in\Pi} d^p\left(Z_t, Z_{t_-}\right)\right)^{1/p}. \tag{2.1}$$

*Furthermore, let $\mathcal{P}(a, b)$ denote the collection of partitions of $[a, b]$. Also let $\mathrm{mesh}(\Pi) := \max_{t\in\Pi} |t - t_-|$ be the **mesh** of the partition, Π.*

Definition 2.2. *and $Z \in C([a, b], E)$. For $1 \leq p < \infty$, the **p - variation** of Z is;*

$$V_p(Z) := \sup_{\Pi\in\mathcal{P}(a,b)} V_p(Z:\Pi) = \sup_{\Pi\in\mathcal{P}(a,b)} \left(\sum_{j=1}^{n} d^p\left(Z_{t_j}, Z_{t_{j-1}}\right)\right)^{1/p}. \tag{2.2}$$

Moreover if $Z \in C([0, T], E)$ and $0 \leq a \leq b \leq T$, we let

$$\omega_{Z,p}(a, b) := \left[\nu_p\left(Z|_{[a,b]}\right)\right]^p = \sup_{\Pi\in\mathcal{P}(a,b)} \sum_{j=1}^{n} d^p\left(Z_{t_j}, Z_{t_{j-1}}\right). \tag{2.3}$$

Remark 2.3. We can define $V_p(Z)$ for $p \in (0, 1)$ as well but this is not so interesting. Indeed if $0 \leq s \leq T$ and $\Pi \in \mathcal{P}(0, T)$ is a partition such that $s \in \Pi$, then

$$\begin{aligned} d(Z(s), Z(0)) &\leq \sum_{t\in\Pi} d(Z(t), Z(t_-)) = \sum_{t\in\Pi} d^{1-p}(Z(t), Z(t_-))\, d^p(Z(t), Z(t_-)) \\ &\leq \max_{t\in\Pi} d^{1-p}(Z(t), Z(t_-)) \cdot V_p^p(Z:\Pi) \\ &\leq \max_{t\in\Pi} d^{1-p}(Z(t), Z(t_-)) \cdot V_p^p(Z). \end{aligned}$$

Using the uniform continuity of Z (or $d(Z(s), Z(t))$ if you wish) we know that $\lim_{|\Pi|\to 0} \max_{t\in\Pi} d^{1-p}(Z(t), Z(t_-)) = 0$ and hence that,

$$d(Z(s), Z(0)) \leq \lim_{|\Pi|\to 0} \max_{t\in\Pi} d^{1-p}(Z(t), Z(t_-)) \cdot V_p^p(Z) = 0.$$

Thus we may conclude $Z(s) = Z(0)$, i.e. Z must be constant.

Lemma 2.4. *Let $\{a_i > 0\}_{i=1}^n$, then*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \text{ is decreasing in } p \text{ and}$$

$$\varphi(p) := \ln\left(\sum_{i=1}^{n} a_i^p\right) \text{ is convex in } p.$$

Proof. Let $f(i) = a_i$ and $\mu(\{i\}) = 1$ be counting measure so that

$$\sum_{i=1}^{n} a_i^p = \mu(f^p) \text{ and } \varphi(p) = \ln \mu(f^p).$$

Using $\frac{d}{dp} f^p = f^p \ln f$, it follows that and

$$\varphi'(p) = \frac{\mu(f^p \ln f)}{\mu(f^p)} \text{ and}$$
$$\varphi''(p) = \frac{\mu\left(f^p \ln^2 f\right)}{\mu(f^p)} - \left[\frac{\mu(f^p \ln f)}{\mu(f^p)}\right]^2.$$

Thus if we let $\mathbb{E}X := \mu(f^p X)/\mu(f^p)$, we have shown, $\varphi'(p) = \mathbb{E}[\ln f]$ and

$$\varphi''(p) = \mathbb{E}\left[\ln^2 f\right] - (\mathbb{E}[\ln f])^2 = \mathrm{Var}(\ln f) \geq 0$$

which shows that φ is convex in p.

Now let us shows that $\|f\|_p$ is decreasing in in p. To this end we compute,

$$\frac{d}{dp}\left[\ln\|f\|_p\right] = \frac{d}{dp}\left[\frac{1}{p}\varphi(p)\right] = \frac{1}{p}\varphi'(p) - \frac{1}{p^2}\varphi(p)$$
$$= \frac{1}{p^2\mu(f^p)}\left[p\mu(f^p\ln f) - \mu(f^p)\ln\mu(f^p)\right]$$
$$= \frac{1}{p^2\mu(f^p)}\left[\mu(f^p\ln f^p) - \mu(f^p)\ln\mu(f^p)\right]$$
$$= \frac{1}{p^2\mu(f^p)}\left[\mu\left(f^p\ln\frac{f^p}{\mu(f^p)}\right)\right].$$

Up to now our computation has been fairly general. The point where μ being counting measure comes in is that in this case $\mu(f^p) \geq f^p$ everywhere and therefore $\ln\frac{f^p}{\mu(f^p)} \leq 0$ and therefore, $\frac{d}{dp}\left[\ln\|f\|_p\right] \leq 0$ as desired.

Alternative proof that $\|f\|_p$ is decreasing in p. If we let $q = p + r$, then

$$\|a\|_q^q = \sum_{j=1}^n a_j^{p+r} \leq \left(\max_j a_j\right)^r \cdot \sum_{j=1}^n a_j^p \leq \|a\|_p^r \cdot \|a\|_p^p = \|a\|_p^q,$$

wherein we have used,

$$\max_j a_j = \left(\max_j a_j^p\right)^{1/p} \leq \left(\sum_{j=1}^n a_j^p\right)^{1/p} = \|a\|_p.$$

∎

Remark 2.5. It is not too hard to see that the convexity of φ is equivalent to the interpolation inequality,

$$\|f\|_{p_s} \leq \|f\|_{p_0}^{1-s} \cdot \|f\|_{p_1}^s,$$

where $0 \leq s \leq 1$, $1 \leq p_0, p_1$, and

$$\frac{1}{p_s} := (1-s)\frac{1}{p_0} + s\frac{1}{p_1}.$$

This interpolation inequality may be proved via Hölder's inequality.

Corollary 2.6. *The function $V_p(Z)$ is a decreasing function of p and $\ln V_p^p(Z)$ is a convex function of p where they are finite. Moreover, for all $p_0 > 1$,*

$$\lim_{p\downarrow p_0} V_p(Z) = V_{p_0}(Z). \tag{2.4}$$

and $p \to V_p(Z)$ is continuous on the set of p's where $V_p(Z)$ is finite.

Proof. Given Lemma 2.4, it suffices to prove Eq. (2.4) and the continuity assertion on $p \to V_p(Z)$. Since $p \to V_p(Z)$ is a decreasing function, we know that $\lim_{p\uparrow p_0} V_p(Z)$ and $\lim_{p\downarrow p_0} V_p(Z)$ always exists and also that $\lim_{p\downarrow p_0} V_p(Z) = \sup_{p>p_0}\sup_{\Pi} V_p(Z:\Pi)$. Therefore,

$$\lim_{p\downarrow p_0} V_p(Z) = \sup_{p>p_0}\sup_{\Pi} V_p(Z:\Pi) = \sup_{\Pi}\sup_{p>p_0} V_p(Z:\Pi) = \sup_{\Pi} V_{p_0}(Z:\Pi) = V_{p_0}(Z)$$

which proves Eq. (2.4). The continuity of $V_p(Z) = \exp\left(\frac{1}{p}\ln V_p(Z)^p\right)$ follows directly from the fact that $\ln V_p(Z)^p$ is convex in p and that convex functions are continuous (where finite).

Here is a proof for this case. Let $\varphi(p) := \ln V_p(Z)^p$, $1 \leq p_0 < p_1$ such that $V_{p_0}(Z) < \infty$, and $p_s := (1-s)p_0 + sp_1$, then

$$\varphi(p_s) \leq (1-s)\varphi(p_0) + s\varphi(p_1).$$

Letting $s \uparrow 1$ then implies $p_s \uparrow p_1$ and $\varphi(p_1-) \leq \varphi(p_1)$, i.e. $V_{p_1-} \leq V_{p_1} \leq V_{p_1-}$. Therefore $V_{p_1-} = V_{p_1}$ and along with Eq. (2.4) proves the continuity of $p \to V_p(Z)$. ∎

2.1 Computing $V_p(x)$

How do we actually compute $V_p(x) := V_p(x;0,T)$ for a given path $x \in C([0,T],\mathbb{R})$, even a very simple one? Suppose x is piecewise linear, with corners at the points $0 = s_0, s_1, \ldots, s_m = T$. Intuitively it would seem that the p-variation should be given by choosing the corners to be the partition points. That is, if $S = \{s_0, \ldots, s_n\}$ is the partition of corner points, we might think that $V_p(x) = V_p(x;S)$. Well, first we would have to leave out any corner which is not a local extremum (because of Lemma 2.8 below). But even then, this is not generally true as is seen in Example 2.9 below.

Lemma 2.7. *For all $a, b \geq 0$ and $p \geq 1$,*

$$(a+b)^p \geq a^p + b^p \tag{2.5}$$

and the inequality is strict if $a, b > 0$ and $p > 1$.

Proof. Observe that $(a+b)^p \geq a^p + b^p$ happens iff

$$1 \geq \left(\frac{a}{a+b}\right)^p + \left(\frac{b}{a+b}\right)^p$$

which obviously holds since

$$\left(\frac{a}{a+b}\right)^p + \left(\frac{b}{a+b}\right)^p \leq \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Moreover the latter inequality is strict if if $a, b > 0$ and $p > 1$. ∎

Lemma 2.8. *Let x be a path, and $D = \{t_0, \ldots, t_n\}$ be a partition. Suppose x is monotone increasing (decreasing) on $[t_{i-1}, t_{i+1}]$. Then if $D' = D \backslash \{t_i\}$, $V_p(x : D\prime) \geq V_p(x : D)$. If x is strictly increasing and $p > 1$, the inequality is strict.*

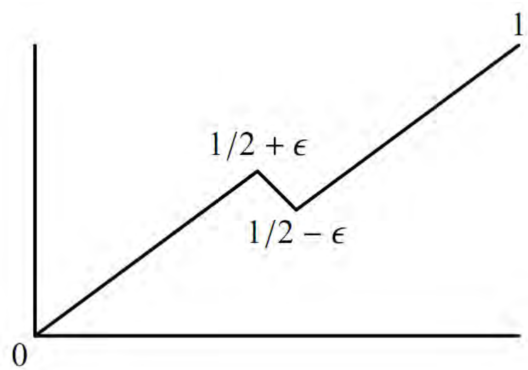
Proof. From Eq. (2.5) it follows

$$\begin{aligned} V_p(x : D\prime)^p - V_p(x : D)^p &= (x(t_{i+1}) - x(t_{i-1}))^p - (x(t_{i+1}) - x(t_i))^p - (x(t_i) - x(t_{i-1}))^p \\ &= \left(\Delta_{t_i} x + \Delta_{t_{i+1}} x\right)^p - (\Delta_{t_i} x)^p - \left(\Delta_{t_{i+1}} x\right)^p \geq 0 \end{aligned}$$

and the inequality is strict if $\Delta_{t_i} x > 0$, $\Delta_{t_{i+1}} x > 0$ and $p > 1$. ■

In other words, on any monotone increasing segment, we should not include any intermediate points, because they can only hurt us.

Example 2.9. Consider a path like the following: If we partition $[0, T]$ at the



corner points, then

$$V_p(x : S)^p = (\frac{1}{2} + \epsilon)^p + (2\epsilon)^p + (\frac{1}{2} + \epsilon)^p \approx 2(\frac{1}{2})^p < 1$$

by taking ϵ small. On the other hand, taking the trivial partition $D = \{0, T\}$, $V_p(x : D) = 1$, so $V_p(x : S) < 1 \leq V_p(x)$ and in this case using all of local minimum and maximum does not maximize the $p -$ variation.

The clean proof of the following theorem is due to Thomas Laetsch.

Theorem 2.10. *If $x : [0, T] \to \mathbb{R}$ having only finitely many local extremum in $(0, T)$ located at $\{s_1 < \cdots < s_{n-1}\}$. Then*

$$V_p(x) = \sup\{V_p(x : D) : \{0, T\} \subset D \subset S\},$$

where $S = \{0 = s_0 < s_1 < \cdots < s_n = T\}$.

Proof. Let $D = \{0 = t_0 < t_1 < \cdots < t_r = T\} \in \mathcal{P}(0, T)$ be an arbitrary partition of $[0, T]$. We are going to prove by induction that there is a partition $\Pi \subset S$ such that $V_p(x : D) \leq V_p(x : \Pi)$. The proof will be by induction on $n := \#(D \setminus S)$. If $n = 0$ there is nothing to prove. So let us now suppose that the theorem holds at some level $n \geq 0$ and suppose that $\#(D \setminus S) = n + 1$. Let $1 \leq k < r$ be chosen so that $t_k \in D \setminus S$. If $x(t_k)$ is between $x(t_{k-1})$ and $x(t_{k+1})$ (i.e. $(x(t_{k-1}), x(t_k), x(t_{k+1}))$ is a monotonic triple), then according Lemma 2.8 we will have $V_p(x : D) \leq V_p(x : D \setminus \{t_k\})$ and since $\#[(D \setminus \{t_k\}) \setminus S] = n$, the induction hypothesis implies there exists a partition, $\Pi \subset S$ such that

$$V_p(x : D) \leq V_p(x : D \setminus \{t_k\}) \leq V_p(x : \Pi).$$

Hence we may now assume that either $x(t_k) < \min(x(t_{k-1}), x(t_{k+1}))$ or $x(t_k) > \max(x(t_{k-1}), x(t_{k+1}))$. In the first case we let $t_k^* \in (t_{k-1}, t_{k+1})$ be a point where $x|_{[t_{k-1}, t_{k+1}]}$ has a minimum and in the second let $t_k^* \in (t_{k-1}, t_{k+1})$ be a point where $x|_{[t_{k-1}, t_{k+1}]}$ has a maximum. In either case if $D^* := (D \setminus \{t_k\}) \cup \{t_k^*\}$ we will have $V_p(x : D) \leq V_p(x : D^*)$ and $\#(D^* \setminus S) = n$. So again the induction hypothesis implies there exists a partition $\Pi \subset S$ such that

$$V_p(x : D) \leq V_p(x : D^*) \leq V_p(x : \Pi).$$

From these considerations it follows that

$$V_p(x : D) \leq \sup\{V_p(x : \Pi) : \Pi \in \mathcal{P}(0, T) \text{ s.t. } \Pi \subset S\}$$

and therefore

$$\begin{aligned} V_p(x) &= \sup\{V_p(x : D) : D \in \mathcal{P}(0, T)\} \\ &\leq \sup\{V_p(x : \Pi) : \Pi \in \mathcal{P}(0, T) \text{ s.t. } \Pi \subset S\} \leq V_p(x). \end{aligned}$$

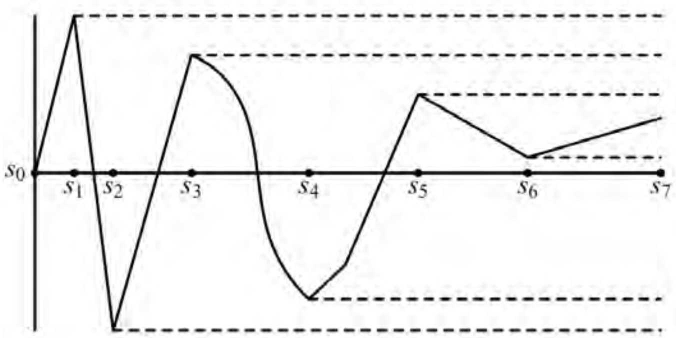
■

Let us now suppose that x is (say) monotone increasing (not strictly) on $[s_0, s_1]$, monotone decreasing on $[s_1, s_2]$, and so on. Thus $s_0, s_2, \ldots$ are local minima, and $s_1, s_3, \ldots$ are local maxima. (If you want the reverse, just replace x with $-x$, which of course has the same p-variation.)

Definition 2.11. *Say that $s \in [0, T]$ is a* ***forward maximum*** *for x if $x(s) \geq x(t)$ for all $t \geq s$. Similarly, s is a* ***forward minimum*** *if $x(s) \leq x(t)$ for all $t \geq s$.*

Definition 2.12. *Suppose x is piecewise monotone, as above, with extrema $\{s_0, s_1, \ldots\}$. Suppose further that $s_2, s_4, \ldots$ are not only local minima but also forward minima, and that $s_1, s_3, \ldots$ are both local and forward maxima. Then we will say that x is* ***jog-free****.*

Note that $s_0 = 0$ does not *have to be a forward extremum. This is in order to admit a path with $x(0) = 0$ which can change signs.*



Here is an example.

Remark 2.13. Here is another way to state the jog-free condition. Let x be piecewise monotone with extrema $s_0, s_1, \ldots$. Let $\xi_i = |x(s_{i+1}) - x(s_i)|$. Then x is jog-free iff $\xi_1 \geq \xi_2 \geq \ldots$. The idea is that the oscillations are shrinking. (Notice that we don't need $\xi_0 \geq \xi_1$; this is because $s_0 = 0$ is not required to be a forward extremum.)

Remark 2.14. It is also okay if $s_1, s_2, \ldots$ are *backwards* extrema; this corresponds to the oscillations getting larger. Just reverse time, replacing $x(t)$ by $x(T-t)$, which again doesn't change the p-variation. Note that if ξ_i are as above, this corresponds to having $\xi_0 \leq \xi_1 \leq \xi_2 \leq \ldots$ (note that ξ_0 is included now, but ξ_{m-1} would not be). This case seems less useful, however.

Lemma 2.15. *Let x be jog-free with extrema $s_0, \ldots, s_m$. Let $D = \{t_0, \ldots, t_n\}$ be any partition not containing all the s_j. Then there is some $s_j \notin D$ such that if $D' = D \cup \{s_j\}$, $V_p(x : D\prime) \geq V_p(x : D)$.*

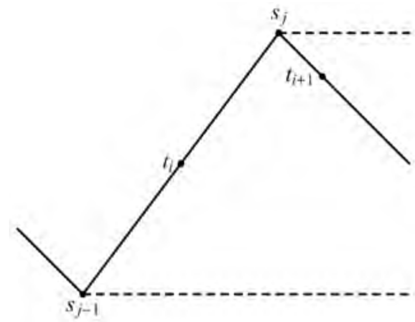
Proof. Let s_j be the first extremum not contained in D (note $s_0 = 0 \in D$ already, so j is at least 1 and s_j is also a forward extremum). Let t_i be the last element of D less than s_j. Note that $s_{j-1} \leq t_i < s_j < t_{i+1}$.

Now x is monotone on $[s_{j-1}, s_j]$; say WLOG it's monotone increasing, so that s_j is a local maximum and also a forward maximum. Since $t_i \in [s_{j-1}, s_j]$, where x is monotone increasing, $x(s_j) \geq x(t_i)$. And since s_j is a forward maximum, $x(s_j) \geq x(t_{i+1})$.

Therefore we have

$$x(s_j) - x(t_i) \geq x(t_{i+1}) - x(t_i)$$
$$x(s_j) - x(t_{i+1}) \geq x(t_i) - x(t_{i+1}).$$

One of the quantities on the right is equal to $|x(t_{i+1}) - x(t_i)|$, and so it follows that



$$|x(s_j) - x(t_i)|^p + |x(s_j) - x(t_{i+1})|^p \geq |x(t_{i+1}) - x(t_i)|^p$$

since one of the terms on the left is already $\geq$ the term on the right. This shows that $V_p(x : D\prime)^p \geq V_p(x : D)^p$. ∎

In other words, we should definitely include the extreme points, because they can only help.

Putting these together yields the desired result.

Proposition 2.16. *If x is jog-free with extrema $S = \{s_0, \ldots, s_m\}$, then $V_p(x) = V_p(x : S) = (\sum \xi_i^p)^{1/p}$.*

Proof. Fix $\epsilon > 0$, and let D be a partition such that $V_p(x : D) \geq V_p(x) - \epsilon$. By repeatedly applying Lemma 2.15, we can add the points of S to D one by one (in some order), and only increase the p-variation. So $V_p(x : D \cup S) \geq V_p(x : D)$. Now, if $t \in D \backslash S$, it is inside some interval $[s_j, s_{j+1}]$ on which x is monotone, and so by Lemma 2.8 t can be removed from $D \cup S$ to increase the p-variation. Removing all such points one by one (in any order), we find that $V_p(x : S) \geq V_p(x : D \cup S)$. Thus we have $V_p(x : S) \geq V_p(x : D) \geq V_p(x) - \epsilon$; since ϵ was arbitrary we are done. ∎

Notice that we only considered the case of jog-free paths with only finitely many extrema. Of course, in order to get infinite p-variation for any p we would need infinitely many extrema. Let's just check that the analogous result holds there.

Proposition 2.17. *Suppose we have a sequence $s_0, s_1, \ldots$ increasing to T, where x is alternately monotone increasing and decreasing on the intervals $[s_j, s_{j+1}]$. Suppose also that the s_j are forward extrema for x. Letting $\xi_j = |x(s_{j+1}) - x(s_j)|$ as before, we have*

$$V_p(x) = \left(\sum_{j=0}^{\infty} \xi_j^p \right)^{1/p}.$$

Actually, the extreme points s_j can converge to some earlier time than T, but x will have to be constant after that time.

Proof. For any m, we have $\sum_{j=0}^{m} \xi_j^p = V_p(x:D)^p$ for $D = \{s_0, \ldots, s_{m+1}\}$, so $V_p(x)^p \geq \sum_{j=0}^{m} \xi_j^p$. Passing to the limit, $V_p(x)^p \geq \sum_{j=0}^{\infty} \xi_j^p$.

For the reverse inequality, let $D = \{0 = t_0, t_1, \ldots, t_n = T\}$ be a partition with $V_p(x:D) \geq V_p(x) - \epsilon$. Choose m so large that $s_m > t_{n-1}$. Let $S = \{s_0, \ldots, s_m, T\}$, then by the same argument as in Proposition 2.16 we find that $V_p(x:S) \geq V_p(x:D)$. (Previously, the only way we used the assumption that S contained *all* extrema $s_j$was in order to have every $t_i \in D \backslash S$ contained in some monotone interval $[s_j, s_{j+1}]$. That is still the case here; we just take enough s_j's to ensure that we can surround each t_i. We do not need to surround $t_n = T$, since it is already in S.)

But $V_p(x:S)^p = \sum_{j=0}^{m-1} \xi_j^p \leq \sum_{j=0}^{\infty} \xi_j^p$, and so we have that

$$\left(\sum_{j=0}^{\infty} \xi_j^p\right)^{1/p} \geq V_p(x:D) \geq V_p(x) - \epsilon.$$

ϵ was arbitrary and we are done. ■

2.2 Brownian Motion in the Rough

Corollary 2.18. *For all $p > 2$ and $T < \infty$, $V_p\left(B|_{[0,T]}\right) < \infty$ a.s. (We will see later that $V_p\left(B|_{[0,T]}\right) = \infty$ a.s. for all $p < 2$.)*

Proof. By Corollary 1.8, there exists $K_p < \infty$ a.s. such that

$$|B_t - B_s| \leq K_p |t-s|^{1/p} \text{ for all } 0 \leq s, t \leq T. \tag{2.6}$$

Thus we have

$$\sum_i |\Delta_i B|^p \leq \sum_i \left(K_p |t_i - t_{i-1}|^{1/p}\right)^p \leq \sum_i K_p^p |t_i - t_{i-1}| = K_p^p T$$

and therefore, $V_p\left(B|_{[0,T]}\right) \leq K_p^p T < \infty$ a.s. ■

Proposition 2.19 (Quadratic Variation). *Let $\{\Pi_m\}_{m=1}^{\infty}$ be a sequence of partition of $[0,T]$ such that $\lim_{m\to\infty} |\Pi_m| = 0$ and define $Q_m := V_2^2(B:\Pi_m)$. Then*

$$\lim_{m\to\infty} \mathbb{E}\left[(Q_m - T)^2\right] = 0 \tag{2.7}$$

and if $\sum_{m=1}^{\infty} \mathrm{mesh}(\Pi_m) < \infty$ then $\lim_{m\to\infty} Q_m = T$ a.s. This result is often abbreviated by the writing, $dB_t^2 = dt$.

Proof. Let N be an $N(0,1)$ random variable, $\Delta t := t - t_-$, $\Delta_t B := B_t - B_{t_-}$ and observe that $\Delta_t B \sim \sqrt{\Delta t} N$. Thus we have,

$$\mathbb{E}Q_m = \sum_{t\in\Pi_m} \mathbb{E}(\Delta_t B)^2 = \sum_{t\in\Pi_m} \Delta t = T.$$

Let us define

$$\mathrm{Cov}(A,B) := \mathbb{E}[AB] - \mathbb{E}A \cdot \mathbb{E}B \text{ and}$$
$$\mathrm{Var}(A) := \mathrm{Cov}(A,A) = \mathbb{E}A^2 - (\mathbb{E}A)^2 = \mathbb{E}\left[(A - \mathbb{E}A)^2\right].$$

and observe that

$$\mathrm{Var}\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \mathrm{Var}(A_i) + \sum_{i\neq j} \mathrm{Cov}(A_i, A_j).$$

As $\mathrm{Cov}(\Delta_t B, \Delta_s B) = 0$ if $s \neq t$, we may use the above computation to conclude,

$$\begin{aligned}\mathrm{Var}(Q_m) &= \sum_{t\in\Pi} \mathrm{Var}((\Delta_t B)^2) = \sum_{t\in\Pi} \mathrm{Var}(\Delta t \cdot N^2) \\ &= \mathrm{Var}(N^2) \sum_{t\in\Pi} (\Delta t)^2 \leq \mathrm{Var}(N^2)\,|\Pi_m| \sum_{t\in\Pi} \Delta t \\ &= T \cdot \mathrm{Var}(N^2)\,|\Pi_m| \to 0 \text{ as } m \to \infty.\end{aligned}$$

(By explicit Gaussian integral computations,

$$\mathrm{Var}(N^2) = \mathbb{E}N^4 - \left(\mathbb{E}N^2\right)^2 = 3 - 1 = 2 < \infty.)$$

Thus we have shown

$$\lim_{m\to\infty} \mathbb{E}\left[(Q_m - T)^2\right] = \lim_{m\to\infty} \mathbb{E}\left[(Q_m - \mathbb{E}Q)^2\right] = \lim_{m\to\infty} \mathrm{Var}(Q_m) = 0.$$

If $\sum_{m=1}^{\infty} |\Pi_m| < \infty$, then

$$\begin{aligned}\mathbb{E}\left[\sum_{m=1}^{\infty} (Q_m - T)^2\right] &= \sum_{m=1}^{\infty} \mathbb{E}(Q_m - T)^2 = \sum_{m=1}^{\infty} \mathrm{Var}(Q_m) \\ &\leq \mathrm{Var}(N^2) \cdot T \cdot \sum_{m=1}^{\infty} \mathrm{mesh}(\Pi_m) < \infty\end{aligned}$$

from which it follows that $\sum_{m=1}^{\infty} (Q_m - T)^2 < \infty$ a.s. In particular $(Q_m - T) \to 0$ almost surely. ■

Proposition 2.20. *If $p > q \geq 1$ and $V_q(Z) < \infty$, then $\lim_{|\Pi|\to 0} V_p(Z : \Pi) = 0$.*

Proof. Let $\Pi \in \mathcal{P}(0,T)$, then

$$\begin{aligned} V_p^p(Z:\Pi) &= \sum_{t\in\Pi} d^p(Z(t), Z(t_-)) = \sum_{t\in\Pi} d^{p-q}(Z(t), Z(t_-))\, d^q(Z(t), Z(t_-)) \\ &\leq \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot \sum_{t\in\Pi} d^q(Z(t), Z(t_-)) \\ &\leq \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot V_q^q(Z:\Pi) \\ &\leq \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot V_q^q(Z). \end{aligned}$$

Thus, by the uniform continuity of $Z|_{[0,T]}$ we have

$$\limsup_{|\Pi|\to 0} V_p(Z:\Pi) \leq \limsup_{|\Pi|\to 0} \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot V_q^q(Z) = 0.$$

■

Corollary 2.21. *If $p < 2$, then $V_p\left(B|_{[0,T]}\right) = \infty$ a.s.*

Proof. Choose partitions, $\{\Pi_m\}$, of $[0,T]$ such that $\lim_{m\to\infty} Q_m = T$ a.s. where $Q_m := V_2^2(B:\Pi_m)$ and let $\Omega_0 := \{\lim_{m\to\infty} Q_m = T\}$ so that $P(\Omega_0) = 1$. If $V_p\left(B|_{[0,T]}(\omega)\right) < \infty$ for then by Proposition 2.20,

$$\lim_{m\to\infty} Q_m(\omega) = \lim_{m\to\infty} V_2^2(B(\omega):\Pi_m) = 0$$

and hence $\omega \notin \Omega_0$, i.e. $\left\{V_p\left(B|_{[0,T]}(\cdot)\right) < \infty\right\} \subset \Omega_0^c$. Therefore $\Omega_0 \subset \left\{V_p\left(B|_{[0,T]}(\cdot)\right) = \infty\right\}$ and hence

$$P\left(\left\{V_p\left(B|_{[0,T]}(\cdot)\right) = \infty\right\}\right) \geq P(\Omega) = 1.$$

■

Fact 2.22 *If $\{B_t\}_{t\geq 0}$ is a Brownian motion, then*

$$P(V_p(B) < \infty) = \begin{cases} 1 \text{ if } p > 2 \\ 0 \text{ if } p \leq 2 \end{cases}.$$

See for example [17, Exercise 1.14 on p. 36].

Corollary 2.23 (Roughness of Brownian Paths). *A Brownian motion, $\{B_t\}_{t\geq 0}$, is **not** almost surely α – Hölder continuous for any $\alpha > 1/2$.*

Proof. According to Proposition 2.19 we may choose partition, Π_m, such that $\text{mesh}(\Pi_m) \to 0$ and $Q_m \to T$ a.s. If B were α – Hölder continuous for some $\alpha > 1/2$, then

$$\begin{aligned} Q_m &= \sum_{t\in\Pi_m} (\Delta_t B)^2 \leq C \sum_{t\in\Pi_m} (\Delta t)^{2\alpha} \leq C \max\left([\Delta t]^{2\alpha-1}\right) \sum_{t\in\Pi_m} \Delta t \\ &\leq C\,[|\Pi_m|]^{2\alpha-1}\, T \to 0 \text{ as } m\to\infty \end{aligned}$$

which contradicts the fact that $Q_m \to T$ as $m \to \infty$. ■

2.3 The Bounded Variation Obstruction

Proposition 2.24. *Suppose that $Z(t)$ is a real continuous function such that $Z_0 = 0$ for simplicity. Define*

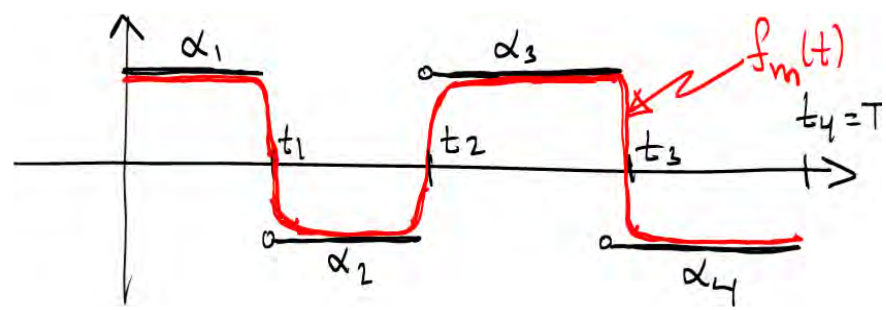
$$\int_0^T f(\tau)\, dZ(\tau) := -\int_0^T \dot{f}(\tau)\, Z(t)\, d\tau + f(t)\, Z(t)|_0^T$$

whenever f is a C^1 – function. If there exists, $C < \infty$ such that

$$\left|\int_0^T f(\tau)\, dZ(\tau)\right| \leq C \cdot \max_{0\leq\tau\leq T} |f(\tau)|, \qquad (2.8)$$

then $V_1(Z) < \infty$ (See Definition 2.2 above) and the best possible choice for C in Eq. (2.8) is $V_1(Z)$.

Proof. Given a partition, $\Pi := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$, $\{\alpha_k\}_{k=1}^n \subset \mathbb{R}$, and $f(t) := \alpha_1 1_{\{0\}} + \sum_{k=1}^n \alpha_k 1_{(t_{k-1},t_k]}$. Choose $f_m(t)$ in $C^1([0,T],\mathbb{R})$ "well approximating" $f(t)$ as in Figure 2.3. It then is fairly



easy to show,

$$\int_0^T \dot{f}_m(\tau) Z(t) d\tau \to \sum_{k=1}^{n-1} (\alpha_{k+1} - \alpha_k) Z(t_k)$$

and therefore,

$$\lim_{m\to\infty} \int_0^T f_m(t) dZ(t) = -\sum_{k=1}^{n-1} (\alpha_{k+1} - \alpha_k) Z(t_k) + \alpha_n Z(t_n) - \alpha_1 Z(t_0)$$
$$= \sum_{k=1}^{n} \alpha_k (Z(t_k) - Z(t_{k-1})).$$

Therefore we have,

$$\left|\sum_{k=1}^{n} \alpha_k (Z(t_k) - Z(t_{k-1}))\right| = \lim_{m\to\infty} \left|\int_0^T f_m(\tau) dZ(\tau)\right|$$
$$\le C \cdot \limsup_{m\to\infty} \max_{0\le\tau\le T} |f_m(\tau)| = C \max_k |\alpha_k|.$$

Taking $\alpha_k = \operatorname{sgn}(Z(t_k) - Z(t_{k-1}))$ for each k, then shows $\sum_{k=1}^n |Z(t_k) - Z(t_{k-1})| \le C$. Since this holds for any partition Π, it follows that $V_1(Z) \le C$.

If $V_1(Z) < \infty$, then

$$\int_0^T \dot{f}(\tau) Z(t) d\tau = -\int_0^T f(t) d\lambda_Z(t) + f(t) Z(t)|_0^T$$

where λ_Z is the Lebesgue Stieltjes measure associated to Z. From this identity and integration by parts for such finite variation functions, it follows that

$$\int_0^T f(t) dZ(t) = \int_0^T f(t) d\lambda_Z(t)$$

and

$$\left|\int_0^T f(t) dZ(t)\right| = \left|\int_0^T f(t) d\lambda_Z(t)\right| \le \int_0^T |f(t)| d\|\lambda_Z\|(t)$$
$$\le \max_{0\le\tau\le T} |f(\tau)| \cdot \|\lambda_Z\|([0,T]) = V_1(Z) \cdot \max_{0\le\tau\le T} |f(\tau)|$$

Therefore C can be taken to be $V_1(Z)$ in Eq. (2.8) and hence $V_1(Z)$ is the best possible constant to use in this equation. ■

Combining Fact 2.22 with Proposition 2.24 explains why we are going to have trouble defining $\int_0^t f_s dB_s$ when B is a Brownian motion. However, one might hope to use Young's integral in this setting.

Theorem 2.25 (L. C. Young 1936). *Suppose that $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} =: \theta > 1$. Then there exists a constant, $C(\theta) < \infty$ such that*

$$\left|\int_0^T f(t) dZ(t)\right| \le C(\theta) (\|f\|_\infty + V_q(f)) \cdot V_p(Z)$$

for all $f \in C^1$. Thus if $V_p(Z) < \infty$ the integral extends to those $f \in C([0,T])$ such that $V_q(f) < \infty$.

Unfortunately, Young's integral is still not sufficiently general to allow us to solve the typical SDE that we would like to consider. For example, consider the "simple" SDE,

$$\dot{y}(t) = B(t) \dot{B}(t) \text{ with } y(0) = 0.$$

The solution to this equation should be,

$$y(T) = \int_0^T B(t) dB(t)$$

which still does not make sense as a Young's integral when B is a Brownian motion because for any $p > 2$, $\frac{1}{p} + \frac{1}{p} =: \theta < 1$. For more on this point view see the very interesting work of Terry Lyons on "rough path analysis," [13].

2.4 Controls

Notation 2.26 (Controls) *Let*

$$\Delta = \{(s,t) : 0 \le s \le t \le T\}.$$

A ***control****, is a continuous function $\omega : \Delta \to [0,\infty)$ such that*

1. *$\omega(t,t) = 0$ for all $t \in [0,T]$,*
2. *ω is super-additive, i.e., for all $s \le t \le v$ we have*

$$\omega(s,t) + \omega(t,v) \le \omega(s,v). \tag{2.9}$$

Remark 2.27. If ω is a control then and $\omega(s,t)$ is increasing in t and decreasing in s for $(s,t) \in \Delta$. For example if $s \le \sigma \le t$, then $\omega(s,\sigma) + \omega(\sigma,t) \le \omega(s,t)$ and therefore, $\omega(\sigma,t) \le \omega(s,t)$. Similarly if $s \le t \le \tau$, then $\omega(s,t) + \omega(t,\tau) \le \omega(s,\tau)$ and therefore $\omega(s,t) \le \omega(s,\tau)$.

Lemma 2.28. *If ω is a control and $\varphi \in C([0,\infty) \to [0,\infty))$ such that $\varphi(0) = 0$ and φ is convex and increasing*[1]*, then $\varphi \circ \omega$ is also a control.*

[1] The assumption that φ is increasing is redundant here since we are assuming $\varphi'' \ge 0$ and we may deduce that $\varphi'(0) \ge 0$, it follows that $\varphi'(x) \ge 0$ for all x. This assertion also follows from Eq. (2.11).

Proof. We must show $\varphi \circ \omega$ is still superadditive. and this boils down to showing if $0 \le a, b, c$ with $a + b \le c$, then

$$\varphi(a) + \varphi(b) \le \varphi(c).$$

As φ is increasing, it suffices to show,

$$\varphi(a) + \varphi(b) \le \varphi(a+b). \tag{2.10}$$

Making use of the convexity of φ, we have,

$$\begin{aligned}\varphi(b) &= \varphi\left(\frac{a}{a+b}\cdot 0 + \frac{b}{a+b}(a+b)\right) \\ &\le \frac{a}{a+b}\varphi(0) + \frac{b}{a+b}\varphi(a+b) = \frac{b}{a+b}\varphi(a+b)\end{aligned}$$

and interchanging the roles of a and b gives,

$$\varphi(a) \le \frac{a}{a+b}\varphi(a+b). \tag{2.11}$$

.

Adding these last two inequalities then proves Eq. (2.10). ■

Example 2.29. Suppose that $u(t)$ is any increasing continuous function of t, then $\omega(s,t) := u(t) - u(s)$ is a control which is in fact additive, i.e.

$$\omega(s,t) + \omega(t,v) = \omega(s,v) \text{ for all } s \le t \le v.$$

So for example $\omega(s,t) = t - s$ is an additive control and for any $p > 1$, $\omega(s,t) = (t-s)^p$ or more generally, $\omega(s,t) = (u(t) - u(s))^p$ is a control.

Lemma 2.30. *Suppose that ω is a control, $p \in [1,\infty)$, and $Z \in C([0,T], E)$ is a function satisfying,*

$$d(Z_s, Z_t) \le \omega(s,t)^{1/p} \text{ for all } (s,t) \in \Delta,$$

then $V_p^p(Z) \le \omega(0,T) < \infty$. More generally,

$$\omega_{p,Z}(s,t) := V_p^p\left(Z|_{[s,t]}\right) \le \omega(s,t) \text{ for all } (s,t) \in \Delta.$$

Proof. Let $(s,t) \in \Delta$ and $\Pi \in \mathcal{P}([s,t])$, then using the superadditivity of ω we find

$$V_p^p\left(Z|_{[s,t]} : \Pi\right) = \sum_{t \in \Pi} d^p\left(Z_t, Z_{t_-}\right) \le \sum_{t \in \Pi} \omega\left(Z_t, Z_{t_-}\right) \le \omega(s,t).$$

Therefore,

$$\omega_{p,Z}(s,t) := V_p^p\left(Z|_{[s,t]}\right) = \sup_{\Pi \in \mathcal{P}([s,t])} V_p^p\left(Z|_{[s,t]} : \Pi\right) \le \omega(s,t).$$

■

Notation 2.31 *Given $o \in E$ and $p \in [1,\infty)$, let*

$$C_p([0,T], E) := \{Z \in C([0,T], E) : V_p(Z) < \infty\} \text{ and}$$

$$C_{0,p}([0,T], E) := \{Z \in C_p([0,T], E) : Z(0) = o\}.$$

Theorem 2.32. *Let $\rho : \Delta \to [0,\infty)$ be a function and define,*

$$\omega(s,t) := \omega_\rho(s,t) := \sup_{\Pi \in \mathcal{P}(s,t)} V_1(\rho : \Pi), \tag{2.12}$$

where for any $\Pi \in \mathcal{P}(s,t)$,

$$V_1(\rho : \Pi) = \sum_{t \in \Pi} \rho(t_-, t). \tag{2.13}$$

We now assume:

1. *ρ is continuous,*
2. *$\rho(t,t) = 0$ for all $t \in [0,T]$ (This condition is redundant since next condition would fail if it were violated.), and*
3. *$V_1(\rho) := \omega(0,T) := \sup_{\Pi \in \mathcal{P}(0,T)} V_1(\rho : \Pi) < \infty$.*

Under these assumptions, $\omega : \Delta \to [0,\infty)$ is a control.

We will give the proof of Theorem 2.32 after a Lemma 2.37 below.

Corollary 2.33 (The variation control). *Let $p \in [1,\infty)$ and suppose that $Z \in C_p([0,T], E)$. Then $\omega_{Z,p} : \Delta \to [0,\infty)$ defined in Eq. (2.3) is a control satisfying, $d(Z(s), Z(t)) \le \omega_{Z,p}(s,t)^{1/p}$ for all $(s,t) \in \Delta$.*

Proof. Apply Theorem 2.32 with $\rho(s,t) := d^p(Z(s), Z(t))$ and observe that with this definition, $\omega_{Z,p} = \omega_\rho$. ■

Lemma 2.34. *Let $\rho : \Delta \to [0,\infty)$ satisfy the hypothesis in Theorem 2.32, then $\omega = \omega_\rho$ (defined in Eq. (2.12)) is superadditive.*

Proof. If $0 \le u \le s \le v \le T$ and $\Pi_1 \in \mathcal{P}(u,s)$, $\Pi_2 \in \mathcal{P}(s,v)$, then $\Pi_1 \cup \Pi_2 \in \mathcal{P}(u,v)$. Thus we have,

$$V_1(\rho : \Pi_1) + V_1(\rho : \Pi_2) = V_1(\rho : \Pi_1 \cup \Pi_2) \le \omega(u,v).$$

Taking the supremum over all Π_1 and Π_2 then implies,

$$\omega(u,s) + \omega(s,v) \le \omega(u,v) \text{ for all } u \le s \le v,$$

i.e. ω is superadditive. ■

Lemma 2.35. *Let $Z \in C_p([0,T],E)$ for some $p \in [1,\infty)$ and let $\omega := \omega_{Z,p} : \Delta \to [0,\infty)$ defined in Eq. (2.3). Then ω is superadditive. Furthermore if $p = 1$, ω is additive, i.e. Equality holds in Eq. (2.9).*

Proof. The superadditivity of $\omega_{Z,p}$ follows from Lemma 2.34 and since $\omega_{Z,p}(s,t) = \omega_\rho(s,t)$ where $\rho(s,t) := d^p(Z(s), Z(t))$. In the case $p = 1$, it is easily seen using the triangle inequality that if $\Pi_1, \Pi_2 \in \mathcal{P}(s,t)$ and $\Pi_1 \subset \Pi_2$, then $V_1(X : \Pi_1) \le V_1(X, \Pi_2)$. Thus in computing the supremum of $V_1(X : \Pi)$ over all partition in $\mathcal{P}(s,t)$ it never hurts to add more points to a partition. Using this remark it is easy to show,

$$\begin{aligned}\omega(u,s) + \omega(s,v) &= \sup_{\Pi_1 \in \mathcal{P}(u,s), \Pi_2 \in \mathcal{P}(s,v)} [V_1(X : \Pi_1) + V_1(X : \Pi_2)] \\ &= \sup_{\Pi_1 \in \mathcal{P}(u,s), \Pi_2 \in \mathcal{P}(s,v)} V_1(X : \Pi_1 \cup \Pi_2) \\ &= \sup_{\Pi \in \mathcal{P}(u,v)} V_1(X : \Pi) = \omega(u,v)\end{aligned}$$

as desired. ■

Lemma 2.36. *Let $\rho : \Delta \to [0,\infty)$ and $\omega = \omega_\rho$ be as in Theorem 2.32. Further suppose $(a,b) \in \Delta$, $\Pi \in \mathcal{P}(a,b)$, and let*

$$\varepsilon := \omega(a,b) - V_1(\rho : \Pi) \ge 0.$$

Then for any $\Pi' \in \mathcal{P}(a,b)$ with $\Pi' \subset \Pi$, we have

$$\sum_{t \in \Pi'} [\omega(t_-, t) - V_1(\rho : \Pi \cap [t_-, t])] \le \varepsilon. \tag{2.14}$$

In particular, if $(\alpha, \beta) \in \Delta \cap \Pi^2$ then

$$\omega(\alpha,\beta) \le V_1(\rho : \Pi \cap [\alpha,\beta]) + \varepsilon. \tag{2.15}$$

Proof. Equation (2.14) is a simple consequence of the superadditivity of ω (Lemma 2.34) and the identity,

$$\sum_{t \in \Pi'} V_1(\rho : \Pi \cap [t_-, t]) = V_1(\rho : \Pi)$$

where $t_- := t_-(\Pi')$. Indeed, using these properties we find,

$$\begin{aligned}\sum_{t \in \Pi'} [\omega(t_-, t) - V_1(\rho : \Pi \cap [t_-, t])] &= \sum_{t \in \Pi'} \omega(t_-, t) - V_1(\rho : \Pi) \\ &\le \omega(a,b) - V_1(\rho : \Pi) = \varepsilon.\end{aligned}$$

■

Lemma 2.37. *Suppose that $\rho : \Delta \to [0,\infty)$ is a continuous function such that $\rho(t,t) = 0$ for all $t \in [0,T]$ and $\varepsilon > 0$ is given. Then there exists $\delta > 0$ such that, for every $\Pi \subset\subset [0,T]$ and $u \in [0,T]$ such that $\operatorname{dist}(u, \Pi) < \delta$ we have,*

$$|V_1(\rho : \Pi) - V_1(\rho : \Pi \cup \{u\})| < \varepsilon.$$

Proof. By the uniform continuity of ρ, there exists $\delta > 0$ such that $|\rho(s,t) - \rho(u,v)| < \varepsilon/2$ provided $|(s,t) - (u,v)| < \delta$. Suppose that $\Pi = \{t_0 < t_1 < \cdots < t_n\} \subset [0,T]$ and $u \in [0,T]$ such that $\operatorname{dist}(u,\Pi) < \delta$. There are now three case to consider, $u \in (t_0, t_n)$, $u < t_0$ and $u > t_1$. In the first case, suppose that $t_{i-1} < u < t_i$ and that (for the sake of definiteness) that $|t_i - u| < \delta$, then

$$\begin{aligned}|V_1(\rho : \Pi) - V_1(\rho : \Pi \cup \{u\})| &= |\rho(t_{i-1}, t_i) - \rho(t_{i-1}, u) - \rho(u, t_i)| \\ &\le |\rho(t_{i-1}, t_i) - \rho(t_{i-1}, u)| + |\rho(u, t_i) - \rho(t_i, t_i)| < \varepsilon.\end{aligned}$$

The second and third case are similar. For example if $u < t_0$, we will have,

$$|V_1(\rho : \Pi \cup \{u\}) - V_1(\rho : \Pi)| = \rho(u, t_0) = \rho(u, t_0) - \rho(t_0, t_0) < \varepsilon/2.$$

■

With these lemmas as preparation we are now ready to complete the proof of Theorem 2.32.

Proof. Proof of **Theorem 2.32.** Let $\omega(s,t) := \omega_\rho(s,t)$ be as in Theorem 2.32. It is clear by the definition of ω, the $\omega(t,t) = 0$ for all t and we have already seen in Lemma 2.34 that ω is superadditive. So to finish the proof we must show ω is continuous.

Using Remark 2.27, we know that $\omega(s,t)$ is increasing in t and decreasing in s and therefore $\omega(u+, v-) = \lim_{s \downarrow u, t \uparrow v} \omega(s,t)$ and $\omega(u-, v+) = \lim_{s \uparrow u, t \downarrow v} \omega(s,t)$ exists and satisfies,

$$\omega(u+, v-) \le \omega(u,v) \le \omega(u-, v+). \tag{2.16}$$

The main crux of the continuity proof is to show that the inequalities in Eq. (2.16) are all equalities.

1. Suppose that $\varepsilon > 0$ is given and $\delta > 0$ is chosen as in Lemma 2.37 and suppose that $u < s < t < v$ with $|s - u| < \delta$ and $|v - t| < \delta$. Further let $\Pi \in \mathcal{P}(u,v)$ be a partition of $[u,v]$, then according to Lemma 2.37,

$$\begin{aligned}V_1(\rho : \Pi) &\le V_1(\rho : \Pi \cup \{s,t\}) + 2\varepsilon \\ &= \rho(u,s) + \rho(t,v) + V_1(\rho : \Pi \cap [s,t] \cup \{s,t\}) + 2\varepsilon \\ &\le \rho(u,s) + \rho(t,v) + \omega(s,t) + 2\varepsilon.\end{aligned}$$

Letting $s \downarrow u$ and $t \uparrow v$ in this inequality shows,

$$V_1(\rho : \Pi) \le \omega(u+, v-) + 2\varepsilon$$

and then taking the supremum over $\Pi \in \mathcal{P}(u,v)$ and then letting $\varepsilon \downarrow 0$ shows $\omega(u,v) \le \omega(u+, v-)$. Combined this with the first inequality in Eq. (2.16) shows, $\omega(u+, v-) = \omega(u,v)$.

2. We will now show $\omega(u,v) = \omega(u-, v+)$ by showing $\omega(u-, v+) \le \omega(u,v)$. Let $\varepsilon > 0$ and $\delta > 0$ be as in Lemma 2.37 and suppose that $s < u$ and $t > v$ with $|u - s| < \delta$ and $|t - v| < \delta$. Let us now choose a partition $\Pi \in \mathcal{P}(s,t)$ such that

$$\omega(s,t) \le V_1(\rho : \Pi) + \varepsilon.$$

Then applying Lemma 2.37 gives,

$$\omega(s,t) \le V_1(\rho : \Pi_1) + 3\varepsilon$$

where $\Pi_1 = \Pi \cup \{u, v\}$. As above, let u_- and v_+ be the elements in Π_1 just before u and just after v respectively. An application of Lemma 2.36 then shows,

$$\begin{aligned}\omega(u-, v+) &\le \omega(u_-, v_+) \le V_1(\rho : \Pi_1 \cap [u_-, v_+]) + 3\varepsilon \\ &= V_1(\rho : \Pi_1 \cap [u, v]) + \rho(u_-, u) + \rho(v, v_+) + 3\varepsilon \\ &\le \omega(u,v) + 5\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary we may conclude $\omega(u-, v+) \le \omega(u,v)$ which completes the proof that $\omega(u-, v+) = \omega(u,v)$.

I now claim all the other limiting directions follow easily from what we have proved. For example,

$$\begin{aligned}\omega(u,v) \le \omega(u, v+) \le \omega(u-, v+) = \omega(u,v) &\implies \omega(u, v+) = \omega(u,v), \\ \omega(u,v) = \omega(u+, v-) \le \omega(u, v-) \le \omega(u,v) &\implies \omega(u, v-) = \omega(u,v),\end{aligned}$$

and similarly, $\omega(u\pm, v) = \omega(u,v)$. We also have,

$$\omega(u,v) = \omega(u+, v-) \le \liminf_{s \downarrow u,\ t \downarrow v} \omega(s,t) \le \limsup_{s \downarrow u,\ t \downarrow v} \omega(s,t) \le \omega(u-, v+) = \omega(u,v)$$

which shows $\omega(u+, v+) = \omega(u,v)$ and

$$\omega(u,v) = \omega(u+, v-) \le \liminf_{s \uparrow u,\ t \uparrow v} \omega(s,t) \le \liminf_{s \uparrow u,\ t \uparrow v} \omega(s,t) \le \omega(u-, v+) = \omega(u,v)$$

so that $\omega(u-, v-) = \omega(u,v)$. ■

Proposition 2.38 (See [6, Proposition 5.15 from p. 83.]). *Let (E,d) be a metric space, and let $x : [0,T] \to E$ be a continuous path. Then x is of finite p-variation if and only if there exists a continuous increasing (i.e. non – decreasing) function $h : [0,T] \to [0, V_p^p(Z)]$ and a $1/p$ – Hölder path $g : [0, V_p^p(Z)] \to E$ such that such that $x = g \circ h$. More explicitly we have,*

$$d(g(v), g(u)) \le |v - u|^{1/p} \text{ for all } u, v \in [0, V_p^p(Z)]. \tag{2.17}$$

Proof. Let $\omega(s,t) := \omega_{p,x}(s,t) = V_p^p\left(x|_{[s,t]}\right)$ be the control associated to x and define $h(t) := \omega(0,t)$. Observe that h is increasing and for $0 \le s \le t \le T$ that $h(s) + \omega(s,t) \le h(t)$, i.e.

$$\omega(s,t) \le h(t) - h(s) \text{ for all } 0 \le s \le t \le T.$$

Let $g : [0, h(T)] \to E$ be defined by $g(h(t)) := x(t)$. This is well defined since if $s \le t$ and $h(s) = h(t)$, then $\omega(s,t) = 0$ and hence $x|_{[s,t]}$ is constant and in particular $x(s) = x(t)$. Moreover it now follows for $s < t$ such that $u := h(s) < h(t) =: v$, that

$$\begin{aligned}d^p(g(v), g(u)) &= d^p(g(h(t)), g(h(s))) = d^p(x(t), x(s)) \\ &\le \omega(s,t) \le h(t) - h(s) = v - u\end{aligned}$$

from which Eq. (2.17) easily follows. ■

2.5 Banach Space Structures

This section needs more work and may be moved later.

