# Part VI

# Lebesgue Integration Theory

# 17

## Introduction: What are measures and why "measurable" sets

**Definition 17.1 (Preliminary).** *A **measure** $\mu$ "on" a set $X$ is a function $\mu : 2^X \to [0,\infty]$ such that*

1. $\mu(\emptyset) = 0$
2. *If $\{A_i\}_{i=1}^N$ is a finite ($N < \infty$) or countable ($N = \infty$) collection of subsets of $X$ which are pair-wise disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then*

$$\mu(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mu(A_i).$$

*Example 17.2.* Suppose that $X$ is any set and $x \in X$ is a point. For $A \subset X$, let

$$\delta_x(A) = \begin{cases} 1 \text{ if } & x \in A \\ 0 \text{ if } & x \notin A. \end{cases}$$

Then $\mu = \delta_x$ is a measure on $X$ called the Dirac delta measure at $x$.

*Example 17.3.* Suppose that $\mu$ is a measure on $X$ and $\lambda > 0$