To put a metric on Hölder spaces seems to require some extra structure on the metric space, E. What is of interest here is the case $E = G$ is a group with a left (right) invariant metric, d. In this case suppose that we consider p - variation paths, x and y starting at $e \in G$ in which case we define,

$$d_{p-var}(x,y) := \sup_{\Pi \in \mathcal{P}(0,T)} \left(\sum_{t \in \Pi} d^p(\Delta_t x, \Delta_t y) \right)^{1/p}$$

where $\Delta_t x := x_{t_-}^{-1} x_t$ for all $t \in \Pi$. The claim is that this should now be a complete metric space.

Lemma 2.39. *$(C_{0,p}(\Delta, T^{(n)}(V)), d_p)$ is a metric space.*

Proof. For each fixed partition D and each $1 \le i \le \lfloor p \rfloor$, we have

$$v_D^i(X) = \left(\sum_{\ell=1}^{r} \left| X^i_{t_{\ell-1} t_\ell} \right|^{p/i} \right)^{i/p}$$

is a semi-norm on $C_{0,p}(\Delta, T^{(n)}(V))$ and in particular satisfies the triangle inequality. Moreover,

$$v_D^i(X + Y) \le \sup_{D'} [v_D^i(X) + v_{D'}^i(Y)] \le \sup_{D'} v_{D'}^i(X) + \sup_{D'} v_{D'}^i(Y)$$

and therefore

$$\sup_D v_D^i(X+Y) \le \sup_D v_D^i(X) + \sup_D v_D^i(Y)$$

which shows $\sup_D v_D^i(X)$ still satisfies the triangle inequality. (i.e., the supremum of a family of semi-norms is a semi-norm). Thus we have

$$d_p(X) := \max_{1 \le i \le \lfloor p \rfloor} \sup_D v_D^i(X)$$

is also a semi-norm on $C_{0,p}(\Delta, T^{(n)}(V))$. Thus $d_p(X,Y) = d_p(X-Y)$ satisfies the triangle inequality. Moreover we have $d_p(X,Y) = 0$ implies that

$$\left|X_{st}^i - Y_{st}^i\right|^{p/i} = 0 \quad \forall \ \ 1 \le i \le \lfloor p \rfloor$$

and $(s,t) \in \Delta$, i.e., $X^i = Y^i$ for all $1 \le i \le \lfloor p \rfloor$ and we have verified $d_p(X,Y)$ is a metric. ∎

3

The Bounded Variation Theory

3.1 Integration Theory

Let $T \in (0,\infty)$ be fixed,

$$\mathcal{S} := \{(a,b] : 0 \le a \le b \le T\} \cup \{[0,b] \cap \mathbb{R} : 0 \le b \le T\}. \tag{3.1}$$

Further let $\mathcal{A}$ be the algebra generated by $\mathcal{S}$. Since $\mathcal{S}$ is an elementary set, $\mathcal{A}$ may be described as the collection of sets which are finite disjoint unions of subsets from $\mathcal{S}$. Given any function, $Z : [0,T] \to V$ with V being a vector define $\mu_Z : \mathcal{S} \to V$ via,

$$\mu_Z((a,b]) := Z_b - Z_a \text{ and } \mu_Z([0,b]) = Z_b - Z_0 \;\forall\; 0 \le a \le b \le T.$$

With this definition we are asserting that $\mu_Z(\{0\}) = 0$. Another common choice is to take $\mu_Z(\{0\}) = Z_0$ which would be implemented by taking $\mu_Z([0,b]) = Z_b$ instead of $Z_b - Z_0$.

Lemma 3.1. *μ_Z is finitely additive on $\mathcal{S}$ and hence extends to a finitely additive measure on $\mathcal{A}$.*

Proof. See Chapter **??** and in particular make the minor necessary modifications to Examples **??**, **??**, and Proposition **??**. ■

Let W be another vector space and $f : [0,T] \to \operatorname{End}(V,W)$ be an $\mathcal{A}$ – **simple function**, i.e. $f([0,T])$ is a finite set and $f^{-1}(\lambda) \in \mathcal{A}$ for all $\lambda \in \operatorname{End}(V,W)$. For such functions we define,

$$\int_{[0,T]} f(t)\,dZ(t) := \int_{[0,T]} f d\mu_Z = \sum_{\lambda \in \operatorname{End}(V,W)^\times} \lambda \mu_Z(f = \lambda) \in W. \tag{3.2}$$

The basic linearity properties of this integral are explained in Proposition **??**. For later purposes, it will be useful to have the following substitution formula at our disposal.

Theorem 3.2 (Substitution formula). *Suppose that f and Z are as above and $Y_t = \int_{[0,t]} f d\mu_Z \in W$. Further suppose that $g : \mathbb{R}_+ \to \operatorname{End}(W,U)$ is another $\mathcal{A}$ – simple function with finite support. Then*

$$\int_{[0,T]} g d\mu_Y = \int_{[0,T]} g f d\mu_Z.$$

Proof. By definition of these finitely additive integrals,

$$\begin{aligned}\mu_Y((a,b]) &= Y_b - Y_a = \int_{[0,b]} f d\mu_Z - \int_{[0,a]} f d\mu_Z \\ &= \int_{[0,T]} \left(1_{[0,b]} - 1_{[0,a]}\right) f d\mu_Z = \int_{[0,T]} 1_{(a,b]} f d\mu_Z.\end{aligned}$$

Therefore, it follows by the finite additivity of μ_Y and linearity $\int_{[0,T]} (\cdot)\, d\mu_Z$, that

$$\mu_Y(A) = \int_A f d\mu_Z = \int_{[0,T]} 1_A f d\mu_Z \text{ for all } A \in \mathcal{A}.$$

Therefore,

$$\begin{aligned}\int_{[0,T]} g d\mu_Y &= \sum_{\lambda \in \operatorname{End}(W,U)^\times} \lambda \mu_Y(g = \lambda) = \sum_{\lambda \in \operatorname{End}(W,U)^\times} \lambda \int_{[0,T]} 1_{\{g=\lambda\}} f d\mu_Z \\ &= \int_{[0,T]} \sum_{\lambda \in \operatorname{End}(W,U)^\times} 1_{\{g=\lambda\}} \lambda f d\mu_Z = \int_{[0,T]} g f d\mu_Z\end{aligned}$$

as desired. ■

Let us observe that

$$\left\| \int_{[0,T]} f(t)\, dZ_t \right\| \le \sum_{\lambda \in \operatorname{End}(V,W)} \|\lambda\| \, \|\mu_Z(f = \lambda)\|.$$

Let us now define,

$$\begin{aligned}\|\mu_Z\|((a,b]) &:= V_1\left(Z|_{[a,b]}\right) \\ &= \sup\left\{\sum_{j=1}^n \left\|Z_{t_j} - Z_{t_{j-1}}\right\| : a = t_0 < t_1 < \cdots < t_n = b \text{ and } n \in \mathbb{N}\right\}\end{aligned}$$

be the variation measure associated to μ_Z.

Lemma 3.3. *If $\|\mu_Z\|((0,T]) < \infty$, then $\|\mu_Z\|$ is a finitely additive measure on $\mathcal{S}$ and hence extends to a finitely additive measure on $\mathcal{A}$. Moreover for all $A \in \mathcal{A}$ we have,*

$$\|\mu_Z(A)\| \le \|\mu_Z\|(A). \tag{3.3}$$

Proof. The additivity on $\mathcal{S}$ was already verified in Lemma 2.34. Here is the proof again for sake of convenience.

Suppose that $\Pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$, $s \in (t_{l-1}, t_l)$ for some l, and $\Pi' := \Pi \cup \{s\}$. Then

$$\begin{aligned}\|\mu_Z\|^{\Pi}((a,b]) &:= \sum_{j=1}^{n} \left\|Z_{t_j} - Z_{t_{j-1}}\right\| \\ &= \sum_{j=1:j\neq l}^{n} \left\|Z_{t_j} - Z_{t_{j-1}}\right\| + \left\|Z_{t_l} - Z_s + Z_s - Z_{t_{l-1}}\right\| \\ &\leq \sum_{j=1:j\neq l}^{n} \left\|Z_{t_j} - Z_{t_{j-1}}\right\| + \|Z_{t_l} - Z_s\| + \left\|Z_s - Z_{t_{l-1}}\right\| \\ &= \|\mu_Z\|^{\Pi'}((a,b]) \leq \|\mu_Z\|((a,s]) + \|\mu_Z\|((s,b]).\end{aligned}$$

Hence it follows that

$$\|\mu_Z\|((a,b]) = \sup_{\Pi} \|\mu_Z\|^{\Pi}((a,b]) \leq \|\mu_Z\|((a,s]) + \|\mu_Z\|((s,b]).$$

Conversely if Π_1 is a partition of $(a,s]$ and Π_2 is a partition of $(s,b]$, then $\Pi := \Pi_1 \cup \Pi_2$ is a partition of $(a,b]$. Therefore,

$$\|\mu_Z\|^{\Pi_1}((a,s]) + \|\mu_Z\|^{\Pi_2}((s,b]) = \|\mu_Z\|^{\Pi}((a,b]) \leq \|\mu_Z\|((a,b])$$

and therefore,

$$\|\mu_Z\|((a,s]) + \|\mu_Z\|((s,b]) \leq \|\mu_Z\|((a,b]).$$

Lastly if $A \in \mathcal{A}$, then A is the disjoint union of intervals, J_i from $\mathcal{S}$ and we have,

$$\|\mu_Z(A)\| = \left\|\sum_i \mu_Z(J_i)\right\| \leq \sum_i \|\mu_Z(J_i)\| \leq \sum_i \|\mu_Z\|(J_i) = \|\mu_Z\|(A).$$

■

Corollary 3.4. *If Z has finite variation on $[0,T]$, then we have*

$$\left\|\int_{[0,T]} f(t)\, dZ_t\right\| \leq \int_{[0,T]} \|f(\lambda)\| \, \|\mu_Z\|(d\lambda) \leq \|f\|_{\infty} \cdot \|\mu_Z\|([0,T]).$$

Proof. Simply observe that $\|\mu_Z(A)\| \leq \|\mu_Z\|(A)$ for all $A \in \mathcal{A}_T$ and hence from Eq. (3.2) and the bound in Eq. (3.3) we have

$$\begin{aligned}\left\|\int_{[0,T]} f(t)\, dZ_t\right\| &\leq \sum_{\lambda \in \mathrm{End}(V,W)} \|\lambda\| \, \|\mu_Z(f = \lambda)\| \\ &\leq \sum_{\lambda \in \mathrm{End}(V,W)} \|\lambda\| \, \|\mu_Z\|(f = \lambda) = \int_{[0,T]} \|f(\lambda)\| \, \|\mu_Z\|(d\lambda) \\ &\leq \|f\|_{\infty} \cdot \|\mu_Z\|([0,T]).\end{aligned}$$

■

Notation 3.5 *In the future we will often write $\|dZ\|$ for $d\|\mu_Z\|$.*

Theorem 3.6. *If V and W are Banach spaces and $V_1(Z) = \|\mu_Z\|([0,T]) < \infty$, we may extend the integral, $\int_{[0,T]} f(t)\, dZ_t$, by continuity to all functions which are in the uniform closure of the $\mathcal{A}$ – simple functions. In fact we may extend the integral to $L^1(\|\mu_Z\|)$ – closure of the $\mathcal{A}$ – simple functions. In particular, if $f : [0,T] \to \mathrm{Hom}(V,W)$ is a continuous function,*

$$\int_{[0,T]} f(t)\, dZ(t) = \lim_{|\Pi| \to 0} \sum_{t \in \Pi} f(t_-)(Z(t) - Z(t_-)). \tag{3.4}$$

Proof. These results are elementary soft analysis except possibly for the last assertion for the statement in Eq. (3.4). To prove this, to any partition, $\Pi \in \mathcal{P}(0,T)$, let

$$f_{\Pi} := \sum_{\tau \in \Pi} f(t_-) 1_{(t_-,t]} + f(0) 1_{\{0\}}$$

in which case,

$$\sum_{t \in \Pi} f(t_-)(Z(t) - Z(t_-)) = \int_{[0,T]} f_{\Pi}(t)\, dZ(t).$$

This completes the proof since $f_{\Pi} \to f$ uniformly on $[0,T]$ as $|\Pi| \to 0$ by the uninform continuity of f. ■

Theorem 3.7 (Substitution formula II). *Let $Z : [0,T] \to V$ be a finite variation process, $f : [0,T] \to \mathrm{End}(V,W)$ and $g : [0,T] \to \mathrm{End}(W,U)$ be continuous maps and define,*

$$Y_t = \int_{[0,t]} f dZ \in W.$$

Then Y is a continuous finite variation process and the following substitution formula holds,

$$\int_{[0,T]} g dY = \int_{[0,T]} g f dZ. \tag{3.5}$$

In short, $dY = f dZ$.

Proof. First off observe that

$$\|Y_t - Y_s\| \leq \int_s^t \|f\| \, \|dZ\| =: \omega(s,t)$$

where the right side is a continuous control. This follows from the fact that $\|dZ\|$ is a continuous measure. Therefore

$$V_1(Y) \leq \int_0^T \|f\| \, \|dZ\| < \infty.$$

If $g = \lambda 1_{(a,b]}$ with $\lambda \in \mathrm{End}(W,U)$, then

$$\int_{[0,T]} g dY = \lambda (Y_b - Y_a) = \lambda \int_a^b f dZ = \int_a^b \lambda f dZ = \int_0^T g f dZ.$$

Thus Eq. (3.5) holds for all $\mathcal{A}$ – simple functions and hence also for all uniform limits of simple functions. In particular this includes all continuous $g : [0,T] \to \mathrm{End}(W,U)$. ■

Remark 3.8. If we keep the same hypothesis as in Theorem 3.7 but now take $Y_t := \int_t^T f dZ$ instead. In this case we have,

$$\int_{[0,T]} g dY = -\int_{[0,T]} g f dZ.$$

To prove this just observe that $Y_t = W_T - W_t$ where $W_t := \int_0^t f dZ$. It is now easy to see that

$$dY_t = d(-W_t) = -dW_t = -f dZ$$

and the claim follows.

3.2 The Fundamental Theorem of Calculus

As above, let V and W be Banach spaces and $0 \leq a < b \leq T$.

Proposition 3.9. *Suppose that $f : [a,b] \to V$ is a continuous function such that $\dot{f}(t)$ exists and is equal to zero for $t \in (a,b)$. Then f is constant.*

Proof. First Proof. For $\ell \in V^*$, we have $f_\ell := \ell \circ f : [a,b] \to \mathbb{R}$ with $\dot{f}_\ell(t) = 0$ for all $t \in (a,b)$. Therefore by the mean value theory, it follows that $f_\ell(t)$ is constant, i.e. $\ell(f(t) - f(a)) = 0$ for all $t \in [a,b]$. Since $\ell \in V^*$ is arbitrary, it follows from the Hahn – Banach theorem that $f(t) - f(a) = 0$, i.e. $f(t) = f(a)$ independent of t.

Second Proof (with out Hahn − Banach). Let $\varepsilon > 0$ and $\alpha \in (a,b)$ be given. (We will later let $\varepsilon \downarrow 0$.) By the definition of the derivative, for all $\tau \in (a,b)$ there exists $\delta_\tau > 0$ such that

$$\|f(t) - f(\tau)\| = \left\| f(t) - f(\tau) - \dot{f}(\tau)(t-\tau) \right\| \leq \varepsilon |t - \tau| \text{ if } |t-\tau| < \delta_\tau. \quad (3.6)$$

Let

$$A = \{t \in [\alpha, b] : \|f(t) - f(\alpha)\| \leq \varepsilon(t - \alpha)\} \quad (3.7)$$

and t_0 be the least upper bound for A. We will now use a standard argument which is sometimes referred to as **continuous induction** to show $t_0 = b$. Eq. (3.6) with $\tau = \alpha$ shows $t_0 > \alpha$ and a simple continuity argument shows $t_0 \in A$, i.e.

$$\|f(t_0) - f(\alpha)\| \leq \varepsilon(t_0 - \alpha). \quad (3.8)$$

For the sake of contradiction, suppose that $t_0 < b$. By Eqs. (3.6) and (3.8),

$$\begin{aligned} \|f(t) - f(\alpha)\| &\leq \|f(t) - f(t_0)\| + \|f(t_0) - f(\alpha)\| \\ &\leq \varepsilon(t_0 - \alpha) + \varepsilon(t - t_0) = \varepsilon(t - \alpha) \end{aligned}$$

for $0 \leq t - t_0 < \delta_{t_0}$ which violates the definition of t_0 being an upper bound. Thus we have shown $b \in A$ and hence

$$\|f(b) - f(\alpha)\| \leq \varepsilon(b - \alpha).$$

Since $\varepsilon > 0$ was arbitrary we may let $\varepsilon \downarrow 0$ in the last equation to conclude $f(b) = f(\alpha)$. Since $\alpha \in (a,b)$ was arbitrary it follows that $f(b) = f(\alpha)$ for all $\alpha \in (a,b]$ and then by continuity for all $\alpha \in [a,b]$, i.e. f is constant. ■

Theorem 3.10 (Fundamental Theorem of Calculus). *Suppose that $f \in C([a,b],V)$, Then*

1. $\frac{d}{dt}\int_a^t f(\tau)\, d\tau = f(t)$ *for all* $t \in (a,b)$.
2. *Now assume that $F \in C([a,b],V)$, F is continuously differentiable on (a,b) (i.e. $\dot{F}(t)$ exists and is continuous for $t \in (a,b)$) and $\dot{F}$ extends to a continuous function on $[a,b]$ which is still denoted by $\dot{F}$. Then*

$$\int_a^b \dot{F}(t)\, dt = F(b) - F(a). \quad (3.9)$$

Proof. Let $h > 0$ be a small number and consider

$$\begin{aligned} \left\| \int_a^{t+h} f(\tau) d\tau - \int_a^t f(\tau) d\tau - f(t) h \right\| &= \left\| \int_t^{t+h} (f(\tau) - f(t))\, d\tau \right\| \\ &\leq \int_t^{t+h} \|(f(\tau) - f(t))\|\, d\tau \leq h\varepsilon(h), \end{aligned}$$

where $\varepsilon(h) := \max_{\tau\in[t,t+h]} \|(f(\tau) - f(t))\|$. Combining this with a similar computation when $h < 0$ shows, for all $h \in \mathbb{R}$ sufficiently small, that

$$\left\| \int_a^{t+h} f(\tau)d\tau - \int_a^t f(\tau)d\tau - f(t)h \right\| \le |h|\varepsilon(h),$$

where now $\varepsilon(h) := \max_{\tau\in[t-|h|,t+|h|]} \|(f(\tau) - f(t))\|$. By continuity of f at t, $\varepsilon(h) \to 0$ and hence $\frac{d}{dt}\int_a^t f(\tau)\,d\tau$ exists and is equal to $f(t)$.

For the second item, set $G(t) := \int_a^t \dot{F}(\tau)\,d\tau - F(t)$. Then G is continuous and $\dot{G}(t) = 0$ for all $t \in (a, b)$ by item 1. An application of Proposition 3.9 shows G is a constant and in particular $G(b) = G(a)$, i.e. $\int_a^b \dot{F}(\tau)\,d\tau - F(b) = -F(a)$.

Alternative proof of Eq. (3.9). It is easy to show

$$\ell\left(\int_a^b \dot{F}(t)\,dt\right) = \int_a^b \ell\circ\dot{F}(t)\,dt = \int_a^b \frac{d}{dt}(\ell\circ F)(t)\,dt \text{ for all } \ell \in V^*.$$

Moreover by the real variable fundamental theorem of calculus we have,

$$\int_a^b \frac{d}{dt}(\ell\circ F)(t)\,dt = \ell\circ F(b) - \ell\circ F(a) \ \text{ for all } \ell \in V^*.$$

Combining the last two equations implies,

$$\ell\left(\int_a^b \dot{F}(t)\,dt - F(b) + F(a)\right) = 0 \text{ for all } \ell \in V^*.$$

Equation (3.9) now follows from these identities after an application of the Hahn – Banach theorem. ■

Corollary 3.11 (Mean Value Inequality). *Suppose that $f : [a, b] \to V$ is a continuous function such that $\dot{f}(t)$ exists for $t \in (a, b)$ and $\dot{f}$ extends to a continuous function on $[a, b]$. Then*

$$\|f(b) - f(a)\| \le \int_a^b \|\dot{f}(t)\|dt \le (b-a)\cdot\left\|\dot{f}\right\|_\infty. \tag{3.10}$$

Proof. By the fundamental theorem of calculus, $f(b) - f(a) = \int_a^b \dot{f}(t)dt$ and then (by the triangle inequality for integrals)

$$\begin{aligned}\|f(b) - f(a)\| &= \left\|\int_a^b \dot{f}(t)dt\right\| \le \int_a^b \|\dot{f}(t)\|dt \\ &\le \int_a^b \left\|\dot{f}\right\|_\infty dt = (b-a)\cdot\left\|\dot{f}\right\|_\infty.\end{aligned}$$

■

Corollary 3.12 (Change of Variable Formula). *Suppose that $f \in C([a, b], V)$ and $T : [c, d] \to (a, b)$ is a continuous function such that $T(s)$ is continuously differentiable for $s \in (c, d)$ and $T'(s)$ extends to a continuous function on $[c, d]$. Then*

$$\int_c^d f(T(s))\,T'(s)\,ds = \int_{T(c)}^{T(d)} f(t)\,dt.$$

Proof. For $t \in (a, b)$ define $F(t) := \int_{T(c)}^t f(\tau)\,d\tau$. Then $F \in C^1((a, b), V)$ and by the fundamental theorem of calculus and the chain rule,

$$\frac{d}{ds}F(T(s)) = F'(T(s))\,T'(s) = f(T(s))\,T'(s).$$

Integrating this equation on $s \in [c, d]$ and using the chain rule again gives

$$\int_c^d f(T(s))\,T'(s)\,ds = F(T(d)) - F(T(c)) = \int_{T(c)}^{T(d)} f(t)\,dt.$$

■

Exercise 3.1 (Fundamental Theorem of Calculus II). Prove the fundamental theorem of calculus in this context. That is; if $f : V \to W$ be a C^1 – function and $\{Z_t\}_{t\ge0}$ is a V – valued function of locally bounded variation, then for all $0 \le a < b \le T$,

$$f(Z_b) - f(Z_a) = \int_a^b f'(Z_\tau)\,dZ_\tau := \int_{[a,b]} f'(Z_\tau)\,dZ_\tau,$$

where $f'(z) \in \mathrm{End}(V, W)$ is defined by, $f'(z)\,v := \frac{d}{dt}|_0 f(z + tv)$. In particular it follows that $f(Z(t))$ has finite variation and

$$df(Z(t)) = f'(Z(t))\,dZ(t).$$

Solution to Exercise (3.1). Let $\Pi \in \mathcal{P}(0, T)$. Then by a telescoping series argument,

$$f(Z_b) - f(Z_a) = \sum_{t\in\Pi} \Delta_t f(Z_\cdot)$$

where

$$\begin{aligned}\Delta_t f(Z_\cdot) &= f(Z_t) - f\left(Z_{t_-}\right) = f\left(Z_{t_-} + \Delta_t Z\right) - f\left(Z_{t_-}\right) \\ &= \int_0^1 f'\left(Z_{t_-} + s\Delta_t Z\right)\Delta_t Z\ ds = f'\left(Z_{t_-}\right)\Delta_t Z + \varepsilon_t^\Pi \Delta_t Z\end{aligned}$$

and

$$\varepsilon_t^{\Pi} := \int_0^1 \left[f'\left(Z_{t_-} + s\Delta_t Z\right) - f'\left(Z_{t_-}\right)\right] \, ds.$$

Thus we have,

$$f(Z_b) - f(Z_a) = \sum_{t\in\Pi} f'\left(Z_{t_-}\right) \Delta_t Z + \delta_\Pi = \int_{[a,b]} f'\left(Z_{t_-}\right) dZ(t) + \delta_\Pi \quad (3.11)$$

where $\delta_\Pi := \sum_{t\in\Pi} \varepsilon_t^{\Pi} \Delta_t Z$. Since,

$$\begin{aligned}\|\delta_\Pi\| &\le \sum_{t\in\Pi} \left\|\varepsilon_t^{\Pi} \Delta_t Z\right\| \le \sum_{t\in\Pi} \left\|\varepsilon_t^{\Pi}\right\| \|\Delta_t Z\| \le \max_{t\in\Pi} \left\|\varepsilon_t^{\Pi}\right\| \cdot \sum_{t\in\Pi} \|\Delta_t Z\| \\ &\le \max_{t\in\Pi} \left\|\varepsilon_t^{\Pi}\right\| \cdot V_1(Z),\end{aligned}$$

and

$$\left\|\varepsilon_t^{\Pi}\right\| := \int_0^1 \left\|\left[f'\left(Z_{t_-} + s\Delta_t Z\right) - f'\left(Z_{t_-}\right)\right]\right\| ds.$$

Since $g(s,\tau,t) := \|[f'(Z_\tau + s(Z_t - Z_\tau)) - f'(Z_\tau)]\|$ is a continuous function in $s \in [0,1]$ and $\tau, t \in [0,T]$ with $g(s,t,t) = 0$ for all s and t, it follows by uniform continuity arguments that $g(s,\tau,t)$ is small whenever $|t-\tau|$ is small. Therefore, $\lim_{|\Pi|\to 0} \left\|\varepsilon_t^{\Pi}\right\| = 0$. Moreover, again by a uniform continuity argument, $f'\left(Z_{t_-}\right) \to f'(Z_t)$ uniformly as $|\Pi| \to 0$. Thus we may pass to the limit as $|\Pi| \to 0$ in Eq. (3.11) to complete the proof.

3.3 Calculus Bounds

For the exercises to follow we suppose that μ is a positive σ − finite measure on $\left([0,\infty), \mathcal{B}_{[0,\infty)}\right)$ such that μ is continuous, i.e. $\mu(\{s\}) = 0$ for all $s \in [0,\infty)$. We will further write,

$$\int_0^t f(s)\, d\mu(s) := \int_{[0,t]} f(s)\, d\mu(s) = \int_{(0,t)} f(s)\, d\mu(s),$$

wherein the second equality holds since μ is continuous. Although it is not necessary, you may use Exercise 3.1 with $Z_t := \mu([0,t])$ to solve the following problems.

Exercise 3.2. Show for all $0 \le a < b < \infty$ and $n \in \mathbb{N}$ that

$$h_n(b) := \int_{a\le s_1\le s_2\le\cdots\le s_n\le b} d\mu(s_1)\dots d\mu(s_n) = \frac{\mu([a,b])^n}{n!}. \quad (3.12)$$

Solution to Exercise (3.2). First solution. Let us observe that $h(t) := h_1(t) = \mu([a,t])$ and $h_n(t)$ satisfies the recursive relation,

$$h_{n+1}(t) := \int_a^t h_n(s)\, d\mu(s) = \int_a^t h_n(s)\, dh(s) \text{ for all } t \ge a.$$

Now let $H_n(t) := \frac{1}{n!} h^n(t)$, by an application of Exercise 3.1 with $f(x) = x^{n+1}/(n+1)!$ implies,

$$H_{n+1}(t) = H_{n+1}(t) - H_{n+1}(a) = \int_a^t f'(h(\tau))\, dh(\tau) = \int_a^t H_n(\tau)\, dh(\tau)$$

and therefore it follows that $H_n(t) = h_n(t)$ for all $t \ge a$ and $n \in \mathbb{N}$.

Second solution. If $i \ne j$, it follows by Fubini's theorem that

$$\begin{aligned}&\mu^{\otimes n}\left(\{(s_1,\dots,s_n) \in [a,b]^n : s_i = s_j\}\right) \\ &= \mu([a,b])^{n-2} \cdot \int_{[a,b]^2} 1_{s_i=s_j} d\mu(s_i)\, d\mu(s_j) \\ &= \mu([a,b])^{n-2} \cdot \int_{[a,b]} \mu(\{s_j\})\, d\mu(s_j) = 0.\end{aligned}$$

From this observation it follows that

$$1_{[a,b]^n}(s_1,\dots,s_n) = \sum_{\sigma\in\mathcal{S}_n} 1_{a\le s_{\sigma 1}\le s_{\sigma 2}\le\cdots\le s_{\sigma n}\le b} - \mu^{\otimes n} - \text{a.e.},$$

where σ ranges over the permutations, $\mathcal{S}_n$, of $\{1,2,\dots,n\}$. Integrating this equation relative with respect to $\mu^{\otimes n}$ and then using Fubini's theorem gives,

$$\begin{aligned}\mu([a,b])^n = \mu^{\otimes n}([a,b]^n) &= \sum_{\sigma\in\mathcal{S}_n} \int 1_{a\le s_{\sigma 1}\le s_{\sigma 2}\le\cdots\le s_{\sigma n}\le b} d\mu^{\otimes n}(\mathbf{s}) \\ &= \sum_{\sigma\in\mathcal{S}_n} \int 1_{a\le s_{\sigma 1}\le s_{\sigma 2}\le\cdots\le s_{\sigma n}\le b} d\mu(s_1)\dots d\mu(s_n) \\ &= \sum_{\sigma\in\mathcal{S}_n} \int_{a\le s_1\le s_2\le\cdots\le s_n\le b} d\mu(s_1)\dots d\mu(s_n) \\ &= n! \int_{a\le s_1\le s_2\le\cdots\le s_n\le b} d\mu(s_1)\dots d\mu(s_n).\end{aligned}$$

Exercise 3.3 (Gronwall's Lemma). If $\varepsilon(t)$ and $f(t)$ are continuous non-negative functions such that

$$f(t) \le \varepsilon(t) + \int_0^t f(\tau)\, d\mu(\tau), \quad (3.13)$$

then

$$f(t) \le \varepsilon(t) + \int_0^t e^{\mu([\tau,t])}\varepsilon(\tau)\, d\mu(\tau). \tag{3.14}$$

If we further assume that ε is increasing, then

$$f(t) \le \varepsilon(t)\, e^{\mu([0,t])}. \tag{3.15}$$

Solution to Exercise (3.3). Feeding Eq. (3.13) back into itself implies

$$\begin{aligned}
f(t) &\le \varepsilon(t) + \int_0^t \left[\varepsilon(\tau) + \int_0^\tau f(s)\, d\mu(s)\right] d\mu(\tau) \\
&= \varepsilon(t) + \int_0^t \varepsilon(s_1)\, d\mu(s_1) + \int_{0\le s_2\le s_1\le t} f(s_2)\, d\mu(s_1)\, d\mu(s_2) \\
&\le \varepsilon(t) + \int_0^t \varepsilon(s_1)\, d\mu(s_1) + \int_{0\le s_2\le s_1\le t} \left[\varepsilon(s_2) + \int_0^{s_2} f(s_3)\, d\mu(s_3)\right] d\mu(s_1)\, d\mu(s_2) \\
&= \varepsilon(t) + \int_0^t \varepsilon(s_1)\, d\mu(s_1) + \int_{0\le s_2\le s_1\le t} \varepsilon(s_2)\, d\mu(s_1)\, d\mu(s_2) \\
&\quad + \int_{0\le s_3\le s_2\le s_1\le t} f(s_3)\, d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3).
\end{aligned}$$

Continuing in this manner inductively shows,

$$f(t) \le \varepsilon(t) + \sum_{k=1}^N \int_{0\le s_k\le\dots\le s_2\le s_1\le t} \varepsilon(s_k)\, d\mu(s_1)\dots d\mu(s_k) + R_N(t) \tag{3.16}$$

where, using Exercise 3.2,

$$\begin{aligned}
R_N(t) &:= \int_{0\le s_{k+1}\le\dots\le s_2\le s_1\le t} f(s_{k+1})\, d\mu(s_1)\dots d\mu(s_k)\, d\mu(s_{k+1}) \\
&\le \max_{0\le s\le t} f(t)\cdot \frac{\mu([0,t])^{N+1}}{(N+1)!} \to 0 \text{ as } N\to\infty.
\end{aligned}$$

So passing to the limit in Eq. (3.16) and again making use of Exercise 3.2 shows,

$$\begin{aligned}
f(t) &\le \varepsilon(t) + \sum_{k=1}^\infty \int_{0\le s_k\le\dots\le s_2\le s_1\le t} \varepsilon(s_k)\, d\mu(s_1)\dots d\mu(s_k) \qquad (3.17)\\
&= \varepsilon(t) + \sum_{k=1}^\infty \int_0^t \varepsilon(s_k)\frac{\mu([s_k,t])^{k-1}}{(k-1)!} d\mu(s_k) \\
&= \varepsilon(t) + \int_0^t \varepsilon(\tau)\cdot \sum_{k=1}^\infty \frac{\mu([\tau,t])^{k-1}}{(k-1)!} d\mu(\tau) \\
&= \varepsilon(t) + \int_0^t e^{\mu([\tau,t])}\varepsilon(\tau)\, d\mu(\tau).
\end{aligned}$$

If we further assume that ε is increasing, then from Eq. (3.17) and Exercise 3.2 we have

$$\begin{aligned}
f(t) &\le \varepsilon(t) + \varepsilon(t)\sum_{k=1}^\infty \int_{0\le s_k\le\dots\le s_2\le s_1\le t} d\mu(s_1)\dots d\mu(s_k) \\
&= \varepsilon(t) + \varepsilon(t)\sum_{k=1}^\infty \frac{\mu([0,t])^k}{k!} = \varepsilon(t)\, e^{\mu([0,t])}.
\end{aligned}$$

Alternatively if we let $Z_t := \mu([0,t])$, then

$$\begin{aligned}
\int_0^t e^{\mu([\tau,t])} d\mu(\tau) &= \int_0^t e^{Z_t - Z_\tau} dZ_\tau = \int_0^t d_\tau\left(-e^{Z_t-Z_\tau}\right) \\
&= \left(-e^{Z_t-Z_\tau}\right)_0^t = e^{Z_t} - 1.
\end{aligned}$$

Therefore,

$$f(t) \le \varepsilon(t) + \varepsilon(t)\left(e^{Z_t}-1\right) = \varepsilon(t)\, e^{Z_t}.$$

Exercise 3.4. Suppose that $\{\varepsilon_n(t)\}_{n=0}^\infty$ is a sequence of non-negative continuous functions such that

$$\varepsilon_{n+1}(t) \le \int_0^t \varepsilon_n(\tau)\, d\mu(\tau) \text{ for all } n\ge 0 \tag{3.18}$$

and $\delta(t) = \max_{0\le\tau\le t}\varepsilon_0(\tau)$. Show

$$\varepsilon_n(t) \le \delta(t)\frac{\mu([0,t])^n}{n!} \text{ for all } n\ge 0.$$

Solution to Exercise (3.4). By iteration of Eq. (3.18) we find,

$$\begin{aligned}
\varepsilon_1(t) &\le \int_0^t \varepsilon_0(\tau)\, d\mu(\tau) \le \delta(t)\int_{0\le s_1\le t} d\mu(s_1), \\
\varepsilon_2(t) &\le \int_0^t \varepsilon_1(s_2)\, d\mu(s_2) \le \delta(t)\int_0^t \left[\int_{0\le s_1\le t} d\mu(s_1)\right] d\mu(s_2) \\
&= \delta(t)\int_{0\le s_2\le s_1\le t} d\mu(s_1)\, d\mu(s_2), \\
&\vdots \\
\varepsilon_n(t) &\le \delta(t)\int_{0\le s_n\le\dots\le s_1\le t} d\mu(s_1)\dots d\mu(s_n).
\end{aligned}$$

The result now follows directly from Exercise 3.2.

3.4 Bounded Variation Ordinary Differential Equations

In this section we begin by reviewing some of the basic theory of ordinary differential equations – O.D.E.s for short. Throughout this chapter we will let X and Y be Banach spaces, $U \subset_o Y$ an open subset of Y, and $y_0 \in U$, $x : [0,T] \to X$ is a continuous process of bounded variation, and $F : [0,T] \times U \to \operatorname{End}(X,Y)$ is a continuous function. (We will make further assumptions on F as we need them.) Our goal here is to investigate the "ordinary differential equation,"

$$\dot{y}(t) = F(t, y(t))\,\dot{x}(t) \text{ with } y(0) = y_0 \in U. \tag{3.19}$$

Since x is only of bounded variation, to make sense of this equation we will interpret it in its integrated form,

$$y(t) = y_0 + \int_0^t F(\tau, y(\tau))\,dx(\tau). \tag{3.20}$$

Proposition 3.13 (Continuous dependence on the data). *Suppose that $G : [0,T] \times U \to \operatorname{End}(X,Y)$ is another continuous function, $z : [0,T] \to X$ is another continuous function with bounded variation, and $w : [0,T] \to U$ satisfies the differential equation,*

$$w(t) = w_0 + \int_0^t G(\tau, w(\tau))\,dz(\tau) \tag{3.21}$$

for some $w_0 \in U$. Further assume there exists a continuous function, $K(t) \ge 0$ such that F satisfies the ***Lipschitz condition,***

$$\|F(t,y) - F(t,w)\| \le K(t)\,\|y - w\| \text{ for all } 0 \le t \le T \text{ and } y, w \in U. \tag{3.22}$$

Then

$$\|y(t) - w(t)\| \le \varepsilon(t) \exp\left(\int_0^t K(\tau)\,\|dx(\tau)\|\right). \tag{3.23}$$

where

$$\varepsilon(t) := \|y_0 - w_0\| + \int_0^t \|F(\tau, w(\tau)) - G(\tau, w(\tau))\|\,\|dx(\tau)\| + \int_0^t \|G(\tau, w(\tau))\|\,\|d(x-z)(\tau)\| \tag{3.24}$$

Proof. Let $\delta(t) := y(t) - w(t)$, so that $y = w + \delta$. We then have,

$$\begin{aligned}\delta(t) &= y_0 - w_0 + \int_0^t F(\tau, y(\tau))\,dx(\tau) - \int_0^t G(\tau, w(\tau))\,dz(\tau)\\ &= y_0 - w_0 + \int_0^t F(\tau, w(\tau) + \delta(\tau))\,dx(\tau) - \int_0^t G(\tau, w(\tau))\,dz(\tau)\\ &= y_0 - w_0 + \int_0^t [F(\tau, w(\tau)) - G(\tau, w(\tau))]\,dx(\tau) + \int_0^t G(\tau, w(\tau))\,d(x-z)(\tau)\\ &+ \int_0^t [F(\tau, w(\tau) + \delta(\tau)) - F(\tau, w(\tau))]\,dx(\tau).\end{aligned}$$

Crashing through this identity with norms shows,

$$\|\delta(t)\| \le \varepsilon(t) + \int_0^t K(\tau)\,\|\delta(\tau)\|\,\|dx(\tau)\|$$

where $\varepsilon(t)$ is given in Eq. (3.24). The estimate in Eq. (3.23) is now a consequence of this inequality and Exercise 3.3 with $d\mu(\tau) := K(\tau)\,\|dx(\tau)\|$. ∎

Corollary 3.14 (Uniquness of solutions). *If F satisfies the Lipschitz hypothesis in Eq. (3.22), then there is at most one solution to the ODE in Eq. (3.20).*

Proof. Simply apply Proposition 3.13 with $F = G$, $y_0 = w_0$, and $x = z$. In this case $\varepsilon \equiv 0$ and the result follows. ∎

Proposition 3.15 (An apriori growth bound). *Suppose that $U = Y$, $T = \infty$, and there are continuous functions, $a(t) \ge 0$ and $b(t) \ge 0$ such that*

$$\|F(t,y)\| \le a(t) + b(t)\,\|y\| \text{ for all } t \ge 0 \text{ and } y \in Y.$$

Then

$$\|y(t)\| \le \left(\|y_0\| + \int_0^t a(\tau)\,d\nu(\tau)\right) \exp\left(\int_0^t b(\tau)\,d\nu(\tau)\right), \text{ where} \tag{3.25}$$

$$\nu(t) := \omega_{x,1}(0,t) = \|s\|_{1\text{-}Var}(t). \tag{3.26}$$

Proof. From Eq. (3.20) we have,

$$\begin{aligned}\|y(t)\| &\le \|y_0\| + \int_0^t \|F(\tau, y(\tau))\|\,d\nu(\tau)\\ &\le \|y_0\| + \int_0^t (a(\tau) + b(\tau)\,\|y(\tau)\|)\,d\nu(\tau)\\ &= \varepsilon(t) + \int_0^t \|y(\tau)\|\,d\mu(\tau)\end{aligned}$$

where

$$\varepsilon(t) := \|y_0\| + \int_0^t a(\tau)\, d\nu(\tau) \text{ and } d\mu(\tau) := b(\tau)\, d\nu(\tau).$$

Hence we may apply Exercise 3.3 to learn $\|y(t)\| \le \varepsilon(t) e^{\mu([0,t])}$ which is the same as Eq. (3.25). ■

Theorem 3.16 (Global Existence). *Let us now suppose $U = X$ and F satisfies the Lipschitz hypothesis in Eq. (3.22). Then there is a unique solution, $y(t)$ to the ODE in Eq. (3.20).*

Proof. We will use the standard method of Picard iterates. Namely let $y_0(t) \in W$ be **any** continuous function and then define $y_n(t)$ inductively by,

$$y_{n+1}(t) := y_0 + \int_0^t F(\tau, y_n(\tau))\, dx(\tau). \tag{3.27}$$

Then from our assumptions and the definition of $y_n(t)$, we find for $n \ge 1$ that

$$\begin{aligned}\|y_{n+1}(t) - y_n(t)\| &= \left\|\int_0^t F(\tau, y_n(\tau))\, dx(\tau) - \int_0^t F(\tau, y_{n-1}(\tau))\, dx(\tau)\right\| \\ &\le \int_0^t \|F(\tau, y_n(\tau)) - F(\tau, y_{n-1}(\tau))\| \|dx(\tau)\| \\ &\le \int_0^t K(\tau) \|y_n(\tau) - y_{n-1}(\tau)\| \|dx(\tau)\|.\end{aligned}$$

Since,

$$\begin{aligned}\|y_1(t) - y_0(t)\| &= \left\|y_0 + \int_0^t F(\tau, y_0(\tau))\, dx(\tau) - y_0(t)\right\| \\ &\le \max_{0\le\tau\le t} \|y_0(\tau) - y_0\| + \int_0^t \|F(\tau, y_0)\| \|dx(\tau)\| =: \delta(t),\end{aligned}$$

it follows by an application of Exercise 3.4 with

$$\varepsilon_n(t) := \|y_{n+1}(t) - y_n(t)\|$$

that

$$\|y_{n+1}(t) - y_n(t)\| \le \delta(t) \cdot \left(\int_0^t K(\tau) \|dx(\tau)\|\right)^n / n!. \tag{3.28}$$

Since the right side of this equation is increasing in t, we may conclude by summing Eq. (3.28) that

$$\sum_{n=0}^{\infty} \sup_{0\le t\le T} \|y_{n+1}(t) - y_n(t)\| \le \delta(T) e^{\int_0^T K(\tau)\|dx(\tau)\|} < \infty.$$

Therefore, it follows that $y_n(t)$ is uniformly convergent on compact subsets of $[0,\infty)$ and therefore $y(t) := \lim_{n\to\infty} y_n(t)$ exists and is a continuous function. Moreover, we may now pass to the limit in Eq. (3.27) to learn this function y satisfies Eq. (3.20). Indeed,

$$\begin{aligned}&\left\|\int_0^t F(\tau, y_n(\tau))\, dx(\tau) - \int_0^t F(\tau, y(\tau))\, dx(\tau)\right\| \\ &\quad\le \int_0^t \|F(\tau, y_n(\tau)) - F(\tau, y(\tau))\| \|dx(\tau)\| \\ &\quad\le \int_0^t K(\tau) \|y_n(\tau) - y(\tau)\| \|dx(\tau)\| \\ &\quad\le \sup_{0\le\tau\le t} \|y_n(\tau) - y(\tau)\| \cdot \int_0^t K(\tau) \|dx(\tau)\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

■

Remark 3.17 (Independence of initial guess). In the above proof, we were allowed to choose $y_0(t)$ as we pleased. In all cases we ended up with a solution to the ODE which we already knew to be unique if it existed. Therefore all initial guesses give rise to the same solution. This can also be see directly. Indeed, if $z_0(t)$ is another continuous path in W and $z_n(t)$ is defined inductively by,

$$z_{n+1}(t) := y_0 + \int_0^t F(\tau, z_n(\tau))\, dx(\tau) \text{ for } n \ge 0.$$

Then

$$z_{n+1}(t) - y_{n+1}(t) = \int_0^t [F(\tau, z_n(\tau)) - F(\tau, y_n(\tau))]\, dx(\tau)$$

and therefore,

$$\begin{aligned}\|z_{n+1}(t) - y_{n+1}(t)\| &\le \int_0^t \|F(\tau, z_n(\tau)) - F(\tau, y_n(\tau))\| \|dx(\tau)\| \\ &\le \int_0^t K(\tau) \|z_n(\tau) - y_n(\tau)\| \|dx(\tau)\|.\end{aligned}$$

Thus it follows from Exercise 3.4 that

$$\|z_n(t) - y_n(t)\| \le \frac{1}{n!}\left(\int_0^t K(\tau) \|dx(\tau)\|\right)^n \cdot \max_{0\le\tau\le t} \|z_0(\tau) - y_0(\tau)\| \to 0 \text{ as } n \to \infty.$$

3.5 Some Linear ODE Results

In this section we wish to consider linear ODE of the form,

$$y(t) = \int_0^t dx(\tau)\, y(\tau) + f(t) \tag{3.29}$$

where $x(t) \in \mathrm{End}(W)$ and $f(t) \in W$ are finite variation paths. To put this in the form considered above, let $V := \mathrm{End}(W)$ and define, for $y \in W$, $F(y) : V \times W \to W$ by,

$$F(y)(x, f) := xy + f \text{ for all } (x, f) \in V \times W = \mathrm{End}(W) \times W.$$

Then the above equation may be written as,

$$y(t) = f(0) + \int_0^t F(y(\tau))\, d(x, f)(\tau).$$

Notice that

$$\|[F(y) - F(y')](x, f)\| = \|x(y - y')\| \le \|x\| \cdot \|y - y'\|$$

and therefore,

$$\|F(y) - F(y')\| \le \|y - y'\|$$

where we use any reasonable norm on $V \times W$, for example $\|(x, w)\| := \|x\| + \|w\|$ or $\|(x, w)\| := \max(\|x\|, \|w\|)$. Thus the theory we have developed above guarantees that Eq. (3.29) has a unique solution which we can construct via the method of Picard iterates.

Theorem 3.18. *The unique solution to Eq. (3.29) is given by*

$$y(t) = f(t) + \sum_{n=1}^{\infty} \int_{0 \le \tau_1 \le \dots \le \tau_n \le t} dx(\tau_n)\, dx(\tau_{n-1})\, dx(\tau_{n-2}) \dots dx(\tau_1)\, f(\tau_1).$$

More generally if $0 \le s \le t \le T$, then the unique solution to

$$y(t) = \int_s^t dx(\tau)\, y(\tau) + f(t) \text{ for } s \le t \le T \tag{3.30}$$

is given by

$$y(t) = f(t) + \sum_{n=1}^{\infty} \int_{s \le \tau_1 \le \dots \le \tau_n \le t} dx(\tau_n)\, dx(\tau_{n-1})\, dx(\tau_{n-2}) \dots dx(\tau_1)\, f(\tau_1). \tag{3.31}$$

Proof. Let us first find the formula for $y(t)$. To this end, let

$$(Ay)(t) := \int_s^t dx(\tau)\, y(\tau).$$

Then Eq. (3.29) may be written as $y - Ay = f$ or equivalently as,

$$(I - A)\, y = f.$$

Thus the solution to this equation should be given by,

$$y = (I - A)^{-1} f = \sum_{n=0}^{\infty} A^n f. \tag{3.32}$$

But

$$\begin{aligned}
(A^n f)(t) &= \int_s^t dx(\tau_n) \left(A^{n-1} f\right)(\tau_n) = \int_s^t dx(\tau_n) \int_s^{\tau_n} dx(\tau_{n-1}) \left(A^{n-2} f\right)(\tau_{n-1}) \\
&\vdots \\
&= \int_s^t dx(\tau_n) \int_s^{\tau_n} dx(\tau_{n-1}) \int_s^{\tau_{n-1}} dx(\tau_{n-2}) \dots \int_s^{\tau_1} dx(\tau_1)\, f(\tau_1) \\
&= \int_{s \le \tau_1 \le \dots \le \tau_n \le t} dx(\tau_n)\, dx(\tau_{n-1})\, dx(\tau_{n-2}) \dots dx(\tau_1)\, f(\tau_1) \qquad (3.33)
\end{aligned}$$

and therefore, Eq. (3.31) now follows from Eq. (3.32) and (3.33).

For those not happy with this argument one may use Picard iterates instead. So we begin by setting $y_0(t) = f(t)$ and then define $y_n(t)$ inductively by,

$$\begin{aligned}
y_{n+1}(t) &= f(s) + \int_s^t F(y_n(\tau))\, d(x, f)(\tau) \\
&= f(s) + \int_s^t [dx(\tau)\, y_n(\tau) + df(\tau)] \\
&= \int_s^t dx(\tau)\, y_n(\tau) + f(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_1(t) &= \int_s^t dx(\tau)\, f(\tau) + f(t) \\
y_2(t) &= \int_s^t dx(\tau_2)\, y_1(\tau_2) + f(t) \\
&= \int_s^t dx(\tau_2) \left[\int_s^{\tau_2} dx(\tau_1)\, f(\tau_1) + f(\tau_2)\right] + f(t) \\
&= \int_{s \le \tau_1 \le \tau_2 \le t} dx(\tau_2)\, dx(\tau_1)\, f(\tau_1) + \int_s^t dx(\tau_2)\, f(\tau_2) + f(t)
\end{aligned}$$

and likewise,

$$y_3(t) = \int_{s\le\tau_1\le\tau_2\le\tau_3\le t} dx(\tau_3)\,dx(\tau_2)\,dx(\tau_1)\,f(\tau_1) + \int_{s\le\tau_1\le\tau_2\le t} dx(\tau_2)\,dx(\tau_1)\,f(\tau_1) + \int_s^t dx(\tau_2)\,f(\tau_2) + f(t).$$

So by induction it follows that

$$y_n(t) = \sum_{k=1}^{n} \int_{s\le\tau_1\le\dots\le\tau_k\le t} dx(\tau_k)\dots dx(\tau_1)\,f(\tau_1) + f(t).$$

Letting $n\to\infty$ making use of the fact that

$$\left\| \int_{s\le\tau_1\le\dots\le\tau_n\le t} dx(\tau_n)\dots dx(\tau_1)\right\| \le \int_{s\le\tau_1\le\dots\le\tau_n\le t} \|dx(\tau_n)\|\dots\|dx(\tau_1)\| = \frac{1}{n!}\left(\int_s^t \|dx\|\right)^n, \tag{3.34}$$

we find as before, that

$$y(t) = \lim_{n\to\infty} y_n(t) = f(t) + \sum_{k=1}^{\infty}\int_{s\le\tau_1\le\dots\le\tau_k\le t} dx(\tau_k)\dots dx(\tau_1)\,f(\tau_1).$$

■

Definition 3.19. *For $0\le s\le t\le T$, let $T_0(t,s) := I$,*

$$T_n^x(t,s) = \int_{s\le\tau_1\le\dots\le\tau_n\le t} dx(\tau_n)\,dx(\tau_{n-1})\,dx(\tau_{n-2})\dots dx(\tau_1)\,,\ and \tag{3.35}$$

$$T^x(t,s) = \sum_{n=0}^{\infty} T_n(t,s) = I + \sum_{n=1}^{\infty}\int_{s\le\tau_1\le\dots\le\tau_n\le t} dx(\tau_n)\,dx(\tau_{n-1})\,dx(\tau_{n-2})\dots dx(\tau_1)\,. \tag{3.36}$$

Example 3.20. Suppose that $x(t) = tA$ where $A\in \mathrm{End}(V)$, then

$$\int_{s\le\tau_1\le\dots\le\tau_n\le t} dx(\tau_n)\,dx(\tau_{n-1})\,dx(\tau_{n-2})\dots dx(\tau_1) = A^n\int_{s\le\tau_1\le\dots\le\tau_n\le t} d\tau_n d\tau_{n-1} d\tau_{n-2}\dots d\tau_1 = \frac{(t-s)^n}{n!}A^n$$

and therefore we may conclude in this case that $T^x(t,s) = e^{(t-s)A}$ where

$$e^{tA} := \sum_{n=0}^{\infty}\frac{t^n}{n!}A^n.$$

Theorem 3.21 (Duhamel's principle I). *As a function of $t\in[s,T]$ or $s\in[0,t]$, $T(t,s)$ is of bounded variation and $T(t,s) := T^x(t,s)$ satisfies the ordinary differential equations,*

$$T(dt,s) = dx(t)\,T(t,s)\ \ \textit{with}\ T(s,s) = I\quad (\textit{in}\ t\ge s), \tag{3.37}$$

and

$$T(t,s) = -T(t,s)\,dx(s)\ \ \textit{with}\ T(t,t) = I\ \ (\textit{in}\ 0\le s\le t). \tag{3.38}$$

Moreover, T, obeys the semi-group property[1]*,*

$$T(t,s)\,T(s,u) = T(t,u)\ \ \textit{for all}\ 0\le u\le s\le t\le T,$$

and the solution to Eq. (3.30) is given by

$$y(t) = f(t) - \int_s^t T(t,d\tau)\,f(\tau). \tag{3.39}$$

In particular when $f(t) = y_0$ is a constant we have,

$$y(t) = y_0 - T(t,\tau)\,y_0|_{\tau=s}^{\tau=t} = T(t,s)\,y_0. \tag{3.40}$$

Proof. 1. One may directly conclude that $T(t,s)$ solves Eq. (3.37) by applying Theorem 3.18 with $y(t)$ and $f(t)$ now taking values in $\mathrm{End}(V)$ with $f(t)\equiv I$. Then Theorem 3.18 asserts the solution to $dy(t) = dx(t)\,y(t)$ with $y(0) = I$ is given by $T^x(t,s)$ with $T^x(t,s)$ as in Eq. (3.36). Alternatively it is possible to use the definition of $T^x(t,s)$ in Eq. (3.36) to give a direct proof the Eq. (3.37) holds. We will carry out this style of proof for Eq. (3.38) and leave the similar proof of Eq. (3.37) to the reader if they so desire to do it.

2. **Proof of the semi-group property.** Simply observe that both $t\to T(t,s)\,T(s,u)$ and $t\to T(t,u)$ solve the same differential equation, namely,

$$dy(t) = dx(t)\,y(t)\ \ \text{with}\ y(s) = T(s,u)\in\mathrm{End}(V)\,,$$

hence by our uniqueness results we know that $T(t,s)\,T(s,u) = T(t,u)$.

3. **Proof of Eq. (3.38).** Let $T_n(t,s) := T_n^x(t,s)$ and observe that

[1] This is a key algebraic idenitity that we must demand in the rough path theory to come later.

$$T_n(t,s) = \int_s^t T_{n-1}(t,\sigma)\, dx(\sigma) \ \text{ for } n \geq 1. \tag{3.41}$$

Thus if let

$$T^{(N)}(t,s) := \sum_{n=0}^{N} T_n(t,s) = I + \sum_{n=1}^{N} T_n(t,s),$$

it follows that

$$\begin{aligned} T^{(N)}(t,s) &= I + \sum_{n=1}^{N} \int_s^t T_{n-1}(t,\sigma)\, dx(\sigma) \\ &= I + \int_s^t \sum_{n=0}^{N-1} T_n(t,\sigma)\, dx(\sigma) = I + \int_s^t T^{(N-1)}(t,\sigma)\, dx(\sigma). \end{aligned} \tag{3.42}$$

We already now that $T^{(N)}(t,s) \to T(t,s)$ uniformly in (t,s) which also follows from the estimate in Eq. (3.34) as well. Passing to the limit in Eq. (3.42) as $N \to \infty$ then implies,

$$T(t,s) = I + \int_s^t T(t,\sigma)\, dx(\sigma)$$

which is the integrated form of Eq. (3.38) owing to the fundamental theorem of calculus which asserts that

$$d_s \int_s^t T(t,\sigma)\, dx(\sigma) = -T(t,s)\, dx(s).$$

4. **Proof of Eq. (3.39).** From Eq. (3.31) and Eq. (3.41) which reads in differential form as, $T_n(t,d\sigma) = -T_{n-1}(t,\sigma)\, dx(\sigma)$, we have,

$$\begin{aligned} y(t) &= f(t) - \sum_{n=1}^{\infty} \int_s^t T_n(t,d\sigma)\, f(\sigma) \\ &= f(t) - \sum_{n=1}^{\infty} \int_s^t T_{n-1}(t,\sigma)\, dx(\sigma)\, f(\sigma) \\ &= f(t) - \int_s^t \sum_{n=1}^{\infty} T_{n-1}(t,\sigma)\, dx(\sigma)\, f(\sigma) \\ &= f(t) - \int_s^t T(t,d\sigma)\, f(\sigma). \end{aligned}$$

■

Corollary 3.22 (Duhamel's principle II). *Equation (3.39) may also be expressed as,*

$$y(t) = T^x(t,s)\, f(s) + \int_s^t T^x(t,\tau)\, df(\tau) \tag{3.43}$$

which is one of the standard forms of Duhamel's principle. In words it says,

$$\begin{aligned} y(t) = &\begin{pmatrix} \text{solution to the homogeneous eq.} \\ dy(t) = dx(t)\, y(t) \ \text{ with } y(s) = f(s) \end{pmatrix} \\ &+ \int_s^t \begin{pmatrix} \text{solution to the homogeneous eq.} \\ dy(t) = dx(t)\, y(t) \ \text{ with } y(\tau) = df(\tau) \end{pmatrix}. \end{aligned}$$

Proof. This follows from Eq. (3.39) by integration by parts (you should modify Exercise 3.5 below as necessary);

$$\begin{aligned} y(t) &= f(t) - T^x(t,\tau)\, f(\tau)|_{\tau=s}^{\tau=t} + \int_s^t T^x(t,\tau)\, df(\tau) \\ &= T^x(t,s)\, f(s) + \int_s^t T^x(t,\tau)\, df(\tau). \end{aligned} \tag{3.44}$$

■

Exercise 3.5 (Product Rule). Suppose that V is a Banach space and $x : [0,T] \to \mathrm{End}(V)$ and $y : [0,T] \to \mathrm{End}(V)$ are continuous finite $1-$variation paths. Show for all $0 \leq s < t \leq T$ that,

$$x(t)\, y(t) - x(s)\, y(s) = \int_s^t dx(\tau)\, y(\tau) + \int_s^t x(\tau)\, dy(\tau). \tag{3.45}$$

Alternatively, one may interpret this an an integration by parts formula;

$$\int_s^t x(\tau)\, dy(\tau) = x(\tau)\, y(\tau)|_{\tau=s}^{\tau=t} - \int_s^t dx(\tau)\, y(\tau)$$

Solution to Exercise (3.5). For $\Pi \in \mathcal{P}(s,t)$ we have,

$$\begin{aligned} x(t)\, y(t) - x(s)\, y(s) &= \sum_{\tau \in \Pi} \Delta_\tau (x(\cdot)\, y(\cdot)) \\ &= \sum_{\tau \in \Pi} \left[(x(\tau_-) + \Delta_\tau x)(y(\tau_-) + \Delta_\tau y) - x(\tau_-)\, y(\tau_-)\right] \\ &= \sum_{\tau \in \Pi} \left[x(\tau_-)\, \Delta_\tau y + (\Delta_\tau x)\, y(\tau_-) + (\Delta_\tau x)\, \Delta_\tau y\right]. \end{aligned} \tag{3.46}$$

The last term is easy to estimate as,

$$\left\| \sum_{\tau \in \Pi} (\Delta_\tau x) \Delta_\tau y \right\| \leq \sum_{\tau \in \Pi} \|\Delta_\tau x\| \|\Delta_\tau y\| \leq \max_{\tau \in \Pi} \|\Delta_\tau x\| \cdot \sum_{\tau \in \Pi} \|\Delta_\tau y\|$$
$$\leq \max_{\tau \in \Pi} \|\Delta_\tau x\| \cdot V_1(y) \to 0 \text{ as } |\Pi| \to 0.$$

So passing to the limit as $|\Pi| \to 0$ in Eq. (3.46) gives Eq. (3.45).

Exercise 3.6 (Inverses). Let V be a Banach space and $x : [0,T] \to \mathrm{End}(V)$ be a continuous finite 1 – variation paths. Further suppose that $S(t,s) \in \mathrm{End}(V)$ is the unique solution to,

$$S(dt,s) = -S(t,s)\, dx(t) \text{ with } S(s,s) = I \in \mathrm{End}(V).$$

Show

$$S(t,s) T^x(t,s) = I = T^x(t,s) S(t,s) \text{ for all } 0 \leq s \leq t \leq T,$$

that is to say $T^x(t,s)$ is invertible and $T^x(t,s)^{-1}$ may be described as the unique solution to the ODE,

$$T^x(dt,s)^{-1} = -T^x(dt,s)^{-1} dx(t) \text{ with } T^x(s,s)^{-1} = I. \tag{3.47}$$

Solution to Exercise (3.6). Using the product rule we find,

$$d_t[S(t,s)T^x(t,s)] = -S(t,s)\,dx(t)\,T^x(t,s) + S(t,s)\,dx(t)\,T^x(t,s) = 0.$$

Since $S(s,s)T^x(s,s) = I$, it follows that $S(t,s)T^x(t,s) = I$ for all $0 \leq s \leq t \leq T$.

For opposite product, let $g(t) := T^x(t,s) S(t,s) \in \mathrm{End}(V)$ so that

$$d_t[g(t)] = d_t[T^x(t,s)S(t,s)] = -T^x(t,s)S(t,s)\,dx(t) + dx(t)\,T^x(t,s)S(t,s)$$
$$= dx(t)\,g(t) - g(t)\,dx(t) \text{ with } g(s) = I.$$

Observe that $g(t) \equiv I$ solves this ODE and therefore by uniqueness of solutions to such linear ODE we may conclude that $g(t)$ must be equal to I, i.e. $T^x(t,s)S(t,s) = I$.

As usual we say that $A, B \in \mathrm{End}(V)$ *commute* if

$$0 = [A,B] := AB - BA. \tag{3.48}$$

Exercise 3.7 (Commute). Suppose that V is a Banach space and $x : [0,T] \to \mathrm{End}(V)$ is a continuous finite 1 – variation paths and $f : [0,T] \to \mathrm{End}(V)$ is continuous. If $A \in \mathrm{End}(V)$ commutes with $\{x(t), f(t) : 0 \leq t \leq T\}$, then A commutes with $\int_s^t f(\tau)\,dx(\tau)$ for all $0 \leq s \leq t \leq T$. Also show that $[A, T^x(s,t)] = 0$ for all $0 \leq s \leq t \leq T$.

Exercise 3.8 (Abelian Case). Suppose that $[x(s), x(t)] = 0$ for all $0 \leq s,t \leq T$, show

$$T^x(t,s) = e^{(x(t)-x(s))}. \tag{3.49}$$

Solution to Exercise (3.8). By replacing $x(t)$ by $x(t) - x(s)$ if necessary, we may assume that $x(s) = 0$. Then by the product rule and the assumed commutativity,

$$d\frac{x^n(t)}{n!} = dx(t)\,\frac{x(t)^{n-1}}{(n-1)!}$$

or in integral form,

$$\frac{x^n(t)}{n!} = \int_s^t dx(\tau)\,\frac{x(\tau)^{n-1}}{(n-1)!}$$

which shows that $\frac{1}{n!}x^n(t)$ satisfies the same recursion relations as $T_n^x(t,s)$. Thus we may conclude that $T_n^x(t,s) = (x(t) - x(s))^n / n!$ and thus,

$$T(t,s) = \sum_{n=0}^\infty \frac{1}{n!}(x(t) - x(s))^n = e^{(x(t)-x(s))}$$

as desired.

Exercise 3.9 (Abelian Factorization Property). Suppose that V is a Banach space and $x : [0,T] \to \mathrm{End}(V)$ and $y : [0,T] \to \mathrm{End}(V)$ are continuous finite 1 – variation paths such that $[x(s), y(t)] = 0$ for all $0 \leq s,t \leq T$, then

$$T^{x+y}(s,t) = T^x(s,t)\,T^y(s,t) \text{ for all } 0 \leq s \leq t \leq T. \tag{3.50}$$

Hint: show both sides satisfy the same ordinary differential equations – see the next problem.

Exercise 3.10 (General Factorization Property). Suppose that V is a Banach space and $x : [0,T] \to \mathrm{End}(V)$ and $y : [0,T] \to \mathrm{End}(V)$ are continuous finite 1 – variation paths. Show

$$T^{x+y}(s,t) = T^x(s,t)\,T^z(s,t),$$

where

$$z(t) := \int_s^t T^x(s,\tau)^{-1}\,dy(\tau)\,T^x(s,\tau)$$

Hint: see the hint for Exercise 3.9.

Solution to Exercise (3.10). Let $g(t) := T^x(s,t)^{-1}\,T^{x+y}(s,t)$. Then making use of Exercise 3.6 we have,

$$\begin{aligned} dg(t) &= T^x(s,t)^{-1}(dx(t)+dy(t))T^{x+y}(s,t) - T^x(s,t)^{-1}dx(t)T^{x+y}(s,t) \\ &= T^x(s,t)^{-1}dy(t)T^{x+y}(s,t) \\ &= \left(T^x(s,t)^{-1}dy(t)T^x(s,t)\right)T^x(s,t)^{-1}T^{x+y}(s,t) \\ &= dz(t)g(t) \text{ with } g(s) = I. \end{aligned}$$

Remark 3.23. If $g(t) \in \mathrm{End}(V)$ is a C^1 – path such that $g(t)^{-1}$ is invertible for all t, then $t \to g(t)^{-1}$ is invertible and

$$\frac{d}{dt}g(t)^{-1} = -g(t)^{-1}\dot{g}(t)g(t)^{-1}.$$

Exercise 3.11. Suppose that $g(t) \in \mathrm{Aut}(V)$ is a continuous finite variation path. Show $g(t)^{-1} \in \mathrm{Aut}(V)$ is again a continuous path with finite variation and that

$$dg(t)^{-1} = -g(t)^{-1}dg(t)g(t)^{-1}. \tag{3.51}$$

Hint: recall that the invertible elements, $\mathrm{Aut}(V) \subset \mathrm{End}(V)$, is an open set and that $\mathrm{Aut}(V) \ni g \to g^{-1} \in \mathrm{Aut}(V)$ is a smooth map.

Solution to Exercise (3.11). Let $V(t) := g(t)^{-1}$ which is again a finite variation path by the fundamental theorem of calculus and the fact that $\mathrm{Aut}(V) \ni g \to g^{-1} \in \mathrm{Aut}(V)$ is a smooth map. Moreover we know that $V(t)g(t) = I$ for all t and therefore by the product rule $(dV)g + Vdg = dI = 0$. Making use of the substitution formula we then find,

$$V(t) = V(0) + \int_0^t dV(\tau) = \int_0^t dV(\tau)g(\tau)g(\tau)^{-1} = -\int_0^t V(\tau)dg(\tau)g(\tau)^{-1}.$$

Replacing $V(t)$ by $g(t)^{-1}$ in this equation then shows,

$$g(t)^{-1} - g(0)^{-1} = -\int_0^t g(\tau)dg(\tau)g(\tau)^{-1}$$

which is the integrated form of Eq. (3.51).

Exercise 3.12. Suppose now that B is a Banach algebra and $x(t) \in B$ is a continuous finite variation path. Let

$$X(s,t) := X^x(s,t) := 1 + \sum_{n=1}^{\infty} X_n^x(s,t),$$

where

$$X_n^x(s,t) := \int_{s \le \tau_1 \le \dots \le \tau_n \le t} dx(\tau_1)\dots dx(\tau_n)$$

Show $t \to X(s,t)$ is the unique solution to the ODE,

$$X(s,dt) = X(s,t)dx(t) \text{ with } X(s,s) = 1$$

and that

$$X(s,t)X(t,u) = X(s,u) \text{ for all } 0 \le s \le t \le u \le T.$$

Solution to Exercise (3.12). This can be deduced from what we have already done. In order to do this, let $y(t) := R_{x(t)} \in \mathrm{End}(B)$ so that for $a \in B$,

$$\begin{aligned} T^y(t,s)a &= a + \sum_{n=1}^{\infty}\int_{s \le \tau_1 \le \dots \le \tau_n \le t} dy(\tau_n)\dots dy(\tau_1)a \\ &= a + \sum_{n=1}^{\infty}\int_{s \le \tau_1 \le \dots \le \tau_n \le t} adx(\tau_1)\dots dx(\tau_n) \\ &= aX(s,t). \end{aligned}$$

Therefore, taking $a = 1$, we find,

$$X(s,dt) = T^y(dt,s)1 = dy(t)T^y(t,s)1 = dy(t)X(s,t) = X(s,t)dx(t)$$

with $X(s,s) = T(s,s)1 = 1$. Moreover we also have,

$$X(s,t)X(t,u) = T^y(u,t)X(s,t) = T^y(u,t)T^y(t,s)1 = T^y(u,s)1 = X(s,u).$$

Alternatively: one can just check all the statements as we did for $T(t,s)$. The main point is that if $g(t)$ solves $dg(t) = g(t)dx(t)$, then $ag(t)$ also solves the same equation.

Remark 3.24. Let $\lambda \in \mathbb{R}$ or $\mathbb{C}$ as the case may be and define,

$$X^\lambda(s,t) := X^{\lambda x}(s,t) \text{ and } X_n^\lambda(s,t) := X_n^{\lambda x}(s,t) = \lambda^n X_n(s,t). \tag{3.52}$$

Then the identity in Eq. (3.52) becomes,

$$\begin{aligned} \sum_{n=0}^{\infty}\lambda^n X_n(s,u) &= X^\lambda(s,u) = X^\lambda(s,t)X^\lambda(t,u) \\ &= \sum_{k,l=0}^{\infty}\lambda^k\lambda^l X_k(s,t)X_l(t,u) \\ &= \sum_{n=0}^{\infty}\lambda^n\sum_{k+l=n}X_k(s,t)X_l(t,u) \end{aligned}$$

from which we conclude,

$$X_n(s,u) = \sum_{k=0}^{n} X_k(s,t)\, X_{n-k}(t,u) \text{ for } n = 0,1,2,\dots \tag{3.53}$$

Terry. Lyons refers the identities in Eq. (3.53) as Chen's identities. These identities may be also be deduced directly by looking at the multiple integral expressions defining $X_k(s,t)$.

3.5.1 Bone Yard

Proof. but we can check this easily directly as well. From Eq. (3.41) we conclude,

$$\|T_n(t,s)\| \le \int_s^t \|T_{n-1}(t,\sigma)\| \, \|dx(\sigma)\|$$

By the fundamental theorem of calculus we have, which we write symbolically as,

$$T_{n+1}(t,s) = -\int_s^t T_n(t,\sigma)\, dx(\sigma)$$

Summing this equation upon n shows,

$$\sum_{n=0}^{N}$$

and hence one learn inductively that $T_n(t,s)$ is a finite variation process in t,

$$\begin{aligned}\|T_n(t,s)\| &\le \int_{s\le\tau_1\le\dots\le\tau_n\le t} \|dx(\tau_n)\|\,\|dx(\tau_{n-1})\|\,\|dx(\tau_{n-2})\|\dots\|dx(\tau_1)\| \\ &= \frac{1}{n!}\left(\int_s^t \|dx\|\right)^n.\end{aligned}$$

and

$$\begin{aligned}V_1\left(T_n(\cdot,s)|_{[s,T]}\right) &\le \int_s^T \|dx(\tau)\|\,\|T_{n-1}(\tau,s)\| \\ &\le \int_s^T \frac{1}{(n-1)!}\left(\int_s^\tau \|dx\|\right)^{n-1} \|dx(\tau)\| = \frac{1}{n!}\left(\int_s^T \|dx\|\right)^n.\end{aligned}$$

In particular it follows that

$$\sum_{n=1}^{\infty} V_1\left(T_n(\cdot,s)|_{[s,T]}\right) \le \exp\left(\int_s^T \|dx\|\right) - 1.$$

Hence we learn that $\sum_{n=0}^{\infty} T_n(t,s)$ converges uniformly to $T(t,s)$ so that $T(t,s)$ is continuous. Moreover, if $\Pi \in \mathcal{P}(s,T)$, then

$$\begin{aligned}\sum_{t\in\Pi} \|T(t,s) - T(t_-,s)\| &\le \sum_{n=1}^{\infty}\sum_{t\in\Pi} \|T_n(t,s) - T_n(t_-,s)\| \\ &\le \sum_{n=1}^{\infty} V_1\left(T_n(\cdot,s)|_{[s,T]}\right) < \infty.\end{aligned}$$

Hence it follows from this that

$$V_1\left(T(\cdot,s)|_{[s,T]}\right) \le \exp\left(\int_s^T \|dx\|\right) - 1 < \infty.$$

Similarly we have,

$$\sum_{t\in\Pi}\left\| T(t,s) - \sum_{n=0}^{N} T_n(t_-,s)\right\| \le \sum_{n=N+1}^{\infty} V_1\left(T_n(\cdot,s)|_{[s,T]}\right)$$

and therefore,

$$V_1\left(\left[T(\cdot,s) - \sum_{n=0}^{N} T_n(\cdot,s)\right]|_{[s,T]}\right) \to 0 \text{ as } N\to\infty.$$

It is now a simple matter to see that for any continuous f,

$$\int_s^t f(\tau)\,T(d\tau,s) = \lim_{N\to\infty}\sum_{n=0}^{N}\int_s^t f(\tau)\,T_n(d\tau,s) = \sum_{n=0}^{\infty}\int_s^t f(\tau)\,T_n(d\tau,s)\,.$$

In particular,

$$\begin{aligned}
T(t,s) - T(s,s) &= \sum_{n=0}^{\infty}\left[T_n(t,s) - T_n(s,s)\right] \\
&= \sum_{n=1}^{\infty}\left[\int_s^t dx(\tau)\,T_{n-1}(t,s)\right] = \left[\int_s^t dx(\tau)\sum_{n=1}^{\infty} T_{n-1}(\tau,s)\right] \\
&= \int_s^t dx(\tau)\,T(\tau,s)\,.
\end{aligned}$$

which is the precise version of Eq. (3.37). ∎

Banach Space p – variation results

In this chapter, suppose that V is a Banach space and assume $x \in C([0,T] \to V)$ with $x(0) = 0$ for simplicity. We continue the notation used in Chapter 2. In particular we have,

$$V_p(x : \Pi) := \left(\sum_{t \in \Pi} \|x_t - x_{t_-}\|^p \right)^{1/p} = \left(\sum_{t \in \Pi} \|\Delta_t x\|^p \right)^{1/p} \text{ and}$$
$$V_p(x) := \sup_{\Pi \in \mathcal{P}(0,T)} V_p(x : \Pi).$$

Lemma 4.1. *Suppose that $\Pi \in \mathcal{P}(s,t)$ and $a \in \Pi \cap (s,t)$, then*

$$V_p^p(x, \Pi \setminus \{a\}) \le 2^{p-1} V_p^p(x, \Pi). \tag{4.1}$$

Proof. Since

$$\begin{aligned} \|x(a_+) - x(a_-)\|^p &\le [\|x(a_+) - x(a)\| + \|x(a) - x(a_-)\|]^p \\ &\le 2^{p-1} (\|x(a_+) - x(a)\|^p + \|x(a) - x(a_-)\|^p) \\ &= 2^{p-1} V_p^p(x : \Pi \cap [a_-, a_+]) \end{aligned}$$

and

$$V_p^p(x, \Pi \setminus \{a\}) = V_p^p(x : \Pi \cap [0, a_-]) + \|x(a_+) - x(a_-)\|^p + V_p^p(x : \Pi \cap [a_+, T])$$

we have,

$$\begin{aligned} V_p^p(x, \Pi \setminus \{a\}) &\le V_p^p(x : \Pi \cap [0, a_-]) + 2^{p-1} V_p^p(x : \Pi \cap [a_-, a_+]) + V_p^p(x : \Pi \cap [a_+, T]) \\ &\le 2^{p-1} V_p^p(x, \Pi). \end{aligned}$$

■

Corollary 4.2. *As above, let $\omega(s,t) := \omega_{x,p}(s,t) := V_p^p\left(x|_{[s,t]}\right)$ be the control associated to x. Then for all $0 \le s < u < t \le T$, we have*

$$\omega(s,u) + \omega(u,t) \le \omega(s,t) \le 2^{p-1} [\omega(s,u) + \omega(u,t)]. \tag{4.2}$$

The second inequality shows that if $x|_{[s,u]}$ and $x|_{[s,t]}$ both have finite p – variation, then $x|_{[s,t]}$ has finite p – variation.

Proof. The first inequality is the superadditivity property of the control ω that we have already proved in Lemma 2.35. For the second inequality, let $\Pi \in \mathcal{P}(s,t)$. If $u \in \Pi$ we have ,

$$\begin{aligned} V_p^p(x, \Pi) &= V_p^p(x, \Pi \cap [s,u]) + V_p^p(x, \Pi \cap [u,t]) \\ &\le \omega(s,u) + \omega(u,t). \end{aligned} \tag{4.3}$$

On the other hand if $u \notin \Pi$ we have, using Eq. (4.3) with Π replaced by $\Pi \cup \{u\}$ and Lemma 4.1,

$$V_p^p(x, \Pi) \le 2^{p-1} V_p^p(x, \Pi \cup \{u\}) \le 2^{p-1} [\omega(s,u) + \omega(u,t)].$$

Thus for any $\Pi \in \mathcal{P}(s,t)$ we may conclude that

$$V_p^p(x, \Pi) \le 2^{p-1} [\omega(s,u) + \omega(u,t)].$$

Taking the supremum of this inequality over $\Pi \in \mathcal{P}(s,t)$ then gives the desired result. ■

These results may be significantly improve upon. For example, we have the following proposition whose proof we leave to the interested reader who may wish to consult Lemma (4.13) below.

Proposition 4.3. *If $\Pi, \Pi' \in \mathcal{P}(s,t)$ with*

$$\Pi = \{s := \tau_0 < \tau_1 < \cdots < \tau_n = t\} \subset \Pi',$$

then

$$\begin{aligned} V_p^p(x, \Pi) &\le \sum_{\tau \in \Pi} \#(\Pi' \cap (\tau_-, \tau])^{p-1} V_p^p(\Pi' \cap (\tau_-, \tau]) \qquad (4.4) \\ &\le k^{p-1} V_p^p(x, \Pi'), \qquad (4.5) \end{aligned}$$

where

$$k := \max\{\#(\tau_{i-1}, \tau_i] \cap \Pi' : i = 1, 2, \ldots, n\}. \tag{4.6}$$

Proof. This follows by the same methods used in the proof of Lemma 4.1 or Lemma (4.13) below. The point is that,

$$\begin{aligned}V_p^p(x,\Pi) &= \sum_{\tau\in\Pi}\|\Delta_\tau x\|^p = \sum_{\tau\in\Pi}\left\|\sum_{s\in\Pi'\cap(\tau_-,\tau]}\Delta_s x\right\|^p \\ &\le \sum_{\tau\in\Pi}\#\left(\Pi'\cap(\tau_-,\tau]\right)^{p-1}\sum_{s\in\Pi'\cap(\tau_-,\tau]}\|\Delta_s x\|^p \\ &\le \sum_{\tau\in\Pi}k^{p-1}V_p^p\left(x;\Pi'\cap(\tau_-,\tau]\right) = k^{p-1}V_p^p\left(x;\Pi'\right).\end{aligned}$$

■

Corollary 4.4. *Suppose that* $\Gamma\in\mathcal{P}(s,t)$, *then*

$$V_p^p(x:[s,t]) \le \#\left(\Gamma\cap(s,t]\right)^{p-1}\sum_{\tau\in\Gamma}V_p^p(x:[\tau_-,\tau]). \tag{4.7}$$

and in particular this show that if $V_p(x:[\tau_-,\tau])<\infty$ *for all* $\tau\in\Gamma$, *then* $V_p(x:[s,t])<\infty$.

Proof. Let $\Pi\in\mathcal{P}(s,t)$ and $\Pi':=\Pi\cup\Gamma$, then k defined in Eq. (4.6) is no greater than $\#(\Gamma\cap(s,t])$ and therefore,

$$V_p^p(x,\Pi) \le \#\left(\Gamma\cap(s,t]\right)^{p-1}V_p^p\left(x,\Pi'\right) \tag{4.8}$$

while

$$V_p^p\left(x,\Pi'\right) = \sum_{\tau\in\Gamma}V_p^p\left(x,\Pi'\cap[\tau_-,\tau]\right) \le \sum_{\tau\in\Gamma}V_p^p(x:[\tau_-,\tau]). \tag{4.9}$$

So combining these two inequalities and then taking the supremum over $\Pi\in\mathcal{P}(s,t)$ gives Eq. (4.7). ■

Definition 4.5. *The normalized space of p – variation is*

$$C_{0,p}([0,T],V) := \{x\in C([0,T]\to V) : x(0)=0 \text{ and } V_p(x)<\infty\}.$$

Proposition 4.6. *The space* $C_{0,p}([0,T],V)$ *is a linear space and* $V_p(\cdot)$ *is a Banach norm on this space.*

Proof. 1. If $t\in[0,T]$ we may take $\Pi:=\{0,t,T\}$ to learn that

$$\|x(t)\| = \|x(t)-x(0)\| \le \left(\|x(t)-x(0)\|^p + \|X(T)-x(t)\|^p\right)^{1/p} \le V_p(x).$$

As $t\in[0,T]$ was arbitrary it follows that

$$\|x\|_u := \max_{0\le t\le T}\|x(t)\| \le V_p(x). \tag{4.10}$$

In particular if $V_p(x)=0$ then $x=0$.

2. For $\lambda\in\mathbb{C}$, $V_p(\lambda x:\Pi) = |\lambda|V_p(\lambda x:\Pi)$ and therefore $V_p(\lambda x) = |\lambda|V_p(x)$.

3. For $x,y\in C([0,T]\to V)$,

$$\begin{aligned}V_p(x+y:\Pi) &= \left(\sum_{t\in\Pi}\|\Delta_t x+\Delta_t y\|^p\right)^{1/p} \le \left(\sum_{t\in\Pi}\left(\|\Delta_t x\|+\|\Delta_t y\|\right)^p\right)^{1/p} \\ &\le \left(\sum_{t\in\Pi}\|\Delta_t x\|^p\right)^{1/p} + \left(\sum_{t\in\Pi}\|\Delta_t y\|^p\right)^{1/p} = V_p(x:\Pi)+V_p(y:\Pi) \\ &\le V_p(x)+V_p(y),\end{aligned}$$

and therefore,

$$V_p(x+y) \le V_p(x)+V_p(y).$$

Hence it follows from this triangle inequality and items 1. and 2. that $C_{0,p}([0,T],V)$ is a linear space and that $V_p(\cdot)$ is a norm on $C_{0,p}([0,T],V)$.

4. To finish the proof we must now show $(C_{0,p}([0,T],V),V_p(\cdot))$ is a complete space. So suppose that $\{x_n\}_{n=1}^\infty\subset C_{0,p}([0,T],V)$ is a Cauchy sequence. Then by Eq. (4.10) we know that x_n converges uniformly to some $x\in C([0,T]\to V)$. Moreover, for any partition, $\Pi\in\mathcal{P}(0,T)$ we have

$$\begin{aligned}V_p(x-x_n:\Pi) &\le V_p(x-x_m:\Pi)+V_p(x_m-x_n:\Pi) \\ &\le V_p(x-x_m:\Pi)+V_p(x_m-x_n).\end{aligned}$$

Taking the limit of this equation as $m\to\infty$ the shows,

$$\begin{aligned}V_p(x-x_n:\Pi) &\le \liminf_{m\to\infty}\left(V_p(x-x_m:\Pi)+V_p(x_m-x_n)\right) \\ &= \liminf_{m\to\infty}V_p(x_m-x_n).\end{aligned}$$

We may now take the supremum over $\Pi\in\mathcal{P}(0,T)$ to learn,

$$V_p(x-x_n) \le \liminf_{m\to\infty}V_p(x_m-x_n) \to 0 \text{ as } n\to\infty.$$

So by the triangle inequality, $V_p(x)\le V_p(x-x_n)+V_p(x_n)<\infty$ for sufficiently large n so that $x\in C_{0,p}([0,T],V)$ and $V_p(x-x_n)\to 0$ as $n\to\infty$. ■

Proposition 4.7 (Interpolation). *Suppose that* $x\in C_{0,p}([0,T]\to V)$ *and* $q>p$, *then*

$$V_q(x) \le 2^{1-p/q}\|x\|_u^{1-p/q}V_p^{p/q}(x). \tag{4.11}$$

Proof. Let $\Pi\in\mathcal{P}(0,T)$, then

$$V_q^q(x:\Pi) = \sum_{t\in\Pi} \|\Delta_t x\|^q = \sum_{t\in\Pi} \|\Delta_t x\|^{q-p} \|\Delta_t x\|^p$$
$$\le \max_{t\in\Pi} \|\Delta_t x\|^{q-p} \cdot \sum_{t\in\Pi} \|\Delta_t x\|^p \le 2\|x\|_u^{q-p} V_p^p(x).$$

Taking the supremum over $\Pi \in \mathcal{P}(0,T)$ and then taking the q^{th} – roots of both sides gives the result. ∎

Notation 4.8 *For $x \in C([0,T] \to V)$ and $\Pi \in P(0,T)$, let $x^\Pi(t)$ be the piecewise linear path defined by*

$$x^\Pi(t) := x(t_-) + \frac{(t-t_-)}{t_+ - t_-}\Delta_t x \text{ for all } t \in [0,T],$$

see Figures 4.1 and 4.2.

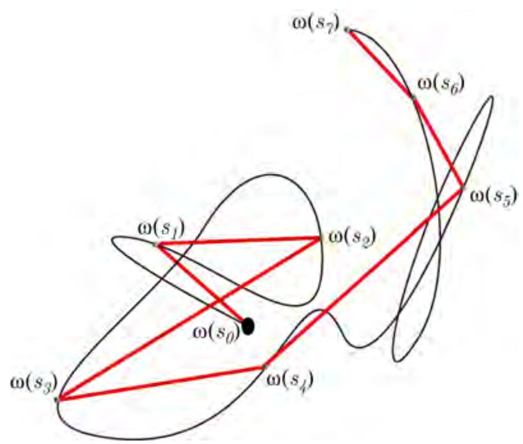


Fig. 4.1. Here $\Pi = \{0 = s_0 < s_1 < \cdots < s_7 = T\}$ and ω should be x. The red lines indicate the image of x^Π.

Proposition 4.9. *For each $x \in C([0,T] \to V)$, $x^\Pi \to x$ uniformly in t as $|\Pi| \to 0$.*

Proof. This is an easy consequence of the uniform continuity of x on the compact interval, $[0,T]$. ∎

Theorem 4.10. *If $x \in C_{0,p}([0,T] \to V)$ and $\Pi \in \mathcal{P}(0,T)$, then*

$$V_p(x^\Pi) \le 3^{1-1/p} V_p(x). \tag{4.12}$$

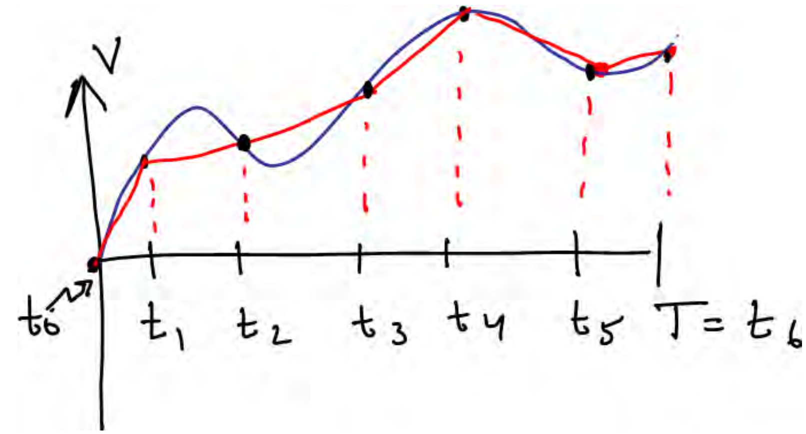


Fig. 4.2. Here $\Pi = \{0 = t_0 < t_1 < \cdots < t6 = T\}$ and x^Π is indicated by the red piecewise linear path.

We will give the proof of this theorem at the end of this section.

Corollary 4.11. *Suppose $x \in C_{0,p}([0,T] \to V)$ and $q \in (p,\infty)$. Then $\lim_{|\Pi|\to 0} V_q(x - x^\Pi) = 0$, i.e. $x^\Pi \to x$ as $|\Pi| \to 0$ in $C_{0,q}$ for any $q > p$.*

Proof. According to Proposition 4.7 and Theorem 4.10,

$$V_q^q(x - x^\Pi) \le \left(2\|x - x^\Pi\|_u\right)^{q-p} V_p^p(x - x^\Pi)$$
$$\le \left(2\|x - x^\Pi\|_u\right)^{q-p} \left[V_p(x) + V_p(x^\Pi)\right]^p$$
$$\le \left(2\|x - x^\Pi\|_u\right)^{q-p} 2^{p-1} \left[V_p^p(x) + V_p^p(x^\Pi)\right]$$
$$\le \left(2\|x - x^\Pi\|_u\right)^{q-p} 2^{p-1} \left[1 + 3^{p-1}\right] V_p^p(x).$$

The latter expression goes to zero because of Proposition 4.9. ∎

We refer the reader to in [5] for more results in this vain. In particular, as a corollary of Theorem 23 and 24 one sees that the finite variation paths are **not** dense in $C_{0,p}([0,T] \to V)$.

Notation 4.12 *Suppose that $\Pi, \Pi' \in \mathcal{P}(0,T)$ the we let*

$$\Lambda = \Lambda(\Pi, \Pi') = \{t \in \Pi \cap (0,T] : \Pi \cap (t_-, t) \ne \emptyset\}$$

$$S := S(\Pi, \Pi') = \cup_{t\in\Lambda} \{t_-, t\},$$

see Figure 4.3 below for an example.

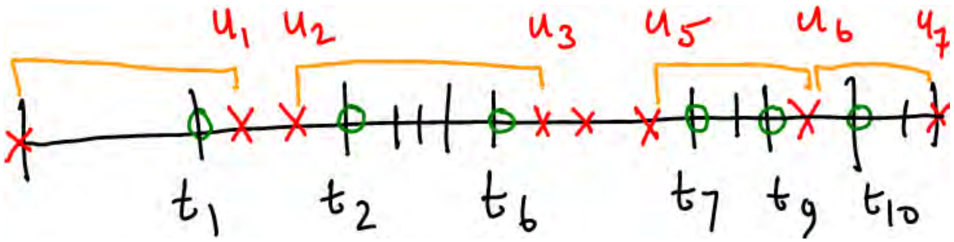


Fig. 4.3. In this figure the red x's correspond to Π' and the vertical hash marks correspond to Π. The green circles indicate the points which make up S.

Lemma 4.13. *If $x \in C([0,T] \to V)$, $\Pi, \Pi' \in \mathcal{P}(0,T)$, and $S = S(\Pi, \Pi')$ as in Notation 4.12, then*

$$V_p^p(x : \Pi') \le 3^{p-1} V_p^p(x : \Pi' \cup S). \tag{4.13}$$

(This is a special case of Proposition 4.3 above.)

Proof. For concreteness, let us consider the scenario in Figure 4.3. The difference between $V_p(x : \Pi')$ and $V_p(x : \Pi' \cup S)$ comes from the terms indicated by the orange brackets above the line. For example consider the $u_2 - u_3$ contribution to $V_p^p(x : \Pi')$ versus the terms in $V_p^p(x : \Pi' \cup S)$ involving $u_2 < t_2 < t_6 < u_3$. We have,

$$\begin{aligned}\|x(u_3) - x(u_2)\|^p &\le (\|x(u_3) - x(t_2)\| + \|x(t_6) - x(t_2)\| + \|x(u_3) - x(t_6)\|)^p \\ &\le 3^{p-1}(\|x(u_3) - x(t_2)\|^p + \|x(t_6) - x(t_2)\|^p + \|x(u_3) - x(t_6)\|^p).\end{aligned}$$

Similar results hold for the other terms. In some case we only get two terms with 3^{p-1} being replaced by 2^{p-1} and where no points are squeezed between the neighbors of Π', the corresponding terms are the same in both $V_p(x : \Pi')$ and $V_p(x : \Pi' \cup S)$. Nevertheless, if we use the crude factor of 3^{p-1} in all cases we arrive at the inequality in Eq. (4.13). ∎

Lemma 4.14. *Suppose that $x(t) = a + tb$ for some $a, b \in \Pi$, then for $\Pi \in \mathcal{P}(u,v)$ we have*

$$V_p(x : \Pi) \le V_p(x : \{u, v\}). \tag{4.14}$$

with the inequality being strict if $p > 1$ and Π is a strict refinement of $\{u, v\}$.

Proof. Here we have,

$$\begin{aligned}V_p^p(x : \Pi) &= \sum_{t \in \Pi} \|x(t) - x(t_-)\|^p = \sum_{t \in \Pi} \|(t - t_-) b\|^p \\ &= \|b\|^p \sum_{t \in \Pi} (t - t_-)^p \le \|b\|^p (v - u)^p = V_p(x : \{u, v\})\end{aligned}$$

wherein the last equality we have used,

$$\sum_{i=1}^n a_i^p \le \left(\sum_{i=1}^n a_i\right)^p.$$

∎

Lemma 4.15. *Let $x \in C([0,T] \to V)$, $\Pi, \Pi' \in \mathcal{P}(0,T)$, and $S = S(\Pi, \Pi')$ be as in Lemma 4.13. Then as in Notation 4.12, then*

$$V_p(x^\Pi : \Pi' \cup S) \le V_p(x^\Pi : S \cup \{0, T\}) = V_p(x : S \cup \{0, T\}) \le V_p(x). \tag{4.15}$$

Proof. Again let us consider the scenario in Figure 4.3. Let $\Gamma := \Pi' \cup S$, then

$$\begin{aligned}V_p^p(x^\Pi : \Pi' \cup S) = {} & V_p^p(x^\Pi : \Gamma \cap [0, t_1]) + V_p^p(x^\Pi : \Gamma \cap [t_1, t_2]) \\ & + V_p^p(x^\Pi : \Gamma \cap [t_2, t_6]) + V_p^p(x^\Pi : \Gamma \cap [t_6, t_7]) \\ & + V_p^p(x^\Pi : \Gamma \cap [t_7, t_9]) + V_p^p(x^\Pi : \Gamma \cap [t_9, t_{10}]) \\ & + V_p^p(x^\Pi : \Gamma \cap [t_{10}, T]).\end{aligned}$$

Since x^Π is linear on each of the intervals $[t_i, t_{i+1}]$, we may apply Lemma 4.14 to find,

$$\begin{aligned}V_p^p(x^\Pi : \Gamma \cap [t_1, t_2]) &+ V_p^p(x^\Pi : \Gamma \cap [t_6, t_7]) + V_p^p(x^\Pi : \Gamma \cap [t_9, t_{10}]) \\ &\le V_p^p(x^\Pi : \{t_1, t_2\}) + V_p^p(x^\Pi : \{t_6, t_7\}) + V_p^p(x^\Pi : \{t_9, t_{10}\})\end{aligned}$$

and for the remaining terms we have,

$$\begin{aligned}&\begin{pmatrix} V_p^p(x^\Pi : \Gamma \cap [0, t_1]) + V_p^p(x^\Pi : \Gamma \cap [t_2, t_6]) \\ + V_p^p(x^\Pi : \Gamma \cap [t_7, t_9]) + V_p^p(x^\Pi : \Gamma \cap [t_{10}, T]) \end{pmatrix} \\ &= \begin{pmatrix} V_p^p(x^\Pi : \{0, t_1\}) + V_p^p(x^\Pi : \{t_2, t_6\}) \\ + V_p^p(x^\Pi : \{t_7, t_9\}) + V_p^p(x^\Pi : \{t_{10}, T\}) \end{pmatrix}\end{aligned}$$

which gives the inequality in Eq. (4.15). ∎

4.0.2 Proof of Theorem 4.10

We are now in a position to prove Theorem 4.10.

Proof. Let $\Pi, \Pi' \in \mathcal{P}(0,T)$. Then by Lemmas 4.13 and Lemma 4.15,

$$V_p(x^\Pi : \Pi') \le 3^{1-1/p} V_p(x^\Pi : \Pi' \cup S) \le V_p(x).$$

Taking the supremum over all $\Pi' \in \mathcal{P}(0,T)$ then gives the estimate in Eq. (4.12). ■

For an alternative proof of Theorem 4.10 the reader is referred to [5] and [6, Chapter 5] where controls are used to prove these results. Most of these results go over to the case where V is replaced by a complete metric space (E,d) which is also geodesic. That is if $a,b \in E$ there should be a path $\sigma : [0,1] \to E$ such that $\sigma(0) = a$, $\sigma(t) = b$ and $d(\sigma(t), \sigma(s)) = |t-s|\, d(a,b)$ for all $s,t \in [0,1]$. More invariantly put, there should be a path $\sigma : [0,1] \to E$ such that if $\ell(t) := d(a, \sigma(t))$, then $d(\sigma(t), \sigma(s)) = |\ell(t) - \ell(s)|\, d(a,b)$ for all $s,t \in [0,1]$.

Young's Integration Theory

Theorem 2.24 above shows that if we insist upon integrating all continuous functions, $f:[0,T]\to\mathbb{R}$, then then the integrator, x, must be of finite variation. This suggests that if we want to allow for rougher integrators, x, then we must in turn require the integrand to be smoother. Young's integral, see [18] is a result along these lines. Our fist goal is to prove some integral bounds.

In this section we will assume that V and W are Banach spaces and $g:[0,T]\to V$ and $f:[0,T]\to \operatorname{End}(V,W)$ are continuous functions. Later we will assume that $V_p(f)+V_q(g)<\infty$ where $p,q\geq 1$ with $\theta:=1/p+1/q>1$.

Notation 5.1 *Given a partition $\Pi\in\mathcal{P}(s,t)$, let*

$$S_\Pi(f,g):=\sum_{\tau\in\Pi} f(\tau_-)\,\Delta_\tau g.$$

So in more detail,

$$\Pi=\{s=\tau_0<\tau_1<\cdots<\tau_r=t\}$$

then

$$S_\Pi(f,g):=\sum_{l=1}^r f(\tau_{l-1})\,\Delta_{\tau_l}g=f(\tau_0)\,\Delta_{\tau_1}g+\cdots+f(\tau_{r-1})\,\Delta_{\tau_r}g. \tag{5.1}$$

Lemma 5.2 (A key identity). *Suppose that $\Pi\in\mathcal{P}(s,t)$ and $u\in\Pi\cap(s,t)$. Then*

$$S_\Pi(f,g)-S_{\Pi\setminus\{u\}}(f,g)=\Delta_u f\Delta_{u+}g \tag{5.2}$$

where

$$\Delta_u f\Delta_{u+}g=[f(u)-f(u_-)]\,(g(u_+)-g(u)).$$

Proof. All terms in $S_\Pi(f,g)$ and $S_{\Pi\setminus\{u\}}(f,g)$ are the same except for those involving the intervals between u_- and u_+. Therefore we have,

$$\begin{aligned}S_{\Pi\setminus\{u\}}(f,g)-S_\Pi(f,g)&=f(u_-)\,[g(u_+)-g(u_-)]-\left[f(u_-)\,\Delta_u g+f(u)\,\Delta_{u+}g\right]\\&=f(u_-)\left[\Delta_u g+\Delta_{u+}g\right]-f(u_-)\,\Delta_u g-f(u)\,\Delta_{u+}g\\&=[f(u_-)-f(u)]\,\Delta_{u+}g\end{aligned}$$

■

Suppose that $\Pi\in\mathcal{P}(s,t)$ with $\#(\Pi):=r$ where $\#(\Pi)$ denotes the number of elements in Π minus one. Let us further suppose that we have chosen $\Pi_i\in\mathcal{P}(s,t)$ such that $\Pi_1\subset\Pi_2\subset\cdots\subset\Pi_{r-1}\subset\Pi_r:=\Pi$ and $\#(\Pi_i)=i$ for each i. Then

$$S_\Pi(f,g)-f(s)\,(g(t)-g(s))=\sum_{i=2}^r S_{\Pi_i}(f,g)-S_{\Pi_{i-1}}(f,g)$$

and thus we find the estimate,

$$\|S_\Pi(f,g)-f(s)\,(g(t)-g(s))\|\leq\sum_{i=2}^r\left\|S_{\Pi_i}(f,g)-S_{\Pi_{i-1}}(f,g)\right\|.$$

To get the best result from this procedure we should choose the sequence, $\{\Pi_i\}$, so as to minimize our estimate for $\left\|S_{\Pi_i}(f,g)-S_{\Pi_{i-1}}(f,g)\right\|$ at each step along the way. The next lemma is key ingredient in this procedure.

Lemma 5.3. *Suppose that $p,q\in(0,\infty)$, $\theta:=\frac{1}{p}+\frac{1}{q}$, and $a_i,b_i\geq 0$ for $i=1,2,\dots,n$, then*

$$\min_{1\leq i\leq n}a_ib_i\leq\left(\frac{1}{n}\right)^\theta\|a\|_p\,\|b\|_q \tag{5.3}$$

where $\|a\|_p:=\left(\sum_{i=1}^n a_i^p\right)^{1/p}$.

Proof. Let $r:=1/\theta$, $s:=p/r$, and $t:=q/r$, then $1/s+1/t=1$ and therefore by Hölder's inequality,

$$\|fg\|_r^r=\langle f^rg^r\rangle\leq\langle f^{rs}\rangle^{1/s}\left\langle g^{rt}\right\rangle^{1/t}=\langle f^p\rangle^{1/s}\langle g^q\rangle^{1/t}$$

which is to say,

$$\|fg\|_r\leq\|f\|_p\,\|g\|_q. \tag{5.4}$$

We also have

$$\min_{1\leq i\leq n}x_i=\left(\min_i x_i^r\right)^{1/r}\leq\left(\frac{1}{n}\sum_{i=1}^n x_i^r\right)^{1/r}=\left(\frac{1}{n}\right)^{1/r}\|x\|_r.$$

Taking $x_i=a_ib_i$ in this inequality and then using Eq. (5.4) implies,

$$\min_{1\le i\le n} a_ib_i \le \left(\frac{1}{n}\right)^{1/r} \|a.b.\|_r \le \left(\frac{1}{n}\right)^{1/r} \|a\|_p \|b\|_q .$$

Recalling that $1/r = \theta$ completes the proof of Eq. (5.3). ■

Alternatively we could use the following form of the geometric – arithmetic mean inequality.

Proposition 5.4 (Geometric-Arithmetic Mean Type Inequalities). *If $\{a_l\}_{l=1}^n$ is a sequence of non-negative numbers, $\{\alpha_l\}_{l=1}^n$ is a sequence of positive numbers such that $\sum_{l=1}^n \alpha_l = 1$, and $p>0$ is given, then*

$$a_1^{\alpha_1}\dots a_n^{\alpha_n} \le \left(\sum_{i=1}^n \alpha_i a_i^p\right)^{1/p} \tag{5.5}$$

and in particular by taking $\alpha_i = 1/n$ for all i we have,

$$[a_1\dots a_n]^{1/n} \le \left(\frac{1}{n}\sum_{i=1}^n a_i^p\right)^{1/p}. \tag{5.6}$$

Moreover, Eq. (5.3) is valid.

Proof. Without loss of generality we may assume that $a_i > 0$ for all i. By Jensen's inequality

$$a_1^{\alpha_1}\dots a_n^{\alpha_n} = \exp\left(\sum_{i=1}^n \alpha_i \ln a_i\right) \le \sum_{i=1}^n \alpha_i \exp(\ln a_i) = \sum_{i=1}^n \alpha_i a_i. \tag{5.7}$$

Replacing a_i by a_i^p in this inequality and then taking the p^{th} – root gives the inequality in Eq. (5.5). (Remark: if $p\ge 1$, Eq. (5.5) follows from Eq. (5.7) by an application of Hölder's inequality.) Making use of Eq. (5.6), we find again that

$$\begin{aligned}\min_{1\le i\le n} a_ib_i &\le [a_1b_1\dots a_nb_n]^{1/n} = [a_1\dots a_n]^{1/n}[b_1\dots b_n]^{1/n}\\ &\le \left(\frac{1}{n}\sum_{i=1}^n a_i^p\right)^{1/p}\left(\frac{1}{n}\sum_{i=1}^n b_i^q\right)^{1/q} = \left(\frac{1}{n}\right)^{\theta}\|a\|_p\|b\|_q .\end{aligned}$$

■

Proposition 5.5 (Young-Love Inequality). *Let $\Pi \in \mathcal{P}(s,t)$ and $r = \#(\Pi)-1$. Then $r :=$If $p,q\in(0,\infty)$ and $\theta := p^{-1}+q^{-1}$, then*

$$\begin{aligned}\|S_\Pi(f,g) - f(s)(g(t)-g(s))\| &\le \zeta_r(\theta) V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right) &(5.8)\\ &\le \zeta(\theta) V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right), &(5.9)\end{aligned}$$

where

$$\zeta_r(\theta) := \sum_{l=1}^{r-1}\frac{1}{l^\theta} \text{ and } \zeta(\theta) = \zeta_\infty(\theta) := \sum_{l=1}^{\infty}\frac{1}{l^\theta}. \tag{5.10}$$

Consequently,

$$\begin{aligned}\|S_\Pi(f,g)\| &\le \|f(s)\|\,\|g(t)-g(s)\| + \zeta_r(\theta)V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right) &(5.11)\\ &\le \left[\|f(s)\| + \zeta_r(\theta)V_p\left(f|_{[s,t]}\right)\right]\cdot V_q\left(g|_{[s,t]}\right). &(5.12)\end{aligned}$$

Proof. Let $\Pi_r := \Pi$ and choose $u \in \Pi_r \cap (s,t)$ such that

$$\begin{aligned}\|\Delta_u f\|\left\|\Delta_{u_+}g\right\| &= \min_{\tau\in\Pi\cap(s,t)}\|\Delta_\tau f\|\left\|\Delta_{\tau_+}g\right\|\\ &\le \left(\frac{1}{r-1}\right)^\theta\left(\sum_{\tau\in\Pi\cap(s,t)}\|\Delta_\tau f\|^p\right)^{1/p}\left(\sum_{\tau\in\Pi\cap(s,t)}\left\|\Delta_{\tau_+}g\right\|^q\right)^{1/q}\\ &\le \left(\frac{1}{r-1}\right)^\theta V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right).\end{aligned}$$

Thus letting $\Pi_{r-1} := \Pi\setminus\{u\}$, we have

$$\begin{aligned}\left\|S_\Pi(f,g) - S_{\Pi_{r-1}}(f,g)\right\| &= \left\|\Delta_u f\Delta_{u_+}g\right\| \le \|\Delta_u f\|\left\|\Delta_{u_+}g\right\|\\ &\le \left(\frac{1}{r-1}\right)^\theta V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right).\end{aligned}$$

Continuing this way inductively, we find $\Pi_i \in \mathcal{P}(s,t)$ such that $\Pi_1\subset\Pi_2\subset\dots\subset\Pi_{r-1}\subset\Pi_r := \Pi$ and $\#(\Pi_i) = i$ for each i and

$$\left\|S_{\Pi_i}(f,g) - S_{\Pi_{i-1}}(f,g)\right\| \le \left(\frac{1}{i-1}\right)^\theta V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right).$$

Thus using

$$S_\Pi(f,g) - f(s)(g(t)-g(s)) = \sum_{i=2}^r\left[S_{\Pi_i}(f,g) - S_{\Pi_{i-1}}(f,g)\right]$$

and the triangle inequality we learn that

$$\begin{aligned}\|S_\Pi(f,g) - f(s)(g(t)-g(s))\| &\le \sum_{i=2}^r\left\|S_{\Pi_i}(f,g) - S_{\Pi_{i-1}}(f,g)\right\|\\ &\le \sum_{i=2}^r\left(\frac{1}{i-1}\right)^\theta V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right)\\ &= \sum_{i=1}^{r-1}\left(\frac{1}{i}\right)^\theta V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right)\\ &= \zeta_r(\theta)V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right).\end{aligned}$$

■

Young gives examples showing that Eq. (5.9) fails if one only assume that $p^{-1}+q^{-1}=1$. See Theorem 4.26 on p. 33 of Dudley 98 [2] and Young (1936) [18] – Young constant is not as good as the one in [2].

Definition 5.6. *Given a function,* $X:\Delta\to V$ *and a partition,* $\Pi\in\mathcal{P}(s,t)$ *with* $(s,t)\in\Delta$*, let*

$$V_p(X:\Pi):=\left(\sum_{\tau\in\Pi}\left\|X_{\tau_-,\tau}\right\|^p\right)^{1/p}.$$

As usual we also let

$$V_p\left(X|_{[s,t]}\right):=\sup_{\Pi\in\mathcal{P}(s,t)}V_p(X:\Pi).$$

If $X_{s,t}=x(t)-x(s)$ for some $x:[0,T]\to V$, then $V_p(X:\Pi)=V_p(x:\Pi)$ and $V_p\left(X|_{[s,t]}\right)=V_p\left(x|_{[s,t]}\right)$.

Lemma 5.7. *If* $0\le u<v\le T$ *and*

$$Y_{st}:=f(s)(g(t)-g(s))\text{ for all }(s,t)\in\Delta, \tag{5.13}$$

then

$$V_q\left(Y|_{[u,v]}\right)\le\left\|f|_{[u,v]}\right\|_u V_q\left(g|_{[u,v]}\right). \tag{5.14}$$

Proof. If $\Pi\in\mathcal{P}(u,v)$, we have,

$$\begin{aligned}V_q^q(Y:\Pi)&=\sum_{\tau\in\Pi}\|f(\tau_-)(g(\tau)-g(\tau_-))\|^q\\&\le\sum_{\tau\in\Pi}\|f(\tau_-)\|^q\|(g(\tau)-g(\tau_-))\|^q\\&\le\left\|f|_{[u,v]}\right\|_u^q\sum_{\tau\in\Pi}\|(g(\tau)-g(\tau_-))\|^q=\left\|f|_{[u,v]}\right\|_u^q V_q^q(g:\Pi).\end{aligned}$$

The result follows by taking the supremum over $\Pi\in\mathcal{P}(u,v)$. ■

Corollary 5.8. *Suppose that* $V_p(f)<\infty$ *and* g *has finite variation, then for any* $q\in[1,\infty)$ *with* $\theta:=1/p+1/q>1$ *we have,*

$$\left\|\int_s^t fdg-f(s)(g(t)-g(s))\right\|\le\zeta(\theta)V_p\left(f|_{[s,t]}\right)\cdot V_q\left(g|_{[s,t]}\right). \tag{5.15}$$

and

$$V_q\left(\int_0^\cdot fdg\right)\le\left[\|f\|_u+\zeta(\theta)V_p(f)\right]\cdot V_q(g) \tag{5.16}$$

$$\le\left[\|f(0)\|+[1+\zeta(\theta)]V_p(f)\right]\cdot V_q(g). \tag{5.17}$$

More generally,

$$V_q\left(\left[\int_0^\cdot fdg\right]|_{[s,t]}\right)\le\left[\left\|f|_{[s,t]}\right\|_u+\zeta(\theta)V_p\left(f|_{[s,t]}\right)\right]\cdot V_q\left(g|_{[s,t]}\right) \tag{5.18}$$

$$\le\left[\|f(s)\|+[1+\zeta(\theta)]V_p\left(f|_{[s,t]}\right)\right]\cdot V_q\left(g|_{[s,t]}\right). \tag{5.19}$$

Proof. Inequality (5.15) follows from Eq. (5.9) upon letting $|\Pi|\to0$. For the remaining inequalities let Y_{st} be as in Eq. (5.13) and define,

$$X_{s,t}:=\int_s^t fdg-f(s)(g(t)-g(s))=\int_s^t fdg-Y_{s,t}.$$

Then according to Proposition 5.12 for any partition, $\Pi\in\mathcal{P}(0,T)$,

$$\begin{aligned}V_q^q(X:\Pi)&=\sum_{\tau\in\Pi}\left\|X_{\tau_-,\tau}\right\|^q=\sum_{\tau\in\Pi}\left\|\int_{\tau_-}^\tau fdg-f(\tau_-)(g(\tau)-g(\tau_-))\right\|^q\\&\le\sum_{\tau\in\Pi}\zeta^q(\theta)V_p^q\left(f|_{[\tau_-,\tau]}\right)\cdot V_q^q\left(g|_{[\tau_-,\tau]}\right)\\&\le\zeta^q(\theta)V_p^q(f)\cdot\sum_{\tau\in\Pi}V_q^q\left(g|_{[\tau_-,\tau]}\right)\le\zeta^q(\theta)V_p^q(f)\cdot V_q^q(g),\end{aligned}$$

wherein we have used $\omega(s,t):=V_q^q\left(g|_{[s,t]}\right)$ is a control for the last inequality. Taking the supremum over $\Pi\in\mathcal{P}(0,T)$ then implies,

$$V_q(X)\le\zeta(\theta)V_p(f)\cdot V_q(g). \tag{5.20}$$

Using the identity,

$$\int_s^t fdg=X_{st}+Y_{st},$$

the triangle inequality, Eq. (5.20), and Lemma 5.7, gives

$$\begin{aligned}V_q\left(\int_0^\cdot fdg\right)&=V_q(X+Y)\le V_q(X)+V_q(Y)\\&\le\left[\|f\|_u+\zeta(\theta)V_p(f)\right]\cdot V_q(g),\end{aligned}$$

which is Eq. (5.16). Equation (5.17) is an easy consequence of Eq. (5.16) and the simple estimate,

$$\|f(t)\|\le\|f(t)-f(0)\|+\|f(0)\|\le V_p(f)+\|f(0)\|. \tag{5.21}$$

The estimates in Eqs. (5.18) and (5.19) follow by the same techniques or a simple reparameterization argument. ■

Theorem 5.9. *If $V_p(g) < \infty$ and $V_q(f) < \infty$ with $\theta := 1/p + 1/q > 1$, then*

$$\int_s^t f dg := \lim_{n\to\infty} \int_s^t f dg_n \text{ exists} \tag{5.22}$$

where $\{g_n\}$ is any sequence of finite variation paths[1] *such that $V_{\tilde{p}}(g - g_n) \to 0$ for all $\tilde{q} > q$ with $\tilde{\theta} := 1/p + 1/\tilde{q} > 1$. This limit satisfies;*

$$\left\| \int_s^t f dg - f(s)(g(t) - g(s)) \right\| \le \zeta(\theta) V_p\left(f|_{[s,t]}\right) \cdot V_q\left(g|_{[s,t]}\right) \text{ for all } (s,t) \in \Delta, \tag{5.23}$$

$\int_s^t f dg$ is a bilinear form in f and g, the estimates in Eqs. (5.18) and (5.19) continue to hold, and

$$\int_s^t f dg := \lim_{\Pi\in\mathcal{P}(s,t) \text{ with } |\Pi|\to 0} S_\Pi(f,g). \tag{5.24}$$

Proof. From Eq. (5.19),

$$\begin{aligned} V_{\tilde{q}}\left(\left[\int_0^\cdot f dg_n - \int_0^\cdot f dg_m\right]|_{[s,t]}\right) &= V_{\tilde{q}}\left(\left[\int_0^\cdot f d(g_n - g_m)\right]|_{[s,t]}\right) \\ &\le \left[\|f(s)\| + [1 + \zeta(\theta)] V_p\left(f|_{[s,t]}\right)\right] \cdot V_{\tilde{q}}\left((g_n - g_m)|_{[s,t]}\right) \end{aligned}$$

which tends to 0 as $n \to \infty$. Therefore the limit in Eq. (5.22). Moreover, passing to the limit in Eq. (5.15) shows,

$$\left\| \int_s^t f dg - f(s)(g(t) - g(s)) \right\| \le \zeta(\theta) V_p\left(f|_{[s,t]}\right) \cdot V_{\tilde{q}}\left(g|_{[s,t]}\right).$$

We may now let $\tilde{q} \downarrow q$ to get the estimate in Eq. (5.23). This estimate gives those in Eqs. (5.18) and (5.19). The independence of the limit on the approximating sequence and the resulting bilinearity statement is left to the reader.

If $\Pi \in \mathcal{P}(s,t)$ it follows from the estimates in Eq. (5.23) and Eq. (5.12) that

$$\begin{aligned} \left\| \int_s^t f dg - S_\Pi(f,g) \right\| &\le \left\| \int_s^t f dg - \int_s^t f dg_n \right\| + \left\| \int_s^t f dg_n - S_\Pi(f,g_n) \right\| \\ &\quad + \|S_\Pi(f,g_n) - S_\Pi(f,g)\| \\ &\le 2\left[\|f(s)\| + \left[1 + \zeta\left(\tilde{\theta}\right)\right] V_p\left(f|_{[s,t]}\right)\right] V_{\tilde{q}}\left((g - g_n)|_{[s,t]}\right) \\ &\quad + \left\| \int_s^t f dg_n - S_\Pi(f,g_n) \right\|. \end{aligned}$$

[1] For exmaple, according to Corollary 4.11 , we can take $g_n := g^{\Pi_n}$ where $\Pi_n \in \mathcal{P}(s,t)$ with $|\Pi_n| \to 0$.

Therefore letting $|\Pi| \to 0$ in this inequality implies,

$$\limsup_{|\Pi|\to 0} \left\| \int_s^t f dg - S_\Pi(f,g) \right\| \le 2\left[\|f(s)\| + \left[1 + \zeta\left(\tilde{\theta}\right)\right] V_p\left(f|_{[s,t]}\right)\right] V_{\tilde{q}}\left((g - g_n)|_{[s,t]}\right)$$

which proves Eq. (5.24). ■

Lemma 5.10. *Suppose $V_p(g) < \infty$, $V_q(f) < \infty$ with $\theta := 1/p + 1/q > 1$. Let $\{\Pi_n\} \subset \mathcal{P}(s,t)$ and suppose that for each $t \in \Pi_n$ we are given $c_n(t) \in [t_-, t]$. Then*

$$\int_s^t f dg = \lim_{n\to\infty} \sum_{t\in\Pi_n} f(c_n(t)) \Delta_t g = \lim_{n\to\infty} \sum_{t\in\Pi_n} f(c_n(t))(g(t) - g(t_-)).$$

Proof. Let

$$\omega(s,t) := V_p^p\left(g|_{[s,t]}\right) + V_q^q\left(f|_{[s,t]}\right),$$

so that ω is a control. We then have,

$$\begin{aligned} \left\| \sum_{t\in\Pi_n} f(c_n(t)) \Delta_t g - S_{\Pi_n}(f,g) \right\| &= \left\| \sum_{t\in\Pi_n} [f(c_n(t)) - f(t_-)] \Delta_t g \right\| \\ &\le \sum_{t\in\Pi_n} \|[f(c_n(t)) - f(t_-)] \Delta_t g\| \\ &\le \sum_{t\in\Pi_n} \|f(c_n(t)) - f(t_-)\| \|\Delta_t g\| \\ &\le \sum_{t\in\Pi_n} \omega(t_-, c_n(t))^{1/p} \omega(t_-, t)^{1/q} \\ &\le \sum_{t\in\Pi_n} \omega(t_-, t)^{1/p} \omega(t_-, t)^{1/q} = \sum_{t\in\Pi_n} \omega(t_-, t)^\theta \\ &\le \sup_{t\in\Pi_n} \omega(t_-, t)^{\theta-1} \sum_{t\in\Pi_n} \omega(t_-, t) \\ &\le \sup_{t\in\Pi_n} \omega(t_-, t)^{\theta-1} \cdot \omega(0,T). \end{aligned} \tag{5.25}$$

Since $\omega(t,t) = 0$ for all $0 \le t \le T$ and $\omega : \Delta \to [0,\infty)$ is uniformly continuous on Δ, the last expression tends to zero as $n \to \infty$.

Alternate Proof. We can avoid the use the control, ω, here by making use of Hölder's inequality instead. To see this, let $q' := q/(q-1)$ be the conjugate exponent to q. Letting,

$$\delta_n := \max_{|t-s|\le|\Pi_n|} \|f(t) - f(s)\|^{\frac{q'-p}{p}} \to 0 \text{ as } n \to \infty,$$

we find,

$$\begin{aligned}\sum_{t\in\Pi_n}\|f(c_n(t))-f(t_-)\|\,\|\Delta_t g\| &\le \left(\sum_{t\in\Pi_n}\|f(c_n(t))-f(t_-)\|^{q'}\right)^{1/q'}\cdot\left(\sum_{t\in\Pi_n}\|\Delta_t g\|^q\right)^{1/q}\\ &\le \delta_n\left(\sum_{t\in\Pi_n}\|f(c_n(t))-f(t_-)\|^{p}\right)^{1/q'}\cdot V_q(g:\Pi)\\ &\le \delta_n\cdot V_q^{p/q'}(f)\cdot V_q(g)\to 0 \text{ as } n\to\infty.\end{aligned}$$

■

Remark 5.11. The same methods used to prove the estimate in Eq. (5.25) allows to give another proof of Eq. (5.24). Observe that

$$\int_s^t f dg^{\Pi} = \sum_{\tau\in\Pi}\int_{\tau_-}^{\tau} f dg^{\Pi} = \sum_{\tau\in\Pi}\int_{\tau_-}^{\tau} f(\sigma)\,\Delta_\tau g\frac{d\sigma}{\Delta\tau}$$

and therefore,

$$\int_s^t f dg^{\Pi} - S_{\Pi}(f,g) = \sum_{\tau\in\Pi}\int_{\tau_-}^{\tau}[f(\sigma)-f(\tau_-)]\,\Delta_\tau g\frac{d\sigma}{\Delta\tau}.$$

Taking norms of this equation and using the obvious inequalities implies,

$$\begin{aligned}\left\|\int_s^t f dg^{\Pi} - S_{\Pi}(f,g)\right\| &\le \sum_{\tau\in\Pi}\int_{\tau_-}^{\tau}\|f(\sigma)-f(\tau_-)\|\,\|\Delta_\tau g\|\frac{d\sigma}{\Delta\tau}\\ &\le \sum_{\tau\in\Pi}\int_{\tau_-}^{\tau}\omega(\tau_-,\tau)^{1/p}\,\omega(\tau_-,\tau)^{1/q}\frac{d\sigma}{\Delta\tau}\\ &= \sum_{\tau\in\Pi}\omega(\tau_-,\tau)^{\theta}\\ &\le \sup_{t\in\Pi}\omega(\tau_-,\tau)^{\theta-1}\cdot\omega(s,t)\to 0 \text{ as } |\Pi|\to 0.\end{aligned}$$

This observation proves Eq. (5.24) since, by definition,

$$\int_s^t f dg = \lim_{\Pi\in\mathcal{P}(s,t),\ |\Pi|\to 0}\int_s^t f dg^{\Pi}.$$

Exercise 5.1 (Product Rule). Suppose that V is a Banach space and $p,q>0$ such that $\theta := \frac{1}{p}+\frac{1}{q} > 1$, $x\in C([0,T]\to\mathrm{End}(V))$ with $V_q(x)<\infty$ and $y\in C([0,T]\to\mathrm{End}(V))$ with $V_p(y)<\infty$. Show for all $0\le s<t\le T$ that,

$$x(t)\,y(t)-x(s)\,y(s) = \int_s^t dx(\tau)\,y(\tau) + \int_s^t x(\tau)\,dy(\tau), \tag{5.26}$$

wherein the integrals are to be interpreted as Young's integrals.

Solution to Exercise (5.1). For $\Pi\in\mathcal{P}(s,t)$,

$$\begin{aligned}x(t)\,y(t)-x(s)\,y(s) &= \sum_{\tau\in\Pi}\Delta_\tau(xy)\\ &= \sum_{\tau\in\Pi}[(x(\tau_-)+\Delta_\tau x)(y(\tau_-)+\Delta_\tau y)-x(\tau_-)\,y(\tau_-)]\\ &= \sum_{\tau\in\Pi}[x(\tau_-)\,\Delta_\tau y+(\Delta_\tau x)\,y(\tau_-)+(\Delta_\tau x)\,\Delta_\tau y].\end{aligned}$$

Taking the limit at $|\Pi|\to 0$, the terms corresponding to the first two summands converge to $\int_s^t x(\tau)\,dy(\tau)$ and $\int_s^t dx(\tau)\,y(\tau)$ respectively. So it suffices to show, $\lim_{|\Pi|\to 0}\sum_{\tau\in\Pi}(\Delta_\tau x)\,\Delta_\tau y = 0$. However for every $\varepsilon>0$ we have,

$$\begin{aligned}\sum_{\tau\in\Pi}\|\Delta_\tau x\|\,\|\Delta_\tau y\| &\le \max_{\tau\in\Pi}\|\Delta_\tau x\|^{\varepsilon}\sum_{\tau\in\Pi}\|\Delta_\tau x\|^{1-\varepsilon}\|\Delta_\tau y\|\\ &\le \max_{\tau\in\Pi}\|\Delta_\tau x\|^{\varepsilon}\left(\sum_{\tau\in\Pi}\|\Delta_\tau x\|^{p'(1-\varepsilon)}\right)^{1/p'}\left(\sum_{\tau\in\Pi}\|\Delta_\tau y\|^{p}\right)^{1/p}\\ &= \max_{\tau\in\Pi}\|\Delta_\tau x\|^{\varepsilon}\cdot V_{p'(1-\varepsilon)}^{(1-\varepsilon)}\left(x|_{[s,t]}\right)V_p\left(y|_{[s,t]}\right),\end{aligned}$$

where $p' = \frac{p}{p-1}$ or equivalently, $\frac{1}{p'}+\frac{1}{p}=1$. As $\frac{1}{p}+\frac{1}{q}=\theta>1$ it follows that $q<p'$ and therefore we may choose $\varepsilon>0$ such that $p'(1-\varepsilon)=q$. For this ε we then have,

$$\sum_{\tau\in\Pi}\|\Delta_\tau x\|\,\|\Delta_\tau y\| \le \max_{\tau\in\Pi}\|\Delta_\tau x\|^{\varepsilon}\cdot V_q^{(1-\varepsilon)}\left(x|_{[s,t]}\right)V_p\left(y|_{[s,t]}\right)\to 0 \text{ as } |\Pi|\to 0.$$

Alternatively: let $\omega(s,t) := V_q^q\left(x|_{[s,t]}\right)+V_p^p\left(y|_{[s,t]}\right)$ which is a control on $[0,T]$. We then have,

$$\begin{aligned}\sum_{\tau\in\Pi}\|\Delta_\tau x\|\,\|\Delta_\tau y\| &\le \sum_{\tau\in\Pi}\omega(\tau_-,\tau)^{1/q}\,\omega(\tau_-,\tau)^{1/p} = \sum_{\tau\in\Pi}\omega(\tau_-,\tau)^{\theta}\\ &\le \max_{\tau\in\Pi}\omega(\tau_-,\tau)^{\theta-1}\sum_{\tau\in\Pi}\omega(\tau_-,\tau)\\ &\le \max_{\tau\in\Pi}\omega(\tau_-,\tau)^{\theta-1}\cdot\omega(s,t)\to 0 \text{ as } |\Pi|\to 0.\end{aligned} \tag{5.27}$$

Exercise 5.2. Find conditions on $\{x_i\}_{i=1}^n$ so as to be able to prove a product rule for $x_1(t)\dots x_n(t)$.

Lemma 5.12. *If $F : V \to W$ is a Lipschitz function with Lip – constant K, then*

$$V_p(F(Z.)) \le K V_p(Z). \tag{5.28}$$

Proof. For $\Pi \in \mathcal{P}(0,T)$, we have

$$\sum_{\tau\in\Pi} \|F(Z(\tau)) - F(Z(\tau_-))\|^p \le K^p \sum_{\tau\in\Pi} \|Z(\tau) - Z(\tau_-)\|^p = K^p V_p^p(Z : \Pi) \le K^p V_p^p(Z).$$

Therefore taking the supremum over all $\Pi \in P(0,T)$ gives Eq. (5.28). ∎

Exercise 5.3 (Fundamental Theorem of Calculus II). Prove the fundamental theorem of calculus in this context. That is; if $f : V \to W$ be a C^1 – function such that f' is Lipschitz and $\{Z_t\}_{t\ge0}$ is a continuous V – valued function such that $V_p(Z) < \infty$ for some $p \in (1,2)$. Then $V_p(f'(Z.)) < \infty$ and for all $0 \le a < b \le T$,

$$f(Z_b) - f(Z_a) = \int_a^b f'(Z_\tau)\, dZ_\tau := \int_{[a,b]} f'(Z_\tau)\, dZ_\tau, \tag{5.29}$$

where $f'(z) \in \mathrm{End}(V,W)$ is defined by, $f'(z)v := \frac{d}{dt}|_0 f(z+tv)$. In particular it follows that $f(Z(t))$ has finite p – variation and

$$df(Z(t)) = f'(Z(t))\, dZ(t).$$

The integrals in Eq. (5.29) are to be interpreted as Young's integrals.

Solution to Exercise (5.3). Let $\Pi \in \mathcal{P}(0,T)$. Because of Lemma 5.12 and the assumption that $p \in (1,2)$ (so that $1/p + 1/p = 2/p =: \theta > 1$), we see that the integral in Eq. (5.29) is well defined as a Young's integral. By a telescoping series argument,

$$f(Z_b) - f(Z_a) = \sum_{t\in\Pi} \Delta_t f(Z.)$$

where

$$\Delta_t f(Z.) = f(Z_t) - f(Z_{t_-}) = f(Z_{t_-} + \Delta_t Z) - f(Z_{t_-}) = \int_0^1 f'(Z_{t_-} + s\Delta_t Z)\, \Delta_t Z\, ds = f'(Z_{t_-})\, \Delta_t Z + \varepsilon_t^\Pi \Delta_t Z$$

and

$$\varepsilon_t^\Pi := \int_0^1 \left[f'(Z_{t_-} + s\Delta_t Z) - f'(Z_{t_-})\right] ds.$$

Thus we have,

$$f(Z_b) - f(Z_a) = \sum_{t\in\Pi} f'(Z_{t_-})\, \Delta_t Z + \delta_\Pi \tag{5.30}$$

where

$$\delta_\Pi := \sum_{t\in\Pi} \varepsilon_t^\Pi \Delta_t Z.$$

Letting $\omega(s,t) := V_p^p\left(Z|_{[s,t]}\right)$, we have,

$$\left\|\varepsilon_t^\Pi\right\| \le \int_0^1 \left\|f'(Z_{t_-} + s\Delta_t Z) - f'(Z_{t_-})\right\| ds \le K \int_0^1 s \|\Delta_t Z\|\, ds \le K \frac{1}{2} \omega(t_-,t)^{1/p}$$

and hence

$$\|\delta_\Pi\| \le \sum_{t\in\Pi} \left\|\varepsilon_t^\Pi\right\| \|\Delta_t Z\| \le \frac{K}{2} \sum_{t\in\Pi} \omega(t_-,t)^{1/p} \cdot \omega(t_-,t)^{1/p} \le \frac{K}{2} \sum_{t\in\Pi} \omega(t_-,t)^\theta \to 0 \text{ as } |\Pi| \to 0$$

as we saw in Eq. (5.27). Thus letting $|\Pi| \to 0$ in Eq. (5.30) completes the proof.

Exercise 5.4. See what you can say about a substitution formula in this case. Namely, suppose that

$$y(t) = \int_0^t f(s)\, dx(s)$$

as a Young's integral. Find conditions so that

$$\int_0^t g(t)\, dy(t) = \int_0^t g(s) f(s)\, dx(s).$$

5.1 Additive (Almost) Rough Paths

Remark 5.13. For an alternate approach to this section, see [3].

Notation 5.14 *Suppose that Π is a partition of $[u,v]$, i.e. a finite subset of $[u,v]$ which contains both u and v. For $u \le s < t \le v$, let*

$$\Pi_{[s,t]} = \{s,t\} \cup \Pi \cap [s,t].$$

Typically we will write

$$\Pi_{[s,t]} = \{s = t_0 < t_1 < \cdots < t_r = t\}$$

where

$$\Pi \cap (s,t) =: \{t_1 < t_2 < \cdots < t_{r-1}\}.$$

Notation 5.15 *Given a function, $X : \Delta \to V$ and a partition,*

$$\Pi = \{s = t_0 < t_1 < \cdots < t_r = t\},$$

of $[s,t]$, let

$$X(\Pi) := \sum_{\tau \in \Pi} X_{\tau_-,\tau} = \sum_{i=1}^{r} X_{t_{i-1},t_i}.$$

Furthermore, given a partition, Π, of $[0,T]$ and $(s,t) \in \Delta$ let

$$X(\Pi)_{st} := X(\Pi_{[s,t]}) = \sum_{\tau \in \Pi_{[s,t]}} X_{\tau_-,\tau}.$$

Definition 5.16. *As usual, let $\Delta := \{(s,t) : 0 \le s \le t \le T\}$ and $p \ge 1$. We say that a function, $X : \Delta \to V$ has finite p - variation if X is continuous, $X_{t,t} = 0$[2] for all $t \in [0,T]$, and*

$$V_p(X) := \left(\sup_{\Pi \in \mathcal{P}(0,T)} \sum_{t \in \Pi} \left\| X_{t_-,t} \right\|_V^p \right)^{1/p} < \infty.$$

Definition 5.17. *Let $\theta > 1$. A θ – **almost additive functional** (A.A.F.) is a function $X : \Delta \to V$ of finite p -variation such that there exists a control, ω, $C < \infty$ such that*

$$\|X_{st} - X_{su} - X_{ut}\| \le C\omega(s,t)^\theta \text{ for all } 0 \le s \le u \le t \le T. \tag{5.31}$$

*If Eq. (5.31) holds for some $\theta > 1$ and control ω, we say X is an (ω, p) – **almost additive functional**.*

Example 5.18. Suppose that $V_p(f) + V_q(g) < \infty$ with $\theta := 1/p + 1/q > 1$,

$$X_{st} := f(s)(g(t) - g(s)),$$

and $\omega(s,t)$ be the control defined by

$$\omega(s,t) := V_p^p\left(f|_{[s,t]}\right) + V_q^q\left(g|_{[s,t]}\right).$$

[2] This is redundant since $V_p(X) < \infty$ can only happen if $X_{t,t} = 0$ for all t.

Then

$$\begin{aligned}\|X_{st} - X_{su} - X_{ut}\| &= \left\| \begin{matrix} f(s)(g(t) - g(s)) - f(s)(g(u) - g(s)) \\ -f(u)(g(t) - g(u)) \end{matrix} \right\| \\ &\le \|(f(s) - f(u))(g(t) - g(u))\| \\ &\le \|f(s) - f(u)\| \, \|g(t) - g(u)\| \\ &\le V_p\left(f|_{[s,t]}\right) V_q\left(g|_{[s,t]}\right) \le \omega(s,t)^{1/p}\,\omega(s,t)^{1/q} = \omega(s,t)^\theta.\end{aligned}$$

Thus X_{st} is a θ – A.A.F.

Notation 5.19 *Suppose that $X : \Delta \to V$ is a any function and $\Pi \subset [0,T]$ is a finite set and $(s,t) \in \Delta$. Then define,*

$$X(\Pi)_{s,t} := \sum_{\tau \in \Pi \cap [s,t] \cup \{s,t\}} X_{\tau_-,\tau}.$$

The following lemma explains the reason for introducing this notation.

Lemma 5.20. *Suppose that $X : \Delta \to V$ is a continuous function such that $X_{t,t} = 0$ for all $t \in [0,T]$ and there exists $\Pi_n \in \mathcal{P}(0,T)$ such that $\lim_{n\to\infty} |\Pi_n| = 0$ and*

$$Y_{st} := \lim_{n\to\infty} X(\Pi_n)_{s,t} \text{ exists for } (s,t) \in \Delta.$$

Then Y_{st} is an additive functional.

Proof. Suppose that $0 \le s < u < t \le T$, then

$$Y_{su} + Y_{ut} = \lim_{n\to\infty}\left[X(\Pi_n)_{s,u} + X(\Pi_n)_{u,t}\right] = \lim_{n\to\infty} X(\Pi_n \cup \{u\})_{s,t}. \tag{5.32}$$

If $u \in \Pi_n$ we have $X(\Pi_n \cup \{u\})_{s,t} = X(\Pi_n)_{s,t}$ while if $u \notin \Pi_n$, then

$$\begin{aligned}\left\| X(\Pi_n \cup \{u\})_{s,t} - X(\Pi_n)_{s,t} \right\| &= \left\| X_{u_-,u} + X_{u,u_+} - X_{u_-,u} \right\| \\ &\le \left\| X_{u_-,u} \right\| + \left\| X_{u,u_+} \right\| + \left\| X_{u_-,u} \right\| \le 3 \cdot \delta_n\end{aligned}$$

where

$$\delta_n := \max\{\|X_{s,t}\| : (s,t) \in \Delta \ni |t - s| \le |\Pi_n|\}.$$

As $\delta_n \to 0$ by the uniform continuity of $X_{s,t}$, we see that

$$\lim_{n\to\infty} X(\Pi_n \cup \{u\})_{s,t} = \lim_{n\to\infty} X(\Pi_n)_{s,t} = Y_{st}$$

which combined with Eq. (5.32) shows Y_{st} is additive. ■

Lemma 5.21. *If $X : \Delta \to V$ is a (θ, ω) – almost additive functional, then there is at most one additive functional, $Y : \Delta \to V$, such that*

$$\|Y_{st} - X_{st}\| \leq C\omega(s,t)^{\theta} \text{ for all } (s,t) \in \Delta$$

for some $C < \infty$.

Proof. If $Z : \Delta \to V$ is another such additive functional. Then $U_{st} := Y_{st} - Z_{st}$ is an additive functional such that

$$\|U_{st}\| = \|Y_{st} - Z_{st}\| \leq \|Y_{st} - X_{st}\| + \|X_{st} - Z_{st}\| \leq 2C\omega(s,t)^{\theta}.$$

Therefore if $\Pi \in \mathcal{P}(s,t)$, we have

$$\|U_{st}\| = \left\|\sum_{\tau \in \Pi} U_{\tau_-,\tau}\right\| \leq \sum_{\tau \in \Pi} \left\|U_{\tau_-,\tau}\right\| \leq 2C \sum_{\tau \in \Pi} \omega(\tau_-, \tau)^{\theta} \to 0 \text{ as } |\Pi| \to 0.$$

∎

Lemma 5.22. *Suppose $\Pi = \{s = t_0 < \cdots < t_r = t\}$ with $r \geq 2$ and ω is a control. Then there exists $j \in \{1, 2, \ldots, r-1\}$ such that*

$$\omega(t_{j-1}, t_{j+1}) \leq \min\left(\frac{2}{r-1}, 1\right)\omega(s,t) \tag{5.33}$$

Proof. If $r = 2$, we take $j = 1$ in which case $\omega(t_{j-1}, t_j) = \omega(s,t)$ and Eq. (5.33) clearly holds. Now suppose $r \geq 3$. Considering these intervals two at a time, by super-additivity, we have,

$$\sum_{k=0}^{\infty} \omega(t_{2k}, t_{2k+2}) 1_{2k+2 \leq r} = \omega(t_0, t_2) + \omega(t_2, t_4) + \omega(t_4, t_6) + \cdots \leq \omega(s,t)$$

and

$$\sum_{k=0}^{\infty} \omega(t_{2k-1}, t_{2k+1}) 1_{2k+1 \leq r} = \omega(t_1, t_3) + \omega(t_3, t_5) + \omega(t_5, t_7) + \cdots \leq \omega(s,t).$$

Adding these two equations then dividing by $r - 1$ shows

$$\frac{1}{r-1} \sum_{j=1}^{r-1} \omega(t_{j-1}, t_{j+1}) \leq \frac{2}{r-1}\omega(s,t)$$

from which it follows that (5.33) holds for some j. ∎

Proposition 5.23. *Suppose that $X : \Delta \to V$ is an (ω, θ) – almost additive functional. Then for any partition, $\Pi \in \mathcal{P}(s,t)$, we have*

$$\|X(\Pi) - X_{s,t}\| \leq \zeta(\theta)\,\omega^{\theta}(s,t).$$

Proof. For any $\tau \in \Pi \cap (s,t)$, let $\Pi(\tau) := \Pi \setminus \{\tau\}$ and observe that

$$\|X(\Pi) - X(\Pi(\tau))\| = \left\|X_{\tau_-,\tau_+} - X_{\tau_-,\tau} - X_{\tau,\tau_+}\right\| \leq \omega(\tau_-, \tau_+)^{\theta}. \tag{5.34}$$

Thus making use of Lemma 5.22 implies,

$$\min_{\tau \in \Pi \cap (s,t)} \|X(\Pi) - X(\Pi(\tau))\| \leq \min_{\tau \in \Pi \cap (s,t)} \omega(\tau_-, \tau_+)^{\theta} \leq \min\left(\frac{2}{|\Pi| - 2}, 1\right)^{\theta} \omega^{\theta}(s,t).$$

Thus we remove points τ from $\Pi \cap (s,t)$ so as to minimize the error to eventually learn,

$$\|X(\Pi) - X_{s,t}\| \leq \left(\sum_{k=1}^{|\Pi|-2} \frac{1}{k^{\theta}}\right) \omega^{\theta}(s,t) \leq \zeta(\theta)\,\omega^{\theta}(s,t).$$

∎

Theorem 5.24. *Let $X : \Delta \to V$ be a continuous (ω, θ) – almost additive functional and Π denote a partition of $[0,T]$. Then*

$$Y_{st} := \lim_{|\Pi| \to 0} X(\Pi)_{st} \text{ exists uniformly in } (s,t) \in \Delta. \tag{5.35}$$

Moreover, $Y : \Delta \to V$ is a continuous additive functional and Y_{st} is the (unique) additive functional such that

$$\|Y_{st} - X_{st}\| \leq C\omega(s,t)^{\theta} \text{ for all } (s,t) \in \Delta \tag{5.36}$$

for some $C < \infty$. In fact according to Proposition 5.23 we know that C may be chosen to be $\zeta(\theta)$.

Proof. Suppose that $\Pi, \Pi' \in \mathcal{P}(s,t)$ with $\Pi \subset \Pi'$ and for $\varepsilon > 0$ let

$$\delta(\varepsilon) := \max_{|\tau - \sigma| \leq \varepsilon} \omega^{\theta - 1}(\sigma, \tau).$$

(Observe that $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.) Then making use of Proposition 5.23 we find,

$$\begin{aligned}\|X(\Pi') - X(\Pi)\| &= \left\|\sum_{\tau\in\Pi}\left(X(\Pi')_{\tau_-,\tau} - X_{\tau_-,\tau}\right)\right\| \\ &\le \sum_{\tau\in\Pi}\left\|X(\Pi')_{\tau_-,\tau} - X_{\tau_-,\tau}\right\| \\ &\le \zeta(\theta)\sum_{\tau\in\Pi}\omega^{\theta}(\tau_-,\tau) \\ &\le \zeta(\theta)\max_{|\tau-\sigma|\le|\Pi|}\omega(\tau,\sigma)^{\theta-1}\sum_{\tau\in\Pi}\omega(\tau_-,\tau) \\ &= \zeta(\theta)\,\delta(|\Pi|)\max_{|\tau-\sigma|\le|\Pi|}\omega(\tau,\sigma)^{\theta-1}\cdot\omega(s,t)\,.\end{aligned}$$

Now let $\Pi_1, \Pi_2 \in \mathcal{P}(0,T)$ be arbitrary and apply the previous inequality with $\Pi' = [\Pi_1 \cup \Pi_2]_{[s,t]}$ and Π being either $[\Pi_1]_{[s,t]}$ or $[\Pi_2]_{[s,t]}$ to find,

$$\begin{aligned}\|[X(\Pi_1) - X(\Pi_2)]_{st}\| &\le \|[X(\Pi_1) - X(\Pi_1\cup\Pi_2)]_{st}\| \\ &\quad + \|[X(\Pi_1\cup\Pi_2) - X(\Pi_2)]_{st}\| \\ &\le \zeta(\theta)\,\omega(s,t)\cdot[\delta(|\Pi_1|) + \delta(|\Pi_2|)] \\ &\le \zeta(\theta)\,\omega(0,T)\cdot[\delta(|\Pi_1|) + \delta(|\Pi_2|)]\,.\end{aligned}$$

Therefore,

$$\max_{(s,t)\in\Delta}\|[X(\Pi_1) - X(\Pi_2)]_{st}\| \le \zeta(\theta)\,\omega(0,T)\cdot[\delta(|\Pi_1|) + \delta(|\Pi_2|)]$$

which tends to zero as $|\Pi_1|, |\Pi_2| \to 0$. This proves Eq. (5.35). The remaining assertions of the theorem were already proved in Lemma 5.20 and Lemma 5.21. ■

Corollary 5.25. *Let $X : \Delta \to V$ be a continuous (ω, θ) – almost additive functional of finite p – variation, then the unique associated additive functional, Y, of Theorem 5.24 is also of finite p – variation and*

$$V_p(Y) \le V_p(X) + C\omega(0,T)^{\theta}\,. \tag{5.37}$$

Proof. By the triangle inequality,

$$V_p(Y) \le V_p(Y - X) + V_p(X)$$

and using Eq. (5.36), for any $\Pi \in \mathcal{P}(0,T)$,

$$\begin{aligned}V_p^p(Y - X : \Pi) &\le C^p\sum_{\tau\in\Pi}\omega(\tau_-,\tau)^{\theta p} \le C^p\omega(0,T)^{\theta p-1}\sum_{\tau\in\Pi}\omega(\tau_-,\tau) \\ &\le C^p\omega(0,T)^{\theta p-1}\omega(0,T) = C^p\omega(0,T)^{\theta p}\,.\end{aligned}$$

Hence it follows that $V_p(Y - X) \le C\omega(0,T)^{\theta}$. ■

5.2 Young's ODE

Now suppose that $1 < p < 2$, so that $\theta := \frac{1}{p} + \frac{1}{p} = 2/p > 1$. Also let V and W be Banach spaces, $f : W \to \mathrm{End}(V,W)$ be a Lipschitz function, $x \in C_p([0,T], V)$ and consider the ODE,

$$\dot{y}(t) = f(y(t))\,\dot{x}(t) \text{ with } y(0) = y_0. \tag{5.38}$$

Definition 5.26. *We say that a function, $y : [0,T] \to V$, solves Eq. (5.38) if $y \in C_p([0,T], W)$ and y satisfies the integral equation,*

$$y(t) = y_0 + \int_0^t f(y(\tau))\,dx(\tau)\,, \tag{5.39}$$

where the latter integral is a the Young integral.

Recall from Lemma 5.12 that

$$V_p(f(y)) \le kV_p(y)\,, \tag{5.40}$$

where k is the Lipschitz constant for f and hence the integral in Eq. (5.39) is well defined. In order to consider existence, uniqueness, and continuity in the driving path x of Eq. (5.46) we will need a few more facts about p – variations.

5.3 An a priori – Bound

Before going on to existence, uniqueness and the continuous dependence of initial condition and the driving noise for Eq. (5.39), we will pause to prove an a priori bound on the solution to Eq. (5.39) which is valid under less restrictive conditions on f.

Proposition 5.27 (Discrete Gronwall's Inequalities). *Suppose that $u_i, \alpha_i, \beta_i \ge 0$ satisfy*

$$u_{i+1} \le \alpha_i u_i + \beta_i,$$

then

$$u_n \le \alpha_{n-1}\dots\alpha_1\alpha_0 u_0 + \sum_{k=0}^{n-1}\left(\prod_{j=1}^{n-1-k}\alpha_j\right)\beta_k. \tag{5.41}$$

Moreover if $\alpha_i = \alpha$ is constant (so that $u_{i+1} \le \alpha u_i + \beta_i$, then this reduces to

$$u_n \le \alpha^n u_0 + \sum_{i=0}^{n-1}\alpha^{n-1-i}\beta_i \tag{5.42}$$

$$\le \alpha^n u_0 + \alpha^{n-1}\sum_{i=0}^{n-1}\beta_i \text{ if } \alpha \ge 1. \tag{5.43}$$

If we further assume $\beta_i = \beta$ is constant (so that $u_{i+1} \leq \alpha u_i + \beta$), then

$$u_n \leq \alpha^n \left(u_0 + \frac{1-\alpha^{-n}}{\alpha - 1}\beta \right). \tag{5.44}$$

If we further assume that $\alpha > 1$, then

$$u_n \leq \alpha^n \left(u_0 + \frac{\beta}{\alpha - 1} \right). \tag{5.45}$$

Proof. The inequality in Eq. (5.41) is proved inductively as

$$\begin{aligned}
&u_1 \leq \alpha_0 u_0 + \beta_0 \\
&u_2 \leq \alpha_1 u_1 + \beta_1 \leq \alpha_1 (\alpha_0 u_0 + \beta_0) + \beta_1 = \alpha_1 \alpha_0 u_0 + \alpha_1 \beta_0 + \beta_1 \\
&u_3 \leq \alpha_2 u_2 + \beta_2 \leq \alpha_2 (\alpha_1 \alpha_0 u_0 + \alpha_1 \beta_0 + \beta_1) + \beta_2 = \alpha_2 \alpha_1 \alpha_0 u_0 + \alpha_2 \alpha_1 \beta_0 + \alpha_2 \beta_1 + \beta_2,
\end{aligned}$$

etc.

Since the special case where $\alpha_i = \alpha$ is the most important case to us, let us give another proof for this case. If we let $v_i := \alpha^{-i} u_i$, then

$$v_{i+1} = \alpha^{-(i+1)} u_{i+1} \leq \alpha^{-(i+1)} (\alpha u_i + \beta_i) = v_i + \alpha^{-(i+1)} \beta_i$$

which is to say,

$$v_{i+1} - v_i \leq \alpha^{-(i+1)} \beta_i.$$

Summing this expression on i implies,

$$\alpha^{-n} u_n - u_0 = v_n - v_0 = \sum_{i=0}^{n-1} (v_{i+1} - v_i) \leq \sum_{i=0}^{n-1} \alpha^{-(i+1)} \beta_i$$

which upon solving for u_n gives Eq. (5.42). When $\beta_i = \beta$ is constant, we use

$$\sum_{i=0}^{n-1} \alpha^{n-1-i} = \sum_{i=0}^{n-1} \alpha^i = \frac{\alpha^n - 1}{\alpha - 1}$$

in Eq. (5.42) to learn,

$$u_n \leq \alpha^n u_0 + \frac{\alpha^n - 1}{\alpha - 1}\beta = \alpha^n \left(u_0 + \frac{1 - \alpha^{-n}}{\alpha - 1}\beta \right)$$

which is Eq. (5.44). ∎

Theorem 5.28 (A priori Bound). *Let $1 < p < 2$, $f(y)$ be a Lipschitz function, and $x \in C([0,T] \to V)$ be a path such that $V_p(x) < \infty$. Then there exists $C(p) < \infty$ such that for all solutions to Eq. (5.39),*

$$V_p^p(y) \leq C(p) e^{C(p)k^p V_p^p(x)} \left(\|y_0\|^p + \|f(0)\|^p V_p^p(x) \right) \tag{5.46}$$

where k is the Lipschitz constant for f.

Proof. Let $\omega(s,t) := V_p^p(x : [s,t])$ be the control associated to x and define

$$\kappa = \kappa(p) := 1 + \zeta(2/p).$$

If y solves Eq. (5.39), then

$$y(t) = y(s) + \int_s^t f(y(\tau))\, dx(\tau) \text{ for all } 0 \leq s \leq t \leq T.$$

So by By Corollary 5.8 (with $p = q$) along with Eq. (5.40), we learn that

$$\begin{aligned}
V_p(y : [s,t]) &= V_p\left(\int_s^{\cdot} f(y)\, dx : [s,t] \right) \\
&\leq [\|f(y(s))\| + \kappa V_p(f(y) : [s,t])] V_p(x : [s,t]) \\
&\leq [\|f(y(s))\| + \kappa k V_p(y : [s,t])] \omega(s,t)^{1/p}
\end{aligned}$$

or equivalently, with $c := \kappa k$,

$$\left(1 - c\omega(s,t)^{1/p} \right) V_p(y : [s,t]) \leq \|f(y(s))\| \omega(s,t)^{1/p}.$$

Therefore it follows that

$$V_p(y : [s,t]) \leq 2\omega(s,t)^{1/p} \|f(y(s))\| \text{ if } \omega(s,t)^{1/p} \leq 1/2c. \tag{5.47}$$

Since

$$\|f(y)\| \leq \|f(y) - f(0)\| + \|f(0)\| \leq k\|y\| + \|f(0)\|$$

and

$$\|y(t)\| \leq \|y(s)\| + V_p(y : [s,t]),$$

it follows from Eq. (5.47) that

$$\begin{aligned}
V_p(y : [s,t]) &\leq 2\omega(s,t)^{1/p} [k\|y(s)\| + \|f(0)\|] && (5.48) \\
&\leq \frac{1}{\kappa}\|y(s)\| + 2\omega(s,t)^{1/p}\|f(0)\| \text{ if } \omega(s,t)^{1/p} \leq 1/2c && (5.49)
\end{aligned}$$

and

$$\|y(t)\| \leq \left(1 + \frac{1}{\kappa} \right) \|y(s)\| + 2\omega(s,t)^{1/p} \|f(0)\| \text{ if } \omega(s,t)^{1/p} \leq 1/2c. \tag{5.50}$$

In order to make use of this result, let $h(t) := \omega(0,t)$. Write $h(T) = n\left(\frac{1}{2c}\right)^p + r$ where $0 \leq r < \left(\frac{1}{2c}\right)^p$ and then choose $0 = t_0 < t_1 < t_2 < \cdots < t_n \leq t_{n+1} := T$ such that $h(t_i) = i\left(\frac{1}{2c}\right)^p$ for $0 \leq i \leq n$. We then have

$$\omega(t_i, t_{i+1}) \leq h(t_{i+1}) - h(t_i) = \left(\frac{1}{2c} \right)^p \text{ for } 0 \leq i \leq n.$$

Therefore we may conclude from Eq. (5.50) that

$$\|y(t_{i+1})\| \le \left(1+\frac{1}{\kappa}\right)\|y(t_i)\| + 2\omega(t_i,t_{i+1})^{1/p}\|f(0)\| \text{ for } i=0,1,2\dots,n$$

and hence that,

$$\begin{aligned}\|y(t_{i+1})\|^p &\le 2^{p-1}\left(1+\frac{1}{\kappa}\right)^p\|y(t_i)\|^p + 2^{2p-1}\omega(t_i,t_{i+1})\|f(0)\|^p\\ &\le (2e)^p\|y(t_i)\|^p + 2^{2p-1}\omega(t_i,t_{i+1})\|f(0)\|^p\\ &\le 4^p\left(\|y(t_i)\|^p + \omega(t_i,t_{i+1})\|f(0)\|^p\right).\end{aligned}$$

Therefore by an application of the discrete Gronwall inequality in Eq. (5.43) we have,

$$\begin{aligned}\|y(t_i)\|^p &\le 4^{ip}\|y_0\|^p + 4^{(i-1)p}\|f(0)\|^p\sum_{l=0}^{i-1}\omega(t_l,t_{l+1})\\ &\le 4^{ip}\|y_0\|^p + 4^{(i-1)p}\|f(0)\|^p\omega(0,t_i)\end{aligned}$$

and in particular it follows that

$$\|y(T)\|^p \le 4^{(n+1)p}\|y_0\|^p + 4^{np}\|f(0)\|^p\omega(0,T).$$

Going back to Eq. (5.49) we have,

$$V_p^p(y:[s,t]) \le 2^{p-1}\left(\frac{1}{\kappa}\right)^p\|y(s)\|^p + 2^{2p-1}\omega(s,t)\|f(0)\|^p \text{ if } \omega(s,t)^{1/p}\le 1/2c$$

and therefore,

$$\begin{aligned}V_p^p(y:[t_i,t_{i+1}]) \le& 2^{p-1}\left(\frac{1}{\kappa}\right)^p\|y(t_i)\|^p + 2^{2p-1}\omega(t_i,t_{i+1})\|f(0)\|^p\\ \le& 2^{p-1}\left(\frac{1}{\kappa}\right)^p\left[4^{ip}\|y_0\|^p + 4^{(i-1)p}\|f(0)\|^p\omega(0,t_i)\right]\\ &+2^{2p-1}\omega(t_i,t_{i+1})\|f(0)\|^p\\ \le& C(p)\left(\|y_0\|^p + \|f(0)\|^p\omega(0,T)\right)4^{ip}.\end{aligned}$$

Summing this result on i and making use of Corollary 4.4 then implies,

$$\begin{aligned}V_p^p(y:[0,T]) &\le (n+1)^{p-1}\sum_{i=0}^n V_p^p(y:[t_i,t_{i+1}])\\ &\le (n+1)^{p-1}C(p)\left(\|y_0\|^p + \|f(0)\|^p\omega(0,T)\right)\frac{4^{(n+1)p}-1}{4-1}\\ &\le C(p)\left(\|y_0\|^p + \|f(0)\|^p\omega(0,T)\right)4^{(n+1)p}(n+1)^{p-1}.\end{aligned}$$

Finally, observing that $n\left(\frac{1}{2c}\right)^p \le h(T) = \omega(0,T)$ so that $n \le (2c)^p\omega(0,T)$ we have,

$$\begin{aligned}V_p^p(y:[0,T]) &\le C(p)\left(\|y_0\|^p + \|f(0)\|^p\omega(0,T)\right)4^{((2c)^p\omega(0,T)+1)p}\left((2c)^p\omega(0,T)+1\right)^{p-1}\\ &\le C(p)\,e^{C(p)k^p\omega(0,T)}\left(\|y_0\|^p + \|f(0)\|^p\omega(0,T)\right),\end{aligned}$$

which is Eq. (5.46). ■

5.4 Some $p-$ Variation Estimates

Lemma 5.29. *Suppose that $f \in C_p([0,T],\mathrm{End}(V,W))$ and $x \in C_p([0,T],V)$, then $(fx)(t) = f(t)x(t)$ is in $C_p([0,T],W)$ and*

$$V_p(fx) \le 2\left[\|f\|_u V_p(x) + \|x\|_u V_p(f)\right]. \tag{5.51}$$

Proof. Let $\Pi \in \mathcal{P}(0,T)$, $t\in\Pi$ and $f_- := f(t_-)$ and $x_- := x(t_-)$, then

$$\begin{aligned}\|\Delta_t(fx)\| &= \|(f_-+\Delta_t f)(x_-+\Delta_t x) - f_-x_-\|\\ &= \|f_-\Delta_t x + \Delta_t f x_- + \Delta_t f\Delta_t x\|\\ &\le \|f\|_u\|\Delta_t x\| + \|x\|_u\|\Delta_t f\| + \frac{1}{2}\left(\|\Delta_t f\Delta_t x\| + \|\Delta_t f\Delta_t x\|\right)\\ &\le \|f\|_u\|\Delta_t x\| + \|x\|_u\|\Delta_t f\| + \frac{1}{2}\left(2\|f\|_u\|\Delta_t x\| + 2\|x\|_u\|\Delta_t f\|\right)\\ &\le 2\left(\|f\|_u\|\Delta_t x\| + \|x\|_u\|\Delta_t f\|\right).\end{aligned}$$

Therefore it follows that

$$V_p(fx:\Pi) \le 2\|f\|_u V_p(x:\Pi) + 2\|x\|_u V_p(f:\Pi)$$

from which the result follows. ■

Theorem 5.30. *Suppose that W and Z are Banach spaces and $f \in C^2(W\to Z)$ with $f' \in C^1(W\to\mathrm{End}(W,Z))$ and $f'' \in C(W\to\mathrm{End}(W,\mathrm{End}(W,Z)))$ both being bounded functions. If $y_0,y_1 \in C_p([0,T]\to W)$, then*

$$V_p(f(y_1)-f(y_0)) \le 2\|f'\|_u V_p(y_1-y_0) + \|f''\|_u\left[V_p(y_0)+V_p(y_1)\right]\|y_1-y_0\|_u. \tag{5.52}$$

Proof. Let

$$k := \|f'\|_u, \qquad M := \|f''\|_u,$$

and for the moment suppose that y_0, y_1 are elements of W. Letting

$$y_s := y_0 + s(y_1 - y_0) = (1-s)y_0 + sy_1,$$

we have, by the fundamental theorem of calculus,

$$f(y_1) - f(y_0) = \int_0^1 \frac{d}{ds} f(y_s)\,ds = \int_0^1 f'(y_s)(y_1 - y_0)\,ds. =: F(y_0, y_1)(y_1 - y_0)$$

Thus if we define,

$$F(y_0, y_1) := \int_0^1 f'(y_s)\,ds, \tag{5.53}$$

then

$$f(y_1) - f(y_0) = F(y_0, y_1)(y_1 - y_0). \tag{5.54}$$

Let us observe that if $\tilde{y}_0$ and $\tilde{y}_1$ are two more such points in W, then

$$\begin{aligned} \|F(y_0, y_1) - F(\tilde{y}_0, \tilde{y}_1)\| &= \left\| \int_0^1 f'(y_s)\,ds - \int_0^1 f'(\tilde{y}_s)\,ds \right\| \\ &= \left\| \int_0^1 [f'(y_s) - f'(\tilde{y}_s)]\,ds \right\| \\ &\le \int_0^1 \|f'(y_s) - f'(\tilde{y}_s)\|\,ds \le M \int_0^1 \|y_s - \tilde{y}_s\|\,ds \\ &\le M \int_0^1 ((1-s)\|y_0 - \tilde{y}_0\| + s\|y_1 - \tilde{y}_1\|)\,ds \\ &= \frac{1}{2} M [\|y_0 - \tilde{y}_0\| + \|y_1 - \tilde{y}_1\|]. \end{aligned}$$

So in summary we have,

$$f(y_1) - f(y_0) = F(y_0, y_1)(y_1 - y_0)$$

where $F : W \times W \to Z$ is bounded Lipschitz function satisfying,

$$\|F\|_u \le k \text{ and}$$
$$\|F(y_0, y_1) - F(\tilde{y}_0, \tilde{y}_1)\| \le \frac{1}{2} M [\|y_0 - \tilde{y}_0\| + \|y_1 - \tilde{y}_1\|].$$

Therefore

$$\begin{aligned} V_p(f(y_1) - f(y_0)) &= V_p(F(y_0, y_1)(y_1 - y_0)) \\ &\le 2[kV_p(y_1 - y_0) + \|y_1 - y_0\|_u V_p(F(y_0, y_1))]. \end{aligned}$$

Since,

$$\begin{aligned} \|\Delta_t F(y_0, y_1)\| &= \|F(y_0(t), y_1(t)) - F(y_0(t_-), y_1(t_-))\| \\ &\le \frac{1}{2} M [\|\Delta_t y_0\| + \|\Delta_t y_1\|], \end{aligned}$$

we learn that

$$V_p(F(y_0, y_1)) \le \frac{1}{2} M [V_p(y_0) + V_p(y_1)].$$

Putting this all together gives the result in Eq. (5.52). ■

5.5 An Existence Theorem

We are now prepared to prove our basic existence, uniqueness, and continuous dependence on data theorem for Eq. (5.39).

Theorem 5.31 (Local Existence of Solutions). *Let $p \in (1,2)$ and $\kappa := 1 + \zeta(2/p) < \infty$. Suppose that $f : W \to \mathrm{End}(V, W)$ is a C^2 – function such that f' and f'' are both bounded functions. Then there exists $\varepsilon_0 = \varepsilon_0(\|f'\|_u, \|f''\|_u, p, \|f(y_0)\|)$ such that for all $x \in C_p([0,T] \to V)$ with $V_p(x) \le \varepsilon_0$, there exists a solution to Eq. (5.39).*

Proof. Let

$$\mathbb{W}_T := \{y \in C_p([0,T], W) : y(0) = y_0\}.$$

Then by Proposition 4.6, we know $(\mathbb{W}_T, \rho)$ is a complete metric space where $\rho(y, z) := V_p(y - z)$ for all For $y, z \in \mathbb{W}_T$.

We now define $S : \mathbb{W}_T \to \mathbb{W}_T$ via,

$$S(y)(t) := y_0 + \int_0^t f(y)\,dx.$$

Our goal is to show that S is a contraction and then apply the contraction mapping principle to deduce the result. For this to work we are going to have to restrict our attention to some ball, C_δ, about the constant path y_0 and at the same time shrink T in such a way that $S(C_\delta) \subset C_\delta$ and $S|_{C_\delta}$ is a contraction. We now carry out the details.

First off if $y \in C_\delta \subset \mathbb{W}_T$, then

$$\begin{aligned} V_p(S(y)) &\le [\|f(y_0)\| + \kappa V_p(f(y))] V_p(x) \le [\|f(y_0)\| + \kappa \|f'\|_u V_p(y)] V_p(x) \\ &\le \|f(y_0)\| V_p(x) + \kappa \|f'\|_u V_p(x)\,\delta. \end{aligned} \tag{5.55}$$

Secondly, if $y, z \in C_\delta \subset \mathbb{W}_T$, then

$$[S(y) - S(z)](t) := \int_0^t (f(y) - f(z))\,dx$$

Therefore, making use of the fact that $f(y) = f(z)$ at $t = 0$, we find that

$$\begin{aligned} V_p(S(y) - S(z)) &\le \kappa V_p(f(y) - f(z)) V_p(x) \\ &\le \kappa \{2\|f'\|_u V_p(y-z) + \|f''\|_u [V_p(z) + V_p(y)] \|y - z\|_u\} V_p(x) \\ &\le \kappa V_p(x) \{2\|f'\|_u + \|f''\|_u [V_p(z) + V_p(y)]\} V_p(y-z) \\ &\le 2\kappa V_p(x) \{\|f'\|_u + \delta \|f''\|_u\} V_p(y-z). \end{aligned} \tag{5.56}$$

So in order for this scheme to work we must require, for some $\alpha \in (0,1)$, that

$$(\|f(y_0)\| + \kappa \|f'\|_u \delta) V_p(x) \le \delta \text{ and} \tag{5.57}$$

$$2\kappa V_p(x) \{\|f'\|_u + \delta \|f''\|_u\} \le \alpha. \tag{5.58}$$

But according to Lemma 5.32 with $\varepsilon := V_p(x)$, $a = \|f(y_0)\|$, $A := \kappa \|f'\|_u$ and $B := \kappa \|f''\|_u$, all of this can be achieved if $V_p(x) = \varepsilon \le \varepsilon_0(\alpha, a, A, B)$. So under this assumption on $V_p(x)$ and with the choice of δ in Lemma 5.32, $S : C_\delta \to C_\delta$ is a contraction and hence the result follows by the contraction mapping principle. ■

Lemma 5.32. *Let a, A, and B be positive constants and $\alpha \in (0,1)$. Then there exists $\varepsilon_0(\alpha, a, A, B) > 0$ such that for $\varepsilon \le \varepsilon_0$, there exists $\delta \in (0,\infty)$ such that*

$$(a + As)\varepsilon \le \delta \text{ and } 2A\varepsilon + 2B\varepsilon s \le \alpha \text{ for } 0 \le s \le \delta. \tag{5.59}$$

Proof. Our goal is to satisfy Eq. (5.59) while allowing for ε to be essentially as large as possible. The worst case scenarios in these inequalities is when $s = \delta$ and in this case the inequalities state,

$$\frac{a\varepsilon}{1 - A\varepsilon} \le \delta \le \frac{\alpha - 2A\varepsilon}{2B\varepsilon}$$

provided $A\varepsilon < 1$.

Letting $M := 1/\varepsilon$ we may rewrite the condition on ε as $M > A$ and

$$\frac{a}{M - A} \le \frac{\alpha M - 2A}{2B}$$

which gives,

$$2aB \le (M - A)(\alpha M - 2A),$$

i.e.

$$\alpha M^2 - (2 + \alpha) AM + 2(A^2 - aB) \ge 0$$

or equivalently,

$$\begin{aligned} 0 &\le M^2 - \frac{2+\alpha}{\alpha} AM + 2\frac{A^2 - aB}{\alpha} \\ &= \left(M - \frac{2+\alpha}{2\alpha}A\right)^2 + 2\frac{A^2 - aB}{\alpha} - \left(\frac{2+\alpha}{2\alpha}A\right)^2. \end{aligned}$$

Thus we must choose (if $2\frac{A^2 - aB}{\alpha} - \left(\frac{2+\alpha}{2\alpha}A\right)^2 \le 0$),

$$\frac{1}{\varepsilon} = M \ge \frac{2+\alpha}{2\alpha}A + \sqrt{\left(\frac{2+\alpha}{2\alpha}A\right)^2 + 2\frac{aB - A^2}{\alpha}}.$$

Thus we need to take

$$\varepsilon_0 := \min\left(\alpha A^{-1}, \left\{\frac{2+\alpha}{2\alpha}A + \sqrt{\left(\frac{2+\alpha}{2\alpha}A\right)^2 + 2\frac{aB - A^2}{\alpha}}\right\}^{-1}\right).$$

■

Corollary 5.33 (Global Existence). *Suppose that $f : W \to \mathrm{End}(V,W)$ is a C^2 – function such that f' and f'' are both bounded functions. Then for all $x \in C_p([0,T] \to V)$, there exists a solution to Eq. (5.39).*

Proof. Let $\varepsilon := \varepsilon_0\left(\|f'\|_u, \|f''\|_u, p, \|f(y_0)\| + KN^{1/p}\right)$ where N denotes the right side of the a priori bound in Eq. (5.46) and ε_0 is the function appearing in Theorem 5.31. Further let $\omega(s,t) := V_p^p(x : [s,t])$ and $h(t) := \omega(0,T)$. If $V_p(x) \le \varepsilon$ we are done by Theorem 5.31. If $V_p(x) > \varepsilon$, then write $h(T) = n\varepsilon^p + r$ with $n \in \mathbb{Z}_+$ and $0 \le r < \varepsilon^p$. Then use the intermediate value theorem to find, $0 = T_0 < T_1 < T_2 < \dots < T_n \le T_{n+1} = T$ such that $h(T_i) = i\varepsilon^p$. With this choice we have $\omega(T_{i-1}, T_i) \le h(T_i) - h(T_{i-1}) \le \varepsilon^p$ for $1 \le i \le n+1$. So by a simple induction argument, there exists $y \in C([0,T] \to W)$ such that, for each $1 \le i \le n+1$, $y \in C_p([T_{i-1}, T_i] \to W)$ and

$$y(t) = y(T_{i-1}) + \int_{T_{i-1}}^t f(y)\, dx \text{ for } t \in [T_{i-1}, T_i]. \tag{5.60}$$

It now follows from Corollary 4.4 that $y \in C_p([0,T] \to W)$. Summing the identity,

$$y(T_k) - y(T_{k-1}) = \int_{T_{k-1}}^{T_{k-1}} f(y)\, dx,$$

on k implies,

$$y(T_{i-1}) - y_0 = \sum_{1 \le k < i} \int_{T_{k-1}}^{T_{k-1}} f(y)\, dx = \int_0^{T_{i-1}} f(y)\, dx$$

which combined with Eq. (5.60) implies,

$$y(t) = y_0 + \int_0^t f(y)\, dx. \tag{5.61}$$

■

Theorem 5.34 (Uniqueness of Solutions). *Keeping the same assumptions as in Corollary 5.33, then the solution to Eq. (5.39) is unique.*

Proof. Suppose that $z \in C_p([0,T] \to V)$ also solves Eq. (5.39), i.e.

$$z(t) = y_0 + \int_0^t f(z(\tau))\, dx(\tau).$$

Then, with $w(t) := z(t) - y(t)$, we have

$$w(t) = \int_0^t [f(z(\tau)) - f(y(\tau))]\, dx(\tau)$$

and therefore,

$$\begin{aligned} V_p(w:[0,t]) &\le \kappa V_p(f\circ z - f\circ y : [0,t])\, V_p(x:[0,t]) \\ &\le C(f, V_p(w), V_p(z))\, V_p(x:[0,t])\, V_p(w:[0,t]), \end{aligned}$$

wherein we have used Eq. (5.52) for the second inequality. Thus if T_1 is chosen so that

$$C(f, V_p(w), V_p(z))\, V_p(x:[0,T_1]) < 1$$

it follows that $V_p(w:[0,T_1]) = 0$ and hence that $w|_{[0,T_1]} = 0$. Similarly we may now show that $w|_{[T_1,T_2]} = 0$ provided

$$C(f, V_p(w), V_p(z))\, V_p(x:[T_1,T_2]) < 1.$$

Working as in the proof of Corollary 5.33, it is now easy to conclude that $w \equiv 0$. ■

5.6 Continuous dependence on the Data

Let us now consider the issue of continuous dependence on the driving path, x. Recall the estimate,

$$\begin{aligned} V_p(f(y_1) - f(y_0)) &\le 2\|f'\|_u V_p(y_1 - y_0) + \|f''\|_u [V_p(y_0) + V_p(y_1)]\, \|y_1 - y_0\|_u \\ &\le C(f, V_p(y_0) + V_p(y_1))\, (\|y_1 - y_0\|_u + V_p(y_1 - y_0)) \\ &\le C(f, V_p(y_0) + V_p(y_1))\, (\|y_1(0) - y_0(0)\| + V_p(y_1 - y_0)) \end{aligned}$$

where $C(f,\xi)$ is a constant depending on $\|f'\|_u$, $\|f''\|_u$, and $\xi \ge 0$.

Theorem 5.35. *Let $Y_t(y_0 : x) := y(t)$ where y solves Eq. (5.39). Then assuming the f' and f'' are bounded, then Y*

$$Y : W \times C_p([0,T], V) \to C_p([0,T], W)$$

is uniformly continuous on bounded subsets of $W \times C_p([0,T], V)$.

Proof. Let that $u \in C_p([0,T], V)$ and $v_0 \in W$ and suppose that v solves,

$$v(t) = v_0 + \int_0^t f(v(\tau))\, du(\tau).$$

Let $w(t) := y(t) - v(t)$ so that

$$\begin{aligned} w(t) &= \int_0^t f(y(\tau))\, dx(\tau) - \int_0^t f(v(\tau))\, du(\tau) \\ &= \int_0^t f(y(\tau))\, d(x-u)(\tau) + \int_0^t [f(y(\tau)) - f(v(\tau))]\, du(\tau) \end{aligned}$$

and therefore,

$$w(t) = w(s) + \int_s^t f(y(\tau))\, d(x-u)(\tau) + \int_s^t [f(y(\tau)) - f(v(\tau))]\, du(\tau)$$

and hence

$$V_p(w:[s,t]) \le A(s,t) + B(s,t)$$

where

$$\begin{aligned} A(s,t) &= V_p\left(\int_s^{\cdot} f(y(\tau))\, d(x-u)(\tau) : [s,t]\right) \text{ and} \\ B(s,t) &= V_p\left(\int_s^{\cdot} [f(y(\tau)) - f(v(\tau))]\, du(\tau) : [s,t]\right). \end{aligned}$$

We now estimate each of these expressions as;

$$\begin{aligned} A(s,t) &\le (\|f(y(s))\| + \kappa V_p(f\circ y:[s,t]))\, V_p(x-u:[s,t]) \\ &\le (\|f(y(s))\| + \kappa K V_p(y:[s,t]))\, V_p(x-u:[s,t]) \\ &\le C(p, f, \|y_0\|, V_p(x))\, V_p(x-u:[s,t]) \end{aligned}$$

and

$$\begin{aligned} B(s,t) &= V_p\left(\int_s^{\cdot} [f(y(\tau)) - f(v(\tau))]\, du(\tau) : [s,t]\right) \\ &\le \|f(y(s)) - f(v(s))\| + \kappa V_p(f\circ y - f\circ v : [s,t])\, V_p(u:[s,t]) \\ &\le C(f, V_p(y) + V_p(v))\, V_p(u:[s,t])\, (\|y(s) - v(s)\| + V_p(y - v:[s,t])) \\ &= C(p, f, \|y_0\|, V_p(x), V_p(u))\, V_p(u:[s,t])\, (\|w(s)\| + V_p(w:[s,t])), \end{aligned}$$

wherein we have used the a priori bounds that we know of y and v. Thus we have,

$$V_p(w:[s,t]) \le C_1 V_p(x-u:[s,t]) + C_2 V_p(u:[s,t])\, (\|w(s)\| + V_p(w:[s,t])).$$

The result follows from this estimate via the usual iteration procedure based on keeping $C_2 V_p(u:[s,t]) \leq 1/2$ so that

$$V_p(w:[s,t]) \leq 2C_1 V_p(x-u:[s,t]) + \|w(s)\|.$$

say. For example choose T_1 so that $C_2 V_p(u:[0,T_1]) = 1/2$, then

$$V_p(w:[0,T_1]) \leq 2C_1 V_p(x-u:[0,T_1]) + \|y_0 - v_0\|.$$

Then choose T_2 such that $C_2 V_p(u:[T_1,T_2]) = 1/2$ to learn,

$$\begin{aligned} V_p(w:[T_1,T_2]) &\leq 2C_1 V_p(x-u:[T_1,T_2]) + \|w(T_1)\| \\ &\leq 2C_1 V_p(x-u:[T_1,T_2]) + \|y_0 - v_0\| + V_p(w:[0,T_1]) \\ &\leq 2C_1 (V_p(x-u:[0,T_1]) + V_p(x-u:[T_1,T_2])) + 2\|y_0 - v_0\|. \end{aligned}$$

Continuing on in this vain and them making use of Corollary 4.4 gives the claimed results. ■

Remark 5.36. This result could be used to give another proof of existence of solutions, namely, we just extend

$$Y : W \times C^1([0,T],V) \to C_p([0,T],W)$$

by continuity to $W \times C_p([0,T],V)$.

5.7 Towards Rougher Paths

Let us now begin to address the question of solving the O.D.E.,

$$dy_t = f(y_t)dx_t \text{ with } y_0 = \xi, \tag{5.62}$$

when $V_p(x) = \infty$ for $p < 2$.

Lemma 5.37. *If $a : \mathbb{R} \to \mathbb{R}$ is a Lipschitz vector field on $\mathbb{R}$, $f(y) = a(y)$, and $x \in C^1([0,T],\mathbb{R})$, then Eq. (5.62) has solution given by*

$$y_t = e^{(x_t - x_0) a \frac{\partial}{\partial x}}(\xi).$$

In particular, $y_t(x,\xi)$ depends continuously on x in the sup-norm topology and hence easily extends to rough paths.

Proof. Let z solve,

$$\dot{z}(\tau) = a(z(\tau)) \text{ with } z(0) = \xi.$$

Then

$$\frac{d}{dt} z(x_t - x_0) = a(z(x_t - x_0))\dot{x}_t \text{ with } z(x_t - x_0)|_{t=0} = \xi.$$

■

The same type of argument works more generally. We state with out proof the following theorem.

Theorem 5.38. *If $\{A_i\}_{i=1}^m$ are commuting Lipschitz vector fields on $\mathbb{R}^d$, $f(y)x := \sum_{i=1}^m x_i A_i(y)$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^d$, and $x \in C^1([0,T],\mathbb{R}^m)$ then Eq. (5.62) becomes,*

$$\dot{y}_t = \sum_{i=1}^m A_i(y_t)\dot{x}_t^i \text{ with } y_0 = \xi.$$

This equation has a unique solution has a unique solution given by

$$y_t = e^{\sum (x_t^i - x_0^i)A_i}(\xi) = e^{(x_t^1 - x_0^1)A_1} \circ \cdots \circ e^{(x_t^m - x_t^m)A_m}(\xi).$$

The next example shows however that life is more complicated when the vector fields do not commute.

Example 5.39. Let A_1 and A_2 be the vector fields on $\mathbb{R}$ defined by $A_1(r) = r$ and $A_2(r) = 1$ or in differential operator form,

$$A_1 := r\frac{\partial}{\partial r} \text{ and } A_2 := \frac{\partial}{\partial r}.$$

Further let $x = (x^1, x^2) \in C^1\left([0,T],\mathbb{R}^2\right)$ so that the corresponding ODE becomes,

$$\begin{aligned} \dot{y}_t &= A_1(y_t)\dot{x}_t^1 + A_2(y_t)\dot{x}_t^2 \\ &= y_t\,\dot{x}_t^1 + \dot{x}_t^2 \text{ with } y_0 = \xi. \end{aligned}$$

The solution to this equation is given by Duhamel's principle as

$$y_t = e^{x_t^1 - x_0^1}\xi + \int_0^t e^{x_t^1 - x_s^1} dx_s^2.$$

To simplify life even further let us now suppose that $\xi = 0$ so that

$$y_t(x) := e^{x_t^1} \int_0^t e^{-x_s^1} dx_s^2.$$

This last expression is **not** continuous in x in the V_p – norm for any $p > 2$ which follows from Lemma 5.40 with $x = (u(n), v(n))$.

Lemma 5.40. *For simplicity suppose that $T = 2\pi$. Suppose $u_t(n) = \frac{1}{n}\cos n^2 t$ and $v_t(n) = \frac{1}{n}\sin\ n^2 t$, then*

$$\int_0^t e^{-u_s(n)} dv_s(n) = o(1) - \frac{1}{2}t \nrightarrow 0 \text{ as } n \to \infty \tag{5.63}$$

while

$$V_p(u(n), v(n)) = O\left(n^{2/p-1}\right) \tag{5.64}$$

which tends to 0 for $p > 2$ as $n \to \infty$.

Proof. For notational simplicity we will suppress the n from our notation. By an integration by parts we have,

$$\begin{aligned}
\int_0^t e^{-u_s} dv_s &= e^{-u_s} v_s\big|_0^t + \int_0^t e^{-u_s} \dot{u}_s v_s ds \\
&= o(1) - \int_0^t e^{-\frac{1}{n}\cos(n^2 s)} \sin^2\left(n^2 s\right)\ ds \\
&= o(1) - \int_0^t \left(e^{-\frac{1}{n}\cos(n^2 s)} - 1\right) \sin^2\left(n^2 s\right)\ ds - \int_0^t \sin^2\left(n^2 s\right)\ ds \\
&= o(1) - \int_0^t O\left(\frac{1}{n}\right) \cdot \sin^2\left(n^2 s\right)\ ds - \int_0^t \sin^2\left(n^2 s\right)\ ds \\
&= o(1) - O\left(\frac{1}{n}\right) - \frac{1}{2}\int_0^t \left(1 - \cos\left(2n^2 s\right)\right)\ ds \\
&= o(1) - \frac{1}{2}t,
\end{aligned}$$

which proves Eq. (5.63).

By Theorem 2.10,

$$V_p(v(n)) \asymp V_p(u(n)) \asymp \left[\left(\frac{2}{n}\right)^p \cdot n^2\right]^{1/p} = O\left(\frac{1}{n} n^{2/p}\right) = O\left(n^{2/p-1}\right).$$

Equation (5.64) now follows from this estimate and the fact that

$$V_p(u, 0) \le V_p(u, v) \le V_p(u, 0) + V_p(0, v).$$

∎

Example 5.41. Let $x(t) := (u(t), v(t))$ be a smooth path such that $x(0) = 0$ and consider the **area** process,

$$A_t := \frac{1}{2}\int_0^t (udv - vdu).$$

By Green's theorem, A_t, is the signed area swept out by $x|_{[0,t]}$ then followed by the straight line path from $x(t)$ back to the origin. Taking

$$x_n(t) := (u_t(n), v_t(n)) = \left(\frac{1}{n}\left(\cos n^2 t - 1\right), \frac{1}{n}\sin\ n^2 t\right)$$

we find,

$$\begin{aligned}
A_t(n) &= \frac{1}{2n^2} n^2 \int_0^t \left[\left(\cos n^2 s - 1\right)\cos n^2 s + \sin\ n^2 s\left(\sin\ n^2 s\right)\right] ds \\
&= \frac{1}{2}\int_0^t \left[1 - \cos n^2 s\right] ds = \frac{1}{2}t - \frac{1}{2n^2}\sin\left(n^2 t\right) \to \frac{1}{2}t \text{ as } n \to \infty.
\end{aligned}$$

Whereas we have seen from above, that $V_p(x_n) \cong O\left(n^{2/p-1}\right)$. This shows that the area process is not continuous in the V_p - norm when $p > 2$ since $V_p(x_n) \to 0$ but $A_t(n) \to \frac{1}{2}t \neq 0$.

Rough Paths with $2 \leq p < 3$

We now are going to consider paths with finite p – variation for some $p \in [2,3)$. As we have seen above we have to add some additional information to get a reasonable theory going. We first need to introduce the appropriate algebra. As usual V with be a Banach space – which we will sometimes assume is finite dimensional in order to avoid technical details.

6.1 Tensor Norms

Throughout the rest of this class, we will assume that $V \otimes V$ has been equipped with a tensor norm satisfying,

$$\|w \otimes v\| = \|v \otimes w\| \leq \|v\| \|w\| \text{ for all } v, w \in V.$$

For example, if V is an inner product space, then there is a unique inner product on $V \otimes V$ determined by

$$(v \otimes w, v' \otimes w') = (v, v')(w, w') \text{ for all } v, w, v', w' \in V.$$

The norm associated to this inner product will satisfy the desired assumptions. Here is another example of such a norm.

Example 6.1 (Projective Norm). Suppose again that V and W are Banach spaces. The **projective norm** of $\xi \in V \otimes W$ is defined by

$$\|\xi\| := \inf \left\{ \sum_i \|v_i\| \ \|w_i\| : \xi = \sum_i v_i \otimes w_i \right\}.$$

It is easy to check that this satisfies the properties of norm modulo showing $\|\xi\| = 0$ implies $\xi = 0$.

Before checking $\|\cdot\|$ is a norm (see Corollary 6.3), let us observe the following property.

Lemma 6.2. *Suppose that E is another Banach space and $Q : V \times W \to E$ is a bilinear form and let $\tilde{Q} : V \otimes W \to E$ be corresponding linear map on $V \otimes W$ to E. Then $\|\tilde{Q}\|_{op} = \|Q\|$. Because of this property, in the future we will no longer distinguish between Q and $\tilde{Q}$.*

Proof. Then for $\xi = \sum_i v_i \otimes w_i$, we have

$$\left|\tilde{Q}(\xi)\right|_E = \left|\sum_i Q(v_i, w_i)\right| \leq \sum_i |Q(v_i, w_i)| \leq \|Q\| \sum_i \|v_i\| \ \|w_i\|,$$

where $\|Q\|$ is the best constant for which the last inequality holds. Taking the infimum over all such decomposition of ξ shows

$$\left|\tilde{Q}(\xi)\right|_E \leq \|Q\| \|\xi\|$$

and therefore $\left\|\tilde{Q}\right\|_{op} \leq \|Q\|$. Let $\alpha \in (0, \|Q\|)$ and choose $v \in V$ and $w \in W$ such that $|Q(v, w)| \geq \alpha |v| |w|$. Then

$$\left|\tilde{Q}(v \otimes w)\right|_E = |Q(v, w)| \geq \alpha |v| |w| \geq \alpha |v \otimes w|$$

from which it follows that $\left\|\tilde{Q}\right\|_{op} \geq \alpha$. Since $\alpha \in (0, \|Q\|)$ was arbitrary, it follows that $\left\|\tilde{Q}\right\|_{op} \geq \|Q\|$. ■

Corollary 6.3. *Assume that $\|\xi\|$ is defined as in Example 6.1. If $\|\xi\| = 0$ then $\xi = 0$.*

Proof. If $\xi \neq 0$ we may write

$$\xi := \sum_{i=1}^n v_i \otimes w_i$$

with $\{v_i\}_{i=1}^n$ being a linearly independent set and each $w_i \neq 0$. Choose $\alpha \in V^*$ such that $\alpha(v_i) = \delta_{i1}$ and $\beta \in W^*$ such that $\beta(w_1) = 1$. Then $Q(v, w) := \alpha(v) \beta(w)$ is a continuous bilinear form on $V \times W$ with $\|Q\| = \|\alpha\| \|\beta\| > 0$. Thus we have

$$1 = |Q(\xi)| \leq \|Q\| \|\xi\|$$

from which it follows that $\|\xi\| > 0$. ■

6.2 Algebraic Preliminaries

Notation 6.4 *let $\sigma : V \otimes V \to V \otimes V$ be the map determined by, $\sigma(v \otimes w) = w \otimes v$ and define,*

$$\Lambda^2(V) := \{\xi \in V \otimes V : \sigma\xi = -\xi\} \text{ and}$$
$$S^2(V) := \{\xi \in V \otimes V : \sigma\xi = \xi\}.$$

Furthermore, for $a, b \in V$, let

$$[a,b] := ab - ba \in \Lambda(V) \text{ and}$$
$$a \vee b := ab + ba \in S^2(V).$$

Observe that if $\xi \in V \otimes V$, then

$$\xi = \frac{1}{2}(\xi + \sigma\xi) + \frac{1}{2}(\xi - \sigma\xi) \in S^2(V) \oplus \Lambda^2(V)$$

and in particular,

$$a \otimes b = \frac{1}{2}(a \vee b + [a,b]),$$

so that

$$V \otimes V = S^2(V) \oplus \Lambda^2(V).$$

Definition 6.5. *Let $\mathcal{A}(V) := T^{(2)}(V) := \mathbb{R} \oplus V \oplus V \otimes V$ which we make into an algebra via the multiplication rule,*

$$(a + v + \xi)(b + w + \eta) = ab + (aw + vb) + (b\xi + a\eta + v \otimes w).$$

*We are now going to **drop** the tensor symbol from the notation.*

Lemma 6.6. *The subset,*

$$G(V) := \{g \in \mathcal{A}(V) : g = 1 + v + \xi \text{ with } v \in V \text{ and } \xi \in V \otimes V\},$$

is a group under the multiplication coming from $\mathcal{A}(V)$. The identity element is 1 and the inverse to g is given by

$$g^{-1} = 1 - v - \xi + v^2.$$

Proof. If $h := 1 + w + \eta$ then

$$gh = (1 + v + \xi)(1 + w + \eta) = 1 + (v + w) + (\xi + \eta + vw) \in G(V)$$

and we see that $gh = 1$ iff

$$w = -v \text{ and } \eta = -\xi - vw = -\xi + v^2.$$

■

Definition 6.7. *A function, $X : \Delta \to G(V)$ is an **algebraic multiplicative functional** if*

$$X_{su} = X_{st}X_{tu} \quad \text{for all } 0 \le s \le u \le t \le T. \tag{6.1}$$

Equation 6.1 is referred to as Chen's identity.

We will write the components of X as $X^1 \in V$ and $X^2 \in V \otimes V$, so that

$$X_{st} = 1 + X^1_{st} + X^2_{st}.$$

Let us observe by the multiplicative property that for all $t \in [0,T]$, $X_{t,t} = X_{t,t}X_{t,t}$, i.e. $X_{t,t} = 1$ for all $t \in [0,T]$.

Lemma 6.8. *Chen's identity (6.1) is equivalent to*

$$X^1_{su} = X^1_{st} + X^1_{tu} \text{ and} \tag{6.2}$$
$$X^2_{su} = X^2_{st} + X^2_{tu} + X^1_{st}X^1_{tu} \tag{6.3}$$

for all $0 \le s \le t \le u \le T$.

Proof. Chen's identity states,

$$\begin{aligned} 1 + X^1_{su} + X^2_{su} &= \left(1 + X^1_{st} + X^2_{st}\right)\left(1 + X^1_{tu} + X^2_{tu}\right) \\ &= 1 + \left[X^1_{st} + X^1_{tu}\right] + \left[X^2_{st} + X^2_{tu} + X^1_{st}X^1_{tu}\right] \end{aligned}$$

which suffices to complete the proof. ■

Example 6.9. If $x \in C_p([0,T], V)$ with $p < 2$, then we let

$$\begin{aligned} X_{st} &= 1 + x(t) - x(s) + \int_{s \le u \le v \le t} dx(u)\,dx(v) \\ &:= 1 + x(t) - x(s) + \int_s^t (x(v) - x(s))\,dx(v) \in G(V) \end{aligned} \tag{6.4}$$

where the latter integral is the Young integral. An alternative way to look at this function is to observe that

$$X_{st} = 1 + \int_s^t X_{s\tau}\,dx(\tau) \tag{6.5}$$

and this differential equation uniquely determines X_{st}. Let us also check directly that Eqs. (6.3) holds. Equation (6.2) holds trivially.

We have in this case $X^1_{st} = x(t) - x(s)$ and $X^2_{st} = \int_s^t (x(v) - x(s))\,dx(v)$, therefore,

$$X_{su}^2 - X_{st}^2 - X_{tu}^2 = \int_s^u (x(v) - x(s))\, dx(v)$$
$$- \int_s^t (x(v) - x(s))\, dx(v) - \int_t^u (x(v) - x(t))\, dx(v)$$
$$= \int_t^u (x(v) - x(s))\, dx(v) - \int_t^u (x(v) - x(t))\, dx(v)$$
$$= \int_t^u (x(t) - x(s))\, dx(v) = X_{st}^1 X_{tu}^1$$

as desired.

Example 6.10. Suppose again that $p < 2$ and $W := x + A$ with $x \in C_p([0,T], V)$ and $A \in C_p([0,T], V \otimes V)$. Then

$$X_{st} = 1 + W(t) - W(s) + \int_{s \le u \le v \le t} dW(u)\, dW(v)$$
$$:= 1 + x(t) - x(s) + A(t) - A(s) + \int_s^t (x(v) - x(s))\, dx(v) \in G(V) \tag{6.6}$$

is the unique solution to

$$X_{st} = 1 + \int_s^t X_{s\tau} dW(\tau) \tag{6.7}$$

and therefore still satisfies Chen's identity and $X_{st}^1 = x(t) - x(s)$. Thus it is not reasonable to try to define

$$\int_s^t (x(v) - x(s))\, dx(v) := X_{st}^2$$

where X_{st}^2 is chosen so that $X_{st} := 1 + x(t) - x(s) + X_{st}^2$ satisfies Chen's identity.

Proposition 6.11. *Let $x \in C_p([0,T], V)$ with $p < 2$ and then we let $X : \Delta \to G(V)$ be as in Eq. (6.4). Then*

$$X_{st}^2 + \sigma X_{st}^2 = \left(X_{st}^1\right)^2 \text{ and}$$
$$X_{st}^2 - \sigma X_{st}^2 = \int_s^t [x(v) - x(s), dx(v)]$$

so that

$$X_{st} = 1 + X_{st}^1 + \frac{1}{2}\left(\left(X_{st}^1\right)^2 + \int_s^t [x(v) - x(s), dx(v)]\right). \tag{6.8}$$

This shows that it is only the anti-symmetric part of $X_{s,t}^2$ which does not depend continuously on x in the sup-norm topology.

Proof. We have,

$$X_{st}^2 + \sigma X_{st}^2 = \int_s^t (x(v) - x(s)) \vee dx(v)$$
$$= \int_s^t [(x(v) - x(s))\, dx(v) + dx(v)(x(v) - x(s))]$$
$$= \int_s^t d_v [x(v) - x(s)]^2 = [x(t) - x(s)]^2 = \left(X_{st}^1\right)^2$$

and

$$X_{st}^2 - \sigma X_{st}^2 = \int_s^t (x(v) - x(s))\, dx(v) - \int_s^t dx(v)(x(v) - x(s))$$
$$= \int_s^t [x(v) - x(s), dx(v)].$$

■

6.3 The Geometric Subgroup

Definition 6.12. *Let*

$$G_{geo}(V) := \left\{g = 1 + a + A \in G(V) : A + \sigma A = a^2\right\},$$

i.e. the symmetric part of A is $\frac{1}{2}a^2$. Alternatively put, the general element of $G_{geo}(V)$ is of the form,

$$g = 1 + a + \frac{1}{2}a^2 + A \text{ with } A \in \Lambda^2(V). \tag{6.9}$$

Lemma 6.13. *$G_{geo}(V)$ is a subgroup of $G(V)$.*

Proof. Let $g, h \in G_{geo}(V)$ with g as in Eq. (6.9) and

$$h = 1 + b + \frac{1}{2}b^2 + B \text{ with } B \in \Lambda^2(V).$$

Then

$$g^{-1} = 1 - a - \frac{1}{2}a^2 - A + a^2 = 1 - a + \frac{1}{2}a^2 - A$$
$$= 1 - a + \frac{1}{2}(-a)^2 - A \in G_{geo}(V)$$

and

$$\begin{aligned} gh &= \left(1 + a + \frac{1}{2}a^2 + A\right)\left(1 + b + \frac{1}{2}b^2 + B\right) \\ &= 1 + a + b + \frac{1}{2}a^2 + \frac{1}{2}b^2 + ab + A + B \\ &= 1 + a + b + \frac{1}{2}(a+b)^2 - \frac{1}{2}(ab + ba) + ab + A + B \\ &= 1 + a + b + \frac{1}{2}(a+b)^2 + \frac{1}{2}(ab - ba) + A + B \\ &= 1 + a + b + \frac{1}{2}(a+b)^2 + \frac{1}{2}[a,b] + A + B \in G_{geo}(V). \end{aligned}$$

■

As with any Lie group, H, we may associate a Lie algebra, $\mathrm{Lie}(H) = T_e H$.

Lemma 6.14. *For $G = G(V)$ and $G_{geo} = G_{geo}(V)$ we have*

$$\mathrm{Lie}\, G = V \oplus V \otimes V \text{ and } \mathrm{Lie}\, G_{geo} = V \otimes \Lambda^2(V). \tag{6.10}$$

Proof. Let $g(t) = 1 + x(t) + A(t)$ be a smooth path in G such that $g(0) = 1$, then

$$\dot{g}(0) = \dot{x}(0) + \dot{A}(0) \in V \oplus V \otimes V.$$

Conversely if $a + A \in V \oplus V \otimes V$ then $g(t) = 1 + t(a + A) \in G$ is a smooth path such that $g(0) = 0$ and $\dot{g}(0) = a + A$. Therefore $\mathrm{Lie}(G) = V \oplus V \otimes V$.

A smooth path $g(t) \in G_{geo}$ may be written as

$$g(t) = 1 + x(t) + \frac{1}{2}x^2(t) + A(t) \text{ with } A(t) \in \Lambda^2(V).$$

Assuming that $g(0) = 0$ so that $x(0) = 0$, it follows that

$$\dot{g}(0) = \dot{x}(0) + \dot{A}(0) \in V \otimes \Lambda^2(V).$$

Conversely if $a + A \in V \oplus \Lambda^2(V)$ then $g(t) = 1 + ta + \frac{1}{2}t^2a^2 + tA \in G_{geo}$ is a smooth path such that $g(0) = 0$ and $\dot{g}(0) = a + A$ and therefore $\mathrm{Lie}\, G_{geo} = V \otimes \Lambda^2(V)$. ■

We now continue on with the Lie group mantra. To this end we associate to element of $\xi \in \mathrm{Lie}\, G$, a left invariant vector field, $\tilde{\xi}(g)$ via,

$$\tilde{\xi}(g) := \frac{d}{dt}|_0 g \cdot h(t)$$

where $h(t)$ is any smooth curve in G such that $h(0) = 0$ and $\dot{h}(0) = \xi$ – for example take $h(t) = 1 + t\xi$. Writing $g = 1 + \eta$, we find,

$$\tilde{\xi}(g) = \frac{d}{dt}|_0 (1 + \eta) \cdot (1 + t\xi) = g \cdot \xi.$$

The Lie bracket is then determined by the formula, $\widetilde{[\xi, \eta]} = \left[\tilde{\xi}, \tilde{\eta}\right]$ which we now work out. Let $f : G \to \mathbb{R}$ be a smooth function, then

$$\left(\tilde{\xi}\tilde{\eta}f\right)(g) = \tilde{\xi}\left(g \to f'(g)(g\eta)\right) = f''(g)(g\xi, g\eta) + f'(g)\,\xi\eta.$$

Since $f''(g)$ is symmetric (mixed partial derivative commute) we find,

$$\left(\left[\tilde{\xi}, \tilde{\eta}\right] f\right)(g) = f'(g)(\xi\eta - \eta\xi) = (\widetilde{\xi\eta - \eta\xi}) f(g).$$

We summarize these results in the following proposition.

Proposition 6.15. *The Lie bracket on* $\mathrm{Lie}(G)$ *is given by* $[\xi, \eta] = \xi\eta - \eta\xi$. *Moreover,* $\mathrm{Lie}(G_{geo})$ *is the Lie sub-algebra (at least when* $\dim V < \infty$*) of* $\mathrm{Lie}(G)$ *generated by* $V \subset \mathrm{Lie}(G) = V \oplus V \otimes V$. *We will denote* $\mathrm{Lie}(G_{geo})$ *by* $\mathcal{L}(V)$.

Proof. It only remains to prove the second assertion. So suppose that $\xi = a + A$ and $\eta = b + B$ with $a, b \in V$ and $A, B \in \Lambda^2(V)$, then

$$[\xi, \eta] = [a + A, b + B] = [a, b] = ab - ba \in \Lambda^2(V) \subset \mathrm{Lie}(G_{geo}).$$

■

Although these are non-trivial Lie algebras they are only slightly non-trivial in the sense that $[[\xi, \eta], \gamma] = 0$ for all $\xi, \eta, \gamma \in \mathrm{Lie}(G)$, i.e. $\mathrm{Lie}(G)$ is nilpotent. The next thing item to compute for these Lie algebras and Lie groups is the group exponential map defined by

$$e^{\xi} = e^{\tilde{\xi}}(1) \text{ for all } \xi \in \mathrm{Lie}(G).$$

Let $g(t) = e^{t\tilde{\xi}}(1)$, so that

$$\dot{g}(t) = \tilde{\xi}(g(t)) = g(t)\,\xi \text{ with } g(0) = 1.$$

Writing $\xi = a + A$ and $g(t) = 1 + x(t) + B(t)$, we learn that

$$\dot{x} + \dot{B} = (1 + x + B)(a + A) = a + xa + A$$

so that $\dot{x} = a$ and $\dot{B} = xa + A$ with $x(0) = 0$ and $B(0) = 0$. The solution to the first equation is $x(t) = at$ and then $\dot{B}(t) = ta^2 + A$ and therefore $B(t) = \frac{1}{2}a^2t^2 + At$. Therefore,

$$e^{t\tilde{\xi}}(1) = g(t) = 1 + at + \frac{t^2}{2}a^2 + At.$$

Thus we have proved

$$e^{\xi}=1+a+\frac{1}{2}a^{2}+A=1+\xi+\frac{1}{2}\xi^{2}=\exp\left(\xi\right).$$

It is not so surprising that e^{ξ} is given by the Taylor's theorem expansion. Indeed if we had defined e^{ξ} by its Taylor's expansion, then

$$\frac{d}{dt}e^{t\xi}=\frac{d}{ds}|_{0}e^{(t+s)\xi}=\frac{d}{ds}|_{0}e^{t\xi}e^{s\xi}=e^{t\xi}\xi\text{ with }e^{0\xi}=1.$$

The other point to notice is that if $\xi=a+A\in\operatorname{Lie}\left(G_{geo}\right)$, then $e^{\xi}=1+a+\frac{1}{2}a^{2}+A\in G_{geo}$ as it should be. Let us summarize what we have done in the following theorem.

Theorem 6.16. *We have,* $\exp\left(\xi\right)=e^{\xi}$, $\exp:\operatorname{Lie}\left(G\right)\rightarrow G$ *and* $\exp:\operatorname{Lie}\left(G_{geo}\right)\rightarrow G_{geo}$ *are diffeomorphism with inverse given by*

$$\log\left(1+\eta\right)=\eta-\frac{1}{2}\eta^{2}.$$

Moreover, $g\in G$ *is in* G_{geo} *iff* $\log\left(g\right)\in\mathcal{L}\left(V\right)=\operatorname{Lie}\left(G_{geo}\right).$

Proof. To prove the last assertion, observe that if

$$1+\eta=e^{\xi}=1+\xi+\frac{1}{2}\xi^{2}$$

then $\eta=\xi+\frac{1}{2}\xi^{2}$ and therefore,

$$\xi=\eta-\frac{1}{2}\xi^{2}=\eta-\frac{1}{2}\left(\eta-\frac{1}{2}\xi\right)^{2}=\eta-\frac{1}{2}\eta^{2}.$$

If $g=1+\eta=1+a+A$, then $g\in G_{geo}$ iff $A-\frac{1}{2}a^{2}\in\Lambda^{2}\left(V\right)$ while

$$\log\left(g\right)=a+A-\frac{1}{2}a^{2}\in\mathcal{L}\left(V\right)\iff A-\frac{1}{2}a^{2}\in\Lambda^{2}\left(V\right).$$

■

Proposition 6.17. *If* $\xi,\eta\in\operatorname{Lie}\left(G\right)$, *then*

$$e^{\xi}e^{\eta}=e^{\xi+\eta+\frac{1}{2}\left[\xi,\eta\right]}.$$

In particular if we define

$$\xi\cdot\eta=\xi+\eta+\frac{1}{2}\left[\xi,\eta\right]\text{ for all }\xi,\eta\in\operatorname{Lie}\left(G\right),$$

then $\operatorname{Lie}\left(G\right)$ *becomes a group such that* $\exp:\operatorname{Lie}\left(G\right)\rightarrow G$ *and* $\exp:\mathcal{L}\left(V\right)\rightarrow G_{geo}$ *are Lie group isomorphisms.*

Proof. We have

$$\begin{aligned}e^{\xi}e^{\eta}&=\left(1+\xi+\frac{1}{2}\xi^{2}\right)\left(1+\eta+\frac{1}{2}\eta^{2}\right)\\&=1+\xi+\eta+\frac{1}{2}\xi^{2}+\frac{1}{2}\eta^{2}+\xi\eta\\&=1+\xi+\eta+\frac{1}{2}\left(\xi+\eta\right)^{2}-\frac{1}{2}\left(\xi\eta+\eta\xi\right)+\xi\eta\\&=1+\xi+\eta+\frac{1}{2}\left(\xi+\eta\right)^{2}+\frac{1}{2}\left[\xi,\eta\right]\\&=e^{\xi+\eta+\frac{1}{2}\left[\xi,\eta\right]}.\end{aligned}$$

Alternatively; compute

$$\begin{aligned}\log\left(\exp\left(\xi\right)\cdot\exp\left(\eta\right)\right)&=\log\left(1+\xi+\eta+\frac{1}{2}\xi^{2}+\frac{1}{2}\eta^{2}+\xi\eta\right)\\&=\xi+\eta+\frac{1}{2}\xi^{2}+\frac{1}{2}\eta^{2}+\xi\eta-\frac{1}{2}\left(\xi+\eta\right)^{2}\\&=\xi+\eta+\frac{1}{2}\left[\xi,\eta\right].\end{aligned}$$

■

Corollary 6.18. *Suppose that* $U\left(t\right)\in\operatorname{Lie}\left(G\right)$ *is a finite variation curve with* $U\left(0\right)=0$ *and* $g\left(t\right)$ *solves,*

$$\dot{g}\left(t\right)=g\left(t\right)U\left(t\right)\text{ with }g\left(0\right)=1,$$

then

$$g\left(t\right)=\exp\left(U\left(t\right)+\frac{1}{2}\int_{0}^{t}\left[U\left(\tau\right),dU\left(\tau\right)\right]\right).$$

If $U\left(t\right)=u\left(t\right)+A\left(t\right)$ *with* $u\left(t\right)\in V$ *and* $A\left(t\right)\in V\otimes V$, *we may write* $g\left(t\right)$ *as,*

$$g\left(t\right)=\exp\left(u\left(t\right)+\frac{1}{2}\int_{0}^{t}\left[u\left(\tau\right),du\left(\tau\right)\right]+A\left(t\right)\right).$$

Proof. We know that the solution to the ODE is given by

$$\begin{aligned}g\left(t\right)&=1+\int_{0}^{t}dU\left(\tau\right)+\int_{0\leq\sigma\leq\tau\leq t}dU\left(\sigma\right)dU\left(\tau\right)\\&=1+U\left(t\right)+\int_{0}^{t}U\left(\tau\right)dU\left(\tau\right)\\&=1+U\left(t\right)+\frac{1}{2}U^{2}\left(t\right)+\frac{1}{2}\int_{0}^{t}\left[U\left(\tau\right),dU\left(\tau\right)\right]\\&=\exp\left(U\left(t\right)+\frac{1}{2}\int_{0}^{t}\left[U\left(\tau\right),dU\left(\tau\right)\right]\right).\end{aligned}$$

■

Corollary 6.19. *If $u, v \in V$, then*

$$e^{[u,v]} = e^{-u}e^{-v}e^{u}e^{v}.$$

Proof. By repeated use of Proposition 6.17,

$$\begin{aligned} e^{-u}e^{-v}e^{u}e^{v} &= e^{-u}e^{-v}e^{u+v+\frac{1}{2}[u,v]} \\ &= e^{-u}e^{u+v+\frac{1}{2}[u,v]-v-\frac{1}{2}\left[v,u+v+\frac{1}{2}[u,v]\right]} \\ &= e^{-u}e^{u+[u,v]} = e^{u+[u,v]-u+\frac{1}{2}[u,u+[u,v]]} = e^{[u,v]}. \end{aligned}$$

■

Theorem 6.20 (Chow's Theorem). *To each $y \in G_{geo}$ there exists a (smooth) finite variation path, $x(t) \in V$ such that $x(0) = 0$ and $g_x(1) = y$, where $g_x(t) = g(t)$ is the solution to,*

$$\dot{g}(t) = g(t)\,\dot{x}(t) \ \ \text{with } g(0) = 1. \tag{6.11}$$

Proof. First Proof. If $A \in \Lambda^2(V)$ is written as $A = \sum_{i=1}^m [u_i, v_i]$ for some $u_i, v_i \in V$, then by Corollary 6.19,

$$e^{A} = e^{\sum_{i=1}^m [u_i,v_i]} = \prod_{i=1}^m e^{[u_i,v_i]} = \prod_{i=1}^m \left(e^{-u_i}e^{-v_i}e^{u_i}e^{v_i}\right).$$

If in addition, $a \in V$, then

$$e^{a+A} = e^{a}e^{A} = e^{a}\prod_{i=1}^m \left(e^{-u_i}e^{-v_i}e^{u_i}e^{v_i}\right).$$

It is now easy to see how to construct the desired path $x(t)$. We determine $x(t)$ so that it is continuous and satisfies, $\dot{x}(t) = a$ for $0 \le t \le 1$, $\dot{x}(t) = -u_i$ for $t \in 4(i-1) + (1,2)$, $\dot{x}(t) = -v_i$ for $t \in 4(i-1) + (2,3)$, $\dot{x}(t) = u_i$ for $t \in 4(i-1) + (3,4)$, and $\dot{x}(t) = v_i$ for $t \in 4(i-1) + (4,5)$ for $i = 1, 2, \ldots, m$. With this definition it follows that

$$g(4(m-1)+5) = e^{a+A}$$

as desired.

Second Proof. We know the solution to Eq. (6.11) is given by

$$g_x(t) = \exp\left(x(t) + \frac{1}{2}\int_0^t [x(\tau), dx(\tau)]\right).$$

Let $x_a(t) = ta$ for $0 \le t \le 1$, then

$$\int_0^t [x_a(\tau), dx_a(\tau)] = \int_0^t [\tau a, a]\, d\tau = 0.$$

Now let

$$x_i(t) = \frac{1}{2\pi}\left(u_i(\cos 2\pi t - 1) + v_i \sin 2\pi t\right) \ \text{for } 0 \le t \le 1.$$

Then

$$\begin{aligned} \int_0^1 [x_i(t), dx_i(t)] &= \int_0^1 [u_i(\cos 2\pi t - 1) + v_i \sin 2\pi t, v_i \cos 2\pi t - u_i \sin 2\pi t]\, dt \\ &= [u_i, v_i]\int_0^1 \left[(\cos 2\pi t - 1)\cos 2\pi t + \sin^2 2\pi t\right] dt \\ &= [u_i, v_i] \end{aligned}$$

Let $x := x_1 * \cdots * x_m * x_a$ by which we mean follows x_1, then x_2, ..., then x_m, and then x_a. Then

$$\begin{aligned} x(1) + &\frac{1}{2}\int_0^1 [x(\tau), dx(\tau)] \\ &= a + \frac{1}{2}\int_0^1 [x_a(\tau), dx_a(\tau)] + \frac{1}{2}\sum_{i=1}^m \int_0^1 [x_i(\tau), dx_i(\tau)] \\ &= a + \frac{1}{2}\sum_{i=1}^m [u_i, v_i] \end{aligned}$$

which again represents an arbitrary element in $\mathcal{L}(V)$. ■

6.4 Characterizations of Algebraic Multiplicative Functionals

Definition 6.21. *A multiplicative functional, $X : \Delta \to G$ such that $X(\Delta) \subset G_{geo}$ is said to be an **algebraic geometric multiplicative functional.***

Example 6.22. To every $x \in C_p([0,T], V)$ with $1 \le p < 2$, the function,

$$\begin{aligned} X_{st} &= 1 + (x(t) - x(s)) + \int_s^t (x(v) - x(s))\, dx(v) \\ &= \exp\left((x(t) - x(s)) + \frac{1}{2}\int_s^t [x(v) - x(s), dx(v)]\right). \end{aligned}$$

Lemma 6.23 (Characterization of MF's). *The space of (algebraic geometric) multiplicative functionals are in one to one correspondence with functions* $(Y : [0,T] \to G_{geo}(V))$ $Y : [0,T] \to G(V)$ *such that* $Y(0) = 1$. *(The latter conditions is an arbitrary normalization.) The correspondence is given by*

$$Y \to X_{st} := Y_s^{-1} Y_t \text{ and } X_{st} \to Y_t := X_{0t}.$$

Proof. Given a multiplicative function, X, let $Y_t := X_{0,t}$. Then for $(s,t) \in \Delta$ we have

$$Y_t = X_{0t} = X_{0s} X_{st} = Y_s X_{st} \implies X_{st} = Y_s^{-1} Y_t.$$

Conversely if $Y : [0,T] \to G(V)$ and $X_{st} := Y_s^{-1} Y_t$, then

$$X_{st} X_{tu} = Y_s^{-1} Y_t \left(Y_t^{-1} Y_u\right) = Y_s^{-1} Y_u = X_{su}$$

as desired. ■

Proposition 6.24. *If* $Y_t = 1 + x_t + A_t \in G(V)$ *then*

$$X_{st} = 1 + (x_t - x_s) + \left(A_t - A_s + x_s^2 - x_s x_t\right) \tag{6.12}$$

$$= \exp\left((x_t - x_s) + \left(A_t - \frac{1}{2} x_t^2\right) - \left(A_s - \frac{1}{2} x_s^2\right) - \frac{1}{2}[x_s, x_t]\right) \tag{6.13}$$

and if $Y_t = 1 + x_t + \frac{1}{2} x_t^2 + \alpha_t \in G_{geo}(V)$, *then*

$$X_{st} = 1 + (x_t - x_s) + \frac{1}{2}(x_t - x_s)^2 + \alpha_t - \alpha_s - \frac{1}{2}[x_s, x_t] \tag{6.14}$$

$$= \exp\left(1 + (x_t - x_s) + \alpha_t - \alpha_s - \frac{1}{2}[x_s, x_t]\right). \tag{6.15}$$

Proof.

$$\begin{aligned}
X_{st} &= Y_s^{-1} Y_t = (1 + x_s + A_s)^{-1}(1 + x_t + A_t) \\
&= \left(1 - x_s - A_s + x_s^2\right)(1 + x_t + A_t) \\
&= 1 + (x_t - x_s) + \left(A_t - A_s + x_s^2 - x_s x_t\right) \\
&= 1 + (x_t - x_s) + \frac{1}{2}(x_t - x_s)^2 \\
&\quad - \frac{1}{2}\left(x_t^2 + x_s^2 - x_s x_t - x_t x_s\right) + \left(A_t - A_s + x_s^2 - x_s x_t\right) \\
&= 1 + (x_t - x_s) + \frac{1}{2}(x_t - x_s)^2 + \left(A_t - \frac{1}{2} x_t^2\right) - \left(A_s - \frac{1}{2} x_s^2\right) - \frac{1}{2}[x_s, x_t]
\end{aligned}$$

For the second assertion, apply Eq. (6.13) with $A_t = \frac{1}{2} x_t^2 + \alpha_t$. ■

Corollary 6.25. *Suppose that* $X, \tilde{X} : \Delta \to G(V)$ *are multiplicative functionals such that* $X^1 = \tilde{X}^1$, *then* $\psi_{st} := X_{st}^2 - \tilde{X}_{st}^2$ *is an additive functional.*

Proof. By Proposition 6.24, X and $\tilde{X}$ may be written as in Eq. (6.12) with A replaced by $\tilde{A}$ for $\tilde{X}$. Therefore,

$$\begin{aligned}
\psi_{st} = X_{st}^2 - \tilde{X}_{st}^2 &= A_t - A_s - \left(\tilde{A}_t - \tilde{A}_s\right) \\
&= A_t - \tilde{A}_t - \left(A_s - \tilde{A}_s\right)
\end{aligned}$$

which is an additive functional.

Alternatively, we make use of Chen's identity to find,

$$\begin{aligned}
\left(X^2 - \tilde{X}^2\right)_{su} &= X_{su}^2 - \tilde{X}_{su}^2 = X_{st}^2 + X_{tu}^2 + X_{st}^1 X_{tu}^1 - \left(\tilde{X}_{st}^2 + \tilde{X}_{tu}^2 + \tilde{X}_{st}^1 \tilde{X}_{tu}^1\right) \\
&= X_{st}^2 - \tilde{X}_{st}^2 + X_{tu}^2 - \tilde{X}_{tu}^2 \\
&= \left(X^2 - \tilde{X}^2\right)_{st} + \left(X^2 - \tilde{X}^2\right)_{tu}.
\end{aligned}$$

■

Definition 6.26. *Given a path* $x : [0,T] \to V$ *we say that* $X : \Delta \to G$ *is a* ***(geometric) lift*** *of* x *if* X *is a (geometric) multiplicative functional and* $X_{st}^1 = x(t) - x(s)$ *for all* $(s,t) \in \Delta$.

Corollary 6.27. *If* X *is a (geometric) lift of* $x : [0,T] \to V$ *then every (geometric) lift,* $\tilde{X}$, *of* x *is of the form,*

$$\tilde{X}_{st} = X_{st} + \psi_{st}$$

where $\left(\psi_{st} \in \Lambda^2(V)\right)$ $\psi_{st} \in V \otimes V$ *is an arbitrary additive functional.*

Example 6.28. Suppose $\dim(V) = 1$, i.e. $V = \mathbb{R}$, and $x : [0,T] \to \mathbb{R}$ is a continuous path. Then

$$\begin{aligned}
Y_t = e^{x_t} &= \left(1, x_t, \frac{x_t^{\otimes 2}}{2!}, \frac{x_t^{\otimes 3}}{3!}, \dots,\right) \\
&= \left(1, x_t, \frac{x_t^2}{2!} 1^{\otimes 2}, \frac{x_t^3}{3!} 1^{\otimes 3}, \dots,\right) \in T(\mathbb{R}),
\end{aligned}$$

$Y_t^{-1} = e^{-x_t}$ because

$$\begin{aligned}
(e^a e^b)_k = \sum_{i=0}^{k} (e^a)^i (e^b)^{k-i} &= \sum_{i=0}^{k} \frac{a^i}{i!} \frac{b^{k-i}}{(k-i)!} \\
&= \frac{(a+b)^k}{k!} = (e^{a+b})_k,
\end{aligned}$$

where we have used

$$\begin{aligned}(a+b)^{\otimes k} &= (a+b)\otimes\cdots\otimes(a+b)\\ &= (a+b)^k\,1^{\otimes k} = \sum_{i=0}^k \frac{a^i}{i!}\frac{b^{k-i}}{(k-i)!}1^{\otimes k}\\ &= \sum_{i=0}^k \frac{a^{\otimes i}}{i!}\frac{b^{\otimes(k-i)}}{(k-i)!}.\end{aligned}$$

Therefore

$$X_{st} = Y_s^{-1}Y_t = e^{-x_s}\,e^{x_t} = e^{(x_t-x_s)}$$

is a multiplicative functional. Indeed if $s<u<t$ then

$$\begin{aligned}\sum_{i=0}^k X^i_{su}X^{k-i}_{ut} &= \sum_{i=0}^k \frac{(x_s-x_u)^j}{i!}\frac{(x_t-x_u)^{k-i}}{(k-i)!}\\ &= \frac{1}{k!}(x_t-x_u+x_s-x_u)^k = X^k_{s,t}\end{aligned}$$

as desired.

Example 6.29. Suppose $\dim(V)=1$, i.e. $V=\mathbb{R}$, and $x:[0,T]\to\mathbb{R}$ is a continuous path and now let $Y_t = 1-x_t$. Then $Y_t^{-1} = 1+\sum_{k=1}^\infty x_t^k$ so that

$$\begin{aligned}X_{st} &:= Y_s^{-1}Y_t = (1+x_s+x_s^2+x_s^3+\dots)(1-x_t)\\ &= 1+(x_s-x_t)+(x_s^2-x_tx_s)+(x_s^3-x_s^2x_t)+\dots\\ &= 1+(x_2-x_t)+x_s(x_s-x_t)+x_s^2(x_2-x_t)+\dots\end{aligned}$$

is a multiplicative functional. Let me check this explicitly at level 2, namely

$$\begin{aligned}&\sum_{i=0}^2 X^i_{su}X^{2-i}_{ut} = X^0_{su}X^2_{ut}+X^1_{su}X^1_{ut}+X^2_{su}X^0_{ut}\\ &= x_u(x_u-x_t)+(x_s-x_u)(x_u-x_t)+x_s(x_s-x_u)\\ &= x_s(x_u-x_t)+x_s(x_s-x_u) = x_s(x_s-x_t)\\ &= X^2_{st}\end{aligned}$$

as desired.

Let us now consider (at level 2) the difference, ψ, between the two multiplicative functionals in Examples 6.28 and 6.29,

$$\begin{aligned}\psi_{st} &:= \frac{(x_t-x_s)^2}{2} - x_s(x_s-x_t) = (x_s-x_t)\left(-x_s+\frac{x_s-x_t}{2}\right)\\ &= (x_s-x_t)\left(-\frac{x_s+x_t}{2}\right) = -\frac{1}{2}(x_s^2-x_t^2)\\ &= \frac{1}{2}(x_t^2-x_s^2).\end{aligned}$$

Definition 6.30. *Let $p\in[1,\infty)$. A p – **(geometric) rough path** is a multiplicative functional, $(X:\Delta\to G_{geo})$ $X:\Delta\to G$ such that;*

1. *X is continuous,*
2. *$V_p\left(X^1\right)+V_{p/2}\left(X^2\right)<\infty$.*

Definition 6.31. *If $x\in C_p([0,T]\to V)$ is given. We say that X is a **(geometric) p - lift** if X is a p – (geometric) rough path such that $X^1_{st} = x(t)-x(s)$ for all $(s,t)\in\Delta$.*

Theorem 6.32. *Let $x\in C_p([0,T],V)$ with $p<2$. Then x has precisely one p – lift which is given by*

$$X^2_{st} = \int_s^t (x(\tau)-x(s))\,dx(\tau) \tag{6.16}$$

$$= \frac{1}{2}(x(t)-x(s))^2+\frac{1}{2}\int_s^t [x(\tau)-x(s),dx(\tau)], \tag{6.17}$$

where all integrals are Young's integrals. Alternatively we may write, X, as

$$X_{st} = \exp\left(x(t)-x(s)+\frac{1}{2}\int_s^t [x(\tau)-x(s),dx(\tau)]\right). \tag{6.18}$$

Moreover, this p – lift is a geometric p – rough path.

Proof. To prove the existence assertion, define X^2_{st} by Eq. (6.16) and recall that Eq. (6.17) follows as in Proposition 6.11. Moreover we have seen in Example 6.9 that X is a lift of x which takes values in G_{geo}. Moreover if we let $\omega(s,t) := V_p^p(x:[s,t])$, then

$$\left\|X^2_{st}\right\| \le \zeta(2/p)\,V_p(x(\cdot)-x(s):[s,t])\,V_p(x:[s,t]) = \zeta(2/p)\,\omega(s,t)^{2/p}.$$

Hence it follows that $V_{p/2}\left(X^2\right)\le\zeta(2/p)\,\omega(0,T)^{2/p}<\infty$ and the existence assertion is proved.

For uniqueness, suppose that Y is another lift. Then we know $\psi_{st} := Y^2_{st}-X^2_{st}\in V\otimes V$ is an additive functional with $V_{p/2}(\psi)<\infty$. As $p/2<1$, this implies that $\psi_{st}=0$ for all $(s,t)\in\Delta$ which gives the uniqueness assertion of the theorem. ∎

Proposition 6.33. *Let $p \geq 1$. A multiplicative functional, $X : \Delta \to G$, is a p – rough path iff there exists a control, ω, such that*

$$\left|X^i_{st}\right| \leq \omega(s,t)^{i/p} \text{ for all } (s,t) \in \Delta,\ 1 \leq i \leq 2. \tag{6.19}$$

Moreover if X is a p – rough path, we may always take,

$$\omega(s,t) := V_p^p\left(X^1 : [s,t]\right) + V_{p/2}^{p/2}\left(X^2 : [s,t]\right). \tag{6.20}$$

Proof. ($\Leftarrow$) This is the easy direction because

$$\sum_\ell \left|X^i_{t_{\ell-1}t_\ell}\right|^{p/i} \leq \sum \omega(t_{\ell-1}, t_\ell) \leq \omega(0,T)$$

and hence $v^i(X) \leq \omega(0,T) < \infty$ for all i. This implies that $X \in \Omega_p(V)$.

For the converse, an application of Theorem 2.32 shows that both $V_p^p\left(X^1 : [s,t]\right)$ and $V_{p/2}^{p/2}\left(X^2 : [s,t]\right)$ are controls. It is now easy to verify that Eq. (6.19) holds for this control. ■

The following theorem due to Lyons and Victoir [14] shows that there are plenty of p – rough paths.

Theorem 6.34 (Extension Theorem). *Let $1 \leq p < \infty$ and $x \in C_p([0,T], V)$, then there always exists a geometric p – lift, $X : \Delta \to G_{geo}$, of x.*

Proposition 6.35 (Non-Uniqueness of Lifts). *If $2 \leq p < \infty$ and $x \in C_p([0,T], V)$, there exists an infinite number of (geometric) p – lifts of x.*

Proof. If X and Y are any two p – lifts of x, then $\psi_{st} := (Y - X)^2_{st}$ is an additive functional with finite $p/2$ – variation. Conversely if $\psi : \Delta \to V \otimes V$ is any continuous additive functional with finite p – variation and X is a fixed p - lift of x, then $Y_{st} := X_{st} + \psi_{st}$ is another p – lift of X. We we assume X and Y are geometric, then we must require ψ takes values in $\Lambda^2(V)$ – otherwise nothing else changes. ■

The following theorem makes use of result from Section 7.3 below.

Theorem 6.36 (A Geometric Rough Path Approximation Theorem). *Suppose $\dim V < \infty$ and $X : \Delta \to G_{geo}$ is a geometric p – rough path for some $1 \leq p < \infty$. Then there exists smooth (or finite variation) paths, $x_n \in C^\infty([0,T] \to V)$ such that for all $q > p$, $\rho_q(X_n, X) \to 0$ as $n \to \infty$, where*

$$X_n(s,t) := 1 + (x_n(t) - x_n(s)) + \frac{1}{2}\int_s^t (x_n(\tau) - x_n(s))\, dx_n(\tau).$$

Proof. The proof is similar to the proof of Corollary 4.11. One needs to now replace the piece x^Π by the horizontal projections of the piecewise geodesics constructed in Theorem 7.15 below. One should also use the Carnot-Caratheodory metric on G_{geo} in the proof. For the details the reader is referred to [6] and [5]. ■

7

Homogeneous Metrics

In this chapter we are going to introduce some metrics on G and G_{geo} which will allow us to relate p – rough paths to the more familiar paths of finite p – variation relative to one of these metrics. We begin with some generalities about left invariant metrics and the associated p – variation spaces.

7.1 Lie group p – variation results

We begin with some generalities about p – variations for group valued functions. In this section, suppose (G, d) is a group equipped with a left invariant metric in which (G, d) is complete. The left invariance assumption on d states that for all $a, b, c \in G$, $d(ca, cb) = d(a, b)$. Equivalently we are assuming that

$$d(a,b) = d\left(e, a^{-1}b\right) = d\left(b^{-1}a, e\right) = d\left(e, b^{-1}a\right),$$

and in particular it follows that $d(e,a) = d\left(e, a^{-1}\right)$. We will write $\|a\|$ for $d(e,a)$.

Suppose that $x \in C([0,T] \to G)$. Given a partition, $\Pi \in \mathcal{P}(s,t)$ and $\tau \in \Pi$, let, $\Delta_\tau x := x_{\tau_-}^{-1} x_\tau$. We continue the notation used in Chapter 2. In particular we have,

$$V_p(x:\Pi) := \left(\sum_{t\in\Pi} d^p\left(x_{t_-}, x_t\right)\right)^{1/p} = \left(\sum_{t\in\Pi} \|\Delta_t x\|^p\right)^{1/p} \text{ and}$$
$$V_p(x) := \sup_{\Pi\in\mathcal{P}(0,T)} V_p(x:\Pi).$$

We also define

$$\rho(x,y:\Pi) := \left(\sum_{t\in\Pi} d^p(\Delta_t x, \Delta_t y)\right)^{1/p} = \left(\sum_{t\in\Pi} \left\|(\Delta_t x)^{-1}\Delta_t y\right\|^p\right)^{1/p}$$

and set,

$$\rho(x,y) := \sup_{\Pi\in\mathcal{P}(0,T)} \rho(x,y:\Pi). \tag{7.1}$$

Observe that $\rho(x,e) = V_p(x) < \infty$ and

$$\rho(x,y:\Pi) = \left(\sum_{t\in\Pi} d^p(\Delta_t x, \Delta_t y)\right)^{1/p} \le \left(\sum_{t\in\Pi} \left[d(\Delta_t x, e) + d(e, \Delta_t y)\right]^p\right)^{1/p}$$
$$\le V_p(x:\Pi) + V_p(y:\Pi) \le V_p(x) + V_p(y)$$

and hence,

$$\rho(x,y) \le V_p(x) + V_p(y) < \infty.$$

Definition 7.1. *Let* $C_{0,p}([0,T], G) := \{x \in C([0,T] \to G) : x(0) = e \text{ and } V_p(x) < \infty\}$

Proposition 7.2. *The function, ρ, defined in Eq. (7.1) is a complete metric on* $C_{0,p}([0,T], G)$.

Proof. Let $x, y, z \in C_0([0,T], G)$.
1. If $t \in [0,T]$ we may take $\Pi := \{0, t, T\}$ to learn that

$$d(x(t), y(t)) = d(\Delta_t x, \Delta_t y)$$
$$\le \left(d^p(\Delta_t x, \Delta_t y) + d^p(\Delta_T x, \Delta_T y)\right)^{1/p} \le \rho(x,y).$$

As $t \in [0,T]$ was arbitrary it follows that

$$d_u(x,y) := \max_{0\le t\le T} d(x(t), y(t)) \le \rho(x,y). \tag{7.2}$$

2. Let $\Pi \in \mathcal{P}(0,T)$ and observe that

$$\rho(x,z:\Pi) = \left(\sum_{t\in\Pi} d^p(\Delta_t x, \Delta_t z)\right)^{1/p}$$
$$\le \left(\sum_{t\in\Pi} \left[d(\Delta_t x, \Delta_t y) + d(\Delta_t y, \Delta_t z)\right]^p\right)^{1/p}$$
$$\le \left(\sum_{t\in\Pi} d^p(\Delta_t x, \Delta_t y)\right)^{1/p} + \left(\sum_{t\in\Pi} d^p(\Delta_t y, \Delta_t z)\right)^{1/p}$$
$$= \rho(x,y:\Pi) + \rho(y,z:\Pi)$$
$$\le \rho(x,y) + \rho(y,z).$$

Hence it follows that

$$\rho(x,z) \le \rho(x,y) + \rho(y,z)$$

which shows that ρ satisfies the triangle inequality. It is clear from the definition of ρ that $\rho(x,y) = \rho(y,x)$ and from Eq. (7.2) that $\rho(x,y) = 0$ implies $x = y$. Moreover we now see that if $x, y \in C_{0,p}([0,T] \to V)$, then $\rho(x,y) \le \rho(x,e) + \rho(e,z) < \infty$ so that ρ is finite on $C_{0,p}([0,T] \to V)$.

3. To finish the proof we must now show ρ is complete. So suppose that $\{x_n\}_{n=1}^{\infty} \subset C_{0,p}([0,T], G)$ is a Cauchy sequence. Then by Eq. (7.2) we know that x_n converges uniformly to some $x \in C([0,T] \to G)$. Moreover, for any partition, $\Pi \in \mathcal{P}(0,T)$ we have

$$\begin{aligned}\rho(x, x_n : \Pi) &\le \rho(x, x_m : \Pi) + \rho(x_m, x_n : \Pi)\\ &\le \rho(x, x_m : \Pi) + \rho(x_m, x_n).\end{aligned}$$

Taking the limit of this equation as $m \to \infty$ the shows,

$$\rho(x, x_n : \Pi) \le \liminf_{m\to\infty} (\rho(x, x_m : \Pi) + \rho(x_m, x_n)) = \liminf_{m\to\infty} \rho(x_m, x_n).$$

We may now take the supremum over $\Pi \in \mathcal{P}(0,T)$ to learn,

$$\rho(x, x_n) \le \liminf_{m\to\infty} \rho(x_m, x_n) \to 0 \text{ as } n \to \infty.$$

So by the triangle inequality, $\rho(e,x) \le \rho(e,x_n) + \rho(x_n,x) < \infty$ for sufficiently large n so that $x \in C_{0,p}([0,T], G)$ and $\rho(x, x_n) \to 0$ as $n \to \infty$. ■

Remark 7.3 (Group Structure Comment). In order to get $C_{0,p}([0,T], G)$ to be a group under pointwise multiplication I think we will need to assume that d satisfies something like,

$$d(y, xy) = d\left(e, y^{-1}xy\right) \le C(y)\, d(e,x).$$

Assuming this to be the case, we then would have,

$$\begin{aligned}\Delta_t(xy) &= (xy)_{t-}^{-1}(xy)_t = y_{t-}^{-1}x_{t-}x_t y_t = y_{t-}^{-1}\Delta_t x y_t\\ &= y_{t-}^{-1}y_t y_t^{-1}\Delta_t x y_t = \Delta_t y \cdot y_t^{-1}\Delta_t x y_t\end{aligned}$$

and therefore,

$$\begin{aligned}d(e, \Delta_t(xy)) &= d\left(e, \Delta_t y \cdot y_t^{-1}\Delta_t x y_t\right) = d\left(\Delta_t y^{-1}, y_t^{-1}\Delta_t x y_t\right)\\ &\le d\left(\Delta_t y^{-1}, e\right) + d\left(e, y_t^{-1}\Delta_t x y_t\right)\\ &= d(e, \Delta_t y) + d\left(e, y_t^{-1}\Delta_t x y_t\right)\\ &\le d(e, \Delta_t y) + C(y_t)\, d(e, \Delta_t x).\end{aligned}$$

Therefore it would follows that

$$\begin{aligned}\rho(e, xy : \Pi) &= \left(\sum_{t\in\Pi} d^p(e, \Delta_t(xy))\right)^{1/p}\\ &\le \max_t C(y_t)\left[\rho(e, x : \Pi) + \rho(e, y : \Pi)\right]\end{aligned}$$

and hence that

$$\rho(e, xy) \le \max_t C(y_t)\left[\rho(e,x) + \rho(e,y)\right] < \infty.$$

7.2 Homogeneous Metrics on $G(V)$ and $G_{geo}(V)$

We now go back to the specific case at hand. In our case the groups G and G_{geo} are equipped with a "dilation" structure.

Definition 7.4. *For $\lambda \in \mathbb{R}^{\times}$ let $\delta_\lambda : \mathcal{A}(V) \to \mathcal{A}(V)$ be defined by $\delta_\lambda(\alpha + a + A) := \alpha + \lambda a + \lambda^2 A$ where $\alpha \in \mathbb{R}$, $a \in V$, and $A \in V \otimes V$. We call δ_λ the* ***dilation isomorphism.***

Proposition 7.5. *For each $\lambda \in \mathbb{R}^{\times}$, $\delta_\lambda : \mathcal{A}(V) \to \mathcal{A}(V)$ is an isomorphism of algebras. Moreover δ_λ restricts to a group isomorphism of $G(V)$ and $G_{geo}(V)$ and to a Lie algebra isomorphism of* Lie G *and* Lie G_{geo}.

Proof. Notice that δ_λ is an algebra homomorphism. Indeed, if $\beta + b + B \in \mathcal{A}$, then

$$(\alpha + a + A)(\beta + b + B) = \alpha\beta + \alpha b + \beta a + \alpha B + \beta A + ab$$

and therefore,

$$\begin{aligned}\delta_\lambda((\alpha + a + A)(\beta + b + B)) &= \alpha\beta + \lambda(\alpha b + \beta a) + \lambda^2[\alpha B + \beta A + ab]\\ &= \alpha\beta + (\alpha(\lambda b) + \beta(\lambda a)) + \left[\alpha\lambda^2 B + \beta\lambda^2 A + (\lambda a)(\lambda b)\right]\\ &= \delta_\lambda(\alpha + a + A)\,\delta_\lambda(\beta + b + B)\end{aligned}$$

as desired. The remaining assertions are easy to prove and our left to the reader. ■

Definition 7.6 (Homogeneous Norm). *A* ***homogeneous norm*** *on G (or G_{geo}) is a continuous function, $\|\cdot\| : G \to [0,\infty)$ such that:*

1. $\|g\|_G = 0$ *iff* $g = 1$,
2. $\|\delta_\lambda(g)\| = |\lambda|\,\|g\|$ *(homogeneous) for all* $\lambda \in \mathbb{R}^{\times}$,
3. $\left\|g^{-1}\right\| = \|g\|$ *(symmetric), and*
4. $\|gh\| \le \|g\| + \|h\|$ *(subadditive).*

We will give one example of such a norm in Corollary 7.8 below. We will give another example on G_{geo} in Section 7.3 below.

Lemma 7.7. *For $g = 1 + g_1 + g_2 \in G = G(V)$, let*

$$\gamma(g) := \max\left(\|g_1\|, \sqrt{2\|g_2\|}\right).$$

Then γ is subadditive and homogeneous, i.e.

$$\gamma(gh) \le \gamma(g) + \gamma(h) \text{ for all } g, h \in G$$

and

$$\gamma(\delta_\lambda(g)) = |\lambda|\,\gamma(g) \text{ for all } g \in G(V) \text{ and } \lambda \in \mathbb{R}^\times.$$

Proof. Only the subadditivity requires any proof here. Let $\alpha := \gamma(g)$ and $\beta := \gamma(h)$ where $h = 1 + h_1 + h_2$. Observe that $\|g_1\| \le \alpha$, $\|h_1\| \le \beta$, $2\|g_2\| \le \alpha^2$, and $2\|h_2\| \le \beta$. With this notation we have,

$$gh = 1 + g_1 + h_1 + (g_2 + h_2 + g_1 h_1),$$

and

$$\begin{aligned}
\gamma(gh) &= \max\left(\|g_1 + h_1\|, \sqrt{2\|g_2 + h_2 + g_1 h_1\|}\right) \\
&\le \max\left(\|g_1\| + \|h_1\|, \sqrt{2\|g_2\| + 2\|h_2\| + 2\|g_1\|\|h_1\|}\right) \\
&\le \max\left(\alpha + \beta, \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta}\right) = \max(\alpha + \beta, \alpha + \beta) \\
&= \alpha + \beta = \gamma(g) + \gamma(h).
\end{aligned}$$

■

Corollary 7.8. *If we define*

$$\|g\|_G := \gamma(g) + \gamma\left(g^{-1}\right),$$

then $\|\cdot\|_G$ is a homogeneous norm on G and by restriction on G_{geo}. Furthermore we have,

$$\gamma(g) \le \|g\|_G \le \sqrt{3}\gamma(g) \text{ for all } g \in G. \tag{7.3}$$

Proof. It only remains to prove the upper bound in Eq. (7.3). Since $g^{-1} = 1 - g_1 - g_2 + g_1^2$, we find,

$$\begin{aligned}
\gamma\left(g^{-1}\right) &= \max\left(\|g_1\|, \sqrt{2\|g_1^2 - g_2\|}\right) \\
&\le \max\left(\|g_1\|, \sqrt{2\|g_1\|^2 + 2\|g_2\|}\right) \\
&\le \max\left(\gamma(g), \sqrt{2\gamma^2(g) + \gamma^2(g)}\right) = \sqrt{3}\gamma(g).
\end{aligned}$$

■

Proposition 7.9. *If $\dim V < \infty$ then any two homogeneous norms on $G = G(V)$ are equivalent.*

Proof. Suppose that $|\cdot|$ is another homogeneous norm on G and then define

$$c := \min_{\|g\|=1} |g| \text{ and } C := \max_{\|g\|=1} |g|.$$

By compactness, $0 < c < C < \infty$. For general $g \in G \setminus \{1\}$, choose $\lambda > 0$ such that $\|\delta_\lambda(g)\|_G = 1$, i.e. take $\lambda := 1/\|g\|_G$. Then we know that

$$c \le |\delta_\lambda(g)| = \frac{|g|}{\|g\|_G} \le C$$

and therefore,

$$c\|g\|_G \le |g| \le C\|g\|_G \text{ for all } g \in G.$$

■

Proposition 7.10. *If $\|\cdot\|_G$ is a homogeneous norm on G then*

$$d(g,h) := \left\|g^{-1}h\right\|_G \text{ for } g, h \in G \tag{7.4}$$

defines a left invariant homogeneous (i.e. $d(\delta_\lambda(g), \delta_\lambda(h)) = |\lambda|\, d(g,h)$) metric on G.

Proof. The proof of this proposition is easy. For example, $d(g,h) = 0$ iff $\left\|g^{-1}h\right\|_G = 0$ iff $g^{-1}h = 1$ iff $g = h$;

$$\begin{aligned}
d(h,g) &= \left\|h^{-1}g\right\|_G = \left\|g^{-1}h\right\|_G = d(g,h), \text{ and} \\
d(g,k) &= \left\|g^{-1}k\right\|_G = \left\|g^{-1}h\ h^{-1}k\right\|_G \\
&\le \left\|g^{-1}h\right\|_G + \left\|h^{-1}k\right\|_G = d(g,h) + d(h,k).
\end{aligned}$$

■

For the rest of this section we will assume that $\gamma(\cdot)$ and $\|\cdot\|_G$ are as defined in Lemma 7.7 and 7.8 above.

Theorem 7.11. *Let $X : \Delta \to G$ (or G_{geo}) be a continuous multiplicative functional and $Y \in C([0,T] \to G)$ be associated to X via, $Y(t) := X_{0,t}$ or equivalently by, $X_{st} = Y(s)^{-1} Y(t)$ for all $(s,t) \in \Delta$. Then X is a p – rough path iff Y has finite p – variation.*

Proof. The point is that

$$V_p^p(Y:\Pi) = \sum_{t\in\Pi} d^p(Y(t_-),Y(t)) = \sum_{t\in\Pi}\left\|Y(t_-)^{-1}Y(t)\right\|_G^p$$
$$= \sum_{t\in\Pi}\|X_{t_-,t}\|_G^p \asymp \sum_{t\in\Pi}\left(\left\|X^1_{t_-,t}\right\|^p + \left(\sqrt{\left\|X^2_{t_-,t}\right\|}\right)^p\right)$$
$$= V_p^p\left(X^1:\Pi\right) + V_{p/2}^{p/2}\left(X^2:\Pi\right).$$

Therefore it follows that

$$V_p^p(Y) \le C\left(V_p^p\left(X^1\right) + V_{p/2}^{p/2}\left(X^2\right)\right),$$
$$V_p^p\left(X^1\right) \le CV_p^p(Y), \text{ and } V_{p/2}^{p/2}\left(X^2\right) \le CV_p^p(Y)$$

which certainly implies,

$$V_p^p(Y) \asymp V_p^p\left(X^1\right) + V_{p/2}^{p/2}\left(X^2\right).$$

■

Using this theorem we can see fairly easily that finite dimensional Brownian motions have geometric p - lifts for all $p>2$. This is the content of the next theorem.

Theorem 7.12 (Enhance Brownian Motion). *Let $\{B_t\}_{t\ge0}$ be a $\mathbb{R}^d$ – valued Brownian motion. Then for all $\alpha\in(0,1/2)$, there exists a $X_{st} = 1 + B_t - B_s + X^2_{st} \in G_{geo}\left(\mathbb{R}^d\right)$ such that*

$$|B_t - B_s| + \left|X^2_{st}\right|^{1/2} \le C_\alpha |t-s|^\alpha \ \ a.s.,$$

where C_α is a random finite (a.s.) constant.

Proof. Let

$$Y_t := 1 + B_t + \int_0^t B_\tau\otimes\circ dB_\tau = Y_t := 1 + B_t + \int_0^t B_\tau\otimes dB_\tau + tC$$

where $C := \sum_{i=1}^d e_i\otimes e_i$ which we assume to be chosen to be continuous. We then define $X_{st} := Y_s^{-1}Y_t$. Since

$$Y_s^{-1} = 1 - B_s - \int_0^s B_\tau\circ dB_\tau + B_s^2$$

we have,

$$X_{st} = \left(1 - B_s - \int_0^s B_\tau\circ dB_\tau + B_s^2\right)\left(1 + B_t + \int_0^t B_\tau\otimes\circ dB_\tau\right)$$
$$= 1 + B_t - B_s + \int_0^t B_\tau\otimes\circ dB_\tau - \int_0^s B_\tau\circ dB_\tau + B_s^2 - B_sB_t$$
$$= 1 + B_t - B_s + \int_s^t B_\tau\otimes\circ dB_\tau - B_s(B_t - B_s)$$
$$= 1 + B_t - B_s + \int_s^t (B_\tau - B_s)\circ dB_\tau \text{ a.s.}$$

Therefore,

$$\mathbb{E}\left[d^p(Y_s,Y_t)\right] = \mathbb{E}\left[d^p\left(1,Y_s^{-1}Y_t\right)\right] = \mathbb{E}\left[d^p(1,X_{st})\right]$$
$$\le C\mathbb{E}\left[\left(|B_t - B_s| + \left|\int_s^t (B_\tau - B_s)\circ dB_\tau\right|^{1/2}\right)^p\right].$$

Let $b_\sigma := B_{s+\sigma} - B_s$ – a new Brownian motion and $T := t-s$, then the above equation may be written as,

$$\mathbb{E}\left[d^p(Y_s,Y_t)\right] \le C\mathbb{E}\left[\left(|b_T| + \left|\int_0^T b_\sigma\circ db_\sigma\right|^{1/2}\right)^p\right]$$
$$= C\mathbb{E}\left[\left(\sqrt{T}\,|b_1| + \left|T\int_0^1 b_\sigma\circ db_\sigma\right|^{1/2}\right)^p\right]$$
$$= C(p,n)\,T^{p/2} = C(p,n)\,|t-s|^{p/2}.$$

For the second line we have used the Brownian scaling, $b_\cdot \overset{d}{=} \sqrt{T}b_{T^{-1}(\cdot)}$, to conclude that

$$\left(b_\sigma + b_{\sigma_+}\right)\left(b_{\sigma_+} - b_\sigma\right) \overset{d}{=} T\left(b_{T^{-1}\sigma} + b_{T^{-1}\sigma_+}\right)\left(b_{T^{-1}\sigma_+} - b_{T^{-1}\sigma}\right)$$

and therefore,

$$\int_0^T b_\sigma\circ db_\sigma \overset{d}{=} T\int_0^1 b_\sigma\circ db_\sigma.$$

As p is arbitrary, it now follows by an application of Kolmogorov's continuity criteria Theorem 1.7 as in the proof of Corollary 1.8, that almost surely,

$$d(Y_s,Y_t) \le C_\alpha |t-s|^\alpha$$

where α can be chosen to be any point in $(0,1/2)$. ■

7.3 Carnot Caratheodory Distance

Definition 7.13. *We say a smooth path, $g : [0,T] \to G_{geo}$ is horizontal if $g^{-1}(t)\dot{g}(t) \in V$ for all t. We define the* ***length of a horizontal path*** *to be given by,*

$$\ell(g) := \int_0^T \left\| g^{-1}(t)\dot{g}(t) \right\| dt.$$

Moreover for $x, y \in G_{geo}$, let

$$d(x,y) := \inf\{\ell(g) : g(0) = x,\ g(T) = y \text{ \& } g \text{ is horizontal}\}.$$

By Chow's theorem, we know that the set of horizontal paths joining x to y is not empty.

Lemma 7.14. *Suppose that $g : [0,T] \to G_{geo}$ is a smooth horizontal path, then there exists a unique path, $\sigma : [0, \ell(g)] \to G_{geo}$, such that $g(t) = \sigma(S(t))$ for all $0 \le t \le T$ where $S(t) := \int_0^t \left\| g^{-1}(\tau)\dot{g}(\tau) \right\| d\tau$ – arc-length $g|_{[0,t]}$. Moreover, σ is absolutely continuous, horizontal, and $\left\| \sigma^{-1}(s)\sigma'(s) \right\| = 1$ for a.e. s.*

Proof. Notice that S is a continuously differentiable function such that $\dot{S}(t) = \left\| g^{-1}(t)\dot{g}(t) \right\|$. Moreover if $S(t_0) = S(t_1)$ for some $0 \le t_0 < t_1 \le T$, then $g^{-1}(\tau)\dot{g}(\tau) = 0$ for $\tau \in [t_0, t_1]$ and therefore $g(\tau)$ is constants on $[t_0, t_1]$. Thus it makes sense to define, $\sigma : [0, \ell(g)] \to G_{geo}$ by the equation,

$$\sigma(S(t)) := g(t) \text{ for all } 0 \le t \le T.$$

Now suppose that $0 \le s_0 < s_1 \le \ell(g)$. Using the intermediate value theorem, there exists $0 \le t_0 < t_1 \le T$ such that $S(t_0) = t_0$ and $S(t_1) = s_1$. Therefore,

$$d(\sigma(S(t_1)), \sigma(S(t_0))) \le \ell\left(g|_{[t_0,t_1]}\right) = \int_{t_0}^{t_1} \left\| g^{-1}(\tau)\dot{g}(\tau) \right\| d\tau = S(t_1) - S(t_0).$$

From this it follows that σ is $d-$Lipschitz. As d dominated the metric associated to a certain Riemannian metric (see the proof of Theorem 7.15 below) we may conclude that σ is absolutely continuous. So on one hand we have, by making use of the change of variables theorem,

$$\begin{aligned} g(t_1) - g(t_0) &= \sigma(S(t_1)) - \sigma(S(t_0)) \\ &= \int_{S(t_0)}^{S(t_1)} \sigma'(s)\, ds = \int_{t_0}^{t_1} \sigma'(S(\tau))\dot{S}(\tau)\, d\tau \end{aligned}$$

while on the other,

$$g(t_1) - g(t_0) = \int_{t_0}^{t_1} \dot{g}(\tau)\, d\tau.$$

Since $0 \le t_0 \le t_1 \le T$ are arbitrary, it follows that (after choosing a particular version of σ')

$$\dot{g}(\tau) = \sigma'(S(\tau))\dot{S}(\tau) \text{ for a.e. } \tau.$$

Hence, using the change of variables theorem one more time, we find,

$$\begin{aligned} S(t_1) - S(t_0) &= \int_{t_0}^{t_1} \left\| g(\tau)^{-1}\dot{g}(\tau) \right\| d\tau \\ &= \int_{t_0}^{t_1} \left\| \sigma(S(\tau))^{-1}\sigma'(S(\tau)) \right\| \dot{S}(\tau)\, d\tau = \int_{S(t_0)}^{S(t_1)} \left\| \sigma(s)^{-1}\sigma'(s) \right\| ds \end{aligned}$$

from which we may conclude,

$$\int_{s_0}^{s_1} \left\| \sigma(s)^{-1}\sigma'(s) \right\| ds = s_1 - s_0 \text{ for all } 0 \le s_0 < s_1 \le \ell(g).$$

This then implies that $\left\| \sigma(s)^{-1}\sigma'(s) \right\| = 1$ for a.e. s showing that σ is parametrized by arc-length as desired. ■

Theorem 7.15. *Assuming* $\dim(V) < \infty$, *we have:*

1. *d is a metric on G_{geo} compatible with the natural induced topology.*
2. *d is left invariant, i.e. $d(uw, u\,y) = d(w,y)$ for all $u, w, y \in G_{geo}$.*
3. *d is homogeneous, i.e. for all $\lambda \in \mathbb{R}$ and $w, y \in G_{geo}$,*
$$d(\delta_\lambda(w), \delta_\lambda(y)) = |\lambda|\, d(w,y). \tag{7.5}$$
4. *Let $\rho(w) := d(e,w)$. Then there are constants, $0 < c < C < \infty$ such that,*
$$c\left(\|a\| + \sqrt{\|A\|}\right) \le \rho\left(e^{a+A}\right) \le C\left(\|a\| + \sqrt{\|A\|}\right). \tag{7.6}$$
5. *(G_{geo}, d) is a complete metric space.*
6. *For all $w, y \in G_{geo}$ there is an absolutely continuous path, $g : [0,1] \to G_{geo}$ such that $\left| g^{-1}(t)\dot{g}(t) \right| = d(w,y)$ a.e. t, $g(0) = w$, and $g(1) = y$. Since $\ell(g) = d(w,y)$, this path is a length minimizing geodesic joining w to y.*

Proof. Since $ug(t)$ is a horizontal path joining uw to uy and $\ell(ug) = \ell(g)$, it follows fairly easily that d is left invariant. Let $w(t) := g(t)^{-1}\dot{g}(t)$ and $g_\lambda(t) := \delta_\lambda(g(t))$. Then

$$g_\lambda(t)^{-1}\dot{g}_\lambda(t) = \frac{d}{ds}|_0 g_\lambda(t)^{-1} g_\lambda(t+s) = \frac{d}{ds}|_0 \delta_\lambda\left(g(t)^{-1} g(t+s)\right) = \lambda w(t)$$

which shows that g_λ is a horizontal path joining $\delta_\lambda(w)$ to $\delta_\lambda(y)$ and moreover $\ell(g_\lambda) = |\lambda|\,\ell(g)$. Equation (7.5) follows easily from this observation.

We now check that d is a metric. Since $g(T-t)$ is a path taking y to w with $\ell(g(T-\cdot)) = \ell(g)$, it follows that $d(w,y) = d(y,w)$. If g is a horizontal path from w to y and k is a horizontal path from y to z, then $g * k$ is a horizontal path from w to z such that

$$d(w,z) \le \ell(g*k) = \ell(g) + \ell(k).$$

Taking the infimum over g and k joining w to y and y to z respectively shows that d satisfies the triangle inequality.

We now must still show that $d(w,y) = 0$ implies $w = y$. To prove this let us consider another metric,

$$d_0(w,y) := \inf\{\ell_0(g) : g(0) = w, \text{ and } g(T) = y\}$$

where now; g is not assumed to be horizontal and

$$\ell_0(g) := \int_0^T \left\|g^{-1}(t)\,\dot g(t)\right\|_{V\oplus V\otimes V} dt.$$

Let $g(t) = 1 + x(t) + A(t)$, then

$$g(t)^{-1}\dot g(t) = \left(1 - x - A + \frac{1}{2}x^2\right)\left(\dot x + \dot A\right) = \dot x + \dot A - x\,\dot x$$

so that, for any $\alpha \in (0,1)$,

$$\begin{aligned}
\left\|g^{-1}\dot g\right\|^2 &= \|\dot x\|^2 + \left\|\dot A - x\,\dot x\right\|^2 \\
&= \|\dot x\|^2 + \left\|\dot A\right\|^2 + \|x\,\dot x\|^2 - 2\left(\dot A, x\,\dot x\right) \\
&\ge \|\dot x\|^2 + \left\|\dot A\right\|^2 + \|x\,\dot x\|^2 - \alpha\left\|\dot A\right\|^2 - \frac{1}{\alpha}\|x\,\dot x\|^2 \\
&= \|\dot x\|^2 - \left(\alpha^{-1} - 1\right)\|x\,\dot x\|^2 + (1-\alpha)\left\|\dot A\right\|^2 \\
&= \|\dot x\|^2\left\{1 - \left(\alpha^{-1} - 1\right)\|x\|^2\right\} + (1-\alpha)\left\|\dot A\right\|^2.
\end{aligned}$$

So if $g(0) = w = 1 + x_0 + A_0$ let t_1 be the first time that $\|x(t) - x_0\| \ge 1$ Then for $t \le t_1$, we can choose α so close to one such that $\inf_{t\le t_1}\left[1 - \left(\alpha^{-1}-1\right)\|x(t)\|^2\right] = \gamma > 0$, then

$$\left\|g^{-1}\dot g\right\|^2 \ge \gamma\|\dot x\|^2 + (1-\alpha)\left\|\dot A\right\|^2$$

and therefore,

$$\begin{aligned}
\ell(g) = \int_0^T \left\|g^{-1}\dot g\right\| dt &\ge \int_0^{t_1}\left\|g^{-1}\dot g\right\| dt \ge C(\gamma,\alpha)\int_0^{t_1}\left(\|\dot x\|^2 + \left\|\dot A\right\|\right) dt \\
&\ge C(\gamma,\alpha)\left(\|x(t_1) - x(0)\| + \|A(t_1) - A(0)\|\right) \\
&\ge C(\gamma,\alpha)\min\left(1, \|x(T) - x(0)\| + \|A(T) - A(0)\|\right)
\end{aligned}$$

where the second bound comes when $t_1 = T$. Thus it follows that

$$d(w,z) \ge d_0(w,z) \ge C(w)\min\left(1, \|x(T) - x(0)\| + \|A(T) - A(0)\|\right).$$

From this it follows that $d(w,z) = 0$ iff $w = z$. Hence we have shown that d is a metric.

Let $\{u,v\}$ be an orthonormal subset of V and $A \in \mathbb{R}$. Letting $x(t) := \frac{1}{2\pi}\left[(\cos 2\pi t - 1)\sqrt{|A|}u + \operatorname{sgn}(A)\sin 2\pi t\cdot\sqrt{|A|}v\right]$, we then have $g_x(1) = \exp\left(\frac{1}{2}A[u,v]\right)$ and therefore,

$$\rho\left(\exp\left(\frac{1}{2}A[u,v]\right)\right) \le \ell(g_x) = \int_0^1 \|\dot x(t)\|\,dt \le \frac{1}{\pi}\sqrt{|A|}.$$

Similarly if $a \in V$, then let $x_a(t) := ta$ so that

$$\rho(e^a) \le \ell(g_{x_a}) = \|a\|.$$

Therefore if $A = \sum_{i<j} A_{ij}[e_i, e_j]$, then

$$\begin{aligned}
\rho\left(e^{a+A}\right) = \rho\left(e^a e^A\right) &\le \rho(e^a) + \rho\left(e^A\right) \\
&\le \|a\| + \rho\left(\prod_{i<j} e^{A_{ij}[e_i,e_j]}\right) \\
&\le \|a\| + \sum_{i<j}\rho\left(e^{A_{ij}[e_i,e_j]}\right) \\
&\le \|a\| + \frac{1}{\pi}\sum_{i<j}\sqrt{2|A_{ij}|} \le C\gamma\left(e^{a+A}\right) \\
&\le C\left(\|a\| + \sqrt{\|A\|}\right) = C\gamma\left(e^{a+A}\right)
\end{aligned}$$

where

$$\gamma\left(e^{a+A}\right) = \|a\| + \sqrt{\|A\|}.$$

This gives the upper bound in Eq. (7.6). This bound also shows ρ is continuous. Indeed,

$$|\rho(w) - \rho(z)| = |d(1,w) - d(1,z)| \le d(w,z) = \rho\left(w^{-1}z\right).$$

So if we write $w = e^{a+A}$ and $z = e^{b+B}$, then

$$\begin{aligned} w^{-1}z &= \exp\left(b - a + B - A - \frac{1}{2}[a,b]\right) \\ &= \exp\left(b - a + B - A - \frac{1}{2}[a, b-a]\right) \end{aligned}$$

and hence,

$$|\rho(w) - \rho(z)| \le C\left(\|b-a\| + \sqrt{\left\|B - A - \frac{1}{2}[a,b-a]\right\|}\right) \to 0 \text{ as } z \to w.$$

We are now ready to use the dilation invariance to prove the lower bound. By compactness, we have,

$$\inf\{\rho(w) : w \text{ s.t. } \gamma(w) = 1\} = c > 0.$$

Thus we know that $\rho(w) \ge c$ whenever $\gamma(w) = 1$. For general $w \in G_{geo}$, let $\lambda > 0$ be chosen so that $\gamma(\delta_\lambda(w)) = 1$. Since $\gamma(\delta_\lambda(w)) = \lambda\gamma(w)$, this means that $\lambda = 1/\gamma(w)$. Then

$$c \le \rho(\delta_\lambda(w)) = \lambda\rho(w) = \frac{\rho(w)}{\gamma(w)}$$

which gives the lower bound in Eq. (7.6). The bounds in Eq. (7.6) shows that the metric topology associated to d is the same as the vector space topology.

Now suppose that $\{w_n\}_{n=1}^\infty$ is a d – Cauchy sequence, i.e.

$$\rho\left(w_n^{-1}w_m\right) = d(w_n, w_m) \to 0 \text{ as } m,n \to \infty.$$

In particular, from Eq. (7.6), we know that $\{w_n\}_{n=1}^\infty$ is a bounded sequence and therefore has a convergent subsequence in the usual topology and therefore in the d – topology. It is now easy to conclude that $\{w_n\}$ is d – convergent. Therefore (G_{geo}, d) is a complete metric space.

We now prove the last assertion about geodesics we will follow Montgomery [16]. Using Lemma 7.14, we may choose $\sigma_n : [0, \ell_n] \to G_{geo}$ which are absolutely continuous horizontal paths with unit speed a.e. such that $\sigma(0) = x$ and $\sigma(\ell_n) = y$ and $\ell_n \downarrow d(x,y)$ as $n \to \infty$. By letting $g_n(t) := \sigma_n(t\ell_n/d(x,y))$, we then have $g_n(0) = x$ and $g_n(\ell) = y$ with $g_n(t)^{-1}\dot{g}_n(t) = \ell_n/d(x,y)$ for a.e. t. It now follows by the Ascolli–Arzela and Banach Anolouge theorem, that after passing to a subsequence if necessary, we may assume that $g_n \to g$ uniformly on $[0, \ell_n]$ and $\dot{g}_n(t) \to u(t)$ weakly in $L^2([0,\ell])$. For any $\lambda \in \mathcal{A}^*$, we have for any bounded measurable φ, that

$$\begin{aligned} \int_0^\ell \varphi(t)\lambda\left(g(t)^{-1}u(t)\right)dt &= \lim_{n\to\infty}\int_0^\ell \varphi(t)\lambda\left(g(t)^{-1}\dot{g}_n(t)\right)dt \\ &= \lim_{n\to\infty}\int_0^\ell \varphi(t)\lambda\left(g_n(t)^{-1}\dot{g}_n(t)\right)dt \end{aligned}$$

where in the we have used the fact that $g_n \to g$ uniformly and $\sup_n \|\dot{g}_n\|_\infty < \infty$. Now if $\lambda(V) = 0$, we find that

$$\int_0^\ell \varphi(t)\lambda\left(g(t)^{-1}u(t)\right)dt = 0$$

allowing us to conclude that $\lambda\left(g(t)^{-1}u(t)\right) = 0$ a.e. t and therefore $g(t)^{-1}u(t) \in V$ for a.e. t. Now taking $\lambda \in V^*$ and $\varphi(t) \ge 0$, it follows that

$$\int_0^\ell \varphi(t)\lambda\left(g(t)^{-1}u(t)\right)dt \le \int_0^\ell \varphi(t)\|\lambda\|\,dt$$

and hence that

$$\lambda\left(g(t)^{-1}u(t)\right) \le \|\lambda\| \text{ a.e. } t.$$

Since $\lambda \in V^*$ was arbitrary we may conclude that $\left\|g(t)^{-1}u(t)\right\|_V \le 1$ for a.e. t. Moreover we have,

$$\begin{aligned} g(t_1) - g(t_0) &= \lim_{n\to\infty}[g_n(t_1) - g_n(t_0)] \\ &= \lim_{n\to\infty}\int_0^\ell 1_{(t_0,t_1]}(\tau)\dot{g}_n(\tau)\,d\tau = \int_0^\ell 1_{(t_0,t_1]}(\tau)u(\tau)\,d\tau \end{aligned}$$

from which it follows that g is absolutely continuous and $\dot{g}(t) = u(t)$ a.e. t. Thus g is a horizontal absolutely continuous path such that $\left\|g(t)^{-1}\dot{g}(t)\right\| \le 1$ for a.e. t. Therefore we may conclude that

$$d(x,y) \le \ell(g) = \int_0^\ell \left\|g(t)^{-1}\dot{g}(t)\right\|dt \le \ell = d(x,y).$$

Thus we must in fact $d(x,y) = \ell(g)$ and $\left\|g(t)^{-1}\dot{g}(t)\right\| = 1$ for a.e. t as desired. ■

Rough Path Integrals

Throughout this chapter we will be assuming that $2 \leq p < 3$. Our first goal is to show how to make p – rough paths out of almost p – rough paths.

8.1 Almost Multiplicative Functionals

The results of this section will be a fairly straight forward generalization of the results in Section 5.1.

Definition 8.1. *Let $\theta > 1$. A θ –* ***almost multiplicative functional*** *(A.M.F.) is a function $X : \Delta \to G$ of finite p -variation such that there exists a control, ω and $C < \infty$ such that*

$$\left|X^i_{st} - [X_{su} X_{ut}]^i\right| \leq C\omega(s,t)^\theta \text{ for all } 0 \leq s \leq u \leq t \leq T \text{ and } 1 \leq i \leq 2. \quad (8.1)$$

If Eq. (8.1) holds for some $\theta > 1$ and control ω, we say X is an (ω, p) – ***almost multiplicative functional*** *or sometimes an (ω, p) –* ***almost rough path****.*

We will see plenty of example of almost rough paths later.

Notation 8.2 *Given a function, $X : \Delta \to G$ and a partition,*

$$\Pi = \{s = t_0 < t_1 < \cdots < t_r = t\},$$

of $[s,t]$, let

$$X(\Pi) := \prod_{\tau \in \Pi} X_{\tau_-,\tau} = X_{t_0,t_1} X_{t_1,t_2} \dots X_{t_{r-1},t_r}.$$

Furthermore, given a partition, Π, of $[0,T]$ and $(s,t) \in \Delta$ let

$$X(\Pi)_{st} := X(\Pi_{[s,t]}) = \prod_{\tau \in \Pi_{[s,t]}} X_{\tau_-,\tau}.$$

Theorem 8.3. *If $X : \Delta \to G$ is a an (ω, p) – almost rough path then there exists a unique p – rough path, $\tilde{X} : \Delta \to G$ such that*

$$\left\|X^i_{st} - \tilde{X}^i_{st}\right\| \leq C K^i_\theta(\omega(0,T))\, \omega(s,t)^\theta \text{ for all } (s,t) \in \Delta \text{ and } i = 1, 2, \quad (8.2)$$

where K^1_θ is independnent of ω.

Proof. (**Uniqueness.**) Suppose Z_{st} is another such rough path so that Eq. (8.2) holds with $\tilde{X}$ replaced by Z. . Then by the triangle inequality we have

$$\left\|\left[Z_{st} - \tilde{X}_{st}\right]^i\right\| \leq 2C_i \omega(s,t)^\theta. \quad (8.3)$$

As $Z^1_{st} - \tilde{X}^1_{st}$ is an additive functional, it follows from Lemma 5.21 and Eq. (8.3) that $Z^1 = \tilde{X}^1$. Now that $\tilde{X}^1 = Z^1$ we further know that $\tilde{X}^2 - Z^2$ is also an additive functional. So another application of Lemma 5.21 along with Eq. (8.3) implies that $\tilde{X}^2 = Z^2$. Thus we have shown $\tilde{X} = Z$ as desired.

(**Existence.**) 1. Notice that the condition in Eq. (8.1) for $i = 1$ is the statement that X^1_{st} is an almost (ω, p) – additive functional. Therefore we may apply Theorem 5.24 to find a finite p – variation additive functional, $\tilde{X}^1_{st}$, such that Eq. (8.2) holds for $i = 1$, with $K_\theta(\omega) = \zeta(\theta)$ in this case.

2. Let $Z_{st} := 1 + \tilde{X}^1_{st} + X^2_{st}$. I now claim that Z is still an (ω, p) – almost rough path. Indeed,

$$\begin{aligned}
\left\|[Z_{su} - Z_{st} Z_{tu}]^2\right\| &= \left\|X^2_{su} - X^2_{st} - X^2_{tu} - \tilde{X}^1_{st}\tilde{X}^1_{tu}\right\| \\
&\leq \left\|X^2_{su} - X^2_{st} - X^2_{tu} - X^1_{st}X^1_{tu}\right\| + \left\|X^1_{st}X^1_{tu} - \tilde{X}^1_{st}\tilde{X}^1_{tu}\right\| \\
&\leq C\omega(s,u)^\theta + \left\|X^1_{st}X^1_{tu} - X^1_{st}\tilde{X}^1_{tu}\right\| + \left\|X^1_{st}\tilde{X}^1_{tu} - \tilde{X}^1_{st}\tilde{X}^1_{tu}\right\| \\
&\leq C\omega(s,u)^\theta + \left\|X^1_{st}\right\| \left\|X^1_{tu} - \tilde{X}^1_{tu}\right\| + \left\|X^1_{st} - \tilde{X}^1_{st}\right\| \left\|\tilde{X}^1_{tu}\right\| \\
&\leq C\omega(s,u)^\theta + \left(\left\|X^1_{st}\right\| + \left\|\tilde{X}^1_{tu}\right\|\right) C\zeta(\theta)\,\omega(s,u)^\theta \\
&\leq C\left[1 + \begin{pmatrix} C\omega(s,t)^{1/p} + C\omega(t,u)^{1/p} \\ +C\zeta(\theta)\,\omega(t,u)^\theta \end{pmatrix} \zeta(\theta)\right] \omega(s,u)^\theta \\
&\leq K(\theta, \omega(0,T))\,\omega(s,t)^\theta.
\end{aligned}$$

3. Now suppose that $\Pi \in \mathcal{P}(s,t)$ and $\tau \in \Pi$. Then

$$\begin{aligned}
Z(\Pi \setminus \{\tau\}) - Z(\Pi) &= Z\left(\Pi_{[s,\tau_-]}\right)\left[Z_{\tau_-,\tau_+} - Z_{\tau_-,\tau} Z_{\tau,\tau_+}\right] Z\left(\Pi_{[\tau_+,t]}\right) \\
&= Z\left(\Pi_{[s,\tau_-]}\right)\left[Z_{\tau_-,\tau_+} - Z_{\tau_-,\tau} Z_{\tau,\tau_+}\right]^2 Z\left(\Pi_{[\tau_+,t]}\right) \\
&= \left[Z_{\tau_-,\tau_+} - Z_{\tau_-,\tau} Z_{\tau,\tau_+}\right]^2,
\end{aligned}$$

wherein we have used,

$$\left(Z_{\tau_-,\tau_+} - Z_{\tau_-,\tau} Z_{\tau,\tau_+}\right)^i = 0 \text{ for } i = 0, 1.$$

Therefore it follows that

$$\left\| Z\left(\Pi \setminus \{\tau\}\right) - Z\left(\Pi\right) \right\| = \left\| \left[Z_{\tau_-,\tau_+} - Z_{\tau_-,\tau} Z_{\tau,\tau_+} \right]^2 \right\| \le C\omega\left(\tau_-, \tau_+\right)^\theta .$$

Comparing this identity with that of Eq. (5.34), we now see that we may follow the proof of Proposition 5.23 and Theorem 5.24 verbatim with X replaced by $Z^2 = X^2$ in order to learn, $\lim_{|\Pi|\to 0} X\left(\Pi\right)^2_{st} := \tilde{X}^2_{st}$ exists and satisfies,

$$\left\| \tilde{X}^2_{st} - X^2_{st} \right\| \le CK\left(\theta, \omega\left(0, T\right)\right) \omega\left(s, t\right)^\theta \text{ for all } (s,t) \in \Delta.$$

It is straight forward to now show that $\tilde{X} : \Delta \to G$ is the the desired multiplicative functional which completes the proof of existence. ■

Lemma 8.4. *Suppose $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are elements of an associative algebra, $\mathcal{A}$, then*

$$a_1 \ldots a_r - b_1 \ldots b_r = \sum_{i=1}^r b_1 \ldots b_{i-1} \left(a_i - b_i\right) a_{i+1} \ldots a_r \tag{8.4}$$

where $b_1 \ldots b_{i-1} := 1$ when $i = 1$ and $a_{i+1} \ldots a_r = 1$ when $i = r$. If $\mathcal{A}$ is a normed algebra with the property that $|ab| \le |a||b|$ for all $a, b \in \mathcal{A}$, then

$$\left|a_1 \ldots a_r - b_1 \ldots b_r\right| \le \sum_{i=1}^r \left|b_1\right| \ldots \left|b_{i-1}\right| \left|a_{i+1}\right| \ldots \left|a_r\right| \left|a_i - b_i\right| .$$

In particular if $|a_i| \le \delta$, $|b_i| \le \delta$ and $|a_i - b_i| \le \varepsilon\delta$ for all i, then

$$\left|a_1 \ldots a_r - b_1 \ldots b_r\right| \le r\varepsilon\delta^r. \tag{8.5}$$

Proof. This is easily proved by induction. Indeed, for $i = 1$, the right side of Eq. (8.4) is $a_1 - b_1$ and by induction,

$$\begin{aligned} a_1 \ldots a_{r+1} &- b_1 \ldots b_{r+1} \\ &= \left(a_1 \ldots a_r - b_1 \ldots b_r\right) a_{r+1} + b_1 \ldots b_r \left(a_{r+1} - b_{r+1}\right) \\ &= \sum_{i=1}^r b_1 \ldots b_{i-1} \left(a_i - b_i\right) a_{i+1} \ldots a_r a_{r+1} + b_1 \ldots b_r \left(a_{r+1} - b_{r+1}\right) \\ &= \sum_{i=1}^{r+1} b_1 \ldots b_{i-1} \left(a_i - b_i\right) a_{i+1} \ldots a_r a_{r+1}. \end{aligned}$$

■

Lemma 8.5. *Let a, b be elements of a Banach algebra, i.e., $|ab| \le |a||b|$, then*

$$\left|a^2 - b^2\right| \le |a + b| \; |a - b|.$$

Proof. This is a consequence of the simple algebraic relation,

$$a^2 - b^2 = \frac{1}{2}\left[(a + b)(a - b) + (a - b)(a + b)\right]$$

from which it follows that

$$\begin{aligned} |a^2 - b^2| &= \frac{1}{2}\left|(a + b)(a - b) + (a - b)(a + b)\right| \\ &\le \frac{1}{2}\left[|a + b| \; |a - b| + |a - b| \; |a + b|\right] = |a + b| \; |a - b|. \end{aligned}$$

■

Remark 8.6. Suppose that $g_i = 1 + a_i + A_i \in G$, then

$$\begin{aligned} g_1 \ldots g_n &= \left(1 + a_1 + A_1\right) \ldots \left(1 + a_n + A_n\right) \\ &= 1 + \sum_{i=1}^n a_i + \left(\sum_{i=1}^n A_i + \sum_{i<j} a_i a_j \right). \end{aligned}$$

Therefore if $\Pi \in \mathcal{P}\left(s, t\right)$, it follows that

$$X\left(\Pi\right)^1 = \sum_{\tau \in \Pi} X^1_{\tau_-,\tau} \text{ and} \tag{8.6}$$

$$X\left(\Pi\right)^2 = \sum_{\tau \in \Pi} X^2_{\tau_-,\tau} + \sum_{\sigma,\tau \in \Pi:\ \sigma < \tau} X^1_{\sigma_-,\sigma} X^1_{\tau_-,\tau}. \tag{8.7}$$

Theorem 8.7. *Suppose $X : \Delta \to G$ is a θ – A.M.F with finite p – variation and $\tilde{X} : \Delta \to G$ is the unique M.F. such that*

$$\left\| X_{st} - \tilde{X}_{st} \right\| \le C\omega(s,t)^\theta \quad \forall \;\; (s,t) \in \Delta.$$

Then $\tilde{X}_{st} = \lim_{|\Pi|\to 0} X\left(\Pi\right)_{st}$ which in components reads,

$$\tilde{X}^1_{st} = \lim_{|\Pi|\to 0} \sum_{\tau \in \Pi} X^1_{\tau_-,\tau} \text{ and} \tag{8.8}$$

$$\tilde{X}^2_{st} = \lim_{|\Pi|\to 0} \left(\sum_{\tau \in \Pi} X^2_{\tau_-,\tau} + \sum_{\sigma,\tau \in \Pi:\ \sigma < \tau} X^1_{\sigma_-,\sigma} X^1_{\tau_-,\tau} \right). \tag{8.9}$$

Proof. Let $Z_{st} = 1 + \tilde{X}^1_{st} + X^2_{st}$, then as was shown in the proof of Theorem 8.3,

$$\tilde{X}^1_{st} = \lim_{|\Pi|\to 0} X(\Pi)^1_{st} \text{ and } \tilde{X}^2_{st} = \lim_{|\Pi|\to 0} Z(\Pi)^2_{st}.$$

So to finish the proof of this theorem, it suffices to prove

$$\lim_{|\Pi|\to 0} [X(\Pi) - Z(\Pi)]^2_{st} = 0.$$

If $\Pi := \{s = t_0 < t_1 < \cdots < t_r = t\}$, then

$$X(\Pi) - Z(\Pi) = X_{t_0 t_1} \ldots X_{t_{r-1} t_r} - Z_{t_0 t_1} \ldots Z_{t_{r-1} t_r}.$$

An application of Eq. (8.4) of Lemma 8.4 using $X^2_{t_{i-1} t_i} = Z^2_{t_{i-1} t_i}$, then gives,

$$\begin{aligned} X(\Pi) - Z(\Pi) &= \sum_{i=1}^{r} Z_{t_0 t_1} \ldots Z_{t_{i-2} t_{i-1}} \left(X_{t_{i-1} t_i} - Z_{t_{i-1} t_i}\right) X_{t_i t_{i+1}} \ldots X_{t_{r-1} t_r} \\ &= \sum_{\tau\in\Pi} Z(\Pi \cap [s, \tau_-]) \left(X_{\tau_-,\tau} - Z_{\tau_-,\tau}\right) X(\Pi \cap [\tau, t]) \\ &= \sum_{\tau\in\Pi} Z(\Pi \cap [s, \tau_-]) \left(X^1_{\tau_-,\tau} - Z^1_{\tau_-,\tau}\right) X(\Pi \cap [\tau, t]). \end{aligned}$$

Taking the $V \otimes V$ component of this identity then shows,

$$\begin{aligned} &X^2(\Pi) - Z^2(\Pi) \\ &\quad = \sum_{\tau\in\Pi} \left\{ Z(\Pi \cap [s, \tau_-])^1 \left(X^1_{\tau_-,\tau} - Z^1_{\tau_-,\tau}\right) + \left(X^1_{\tau_-,\tau} - Z^1_{\tau_-,\tau}\right) X(\Pi \cap [\tau, t])^1 \right\} \\ &\quad = \sum_{\tau\in\Pi} \left\{ Z^1_{s,\tau_-} \left(X^1_{\tau_-,\tau} - Z^1_{\tau_-,\tau}\right) + \left(X^1_{\tau_-,\tau} - Z^1_{\tau_-,\tau}\right) X^1(\Pi \cap [\tau, t]) \right\}. \end{aligned}$$

Crude estimates then imply,

$$\begin{aligned} \left\|X^2(\Pi) - Z^2(\Pi)\right\| &\le \sum_{\tau\in\Pi} \left(\left\|Z^1_{s,\tau_-}\right\| + \left\|X^1(\Pi \cap [\tau, t])\right\| \right) \left\|X^1_{\tau_-,\tau} - Z^1_{\tau_-,\tau}\right\| \\ &\le \sum_{\tau\in\Pi} \left(\left\|Z^1_{s,\tau_-}\right\| + \left\|X^1_{\tau,t}\right\| + \zeta(\theta)\,\omega(\tau, t)^\theta \right) K_\theta \omega(\tau_-, \tau)^\theta \\ &\le K(\theta, \omega, X) \sum_{\tau\in\Pi} \omega(\tau_-, \tau)^\theta \to 0 \text{ as } |\Pi| \to 0. \end{aligned}$$

■

8.2 Path Integration along Rough Paths

Our next goal is to define integrals with integrators being a p – rough path. To get an idea of what sort of definitions we should be using, let us go back to the smooth case briefly. So suppose that $x : [0, T] \to V$ is a smooth function and $f : V \to \mathrm{End}(V, U)$ is also a smooth function and let

$$\begin{aligned} X^1_{st} &:= x(t) - x(s), \quad X^2_{st} := \int_s^t (x(\tau) - x(s))\, dx(\tau) \\ z(t) &:= \int_0^t f(x(\tau))\, dx(\tau), \end{aligned}$$

$$\begin{aligned} Z^1_{st} &:= z(t) - z(s) = \int_s^t f(x(\tau))\, dx(\tau), \text{ and} \\ Z^2_{st} &:= \int_s^t (z(\tau) - z(s))\, dz(\tau) = \int_s^t (z(\tau) - z(s))\, f(x(\tau))\, dx(\tau). \end{aligned}$$

If $(s, t) \in \Delta$ with $|t - s|$ small, then using Taylor's theorem,

$$\begin{aligned} Z^1_{st} &:= z(t) - z(s) = \int_s^t f(x(\tau))\, dx(\tau) \\ &\cong \int_s^t \left[f(x(s)) + f'(x(s))(x(\tau) - x(s))\right] dx(\tau) \\ &= f(x(s))\, X^1_{st} + f'(x(s))\, X^2_{st} \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} Z^2_{st} &= \int_{s\le\sigma\le\tau\le t} dz(\sigma)\, dz(\tau) = \int_{s\le\sigma\le\tau\le t} f(x(\sigma))\, dx(\sigma)\, f(x(\tau))\, dx(\tau) \\ &= \int_{s\le\sigma\le\tau\le t} f(x(\sigma)) \otimes f(x(\tau))\, dx(\sigma) \otimes dx(\tau) \\ &\cong \int_{s\le\sigma\le\tau\le t} f(x(s)) \otimes f(x(s))\, dx(\sigma) \otimes dx(\tau) = f(x(s))^{\otimes 2} X^2_{st}. \end{aligned} \tag{8.11}$$

Proposition 8.8. *Let $x \in C_p([0, T], V)$ and $X_{st} = 1 + (x(t) - x(s)) + X^2_{st}$ is a p – lift of x to a p – rough path. Further suppose that ω is a control such that*

$$\left\|X^i_{st}\right\| \le \omega(s, t)^{i/p} \text{ for all } (s, t) \in \Delta. \tag{8.12}$$

Then the functions $f_s = f(X_s) \in L(V, U)$ and $\alpha_s = f'(X_s) \in L(V \otimes V, U)$ have finite p – variation and satisfy estimates of the form,

$$\left|f_t - f_s - \alpha_s X^1_{st}\right| \le C\omega(s,t)^{p/2}$$

and

$$|\alpha_t - \alpha_s| \le C\omega(s,t)^{1/p}.$$

Proof. By Taylor's Theorem 8.21,

$$\begin{aligned}\left|f_t - f_s - \alpha_s X^1_{st}\right| &= \left|f(x(s)+X_{st}) - f(x(s)) - f'(x(s))X^1_{st}\right| \\ &= \left|\int_0^1 (1-\tau) f''\left(X_s + \tau X^1_{st}\right) X^1_{st}\otimes X^1_{st} d\tau\right| \\ &\le C(f'')\,\omega(s,t)^{2/p}\end{aligned}$$

and

$$\begin{aligned}|\alpha_t - \alpha_s| &= \left|f'\left(x(s)+X^1_{st}\right) - f'(x(s))\right| \\ &= \left|\int_0^1 f''\left(X_s + \tau X^1_{st}\right) X^1_{st}\otimes(\cdot)\, d\tau\right| \le C(f'')\,\omega(s,t)^{1/p}.\end{aligned}$$

■

Definition 8.9 (Differentiable Pairs). *Let U be another Banach space, $x \in C_p([0,T],V)$, and ω be a control such that $x_{st} := x_t - x_s = O\left(\omega(s,t)^{1/p}\right)$. We way that $(y,\alpha) \in C_p([0,T] \to U \times \mathrm{End}(V,U))$ is an (x,ω) – **differentiable pair** if;*

1. $\alpha_{st} := \alpha_t - \alpha_s = O\left(\omega(s,t)^{1/p}\right)$ *and*
2. $\varepsilon_{st} := y_{st} - \alpha_s x_{st} = O\left(\omega(s,t)^{2/p}\right)$. *(The reader may wish to view α_s as a derivative of y relative to x.*

Let $\mathcal{D}(U) = \mathcal{D}(U,x,\omega)$ denote the space of U – valued (x,ω) – differentiable pairs.

We now wish to introduce a number of norms and semi-norms.

Notation 8.10 *For $(y,\alpha) \in \mathcal{D}(U)$ let $N_1(\alpha)$ and $N_2(Y,\alpha)$ denote the best constants such that*

$$|\alpha_{st}| \le N_1(\alpha)\,\omega(s,t)^{1/p} \text{ and}$$
$$|y_{st} - \alpha_s x_{st}| \le N_2(y,\alpha)\,\omega(s,t)^{2/p}.$$

We further define,

$$\begin{aligned}\|\alpha\|_1 &= |\alpha_0| + N_1(\alpha), \\ \|y\|_1 &= |y_0| + N_1(y), \text{ and} \\ \|(y,\alpha)\|_2 &= |\alpha_0| + |y_0| + N_1(\alpha) + N_2(y,\alpha) \\ &= \|\alpha\|_1 + |y_0| + N_2(y,\alpha).\end{aligned}$$

Similarly, if X is a p – rough path controlled by ω, we define $N_1\left(X^1\right)$ and $N_2\left(X^2\right)$ to be the best constants such that

$$\left|X^1_{st}\right| \le N_1\left(X^1\right)\omega(s,t)^{1/p} \text{ and } \left|X^2_{st}\right| \le N_2\left(X^2\right)\omega(s,t)^{2/p}.$$

We also let

$$\|X\| := N_1\left(X^1\right) + N_2(X).$$

For later purposes let us observe that

$$|\alpha_s| \le |\alpha_0| + N(\alpha)\,\omega(0,s)^{1/p}$$

$$\begin{aligned}|y_{st}| &\le |\alpha_s|\,|x_{st}| + N_2(y,\alpha)\,\omega(s,t)^{2/p} \\ &\le \left[|\alpha_0| + N(\alpha)\,\omega(0,s)^{1/p}\right] N(x)\,\omega(s,t)^{1/p} + N_2(y,\alpha)\,\omega(s,t)^{2/p} \\ &\le \left(|\alpha_0|\,N(x) + [N(\alpha)\,N(x) + N_2(y,\alpha)]\,\omega(0,T)^{1/p}\right)\omega(s,t)^{1/p}.\end{aligned}$$

Proposition 8.11. *If $(f,\alpha) \in \mathcal{D}(\mathrm{End}(V,U))$ and*

$$Z^1_{st} := f_s X^1_{st} + \alpha_s X^2_{st} \text{ and } Z^2_{st} = [f_s \otimes f_s]\, X^2_{st}, \tag{8.13}$$

then $Z_{st} := 1 + Z^1_{st} + Z^2_{st}$ is an $(3/p,\omega)$ – A.M.F. satisfying,

$$\left|Z^1_{su} + Z^1_{ut} - Z^1_{st}\right| \le \left[N_2(f,\alpha)\,N_1(X) + N_2\left(X^2\right)N_1(\alpha)\right]\omega(s,t)^{3/p}. \tag{8.14}$$

$$\left|[Z_{su}Z_{ut}]^2 - Z^2_{st}\right| \le C(\omega,f,X)\,\omega(s,t)^{3/p}$$

such that

$$\left\|Z^1_{st}\right\| = \left\|f_s X^1_{st} + \alpha_s X^2_{st}\right\| \le |f_s|\,N\left(X^1\right)\omega(s,t)^{1/p} + |\alpha_s|\,N\left(X^2\right)\omega(s,t)^{2/p}$$

and

$$\left\|Z^2_{st}\right\| = \left\|[f_s \otimes f_s]\, X^2_{st}\right\| \le |f_s|^2\, N\left(X^2\right)\omega(s,t)^{2/p}.$$

Proof. Let $0 \le s \le u \le t \le T$, then

$$\begin{aligned}Z^1_{su} + Z^1_{ut} &= f_s X^1_{su} + \alpha_s X^2_{su} + f_u X^1_{ut} + \alpha_u X^2_{ut} \\ &= f_s X^1_{su} + \alpha_s X^2_{su} + \left[f_s + \alpha_s X^1_{su} + \varepsilon_{su}\right] X^1_{ut} + [\alpha_s + \alpha_{su}]\, X^2_{ut} \\ &= f_s\left[X^1_{su} + X^1_{ut}\right] + \alpha_s\left[X^2_{su} + X^1_{su}X^1_{ut} + X^2_{ut}\right] + \varepsilon_{su}X^1_{ut} + \alpha_{su}X^2_{ut} \\ &= Z^1_{st} + \varepsilon_{su}X^1_{ut} + \alpha_{su}X^2_{ut}\end{aligned}$$

and therefore,

$$\begin{aligned}\left|Z^1_{su}+Z^1_{ut}-Z^1_{st}\right| &\le |\varepsilon_{su}|\left|X^1_{ut}\right|+|\alpha_{su}|\left|X^2_{ut}\right|\\ &\le N_2(f,\alpha)\,\omega(s,u)^{2/p}N_1(X)\,\omega(u,t)^{1/p}\\ &\quad+N_1(\alpha)\,\omega(s,u)^{1/p}N_2\left(X^2\right)\omega(u,t)^{2/p}\\ &\le\left[N_2(f,\alpha)N_1(X)+N_2\left(X^2\right)N_1(\alpha)\right]\omega(s,t)^{3/p}.\end{aligned}$$

Similarly,

$$\begin{aligned}[Z_{su}Z_{ut}]^2 &= Z^2_{su}+Z^2_{ut}+Z^1_{su}Z^1_{ut}\\ &=[f_s\otimes f_s]X^2_{su}+[f_u\otimes f_u]X^2_{ut}+\left(f_sX^1_{su}+\alpha_sX^2_{su}\right)\left(f_uX^1_{ut}+\alpha_uX^2_{ut}\right)\\ &=[f_s\otimes f_s]X^2_{su}+[(f_s+f_{su})\otimes(f_s+f_{su})]X^2_{ut}+f_s\otimes(f_s+f_{su})X^1_{su}\otimes X^1_{ut}\\ &=[f_s\otimes f_s]\left(X^2_{su}+X^2_{ut}+X^1_{su}\otimes X^1_{ut}\right)+[f_s\otimes f_{su}+f_{su}\otimes f_s+f_{su}\otimes f_{su}]X^2_{ut}\\ &\quad+f_s\otimes f_{su}X^1_{su}\otimes X^1_{ut}\\ &=Z^2_{su}+[f_s\otimes f_{su}+f_{su}\otimes f_s+f_{su}\otimes f_{su}]X^2_{ut}+f_s\otimes f_{su}X^1_{su}\otimes X^1_{ut}.\end{aligned}$$

Thus it follows that

$$\begin{aligned}\left|[Z_{su}Z_{ut}]^2-Z^2_{st}\right| &\le\left|[f_s\otimes f_{su}+f_{su}\otimes f_s+f_{su}\otimes f_{su}]X^2_{ut}+f_s\otimes f_{su}X^1_{su}\otimes X^1_{ut}\right|\\ &\le C(\omega,f,X)\,\omega(s,t)^{3/p}.\end{aligned}$$

The exact form of the constant is a bit of a mess. ■

Definition 8.12 (Integration). *For $X:\Delta\to G$ and f and α as in Proposition 8.11, we let*

$$\left[\int_s^t\left(fdX^1+\alpha dX^2\right)\right]^1:=\lim_{|\Pi|\to0}\sum_{\tau\in\Pi}\left[f_{\tau_-}X^1_{\tau_-,\tau}+\alpha_{\tau_-}X^2_{\tau_-\tau}\right]\ \text{and}\tag{8.15}$$

$$\left[\int_s^t\left(fdX^1+\alpha dX^2\right)\right]^2:=\lim_{|\Pi|\to0}\left\{\begin{matrix}\sum_{\tau\in\Pi}\left[f_{\tau_-}\otimes f_{\tau_-}\right]X^2_{\tau_-,\tau}\\+\sum_{\sigma,\tau\in\Pi:\ \sigma<\tau}\left(f_{\sigma_-}X^1_{\sigma_-,\sigma}+\alpha_{\sigma_-}X^2_{\sigma_-,\sigma}\right)\left(f_{\tau_-}X^1_{\tau_-,\tau}+\alpha_{\tau_-}X^2_{\tau_-,\tau}\right)\end{matrix}\right\}\tag{8.16}$$

so that

$$Z_{st}:=1+\int_s^t\left(fdX^1+\alpha dX^2\right)$$

is the unique p – rough path close to that $(3/p,\omega)$ – A.M.F., Z, defined of Proposition 8.11. We will use the notation,

$$\begin{aligned}\int_s^t(f,\alpha)\cdot dX &=\int_s^t\left(fdX^1+\alpha dX^2\right)\\ &=1+\left[\int_s^t\left(fdX^1+\alpha dX^2\right)\right]^1+\left[\int_s^t\left(fdX^1+\alpha dX^2\right)\right]^2.\end{aligned}$$

(See Remark 8.6 for the formulas stated in Eqs. (8.15) and (8.16).)

Proposition 8.13. *Continuing the notation used above we have*

$$|\alpha_t|\le|\alpha_0|+N_1(\alpha)\,\omega(0,t)^{1/p}\le|\alpha_0|+N_1(\alpha)\,\omega(0,T)^{1/p},\tag{8.17}$$

$$|f_{st}|\le\left[|\alpha_0|+N_1(\alpha)\,\omega(0,T)^{1/p}\right]N_1(X)\,\omega(s,t)^{1/p}+N(f,\alpha)\,\omega(s,t)^{2/p}\tag{8.18}$$

$$\le\left[\max\left(1,\omega(0,T)^{1/p}\right)N_1(X)\,\|\alpha\|_1+N(f,\alpha)\,\omega(0,T)^{1/p}\right]\omega(s,t)^{1/p}\tag{8.19}$$

$$N_1(f)\le\max\left(1,\omega(0,T)^{1/p}\right)\left[N_1(X)\,\|\alpha\|_1+N(f,\alpha)\right],\tag{8.20}$$

$$N_1(f)\le|\alpha_0|\,N_1(X)+[N_1(\alpha)N_1(X)+N(f,\alpha)]\,\omega(0,T)^{1/p},\tag{8.21}$$

$$\|f\|_1\le\max(1,N_1(X))\max\left(1,\omega(0,T)^{1/p}\right)\|(f,\alpha)\|_2\tag{8.22}$$

$$|f_t|\le|f_0|+|\alpha_0|\,N_1(X)\,\omega(s,t)^{1/p}+\left[N_1(\alpha)N_1(X)\,\omega(0,T)^{1/p}+N(f,\alpha)\right]\omega(s,t)^{2/p},\tag{8.23}$$

and

$$\|f\|_\infty\le|f_0|+|\alpha_0|\,N_1(X)\,\omega(0,T)^{1/p}+[N_1(\alpha)N_1(X)+N(f,\alpha)]\,\omega(0,T)^{2/p}.\tag{8.24}$$

Proof. The proofs of these estimate are all fairly straight forward. The first estimate is a trivial consequence of the definition of N_1 and the fact that $\alpha_t=\alpha_0+\alpha_{0,t}$. For the estimate in Eq. (8.18) we have

$$\begin{aligned}|f_{st}| &\le|\alpha_s|\left|X^1_{s,t}\right|+|\varepsilon_{st}|\\ &\le\left[|\alpha_0|+N_1(\alpha)\,\omega(0,T)^{1/p}\right]\left|X^1_{s,t}\right|+|\varepsilon_{st}|\\ &\le\left[|\alpha_0|+N_1(\alpha)\,\omega(0,T)^{1/p}\right]N_1(X)\,\omega(s,t)^{1/p}+N(f,\alpha)\,\omega(s,t)^{2/p}.\end{aligned}$$

Eq. (8.21) is a simple consequence of Eq. (8.18) and Eq. (8.23) is a consequence of Eqs. (8.17) and (8.21) and Eq. (8.24) follows directly from Eq. (8.23). For Eq. (8.22) we have

$$\|f\|_1 = |f_0| + N_1(f) \le |f_0| + \omega(0,T)^{1/p} N(f,\alpha) + N_1(X)\left[|\alpha_0| + \omega(0,T)^{1/p} N_1(\alpha)\right]$$
$$\le \max(1, N_1(X)) \max\left(1, \omega(0,T)^{1/p}\right) \|(f,\alpha)\|_2 .$$

■

Theorem 8.14. *Let $F := (f,\alpha) \in \mathcal{D}(\mathrm{End}(V,U))$, then*

$$\left\|\left(\int F \cdot dX, f\right)\right\|_2 \le |f_0| + \|X\| |\alpha_0| + \left[\left(1 + K_{3/p}\right)\|X\| + 1 \vee N_1(X)\right] \omega(0,T)^{1/p} \|F\|_2 \tag{8.25}$$
$$\le |f_0| + \|X\| |\alpha_0| + C_p(\|X\|)\, \omega(0,T)^{1/p} \|F\|_2 \tag{8.26}$$

where

$$C_p(\|X\|) := 1 + \left(2 + K_{3/p}\right)\|X\| . \tag{8.27}$$

Furthermore,

$$\left|\left(\int_s^t F \cdot dX\right)^1\right| \le K(X,f)\, \omega(s,t)^{1/p} \tag{8.28}$$

where

$$K(X,f) := \|f\|_\infty N_1\left(X^1\right) + \|\alpha\|_\infty N_2(X)\, \omega(0,T)^{1/p}$$
$$+ K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] \omega(0,T)^{2/p} . \tag{8.29}$$

Proof. Combining Eq. (8.14) with Theorem 8.3 shows

$$\left|\left(\int_s^t F \cdot dX\right)^1 - W^1_{st}\right| \le K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] \omega(s,t)^{3/p}$$
$$\le K_{3/p} \|X\| \|F\|_2\, \omega(s,t)^{3/p} .$$

Therefore,

$$\left|\left(\int_s^t F \cdot dX\right)^1 - f_s X^1_{st}\right|$$
$$\le \left|\left(\int_s^t F \cdot dX\right)^1 - W^1_{st}\right| + \left|\alpha_s X^2_{st}\right|$$
$$\le K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] \omega(s,t)^{3/p} + |\alpha_s| \left|X^2_{st}\right|$$
$$\le K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] \omega(s,t)^{3/p} + |\alpha_s| N_2(X)\, \omega(s,t)^{2/p}$$
$$\le K_{3/p} \|X\| \|F\|_2\, \omega(s,t)^{3/p} + |\alpha_s| N_2(X)\, \omega(s,t)^{2/p}$$
$$\le K_{3/p} \|X\| \|F\|_2\, \omega(s,t)^{3/p} + \left[|\alpha_0| + N_1(\alpha)\, \omega(0,t)^{1/p}\right] N_2(X)\, \omega(s,t)^{2/p}$$

it follows that

$$N\left(\int F \cdot dX, f\right)$$
$$\le K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] \omega(0,T)^{1/p} + \|\alpha\|_\infty N_2(X)$$
$$\le K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] \omega(0,T)^{1/p}$$
$$+ \left[|\alpha_0| + N_1(\alpha)\, \omega(0,T)^{1/p}\right] N_2(X)$$
$$= \left(K_{3/p}\left[N(F) N_1\left(X^1\right) + N_1(\alpha) N_2(X)\right] + N_1(\alpha) N_2(X)\right) \omega(0,T)^{1/p}$$
$$+ |\alpha_0| N_2(X)$$
$$\le K_{3/p} \|X\| \|F\|_2\, \omega(0,T)^{1/p} + \left[|\alpha_0| + N_1(\alpha)\, \omega(0,T)^{1/p}\right] N_2(X)$$
$$\le \left(1 + K_{3/p}\right) \|X\| \|F\|_2\, \omega(0,T)^{1/p} + |\alpha_0| N_2(X)$$

and thus that

$$N\left(\int F \cdot dX, f\right) \le |\alpha_0| N_2(X) \tag{8.30}$$
$$+ \left(K_{3/p} N(F) N_1(X) + \left(K_{3/p} + 1\right) N_1(\alpha) N_2(X)\right) \omega(0,T)^{1/p}$$
$$\le \left(1 + K_{3/p}\right) \|X\| \|F\|_2\, \omega(0,T)^{1/p} + |\alpha_0| N_2(X) \tag{8.31}$$

Moreover from Eq. (8.17) we find

$$N_1(f) \le |\alpha_0| N_1(X) + \left[N_1(\alpha) N_1(X) + N(F)\right] \omega(0,T)^{1/p} \tag{8.32}$$
$$\le |\alpha_0| N_1(X) + 1 \vee N_1(X) \cdot \omega(0,T)^{1/p} \|F\|_2 . \tag{8.33}$$

Combining these two estimate shows

$$N_1(f) + N\left(\int F \cdot dX, f\right)$$
$$\le \|X\| |\alpha_0| + C_p\left(N_1(X), N_2(X)\right) \omega(0,T)^{1/p} \left[N_1(\alpha_0) + N\left(F^f\right)\right] \tag{8.34}$$
$$\le \|X\| |\alpha_0| + \left[\left(1 + K_{3/p}\right)\|X\| + 1 \vee N_1(X)\right] \omega(0,T)^{1/p} \|F\|_2 \tag{8.35}$$

from which it follow that

$$\left\| \left(\int F \cdot dX, f \right) \right\|_2$$
$$= |f_0| + N_1(f) + \left| \int_0^0 F \cdot dX \right| + N\left(\int F \cdot dX, f \right)$$
$$\leq |f_0| + N_1(f) + N\left(\int F \cdot dX, f \right)$$
$$\leq |f_0| + \|X\| |\alpha_0| + \left[\left(1 + K_{3/p}\right) \|X\| + 1 \vee N_1(X) \right] \omega(0,T)^{1/p} \|F\|_2 \quad (8.36)$$
$$\leq |f_0| + \|X\| |\alpha_0| + \left[1 + \left(2 + K_{3/p}\right) \|X\| \right] \omega(0,T)^{1/p} \|F\|_2$$

■

8.3 Spaces of Integrands

Warning !!Rough Notes Ahead!!

Remark 8.15 (Co-Cycle Condition). If $s < u < t$, then

$$\varepsilon_{st} = \varepsilon_{su} + \varepsilon_{ut} + \alpha_{su} X^1_{ut}$$

because

$$\begin{aligned}\varepsilon_{st} &= y_{st} - \alpha_s X^1_{s,t} = y_{su} + y_{ut} - \alpha_s X^1_{s,t} \\ &= \alpha_s X^1_{su} + \varepsilon_{su} + \alpha_u X^1_{ut} + \varepsilon_{ut} - \alpha_s X^1_{s,t} \\ &= \alpha_s X^1_{su} + \varepsilon_{su} + [\alpha_s + \alpha_{su}] X^1_{ut} + \varepsilon_{ut} - \alpha_s X^1_{s,t} \\ &= \varepsilon_{su} + \varepsilon_{ut} + \alpha_s \left[X^1_{su} + X^1_{ut} - X^1_{s,t} \right] + \alpha_{su} X^1_{ut} \\ &= \varepsilon_{su} + \varepsilon_{ut} + \alpha_{su} X^1_{ut}.\end{aligned}$$

Theorem 8.16 (Completeness I). *Let $\Omega_1(U,\omega) = \Omega_1(U,\omega,p)$ denote those $\alpha : \Delta \to U$ such that $N_1(\alpha) < \infty$ and let*

$$\|\alpha\|_1 := |\alpha_0| + N_1(\alpha).$$

Then $(\Omega_1(U,\omega), \|\cdot\|_1)$ is a Banach space.

Proof. From Eq. (8.17) it follows that

$$\|\alpha\|_\infty \leq \max\left(1, \omega(0,T)^{1/p}\right) \|\alpha\|_1$$

and hence if $\{\alpha(n)\}_{n=1}^\infty$ is a Cauchy sequence in $\Omega_1(U,\omega)$ then it is also uniformly Cauchy. Therefore there exist a continuous function $\alpha : \Delta \to U$ such that $\alpha(n) \to \alpha$ uniformly in t. Since

$$|\alpha(n)_{st} - \alpha(m)_{st}| \leq N_1(\alpha(n) - \alpha(m))_1 \omega(s,t)^{1/p}$$

we may let $m \to \infty$ in this inequality to learn that

$$|\alpha(n)_{st} - \alpha_{st}| \leq \limsup_{m\to\infty} N_1(\alpha(n) - \alpha(m)) \omega(s,t)^{1/p}$$

and hence that

$$N_1(\alpha(n) - \alpha) \leq \limsup_{m\to\infty} N_1(\alpha(n) - \alpha(m)) \to 0 \text{ as } n \to \infty.$$

Hence it follows that $\alpha(n) \to \alpha$ in $(\Omega_1(U,\omega), \|\cdot\|_1)$ and the proof is complete. ■

Theorem 8.17 (Completeness II). $(\mathcal{D}(U,\omega), \|(\cdot,\cdot)\|_2)$ *is a Banach space. Recall that*

$$\|(y,\alpha)\|_2 := \|\alpha\|_1 + |y_0| + N(y,\alpha). \quad (8.37)$$

Proof. From Eqs. (8.23) and (8.17), it follows that there is a constant $C < \infty$ such that

$$\|(y,\alpha)\|_\infty \leq C \|(y,\alpha)\|_2$$

for all $(y,\alpha) \in \mathcal{D}(U,\omega)$. Hence if $\{(y(n),\alpha(n))\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{D}(U,\omega)$ it is also uniformly Cauchy as well. Moreover $\{\alpha(n)\}_{n=1}^\infty$ is Cauchy in $\Omega_1(\mathrm{End}(V,U),\omega)$ and hence convergent in $\Omega_1(\mathrm{End}(V,U),\omega)$ by Theorem 8.16. Let (y,α) denote the uniform limit of the sequence $\{(y(n),\alpha(n))\}_{n=1}^\infty$. Then we have

$$\begin{aligned}\left| y(n)_{st} - \alpha(n)_s X^1_{st} - \left[y(m)_{st} - \alpha(m)_s X^1_{st} \right] \right| &= \left| \varepsilon^{y(n)}_{st} - \varepsilon^{y(m)}_{st} \right| \\ &\leq N(y(n) - y(m), \alpha(n) - \alpha(m)) \omega(s,t)^{2/p}\end{aligned}$$

and by letting $m \to \infty$ this implies

$$\begin{aligned}&\left| y(n)_{st} - \alpha(n)_s X^1_{st} - \left[y_{st} - \alpha_s X^1_{st} \right] \right| \\ &\quad \leq \limsup_{m\to\infty} N(y(n) - y(m), \alpha(n) - \alpha(m)) \omega(s,t)^{2/p}.\end{aligned}$$

Hence it follows that

$$N(y(n) - y, \alpha(n) - \alpha) \leq \limsup_{m\to\infty} N(y(n) - y(m), \alpha(n) - \alpha(m)).$$

Since the right side of this equation goes to zero as $n \to \infty$, it follows that $\lim_{n\to\infty} N(y(n) - y, \alpha(n) - \alpha) = 0$ and we have shown $(y(n),\alpha(n)) \to (y,\alpha)$ as $n \to \infty$ in $(\mathcal{D}(U,\omega), \|(\cdot,\cdot)\|_2)$. ■

We now wish to consider the mapping properties of the spaces Ω_1 and $\mathcal{D}$.

Proposition 8.18. *Let a and b two finite (ω,p) – variation paths valued in appropriate spaces so that ab is well defined. Then*

$$N_1(ab) \le \|b\|_\infty N_1(a) + \|a\|_\infty N_1(b), \tag{8.38}$$

$$\|ab\|_1 \le |a_0|\,|b_0| + \|b\|_\infty N_1(a) + \|a\|_\infty N_1(b) \tag{8.39}$$
$$\|ab\|_1 \le \|b\|_\infty N_1(a) + \|a\|_\infty \|b\|_1, \text{ and} \tag{8.40}$$
$$\|ab\|_1 \le \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p}+1\right)\|a\|_1\|b\|_1 \tag{8.41}$$
$$\le \left(1+2\omega(0,T)^{1/p}\right)\|a\|_1\|b\|_1 \tag{8.42}$$

and if (u,v) is another pair of such paths, then

$$N_1(ab-uv) \le \|a-u\|_\infty N_1(b) + N_1(a-u)\|b\|_\infty + \|u\|_\infty N_1(b-v) + N_1(u)\|b-v\|_\infty \tag{8.43}$$

and there exists $C = C\left(p,\omega(0,T)^{1/p}\right)$ such that

$$\|ab-uv\|_1 \le C\left[\|a-u\|_1\|b\|_1 + \|u\|_1\|b-v\|_1\right]. \tag{8.44}$$

Proof. The simple estimate,

$$\begin{aligned}|(ab)_{st}| &= |a_t b_t - a_s b_s| = |(a_t-a_s)b_t + a_s(b_t-b_s)| \\ &\le |a_t-a_s|\,|b_t| + |a_s|\,|b_t-b_s| \\ &\le \|b\|_\infty N_1(a)\,\omega(s,t)^{1/p} + \|a\|_\infty N_1(b)\,\omega(s,t)^{1/p},\end{aligned}$$

implies Eq. (8.38). Moreover,

$$\begin{aligned}\|ab\|_1 &\le |a_0|\,|b_0| + N_1(ab) \le |a_0|\,|b_0| + \|b\|_\infty N_1(a) + \|a\|_\infty N_1(b) \\ &\le \|b\|_\infty N_1(a) + \|a\|_\infty [|b_0| + N_1(b)] = \|b\|_\infty N_1(a) + \|a\|_\infty \|b\|_1,\end{aligned}$$

$$\begin{aligned}\|ab\|_1 &\le |a_0|\,|b_0| + \|b\|_\infty N_1(a) + \|a\|_\infty N_1(b) \\ &\le \|b\|_\infty [|a_0| + N_1(a)] + \|a\|_\infty N_1(b) = \|b\|_\infty \|a\|_1 + \|a\|_\infty N_1(b),\end{aligned}$$

and

$$\begin{aligned}\|ab\|_1 &\le \left[|b_0| + N_1(b)\,\omega(0,T)^{1/p}\right]\|a\|_1 + \left[|a_0| + N_1(a)\,\omega(0,T)^{1/p}\right]N_1(b) \\ &\le \left[|b_0| + N_1(b)\,\omega(0,T)^{1/p}\right]\|a\|_1 + \left(1\vee\omega(0,T)^{1/p}\right)\|a\|_1 N_1(b) \\ &= \|a\|_1\left[|b_0| + N_1(b)\left\{\omega(0,T)^{1/p} + \left(1\vee\omega(0,T)^{1/p}\right)\right\}\right] \\ &\le \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p}+1\right)\|a\|_1\|b\|_1.\end{aligned}$$

Using Eq. (8.38), it follows that

$$\begin{aligned}N_1(ab-uv) &= N_1((a-u)b + u(b-v)) \le N_1((a-u)b) + N_1(u(b-v)) \\ &\le \|a-u\|_\infty N_1(b) + N_1(a-u)\|b\|_\infty + \|u\|_\infty N_1(b-v) + N_1(u)\|b-v\|_\infty\end{aligned}$$

which is Eq. (8.43). Using

$$\begin{aligned}|(ab-uv)_0| &= |(a-u)_0 b_0 + u_0(b-v)_0| \\ &\le |(a-u)_0|\,|b_0| + |u_0|\,|(b-v)_0| \\ &\le \|a-u\|_\infty\|b\|_\infty + \|u\|_\infty\|b-v\|_\infty\end{aligned}$$

and working as above one easily proves Eq. (8.44) as well. Alternatively,

$$\begin{aligned}\|ab-uv\|_1 &= \|(a-u)b + u(b-v)\|_1 \le \|(a-u)b\|_1 + \|u(b-v)\|_1 \\ &\le \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p}+1\right)\left[\|(a-u)\|_1\|b\|_1 + \|u\|_1\|b-v\|_1\right].\end{aligned}$$

■

Theorem 8.19. *Suppose that $f : U \to S$ is a smooth map of Banach spaces and for $a \in \Omega_1(U,\omega)$, let $f_*(a) := f\circ a$. Then $f(a) := f_*(a) \in \Omega_1(S,\omega)$ and the map $f_* : \Omega_1(U,\omega) \to \Omega_1(S,\omega)$ satisfies the following estimates,*

$$N_1(f(a)) \le \|f'\|_{\infty,a}\cdot N_1(a) \tag{8.45}$$
$$\|f(a)\|_1 \le \|f'\|_{\infty,a}\cdot\|a\|_1 \tag{8.46}$$

where

$$\begin{aligned}\|f'\|_{\infty,a} &= \sup\{|f'(a_s+\tau a_{st})| : (s,t)\in\Delta \text{ and } \tau\in[0,1]\} \\ &\le \sup\{|f'(u)| : |u| \le \|a\|_\infty\}.\end{aligned} \tag{8.47}$$

Now suppose that a and b are two paths of finite (ω,p) – variation and that f is a smooth function, then

$$\begin{aligned}N_1(f(a)-f(b)) &\le \|f'\|_{\infty,b} N_1(a-b) \\ &\quad + \|f''\|_{\infty,a,b}\left[|a_0-b_0| + N_1(a-b)\,\omega(0,T)^{1/p}\right]N_1(a)\end{aligned} \tag{8.48}$$

and

$$\|f(a)-f(b)\|_1 \le \left[\|f'\|_{\infty,a,b} + \|f''\|_{\infty,a,b} N_1(a)\max\left(1,\omega(0,T)^{1/p}\right)\right]\|a-b\|_1 \tag{8.49}$$

where

$$\begin{aligned}\|f''\|_{\infty,a,b} &= \sup\left\{\begin{matrix}|f''(b_s+\tau b_{st} + r[a_s+\tau a_{st} - (b_s+\tau b_{st})])| : \\ s,t\in[0,T] \;\; \& \; r,\tau\in[0,1]\end{matrix}\right\} \\ &\le \sup\{|f''(\xi)| : |\xi| \le \|a\|_\infty \vee \|b\|_\infty\}.\end{aligned}$$

Proof. By Taylor's Theorem,

$$f(a)_{st} = f(a_t) - f(a_s) = \tilde{f}(a_s, a_t)\, a_{st} \tag{8.50}$$

where

$$\tilde{f}(a_s, a_t) = \int_0^1 f'(a_s + \tau a_{st})\, d\tau \tag{8.51}$$

and

$$\left|\tilde{f}(a_s, a_t)\right| \le \int_0^1 |f'(a_s + \tau a_{st})|\, d\tau \le \|f'\|_{\infty,a}.$$

The second inequality in Eq. (8.47) follows from the fact that

$$\begin{aligned}|a_s + \tau a_{st}| &= |a_s + \tau(a_t - a_s)| = |a_s(1-\tau) + \tau a_t| \\ &\le (1-\tau)|a_s| + \tau|a_t| \le |a_s| \vee |a_t| \le \|a\|_\infty.\end{aligned}$$

Hence it follows that

$$|f(a)_{st}| \le \|f'\|_{\infty,a} |a_{st}| \le \|f'\|_{\infty,a} N_1(a)\, \omega(s,t)^{1/p}$$

from which Eqs. (8.45) and (8.46) easily follow.

From Eq. (8.50) we have

$$\begin{aligned}f(a)_{st} - f(b)_{st} &= \tilde{f}(a_s, a_t)\, a_{st} - \tilde{f}(b_s, b_t)\, b_{st} \\ &= \left[\tilde{f}(a_s, a_t) - \tilde{f}(b_s, b_t)\right] a_{st} + \tilde{f}(b_s, b_t)\left[a_{st} - b_{st}\right].\end{aligned} \tag{8.52}$$

Now

$$\begin{aligned}&\left|\tilde{f}(a_s, a_t) - \tilde{f}(b_s, b_t)\right| \\ &\le \int_0^1 |f'(a_s + \tau a_{st}) - f'(b_s + \tau b_{st})|\, d\tau \\ &\le \int_0^1 d\tau \int_0^1 dr \left| f''\begin{pmatrix} b_s + \tau b_{st} \\ +r\left[a_s + \tau a_{st} - (b_s + \tau b_{st})\right]\end{pmatrix}\right| |a_s - b_s + \tau\left[a_{st} - b_{st}\right]| \\ &\le \|f''\|_{\infty,a,b} |a_s - b_s + \tau\left[a_{st} - b_{st}\right]|\end{aligned} \tag{8.53}$$

where

$$\|f''\|_{\infty,a,b} = \sup\left\{\begin{matrix}|f''(b_s + \tau b_{st} + r\left[a_s + \tau a_{st} - (b_s + \tau b_{st})\right])| : \\ s,t \in [0,T] \ \& \ r,\tau \in [0,1]\end{matrix}\right\}.$$

As above, we have

$$\begin{aligned}|a_s - b_s + \tau\left[a_{st} - b_{st}\right]| &\le |a_s - b_s| \vee |a_t - b_t| \\ |b_s + \tau b_{st} + r\left[a_s + \tau a_{st} - (b_s + \tau b_{st})\right]| &\le |b_s + \tau b_{st}| \vee |a_s + \tau a_{st}| \\ &\le \max(|b_s|, |b_t|, |a_s|, |a_t|) \le \|a\|_\infty \vee \|b\|_\infty\end{aligned}$$

and thus we have

$$\left|\tilde{f}(a_s, a_t) - \tilde{f}(b_s, b_t)\right| \le \|f''\|_{\infty,a,b}\left[|a_s - b_s| \vee |a_t - b_t|\right] \le \|f''\|_{\infty,a,b} \|a - b\|_\infty$$

where

$$\|f''\|_{\infty,a,b} \le \sup\left\{|f''(\xi)| : |\xi| \le \|a\|_\infty \vee \|b\|_\infty\right\}.$$

Using these estimates in Eq. (8.52),

$$\begin{aligned}|f(a)_{st} - f(b)_{st}| &\le \|f''\|_{\infty,a,b} \|a - b\|_\infty |a_{st}| + \|f'\|_{\infty,b} |a_{st} - b_{st}| \\ &\le \|f''\|_{\infty,a,b}\left[|a_0 - b_0| + N_1(a-b)\, \omega(0,T)^{1/p}\right] N_1(a)\, \omega(s,t)^{1/p} \\ &\quad + \|f'\|_{\infty,b} N_1(a-b)\, \omega(s,t)^{1/p}\end{aligned}$$

and therefore we have

$$\begin{aligned}N_1(f(a) - f(b)) &\le \|f'\|_{\infty,b} N_1(a-b) \\ &\quad + \|f''\|_{\infty,a,b}\left[|a_0 - b_0| + N_1(a-b)\, \omega(0,T)^{1/p}\right] N_1(a).\end{aligned}$$

Moreover,

$$\begin{aligned}\|f(a) - f(b)\|_1 &= |f(a_0) - f(b_0)| + N_1(f(a) - f(b)) \\ &\le \|f'\|_{\infty,a,b}\left[|a_0 - b_0| + N_1(a-b)\right] \\ &\quad + \|f''\|_{\infty,a,b}\left[|a_0 - b_0| + N_1(a-b)\, \omega(0,T)^{1/p}\right] N_1(a) \\ &\le \|f'\|_{\infty,a,b} \|a - b\|_1 + \|f''\|_{\infty,a,b} N_1(a) \max\left(1, \omega(0,T)^{1/p}\right) \|a - b\|_1 \\ &\le \left[\|f'\|_{\infty,a,b} + \|f''\|_{\infty,a,b} N_1(a) \max\left(1, \omega(0,T)^{1/p}\right)\right] \|a - b\|_1.\end{aligned}$$

■

We will now see how f acts on the space $\mathcal{D}$.

Theorem 8.20. *Suppose that $f : U \to S$ is a smooth map of Banach spaces and for $(y, \alpha) \in \mathcal{D}(U, \omega)$, let $f_*(y, \alpha) := (f(y), f'(y)\alpha)$. Then $f_*(y, \alpha) \in \mathcal{D}(S, \omega)$ and the map $f_* : \mathcal{D}(U, \omega) \to \mathcal{D}(S, \omega)$ satisfies the following estimates,*

$$\begin{aligned}\|f_*(y, \alpha)\| &\le |f(y_0)| + \|f'\|_{\infty,y} \|(y, \alpha)\|_2 \\ &\quad + 2C\left(\omega(0,T)^{1/p}\right) \max\left(1, N_1(X)^2\right) \|f''\|_{\infty,y} \|(y, \alpha)\|_2^2\end{aligned} \tag{8.54}$$

where

$$C\left(\omega(0,T)^{1/p}\right) = \max\left(1, \omega(0,T)^{1/p}\right) \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p} + 1\right) \tag{8.55}$$

$$\le \left[1 + 2\omega(0,T)^{1/p}\right]^2 \tag{8.56}$$

and

$$\left\| f_*\left(y,\alpha^y\right)-f_*\left(z,\alpha^z\right)\right\|_2 \le \left|f\left(y_0\right)-f\left(z_0\right)\right| + C\left\|\left(y,\alpha^y\right)-\left(z,\alpha^z\right)\right\|_2 \qquad (8.57)$$

where

$$C = C\left(\omega(0,T)^{1/p}, N_1(X), \|f'\|_{\infty,y,z}, \|f''\|_{\infty,y,z}, \|f'''\|_{\infty,y,z}, \|(y,\alpha^y)\|_2, \|(z,\alpha^z)\|_2\right) \qquad (8.58)$$

depends on $\left(\|(y,\alpha^y)\|_2, \|(z,\alpha^z)\|_2\right)$ *quadratically and on* $\left(\|f'\|_{\infty,y,z}, \|f''\|_{\infty,y,z}, \|f'''\|_{\infty,y,z}\right)$ *linearly.*

Proof. Let $f: W \to S$ be a smooth map of Banach space, $\varepsilon_{st} = \varepsilon_{st}^y := y_{st} - \alpha_s X_{st}^1$, and

$$\varepsilon_{st}^{f(y)} := f(y_t) - f(y_s) - f'(y_s)\alpha_s X_{st}^1 = f(y)_{st} - f'(y_s)\alpha_s X_{st}^1, \qquad (8.59)$$

then

$$\begin{aligned} f(y_t) - f(y_s) &= f'(y_s) y_{st} + R(y_s,y_t) y_{st}^{\otimes 2} && (8.60)\\ &= f'(y_s)\left[\alpha_s X_{st}^1 + \varepsilon_{st}\right] + R(y_s,y_t) y_{st}^{\otimes 2} && (8.61)\end{aligned}$$

where

$$R(y_s,y_t) = \int_0^1 f''(y_s + \tau y_{st})(1-\tau)\,d\tau.$$

Thus we have

$$df(y) = f'(y)\alpha dX^1 + d\varepsilon^{f(y)}$$

where

$$\begin{aligned} \varepsilon_{st}^{f(y)} &= f'(y_s)\varepsilon_{st}^y + R(y_s,y_t) y_{st}^{\otimes 2} && (8.62)\\ &= f'(y_s)\varepsilon_{st}^y + R(y_s,y_t)\left[\alpha_s^y X_{st}^1 + \varepsilon_{st}^y\right]^{\otimes 2}\\ &= f'(y_s)\varepsilon_{st}^y + R(y_s,y_t)\left[(\alpha_s^y)^2\left(X_{st}^1\right)^2 + \alpha_s^y X_{st}^1 \vee \varepsilon_{st}^y + \left(\varepsilon_{st}^y\right)^2\right].\end{aligned}$$

We now estimate $\varepsilon^{f(y)}$ as

$$\begin{aligned} \left|\varepsilon_{st}^{f(y)}\right| &= \left|f(y)_{st} - f'(y_s)\alpha_s X_{st}^1\right| \le \left|f'(y_s)\varepsilon_{st}\right| + \left|R(y_s,y_t) y_{st}^{\otimes 2}\right|\\ &\le \|f'\|_{\infty,y}\left|\varepsilon_{st}\right| + \|f''\|_{\infty,y}\left|y_{st}\right|^2\\ &\le \|f'\|_{\infty,y} N(y,\alpha)\,\omega(s,t)^{2/p} + \frac{1}{2}\|f''\|_{\infty,y} N_1(y)^2\,\omega(s,t)^{2/p}, && (8.63)\end{aligned}$$

and so

$$\begin{aligned} N\left(f(y), f'(y)\alpha\right) &\le \|f'\|_{\infty,y} N(y,\alpha) + \|f''\|_{\infty,y} N_1(y)^2\\ &\le \|f'\|_{\infty,y} N(y,\alpha)\\ &\quad + \|f''\|_{\infty,y}\left[|\alpha_0| N_1(X) + \left[N_1(\alpha) N_1(X) + N(y,\alpha)\right]\omega(0,T)^{1/p}\right]^2 && (8.64)\\ &\le \|f'\|_{\infty,y} N(y,\alpha)\\ &\quad + \|f''\|_{\infty,y}\cdot\max\left(1, N_1(X)^2\right)\max\left(1,\omega(0,T)^{2/p}\right)\|(y,\alpha)\|_2^2 && (8.65)\end{aligned}$$

wherein the last two inequalities we have made use of Eqs. (8.21) and (8.22). Moreover by Eqs. (8.41) and (8.46)

$$\begin{aligned} \|f'(y)\alpha\|_1 &\le \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p}+1\right)\|f'(y)\|_1\|\alpha\|_1\\ &\le \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p}+1\right)\|f''\|_{\infty,y}\|y\|_1\|\alpha\|_1\\ &\le \max\left(2\omega(0,T)^{1/p}, \omega(0,T)^{1/p}+1\right)\times && (8.66)\\ &\quad\times\max\left(1,\omega(0,T)^{1/p}\right)\max\left(1,N_1(X)^2\right)\|f''\|_{\infty,y}\|(y,\alpha)\|_2\|\alpha\|_1\\ &\quad = C\left(\omega(0,T)^{1/p}\right)\max\left(1,N_1(X)^2\right)\|f''\|_{\infty,y}\|(y,\alpha)\|_2\|\alpha\|_1\end{aligned}$$

where $C\left(\omega(0,T)^{1/p}\right)$ is as in Eq. (8.55). Combining Eqs. (8.65) and (8.66) then shows

$$\begin{aligned} &\|f_*(y,\alpha)\|\\ &\le |f(y_0)| + \|f'\|_{\infty,y} N(y,\alpha) + \|f''\|_{\infty,y}\cdot\max\left(1,\omega(0,T)^{2/p}\right)\max\left(1,N_1(X)^2\right)\|(y,\alpha)\\ &+ C\left(\omega(0,T)^{1/p}\right)\max\left(1,N_1(X)^2\right)\|f''\|_{\infty,y}\|(y,\alpha)\|_2\|\alpha\|_1\\ &\le |f(y_0)| + \|f'\|_{\infty,y} N(y,\alpha) + 2C\left(\omega(0,T)^{1/p}\right)\max\left(1,N_1(X)^2\right)\|f''\|_{\infty,y}\|(y,\alpha)\|_2^2\\ &\le |f(y_0)| + \|f'\|_{\infty,y}\|(y,\alpha)\|_2 + 2C\left(\omega(0,T)^{1/p}\right)\max\left(1,N_1(X)^2\right)\|f''\|_{\infty,y}\|(y,\alpha)\|_2^2\end{aligned}$$

This proves Eq. (8.54).

Now suppose that Y and z are two X – differentiable paths so that

$$dY = \alpha^Y dX^1 + d\varepsilon^Y \text{ and } dz = \alpha^z dX^1 + d\varepsilon^z$$

where

$$\begin{aligned} \left|d\varepsilon^Y\right| &\le N\left(z,\alpha^Y\right)\omega(s,s+ds)^{2/p} \text{ and}\\ \left|d\varepsilon^z\right| &\le N\left(z,\alpha^z\right)\omega(s,s+ds)^{2/p}.\end{aligned}$$

From Eq. (8.59) it follows that

$$\begin{aligned} f(Y)_{st} - f(z)_{st} &= f'(Y_s)\,\alpha_s^Y X_{st}^1 + \varepsilon_{st}^{f(Y)} - \left[f'(z_s)\,\alpha_s^z X_{st}^1 + \varepsilon_{st}^{f(z)}\right] \\ &= \left(f'(Y_s)\,\alpha_s^Y - f'(z_s)\,\alpha_s^z\right) X_{st}^1 + \varepsilon_{st}^{f(Y)} - \varepsilon_{st}^{f(z)} \\ &= f'(Y_s)\,\alpha_s^Y X_{st}^1 + f'(Y_s)\,\varepsilon_{st}^Y + R(Y_s, Y_t)\,Y_{st}^{\otimes 2} \\ &\quad - \left[f'(z_s)\,\alpha_s^z X_{st}^1 + f'(z_s)\,\varepsilon_{st}^z + R(z_s, z_t)\,z_{st}^{\otimes 2}\right] \end{aligned}$$

and hence from Eq. (8.62)

$$\begin{aligned} \varepsilon_{st}^{f(Y)-f(z)} &= \varepsilon_{st}^{f(Y)} - \varepsilon_{st}^{f(z)} \\ &= f'(Y_s)\,\varepsilon_{st}^Y - f'(z_s)\,\varepsilon_{st}^z + R(Y_s, Y_t)\,Y_{st}^{\otimes 2} - R(z_s, z_t)\,z_{st}^{\otimes 2} \\ &= \left[f'(Y_s) - f'(z_s)\right]\varepsilon_{st}^Y + f'(z_s)\left[\varepsilon_{st}^Y - \varepsilon_{st}^z\right] \\ &\quad + \left[R(Y_s, Y_t) - R(z_s, z_t)\right] Y_{st}^{\otimes 2} + R(z_s, z_t)\left[Y_{st}^{\otimes 2} - z_{st}^{\otimes 2}\right]. \end{aligned}$$

Let us observe

$$R(x, y) = \int_0^1 f''(x + \tau(y - x))(1 - \tau)\,d\tau$$

so that

$$\begin{aligned} \partial_{(v,w)} R(x, y) &= \int_0^1 \partial_{(v,w)}\left[f''(x + \tau(y - x))\right](1 - \tau)\,d\tau \\ &= \int_0^1 \frac{d}{dt}|_0 \left[f''(x + tv + \tau(y - x + t(w - v)))\right](1 - \tau)\,d\tau \\ &= \int_0^1 \left[\left(\partial_{v+\tau(w-v)} f''\right)(x + \tau(y - x))\right](1 - \tau)\,d\tau \\ &= \int_0^1 \left[f'''(x + \tau(y - x))(v + \tau(w - v), \cdot, \cdot)\right](1 - \tau)\,d\tau. \end{aligned}$$

From this it follows that

$$\begin{aligned} \left|\partial_{(v,w)} R(x, y)\right| &\le \int_0^1 \left|f'''(x + \tau(y - x))\right| \left|v + \tau(w - v)\right|(1 - \tau)\,d\tau \\ &\le \|f'''\|_{\infty,x,y} \int_0^1 \left[|v| + \tau|w - v|\right](1 - \tau)\,d\tau \\ &= \|f'''\|_{\infty,x,y} \left\{\frac{1}{2}|v| + \left(\frac{1}{2} - \frac{1}{3}\right)|w - v|\right\} \\ &= \|f'''\|_{\infty,x,y} \left\{\frac{1}{2}|v| + \frac{1}{6}|w - v|\right\} \le \|f'''\|_{\infty,x,y} \left\{\frac{1}{2}|v| + \frac{1}{6}|v| + \frac{1}{6}|w|\right\} \\ &\le \|f'''\|_{\infty,x,y} \left\{\frac{2}{3}|v| + \frac{1}{6}|w|\right\} \le \frac{2}{3}\|f'''\|_{\infty,x,y}(|v| + |w|) \end{aligned}$$

and hence we have shown

$$|R'(x, y)| \le \frac{2}{3}\|f'''\|_{\infty,x,y}.$$

It now follows that

$$\begin{aligned} \left|\varepsilon_{st}^{f(y)-f(z)}\right| &= |f'(y_s) - f'(z_s)|\,|\varepsilon_{st}^y| + |f'(z_s)|\,|\varepsilon_{st}^y - \varepsilon_{st}^z| \\ &\quad + |R(y_s, y_t) - R(z_s, z_t)|\,\left|y_{st}^{\otimes 2}\right| + |R(z_s, z_t)|\,\left|y_{st}^{\otimes 2} - z_{st}^{\otimes 2}\right| \\ &\le \|f''\|_{\infty,y,z}\,|y_s - z_s|\,|\varepsilon_{st}^y| + \|f'\|_{\infty,z}\,|\varepsilon_{st}^y - \varepsilon_{st}^z| \\ &\quad + \frac{2}{3}\|f'''\|_{\infty,y,z}\,|(y_s - z_s, y_t - z_t)|\,\left|y_{st}^{\otimes 2}\right| + \frac{1}{2}\|f''\|_{\infty,z}\,\left|y_{st}^{\otimes 2} - z_{st}^{\otimes 2}\right|. \end{aligned} \tag{8.67}$$

From Eq. (8.21)

$$\begin{aligned} N_1(y - z) &\le |\alpha_0^y - \alpha_0^z|\,N_1(X) \\ &\quad + \left[N_1(\alpha^y - \alpha^z)\,N_1(X)\,\omega(0, T)^{1/p} + N(y - z, \alpha^y - \alpha^z)\right]\omega(0, T)^{1/p} \end{aligned} \tag{8.68}$$

and from this it also follows that

$$\begin{aligned} |y_t - z_t| &\le |y_0 - z_0| + N_1(y - z)\,\omega(0, t)^{1/p} \\ &\le |y_0 - z_0| + |\alpha_0^y - \alpha_0^z|\,N_1(X)\,\omega(0, t)^{1/p} \\ &\quad + \left[N_1(\alpha^y - \alpha^z)\,N_1(X)\,\omega(0, T)^{1/p} + N(y - z, \alpha^y - \alpha^z)\right]\omega(0, t)^{2/p}. \end{aligned} \tag{8.69}$$

Combining Eqs. (8.67 – 8.69) shows

$$\begin{aligned} \left|\varepsilon_{st}^{f(y)-f(z)}\right| &\le \|f''\|_{\infty,y,z}\,N(y, \alpha^y)\,|y_s - z_s|\,\omega(s, t)^{2/p} \\ &\quad + \|f'\|_{\infty,z}\,N(y - z, \alpha^y - \alpha^z)\,\omega(s, t)^{2/p} \\ &\quad + \frac{2}{3}\|f'''\|_{\infty,y,z}\,|(y_s - z_s, y_t - z_t)|\,\left|y_{st}^{\otimes 2}\right| \\ &\quad + \frac{1}{2}\|f''\|_{\infty,z}\,\left|y_{st}^{\otimes 2} - z_{st}^{\otimes 2}\right|. \end{aligned}$$

Moreover

$$\begin{aligned} \left|y_{st}^{\otimes 2} - z_{st}^{\otimes 2}\right| &\le |y_{st} + z_{st}|\,|y_{st} - z_{st}| \le \left[|y_{st}| + |z_{st}|\right]|y_{st} - z_{st}| \\ &\le \left[N_1(y) + N_1(z)\right]\omega(s, t)^{1/p}\,N_1(y - z)\,\omega(s, t)^{1/p} \\ &= \left[N_1(y) + N_1(z)\right]N_1(y - z)\,\omega(s, t)^{2/p} \end{aligned}$$

and

$$|y_t - z_t| \le |y_0 - z_0| + N_1(y-z)\,\omega(0,t)^{1/p}.$$

Putting this all together shows

$$\begin{aligned}\left|\varepsilon_{st}^{f(y)-f(z)}\right| &\le \|f''\|_{\infty,y,z}\, N(y,\alpha^y)\left[|y_0-z_0| + N_1(y-z)\,\omega(s,t)^{1/p}\right]\omega(s,t)^{2/p}\\ &+ \|f'\|_{\infty,z}\, N(y-z,\alpha^y-\alpha^z)\,\omega(s,t)^{2/p}\\ &+ \frac{4}{3}\|f'''\|_{\infty,y,z}\left[|y_0-z_0| + N_1(y-z)\,\omega(0,T)^{1/p}\right] N_1(y)^2\,\omega(s,t)^{2/p}\\ &+ \frac{1}{2}\|f''\|_{\infty,z}\left[N_1(y)+N_1(z)\right] N_1(y-z)\,\omega(s,t)^{2/p}\end{aligned}$$

from which it follows that

$$\begin{aligned}&N\left(f(y)-f(z), f'(y)\alpha^y - f'(z)\alpha^z\right)\\ &\le \|f''\|_{\infty,y,z}\, N(y,\alpha^y)\left[|y_0-z_0| + N_1(y-z)\,\omega(0,T)^{1/p}\right]\\ &+ \|f'\|_{\infty,z}\, N(y-z,\alpha^y-\alpha^z)\\ &+ \frac{4}{3}\|f'''\|_{\infty,y,z}\left[|y_0-z_0| + N_1(y-z)\,\omega(0,T)^{1/p}\right] N_1(y)^2\\ &+ \|f''\|_{\infty,z}\left[N_1(y)+N_1(z)\right] N_1(y-z)\end{aligned}$$

or after some rearranging that

$$\begin{aligned}&N\left(f(y)-f(z), f'(y)\alpha^y - f'(z)\alpha^z\right)\\ &\le \|f'\|_{\infty,z}\, N(y-z,\alpha^y-\alpha^z)\\ &+ \|f''\|_{\infty,y,z}\left[\begin{array}{c}|y_0-z_0|\,N(y,\alpha^y) + \\ \left(\frac{1}{2}\left[N_1(y)+N_1(z)\right] + N(y,\alpha^y)\,\omega(0,T)^{1/p}\right) N_1(y-z)\end{array}\right]\\ &+ \frac{4}{3}\|f'''\|_{\infty,y,z}\left[|y_0-z_0| + N_1(y-z)\,\omega(0,T)^{1/p}\right] N_1(y)^2.\end{aligned}$$

Moreover, by Eq. (8.44) with $C = C\left(p, \omega(0,T)^{1/p}\right)$, we have

$$\|f'(y)\alpha^y - f'(z)\alpha^z\|_1 \le C(p,\omega)\left[\|f'(y)-f'(z)\|_1\,\|\alpha^y\|_1 + \|f'(z)\|_1\,\|\alpha^y-\alpha^z\|_1\right]$$

and by Theorem 8.19 we have

$$\|f'(y)-f'(z)\|_1 \le \left[\|f''\|_{\infty,y,z} + \|f'''\|_{\infty,y,z}\, N_1(y)\max\left(1,\omega(0,T)^{1/p}\right)\right]\|y-z\|_1$$

and

$$\|f'(z)\|_1 \le \|f''\|_{\infty,z}\cdot\|z\|_1.$$

Combining the last three equations and using Proposition 8.13 shows

$$\begin{aligned}&\|f'(y)\alpha^y - f'(z)\alpha^z\|_1\\ &\le C(p,\omega)\left[\|f''\|_{\infty,y,z} + \|f'''\|_{\infty,y,z}\, N_1(y)\max\left(1,\omega(0,T)^{1/p}\right)\right]\|y-z\|_1\,\|\alpha^y\|_1\\ &+ C(p,\omega)\,\|f''\|_{\infty,z}\cdot\|z\|_1\,\|\alpha^y-\alpha^z\|_1\\ &\le C(p,\omega,N_1(X))\left[\begin{array}{c}\|f''\|_{\infty,y,z}\\ +\|f'''\|_{\infty,y,z}\,\|(y,\alpha^y)\|_2\end{array}\right]\|(y-z,\alpha^y-\alpha^z)\|_2\,\|(y,\alpha^y)\|_2\\ &+ C(p,\omega)\,\|f''\|_{\infty,z}\cdot\|(z,\alpha^z)\|_2\,\|(y-z,\alpha^y-\alpha^z)\|_2\\ &\le C(p,\omega,N_1(X))\left[\begin{array}{c}\|f''\|_{\infty,y,z}\left(\|(y,\alpha^y)\|_2 + \|(z,\alpha^z)\|_2\right)\\ +\|f'''\|_{\infty,y,z}\,\|(y,\alpha^y)\|_2^2\end{array}\right]\|(y-z,\alpha^y-\alpha^z)\|_2.\end{aligned}$$

Assembling all of these estimates shows Eq. (8.57) with a constant as described in Eq. (8.58). ■

8.4 Appendix on Taylor's Theorem

Theorem 8.21 (Taylor's Theorem). *If $f: V \to E$ is a C^{n+1}-smooth map between Banach space, then*

$$f(v+h) = \sum_{k=0}^{n}\frac{1}{k!}\,D^k f(v)(h,\dots,h) + R_{n+1}(v,h)$$

where

$$R_{n+1}(v,h) = \frac{1}{(n+1)!}\int_0^1 D^{n+1}f(v+rh)(\overbrace{h,\dots,h}^{n+1\ times})\,d\nu_n(r)$$

and

$$d\nu_n(r) = (n+1)(1-r)^n dr.$$

Notice that ν_n is a probability measure on $[0,1]$ for each $n = 0,1,2,\dots$ In particular we have

$$f(v+h) = f(v) + \int_0^1 f'(v+rh)\,dr$$

$$f(v+h) = f(v) + f'(v)h + \int_0^1 f''(v+rh)(h,h)(1-r)\,dr$$

and

$$f(v+h) = f(v) + f'(v)h + \frac{1}{2}f''(v)(h,h) + \frac{1}{2!}\int_0^1 D^3 f(v+rh)(h,h,h)(1-r)^2\,dr.$$

Corollary 8.22. *Keeping the same notation as in Taylor's Theorem 8.21, we have*

$$\left| f(v+h) - \sum_{k=0}^{n} \frac{1}{k!} \; D^k f(v)(h,\ldots,h) \right| \leq \frac{M^{n+1}}{(n+1)!} \left|h\right|^{n+1}$$

where $M := \sup_{0\leq r\leq 1} \left|D^{n+1}f(v+rh)\right|$.

Notation 8.23 *Let $B = B(0,R)$ be a ball in V which contains $X_s := X^1_{0,s}$ for $s \in [0,T]$ and let M be a bound on α, α' and α'' on this ball.*

Example 8.24. Suppose $\alpha \in C^2 \left(V \to L\left(V,W\right)\right)$. Then we have the following factorization results:

1. For $\xi,\eta \in V$ we have, with $h = \eta - \xi$

$$\alpha\left(\eta\right) = \alpha\left(\xi\right) + \alpha'\left(\xi\right)\left(\eta-\xi\right) + R_1\left(\xi,\eta\right)$$

where

$$R_1(\xi,\eta) = \int_0^1 (1-r) \; D^2\alpha(\xi + r(\eta-\xi))(\eta-\xi,\eta-\xi)dr$$

and in particular it follows that

$$\alpha\left(X_t\right) = \alpha\left(X_s\right) + \alpha'\left(X_s\right)X^1_{st} + R_1(X_s,X_t)$$

where

$$R_1(X_s,X_t) = \int_0^1 (1-r) \; D^2\alpha(X_s + rX^1_{st})(X^1_{st},X^1_{st})dr.$$

2. For $\xi,\eta \in V$ we have, with $h = \eta - \xi$

$$\alpha'\left(\eta\right) = \alpha'\left(\xi\right) + R_2\left(\xi,\eta\right)$$

where

$$R_2(\xi,\eta) = \int_0^1 D^2\alpha(\xi + r(\eta-\xi))(\eta-\xi,\cdot,\cdot)dr$$

and in particular

$$\alpha'\left(X_t\right) = \alpha'\left(X_s\right) + R_2(X_s,X_t)$$

where

$$R_2\left(X_s,X_t\right) = \int_0^1 D^2\alpha(X_s + rX^1_{st})(X^1_{st},\cdot,\cdot)dr.$$

3. For $\xi,\eta \in B$,

$$\left|R_1(\xi,\eta)\right| \leq \frac{M}{2}\left|\eta-\xi\right|^2$$

and

$$\left|R_2(\xi,\eta)\right| = M\left|\eta-\xi\right|.$$

and in particular we have

$$\left|R_1(X_s,X_t)\right| = \frac{M}{2}\left|X^1_{st}\right|^2 \leq \frac{M}{2}C^2\omega\left(s,t\right)^{2/p}$$

and

$$\left|R_2\left(X_s,X_t\right)\right| \leq MC\omega\left(s,t\right)^{1/p}.$$

9

Rough ODE

Let us not go on to understanding the meaning of the differential equation,

$$y(t) = y_0 + \int_0^t f(y(\tau))\, dx(\tau), \tag{9.1}$$

where $x \in C_p([0,T] \to V)$ and $y \in C_p([0,T], W)$. As we do not know how to do this integral we work heuristically for the moment. As before, let

$$\begin{aligned}
Y_{st} = y(t) - y(s) &= \int_s^t f(y(\tau))\, dx(\tau) \\
&\cong \int_s^t [f(y(s)) + f'(y(s))(y(\tau) - y(s))]\, dx(\tau) \\
&\cong \int_s^t [f(y(s)) + f'(y(s))\, f(y(s))(x(\tau) - x(s))]\, dx(\tau) \\
&= f(y(s))\, X_{st}^1 + f'(y(s))\, f(y(s))\, X_{st}^2.
\end{aligned}$$

This suggests that if X is a $p-$lift of x, we should reinterpret Eq. (9.1) as

$$y(t) = y_0 + \int_0^t \left[f(y)\, dX^1 + f'(y)\, f(y)\, dX^2\right]. \tag{9.2}$$

Notice that if y is such a solution we would have,

$$y_{st} = \int_s^t \left[f(y)\, dX^1 + f'(y)\, f(y)\, dX^2\right] \cong f(y(s))\, X_{st}^1 + f'(y(s))\, f(y(s))\, X_{st}^2$$

and in particular,

$$\left\| y_{st} - f(y(s))\, X_{st}^1 \right\| \le C\omega(s,t)^{2/p}. \tag{9.3}$$

The following lemmas shows the right side of Eq. (9.2) makes sense provided y satisfies the constraint in Eq. (9.3).

Lemma 9.1. *Suppose that $p \in [1,3)$, $y \in C_p([0,T] \to W)$, and $X : \Delta \to G$ is a $p-$rough path. Further suppose that*

$$y(t) - y(s) - f(y(s))\, X_{st}^1 = O\left(\omega(s,t)^{2/p}\right).$$

Then $(\alpha(t), \beta(t)) = (f(y(t)), f'(y(t))\, f(y(t)))$ is an $X-$integrable function.

Proof. We have,

$$\begin{aligned}
\alpha_{st} = f(y(t)) - f(y(s)) &= \int_0^1 f'(\tau y(t) + (1-\tau)\, y(s))\, y_{st} d\tau \\
&= f'(y(s))\, y_{st} + \delta_{st} = f'(y(s))\, f(y(s))\, X_{st}^1 + O\left(\omega(s,t)^{2/p}\right) + \delta_{st} \\
&= \beta(s)\, X_{st}^1 + O\left(\omega(s,t)^{2/p}\right) + \delta_{st}
\end{aligned}$$

where

$$\delta_{st} := \int_0^1 [f'(\tau y(t) + (1-\tau)\, y(s)) - f'(y(s))]\, y_{st} d\tau.$$

Letting M_2 be a bound on $|f''|$ over $\{(\tau y(t) + (1-\tau)\, y(s)) : s, \tau \in [0,1]\}$, we find

$$|\delta_{st}| := \int_0^1 |f'(\tau y(t) + (1-\tau)\, y(s)) - f'(y(s))|\, |y_{st}|\, d\tau \le \frac{M_2}{2}\, |y_{st}|^2 = O\left(\omega(s,t)^{2/p}\right)$$

so that $\alpha_{st} = \beta(s)\, X_{st}^1 + O\left(\omega(s,t)^{2/p}\right)$. Since $\beta(t) = g(y(t))$ where $g(y) = f'(y)\, f(y)$ is smooth, it follows that $\beta_{st} = O(|y_{st}|) = O\left(\omega(s,t)^{1/p}\right)$ as required. ∎

9.1 Local Existence and Uniqueness

Let $f : U \to L(V,U)$ be a smooth function. We wish to consider the rough path ODE,

$$dy = f(y)dX \text{ with } y_0 = y_0. \tag{9.4}$$

As usual we should interpret this as an integral equation,

$$y_t = y_0 + \int_0^t f(y)dX, \tag{9.5}$$

by which we really mean;

$$y_t = y_0 + \int_0^t f_*(y, \alpha^y) \cdot d\mathbf{X}$$
$$= y_0 + \int_0^t [f(y)dX^1 + f'(y)\alpha^y dX^2] \text{ where} \tag{9.6}$$
$$\alpha^y = f(y) \tag{9.7}$$

Notation 9.2 *In what follows, $Y := (y, \alpha)$ and $\tilde{Y} := (\tilde{y}, \tilde{\alpha})$ will denote elements of $\mathcal{D}(U)$. Moreover, we may write α as α^y and $\tilde{\alpha}$ as $\alpha^{\tilde{y}}$.*

Theorem 9.3 (Global Uniqueness). *Assuming that f is C^3, there is at most one solution to Eq. (9.6).*

Proof. Suppose $Y = (y, \alpha)$ and $\widetilde{Y} = (\tilde{y}, \tilde{\alpha})$ are two solutions to Eq. (9.6). By Eq. (8.57) of Theorem 8.20,

$$\left\| f_*(Y) - f_*(\widetilde{Y}) \right\|_2 \le C \left\| Y - \widetilde{Y} \right\|_2$$

where

$$C = C(\omega(0,T)^{1/p}, \|f\|_{C^3(B(0,\|Y\|_\infty \vee \|\widetilde{Y}\|_\infty))}, \|Y\|_2, \left\|\widetilde{Y}\right\|_2, \|X\|).$$

Using this estimate along with Theorem 8.14 we find,

$$\left\| Y - \widetilde{Y} \right\|_2 = \left\| \int \left[f(Y) - f(\widetilde{Y}) \right] dX \right\|_2 \le C_p(\|X\|)\omega(0,T)^{1/p} \left\| f(Y) - f(\widetilde{Y}) \right\|_2$$
$$\le C \cdot C_p(\|X\|)\omega(0,T)^{1/p} \left\| Y - \widetilde{Y} \right\|_2.$$

Hence on the interval $[0, t]$ such that $C \cdot C_p(\|X\|)\omega(0, t)^{1/p} < 1$, it follows that $Y = \widetilde{Y}$.

Let $\tau := \inf \left\{ t > 0 : Y_t \neq \widetilde{Y}_t \right\}$. By continuity of Y and $\widetilde{Y}$ it follows that $Y_\tau = \widetilde{Y}_\tau$. If $\tau < T$, it follows from the above argument with the time interval shifted to $[\tau, \tau+t]$, then in fact $Y_s = \widetilde{Y}_s$ for all $\tau \le s \le \tau+t$. But this contradicts the definition of τ unless $\tau = T$. ■

When $f = \Lambda \in L(V, U)$ is constant, it follows that the solution to Eq. (9.4) or more precisely Eq. (9.6) is

$$y_t = y_0 + \Lambda X_t := y_0 + \Lambda X^1_{0,t}.$$

We will solve the general case as a perturbation of the solution of the constant case with $\Lambda := f(y_0)$. So let

$$y_t = y_0 + \Lambda X_t + z_t$$

where $Z = (z, \alpha^z) \in \Omega_2(U, \omega)$ with $Z_0 = 0$ and $\alpha^Z_0 = 0$ and let

$$g(y) := f(y_0 + y) - f(y_0).$$

Putting this expression into Eq. (9.5) shows Z must satisfy,

$$y_0 + \Lambda X_t + z_t = y_0 + \int_0^t f(Y)dX = y_0 + \int_0^t f(y_0 + \Lambda X + Z)dX$$
$$= y_0 + \Lambda X_t + \int_0^t g(\Lambda X + Z)dX$$

or equivalently that

$$Z_t = \int_0^t f(y_0 + \Lambda X + Z)dX - \Lambda X_t = \int_0^t [f(y_0 + \Lambda X + Z) - \Lambda]dX$$
$$= \int_0^t g(\Lambda X + Z)dX$$
$$= \int_0^t \left[g(\Lambda X + Z)dX^1 + g'(\Lambda X + Z)\frac{d(\Lambda X + Z)}{dX} dX^2 \right]$$
$$= \int_0^t \left[g(\Lambda X + Z)dX^1 + g'(\Lambda X + Z)(\Lambda + \alpha^Z)dX^2 \right].$$

More precisely, using Eq. (9.6), Z must satisfy

$$y_0 + \Lambda X_t + Z_t = y_0 + \int_0^t [f(y)dX^1 + f'(y)\alpha^y dX^2]$$

or equivalently that

$$Z_t = \int_0^t [f(y)dX^1 + f'[(y)\alpha^y dX^2] - \Lambda X_t$$
$$= \int_0^t ([f(y_0 + \Lambda X + Z) - f(y_0)]dX^1 + f'(y_0 + \Lambda X + Z)(\Lambda + \alpha^Z)dX^2)$$
$$= \int_0^t (g(\Lambda X + Z)dX^1 + g'(\Lambda X + Z)(\Lambda + \alpha^Z)dX^2)$$
$$= \int_0^t g_*(\Lambda X + Z, \Lambda + \alpha^Z) \cdot dX.$$

Theorem 9.4 (Local Existence and Uniqueness). *Assuming that f is C^3, and T is sufficiently small, then Eq. (9.6) has a unique solution. If we further assume that f and its derivatives to order three are bounded, then Eq. (9.6) has a solution defined for $0 \le t \le T$. (BRUCE: Ideally, the assumption that f is bounded should be dropped from the hypothesis.)*

Proof. Let $Z := (z, \alpha^z)$ and then define,

$$\Gamma(Z) = \int g(\Lambda X + z)dX := \int g_*(\Lambda X + Z, \Lambda + \alpha^Z) \cdot d\mathbf{X}$$

or more precisely,

$$\Gamma(Z) := \left(\int g_*(\Lambda X + z, \Lambda + \alpha^Z) \cdot d\mathbf{X}, g(\Lambda X + z)\right)$$

where as usual,

$$g_*(\Lambda X + z, \Lambda + \alpha^Z) = \left(g(\Lambda X + z), g'(\Lambda X + z)\left(\Lambda + \alpha^Z\right)\right)$$

which at $t = 0$ is given by

$$g_*(\Lambda X + z, \Lambda + \alpha^z)|_{t=0} = (g(0), g'(0)\Lambda) = (0, g'(0)\Lambda).$$

From Eq. (8.26) we have, with $C_p(\|X\|) = 1 + K_p\|X\|$ that

$$\begin{aligned}\|\Gamma(Z)\|_2 &\le |g(0) + \|X\|\ \|g'(0)\Lambda| + C_p(\|X\|)\omega(0,T)^{1/p}\|g(\Lambda X + Z)\|_2 \\ &= \|X\|\ |f'(y_0)f(y_0)| + C_p(\|X\|)\omega(0,T)^{1/p}\|g(\Lambda X + Z)\|_2. \end{aligned} \tag{9.8}$$

Moreover, from Theorem 8.20 (using $g([\Lambda X + Z]_0) = g(0) = 0$) we have

$$\begin{aligned}\|g(\Lambda X + Z)\|_2 \le& \|g'\|_{\infty,\Lambda Z+Z}\|\Lambda X + Z\|_2 \\ &+ (\omega(0,T)^{1/p})(1 + \|X\|^2)\|g''\|^2_{\infty,\Lambda X+Z}\|\Lambda X + Z\|_2^2. \end{aligned} \tag{9.9}$$

Let us now assume that

$$\|f'\|_\infty + \|f''\|_\infty + \|f'''\|_\infty \le M < \infty,$$

then upon noting that

$$\|\Lambda X + Z\|_2 \le \|\Lambda X\|_2 + \|Z\|_2 = |\Lambda| + \|Z\|_2$$

it follows from Eq. (9.9) that

$$\|g(\Lambda X+Z)\|_2 \le M[|f(y_0)|+\|Z\|_2]+2C\left(\omega(0,T)^{1/p}\right)\left(1 + \|X\|^2\right)M^2[|f(y_0)|+\|Z\|_2]^2. \tag{9.10}$$

Combining this with Eq. (9.8) shows

$$\begin{aligned}&\|\Gamma(Z)\|_2 \le \|X\||f'(y_0)f(y_0)| \\ &+ C_p(\|X\|)\omega(0,T)^{1/p}\begin{bmatrix} M[|f(y_0)| + \|Z\|_2]+ \\ +2C\left(\omega(0,T)^{1/p}\right)\left(1 + \|X\|^2\right)M^2[|f(y_0)| + \|Z\|_2]^2\end{bmatrix}.\end{aligned}$$

Letting $N := 2\|X\||f'(y_0)f(y_0)|$ and making the assumption that $\|Z\|_2 \le N$, we find

$$\|\Gamma(Z)\|_2 \le \frac{1}{2}N+C_p(\|X\|)\omega(0,T)^{1/p}\begin{bmatrix} M[|f(y_0)| + N]+ \\ +2C\left(\omega(0,T)^{1/p}\right)\left(1 + \|X\|^2\right)M^2[|f(y_0)| + N]^2\end{bmatrix} \tag{9.11}$$

from which it follows that $\|\Gamma(Z)\|_2 \le N$ provided we choose T sufficiently small so that

$$C_p\|X\|\omega(0,T)^{1/p}\begin{bmatrix} M[|f(y_0)| + N]+ \\ +2C\left(\omega(0,T)^{1/p}\right)\left(1 + \|X\|^2\right)M^2[|f(y_0)| + N]^2\end{bmatrix} \le \frac{1}{2}N.$$

(One should keep in mind that by scaling ω, if it is desirable we may assume that $N = 2\|X\|\ |f'(y_0)f(y_0)| = 1$.) In summary, if

$$F := \{Z \in \Omega_1(U,\omega) : Z_0 = 0, \alpha_0^Z = 0, \text{ and } \|Z\|_2 \le N\}$$

and T is sufficiently small, then $\Gamma(F) \subset F$.

Suppose $Z, \widetilde{Z} \in F \subset \Omega_1(U,\omega)$, then

$$\begin{aligned}\left\|\Gamma(Z) - \Gamma(\widetilde{Z})\right\|_2 &= \left\|\Gamma(Z - \widetilde{Z})\right\|_2 = \left\|\int\left[g(\Lambda X + Z) - g(\Lambda X + \widetilde{Z})\right]dX\right\|_2 \\ &\le |g(0) - g(0)| + \|X\||g'(0)\Lambda - g'(0)\Lambda| \\ &\quad + C_p(\|X\|)\omega(0,T)^{1/p}\left\|g(\Lambda X + Z) - g(\Lambda X + \widetilde{Z})\right\|_2 \\ &= C_p(\|X\|)\omega(0,T)^{1/p}\left\|g(\Lambda X + Z) - g(\Lambda X + \widetilde{Z})\right\|_2.\end{aligned}$$

It then follows by an application of Eq. (8.57) of Theorem 8.20 that

$$\begin{aligned}&\left\|g(\Lambda X + Z) - g(\Lambda X + \widetilde{Z})\right\|_2 \le |g(0) - g(0)| \\ &\quad + C\left(M, N, \omega(0,T)^{1/p}, \|X\|\right)\left\|\Lambda X + Z - (\Lambda X + \widetilde{Z})\right\|_2 \\ &= C\left(M, N, \omega(0,T)^{1/p}, \|X\|\right)\left\|Z - \widetilde{Z}\right\|_2.\end{aligned}$$

Combining the last two displayed equations then shows,

$$\left\|\Gamma(Z) - \Gamma(\widetilde{Z})\right\|_2 \le C_p(\|X\|)C\left(M, N, \omega(0,T)^{1/p}, \|X\|\right)\omega(0,T)^{1/p}\left\|Z - \widetilde{Z}\right\|_2.$$

By shrinking T some more if necessary, we may assume

$$C_p(\|X\|)C\left(M, N, \omega(0,T)^{1/p}, \|X\|\right)\omega(0,T)^{1/p} < 1. \tag{9.12}$$

With this choice of T it then follows that $\Gamma|_F : F \to F$ is a contraction. Since F is a closed subset of a Banach space, an application of the contraction mapping principle implies there exists a unique $Z \in F$ such that $\Gamma(Z) = Z$. The desired solution to Eq. (9.4) is then

$$y_t := y_0 + f(y_0)X_t + Z_t.$$

For the last assertion, notice that when f and its derivatives are assumed to be bounded, then for Eqs. (9.11) and (9.12) to be valid, we need only choose T such that $\omega(0,T) \le \varepsilon$ with $\varepsilon > 0$ being a constant independent of y_0. Thus if we want to solve the equation, we may choose a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_r = T\}$ such that $\omega(t_{l-1}, t_l) \le \varepsilon$ for all l. Thus we may inductively construct the solution on $[0, t_l]$ for $l = 1, 2, \ldots, r$. ■

9.2 A priori-Bounds

Theorem 9.5 (A priori Bounds). *Assuming that f is C^2, f is bounded with bounded derivatives to order two and suppose $y(t)$ solves Eq. (9.6), then there exists a constant C and δ depending only on $M_i := \left\|f^{(i)}\right\|_\infty$ for $i = 0, 1, 2$; such that*

$$|y_t| \le |y_0| + C\left(\omega(0,T)/\delta^p + 1\right) \text{ for all } 0 \le t \le T.$$

Proof. Let

$$W_{st} := f(y_s) X^1_{st} + f'(y_s)\alpha_s X^2_{st}$$

where

$$Y_{st} = \alpha_s X^1_{st} + \varepsilon_{st}.$$

Then for $s < u < t$, we have

$$\begin{aligned}
W_{st} - W_{su} - W_{ut} &= f(y_s)\left[X^1_{st} - X^1_{su}\right] - f(y_u) X^1_{ut} \\
&\quad + f'(y_s)\alpha_s\left[X^2_{st} - X^2_{su}\right] - f'(y_u)\alpha_u X^2_{ut} \\
&= f(y_s)\left[X^1_{st} - X^1_{su} - X^1_{ut}\right] + \left[f(y_s) - f(y_u)\right] X^1_{ut} \\
&\quad + f'(y_s)\alpha_s\left[X^2_{st} - X^2_{su} - X^2_{ut}\right] + \left[f'(y_s)\alpha_s - f'(y_u)\alpha_u\right] X^2_{ut} \\
&= \left[f(y_s) - f(y_u)\right] X^1_{ut} + f'(y_s)\alpha_s\left[X^1_{su}X^1_{ut}\right] \\
&\quad + f'(y_s)\alpha_s\left[X^2_{st} - X^2_{su} - X^2_{ut} - X^1_{su}X^1_{ut}\right] \\
&\quad + \left[f'(y_s)\alpha_s - f'(y_u)\alpha_u\right] X^2_{ut} \\
&= \left(\left[f(y_s) - f(y_u)\right] X^1_{ut} + f'(y_s)\alpha_s\left[X^1_{su}X^1_{ut}\right]\right) + \left[f'(y_s)\alpha_s - f'(y_u)\alpha_u\right] X^2_{ut} \\
&=: A + B.
\end{aligned}$$

Since

$$\begin{aligned}
f(y_u) - f(y_s) &= \left[\int_0^1 f'(y_s + \tau Y_{su})\, d\tau\right] Y_{su} \\
&= \left[\int_0^1 f'(y_s + \tau Y_{su})\, d\tau\right] \left(\alpha_s X^1_{su} + \varepsilon_{su}\right)
\end{aligned}$$

we see that

$$\begin{aligned}
A &= f'(y_s)\alpha_s\left[X^1_{su}X^1_{ut}\right] - \left[\int_0^1 f'(y_s + \tau Y_{su})\, d\tau\right]\left(\alpha_s X^1_{su} + \varepsilon_{su}\right) X^1_{ut} \\
&= \left(\int_0^1 \left[f'(y_s) - f'(y_s + \tau Y_{su})\right] d\tau\right)\alpha_s\left[X^1_{su}X^1_{ut}\right] - \left[\int_0^1 f'(y_s + \tau Y_{su})\, d\tau\right]\varepsilon_{su} X^1_{ut}
\end{aligned}$$

and hence that

$$\begin{aligned}
|A| &\le M_2 |Y_{su}| |\alpha_s| \left|X^1_{su}\right| \left|X^1_{ut}\right| + M_1 |\varepsilon_{su}| \left|X^1_{ut}\right| \\
&\le \left[M_2 |\alpha_s| N(Y) + M_1 N(\varepsilon)\right] \omega(s,t)^{3/p}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
B &= \left(f'(y_s)\alpha_s - f'(y_u)\alpha_u\right) X^2_{ut} \\
&= \left(\left[f'(y_s) - f'(y_u)\right]\alpha_s + f'(y_u)\left[\alpha_s - \alpha_u\right]\right) X^2_{ut},
\end{aligned}$$

and thus

$$|B| \le \left(M_2 |Y_{su}| |\alpha_s| + M_1 |\alpha_{su}|\right)\left|X^2_{ut}\right| \le \left(M_2 |\alpha_s| N(Y) + M_1 N(\alpha)\right)\omega(s,t)^{3/p}.$$

Assembling these estimates gives

$$\begin{aligned}
|W_{st} - W_{su} - W_{ut}| &\le |A| + |B| \\
&\le \left[M_2 |\alpha_s| N(Y) + M_1 N(\varepsilon)\right]\omega(s,t)^{3/p} \\
&\quad + \left(M_2 |\alpha_s| N(Y) + M_1 N(\alpha)\right)\omega(s,t)^{3/p} \\
&= \left[2M_2 |\alpha_s| N(Y) + M_1 \left(N(\varepsilon) + N(\alpha)\right)\right]\omega(s,t)^{3/p}
\end{aligned}$$

which allows us to conclude that

$$|Y_{st} - W_{st}| \le K(3/p)\left[2M_2 \|\alpha\|_\infty N(Y) + M_1\left(N(\varepsilon) + N(\alpha)\right)\right]\omega(s,t)^{3/p}.$$

Since $\alpha_s = f(y_s)$, we have $\|\alpha\|_\infty \le M_0$,

$$|\alpha_{st}| = |f(y_t) - f(y_s)| \le M_1 |Y_{st}| \le M_1 N(Y)\,\omega(s,t)^{1/p},$$

and hence

$$N(\alpha) \le M_1 N(Y).$$

Therefore,

$$|Y_{st} - W_{st}| \leq K(3/p)\left[2M_2M_0N(Y) + M_1(N(\varepsilon) + M_1N(Y))\right]\omega(s,t)^{3/p}$$

and this then implies that

$$\begin{aligned}|Y_{st}| &\leq |W_{st}| + |Y_{st} - W_{st}| \leq \left|f(y_s)X_{st}^1\right| + \left|f'(y_s)\alpha_s X_{st}^2\right| + |Y_{st} - W_{st}| \\ &\leq M_0\omega(s,t) + M_1M_0\omega(s,t)^{2/p} \\ &+ K(3/p)\left[2M_2M_0N(Y) + M_1(N(\varepsilon) + M_1N(Y))\right]\omega(s,t)^{3/p}\end{aligned}$$

and therefore that

$$\begin{aligned}N(Y) &\leq M_0 + M_1M_0\omega(0,T)^{1/p} \\ &+ K(3/p)\left[\left(2M_2M_0 + M_1^2\right)N(Y) + M_1N(\varepsilon)\right]\omega(0,T)^{2/p}.\end{aligned}$$

Moreover we also have

$$\begin{aligned}|\varepsilon_{st}| &= \left|Y_{st} - f(y_s)X_{st}^1\right| \leq |Y_{st} - W_{st}| + \left|W_{st} - f(y_s)X_{st}^1\right| \\ &\leq |Y_{st} - W_{st}| + \left|f'(y_s)\alpha_s X_{st}^2\right| \\ &\leq |Y_{st} - W_{st}| + M_1M_0\omega(s,t)^{2/p}\end{aligned}$$

so that

$$N(\varepsilon) \leq M_0M_1 + K(3/p)\left[\left(2M_2M_0 + M_1^2\right)N(Y) + M_1N(\varepsilon)\right]\omega(0,T)^{1/p}.$$

Hence if we choose $1 > \delta > 0$ such that

$$\delta \cdot \max\left(K(3/p)\left(2M_2M_0 + M_1^2\right), M_1\right) =: \frac{\alpha}{2} < \frac{1}{2}$$

and then choose T such that

$$\max\left(\omega(0,T)^{1/p}, \omega(0,T)^{2/p}\right) \leq \delta,$$

(since $\delta < 1$, the condition is really $\omega(0,T) \leq \delta^p$) then we have

$$\begin{aligned}N(Y) &\leq M_0 + M_1M_0\delta + \frac{\alpha}{2}(N(Y) + N(\varepsilon)) \text{ and} \\ N(\varepsilon) &\leq M_0M_1 + \frac{\alpha}{2}(N(Y) + N(\varepsilon)).\end{aligned}$$

Adding these two equations gives the estimate,

$$N(Y) + N(\varepsilon) \leq M_0 + M_1M_0\delta + M_0M_1 + \alpha(N(Y) + N(\varepsilon))$$

from which it follows that

$$N(Y) + N(\varepsilon) \leq \frac{M_0}{1-\alpha}(1 + M_1\delta + M_1).$$

In particular, this gives a bound of the form that

$$|y_t| \leq |y_0| + C \text{ if } \omega(0,t) \leq \delta^p.$$

To be precise, suppose that we take $\alpha = \frac{1}{2}$ in which case (by assuming M_0 is sufficiently large) we have

$$\delta = \frac{1}{4K(3/p)\left(2M_2M_0 + M_1^2\right)}$$

and so for $\omega(0,T) \leq \delta^p$, we have

$$\begin{aligned}N(Y) + N(\varepsilon) &\leq 2M_0(1 + M_1\delta + M_1) \\ &\leq 2M_0\left(1 + \frac{M_1}{4K(3/p)\left(2M_2M_0 + M_1^2\right)} + M_1\right).\end{aligned}$$

In summary, if we define

$$\kappa := 2M_0\left(1 + \frac{M_1}{4K(3/p)\left(2M_2M_0 + M_1^2\right)} + M_1\right) \tag{9.13}$$

and assume

$$\omega(0,t) \leq \delta^p = \left[\frac{1}{4K(3/p)\left(2M_2M_0 + M_1^2\right)}\right]^p,$$

then we have shown

$$|y_t| \leq |y_0| + N(Y)\omega(0,t)^{1/p} \leq |y_0| + \kappa\omega(0,t)^{1/p} \leq |y_0| + \kappa\delta.$$

Choosing $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$ such that $\omega(t_{l-1}, t_l) = \delta^p$ for $l = 1, 2, \ldots, n$ and $\omega(t_n, T) \leq \delta^p$. It then follows that

$$\begin{aligned}&|y_t| \leq |y_0| + \kappa\delta \text{ for } t \in [t_0, t_1], \\ &|y_t| \leq |y_0| + \kappa\delta + \kappa\delta = |y_0| + 2\kappa\delta \text{ for } t \in [t_1, t_2] \\ &\quad \ldots \\ &|y_t| \leq |y_0| + n\kappa\delta \text{ for } t \in [t_{n-1}, t_n] \text{ and} \\ &|y_t| \leq |y_0| + n\kappa\delta + \kappa\omega(t_n, T)^{1/p} \text{ for } t \in [t_n, T].\end{aligned}$$

Since

$$n\delta^p + \omega(t_n, T) = \sum_{l=1}^{n+1}\omega(t_{l-1}, t_l) \leq \omega(0,T)$$

we have $n \leq [\omega(0,T) - \omega(t_n,T)]\,\delta^{-p}$ and therefore we have proven

$$|y_t| \leq |y_0| + \kappa\,[\omega(0,T) - \omega(t_n,T)]\,\delta^{-p}\delta + \kappa\omega(t_n,T)^{1/p} \tag{9.14}$$

where

$$\begin{aligned} C &:= \kappa\delta^{1-p} = \kappa\left[4K(3/p)\left(2M_2M_0 + M_1^2\right)\right]^{p-1} \\ &= 2M_0\left(1 + \frac{M_1}{4K(3/p)\,(2M_2M_0 + M_1^2)} + M_1\right)\left[4K(3/p)\left(2M_2M_0 + M_1^2\right)\right]^{p-1} \end{aligned} \tag{9.15}$$

Since

$$\begin{aligned} \omega(t_n,T)^{1/p} - \omega(t_n,T)\,\delta^{-p}\delta &= \omega(t_n,T)\left[\omega(t_n,T)^{1/p-1} - \delta^{-p}\delta\right] \\ &= \omega(t_n,T)\left[\omega(t_n,T)^{\frac{1-p}{p}} - \delta^{-p}\delta\right] \\ &\leq \omega(t_n,T)\left[\delta^{1-p} - \delta^{1-p}\right] \leq 0, \end{aligned}$$

we may conclude that

$$|y_t| \leq |y_0| + C\omega(0,T) \text{ for all } 0 \leq t \leq T. \tag{9.16}$$

For M_0 large we have the approximate estimate,

$$|y_t| \leq |y_0| + 2(1+M_1)\,[8K(3/p)\,M_2]^{p-1}\,M_0^p\omega(0,T).$$

■

Remark 9.6. From Eqs. (9.16) and (9.15), in the case that f is linear, so that $f'' \equiv 0$, we then have $M_2 = 0$ and in this case we learn that

$$C = 2M_0\left(1 + \frac{M_1}{4K(3/p)\,M_1^2} + M_1\right)\left[4K(3/p)\,M_1^2\right]^{p-1} = \tilde{K}(M_1,p)\,M_0.$$

To make use of this sort of nonsensical statement (nonsensical, since $M_0 = \infty$ if f is linear) we must restrict our attention to solutions in a big open ball. STOP

In more detail, let T be the first exit time of $y(t)$ from $\overline{B(y_0,\Lambda)}$ for some cutoff Λ. In this case we will have for $y \in \overline{B(y_0,\Lambda)}$ that

$$|f(y)| \leq M_1(|y_0| + \Lambda)$$

so that

$$M_0 \leq M_1(|y_0| + \Lambda).$$

Using this estimate back in Eq. (9.16) implies that

$$\begin{aligned} |y_t| &\leq |y_0| + 2[M_1(|y_0|+\Lambda)]\left(1 + \frac{M_1}{4K(3/p)\,M_1^2} + M_1\right)\left[4K(3/p)\,M_1^2\right]^{p-1}\omega(0,T) \\ &\quad + 2[M_1(|y_0|+\Lambda)]\left(1 + \frac{M_1}{4K(3/p)\,M_1^2} + M_1\right)\frac{1}{4K(3/p)\,M_1^2} && (9.17) \\ &\leq C_1(p,M_1)\,[|y_0|+\Lambda] + C_2(p,M_1,) && (9.18) \end{aligned}$$

and hence we get the bound,

$$|y_t| \leq |y_0| + K'(M_1,p,k)\,(1+\Lambda)\omega(0,T) \text{ for all } 0 \leq t \leq T$$

$M_0 \leq k(1+\Lambda)$ where k is a bound on for some constant k and thus we get the bound

$$|y_t| \leq |y_0| + K'(M_1,p,k)\,(1+\Lambda)\omega(0,T) \text{ for all } 0 \leq t \leq T$$

10

Some Open Problems

Problem 10.1 (Measure theoretic approach). Is there are more measure theoretic approach to the rough path theory? I would guess this would require are reasonable notion of simple functions. Also, can one get rid of the continuity assumptions? (I have not done a literature search on this point, so there probably is some work in this direction already.)

Problem 10.2 (Non explosion criteria). It is shown in Fritz and Victoir [6, Exercise 10.61 on p. 259] that if $dy = f(y)\,dx$, x is a geomentric $p-$ variation rough path on $\mathbb{R}^d$ and f has bounded derivatives to sufficiently high order, then the equation has solutions for all time. It is reasonable to ask the following questions;

1. What happens for non-geometric rough paths?
2. What happens in infinite dimensions, i.e. when $d \to \infty$?
3. What are other sufficient conditions for non-explosion?
4. Can one find necessary condititions as well?

Theorem 10.3. *Let $p \in [1, \infty)$ and $n \in \mathbb{Z}_+$ such that $n - 1 \le p < n$. Suppose that $X : \Delta \to G^{(n)}(V)$ is a p – rough path. Then for any $m \ge n$ there is a unique extension of X to a p – rough path, $\tilde{X} : \Delta \to G^{(m)}(V)$. In particular, we may extend X to a p – rough path,*

$$\tilde{X} = \sum_{k=0}^{\infty} \tilde{X}^{(k)} : \Delta \to G^{(\infty)}(V) := \lim_{m \uparrow \infty} G^{(m)}(V).$$

Definition 10.4. *Let $p \in [1, \infty)$ and $n \in \mathbb{Z}_+$ such that $n - 1 \le p < n$. Suppose that $X : \Delta \to G^{(n)}(V)$ is a p – rough path. The signature of X is defined by*

$$\operatorname{sgn}(X) = \tilde{X}_{0,T} \in G^{(\infty)}(V).$$

Problem 10.5. How much of X can be recovered from $\operatorname{sgn}(X)$? When X is finite variation, Hambly and Lyon's [10] give a detailed answer to this question.

References

1. A. M. Davie, *Differential equations driven by rough paths: an approach via discrete approximation*, Appl. Math. Res. Express. AMRX (2007), no. 2, Art. ID abm009, 40. MR MR2387018
2. R.M. Dudley and R. Norvaiša, *An introduction to p-variation and Young integrals – with emphasis on sample functions of stochastic processes*, Lecture given at the Centre for Mathematical Physics and Stochastics, Department of Mathematical Sciences, University of Aarhus, 1998.
3. Denis Feyel and Arnaud de La Pradelle, *Curvilinear integrals along enriched paths*, Electron. J. Probab. **11** (2006), no. 34, 860–892 (electronic). MR MR2261056 (2007k:60112)
4. Franco Flandoli, Massimiliano Gubinelli, Mariano Giaquinta, and Vincenzo M. Tortorelli, *Stochastic currents*, Stochastic Process. Appl. **115** (2005), no. 9, 1583–1601. MR MR2158261 (2006e:60073)
5. Peter Friz and Nicolas Victoir, *A note on the notion of geometric rough paths*, Probab. Theory Related Fields **136** (2006), no. 3, 395–416. MR MR2257130 (2007k:60114)
6. ———, *Multidimensional stochastic processes as rough paths: Theory and applications*, Springer ??, 2008.
7. M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140. MR MR2091358 (2005k:60169)
8. M. Gubinelli, *Ramification of rough paths*, 2006.
9. Massimiliano Gubinelli, Antoine Lejay, and Samy Tindel, *Young integrals and SPDEs*, Potential Anal. **25** (2006), no. 4, 307–326. MR MR2255351 (2007k:60182)
10. B. M. Hambly and T. J. Lyons, *Uniqueness for the signature of a path of bounded variation and the reduced path group*, Ann. Math. **??** (2009), no. 1, 50.
11. Antoine Lejay, *On rough differential equations*, Electron. J. Probab. **14** (2009), no. 12, 341–364 (electronic).
12. Terry Lyons and Zhongmin Qian, *System Control and Rough Paths*, Oxford University Press, 2002, Oxford Mathematical Monographs.
13. ———, *System control and rough paths*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2002, Oxford Science Publications. MR MR2036784 (2005f:93001)
14. Terry Lyons and Nicolas Victoir, *An extension theorem to rough paths*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 5, 835–847. MR MR2348055 (2008h:60229)
15. Terry J. Lyons, Michael Caruana, and Thierry Lévy, *Differential equations driven by rough paths*, Lecture Notes in Mathematics, vol. 1908, Springer, Berlin, 2007, Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard. MR MR2314753
16. Richard Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002. MR MR1867362 (2002m:53045)
17. Peter Mrters, Yuval Peres with appendix by Schramm, and Werner, *Brownian motion*, http://people.bath.ac.uk/maspm/book.pdf, 2006.
18. L. C. Young, *An inequality of Hölder type connected with Stieltjes integration*, Acta Math. (1936), no. 67, 251–282.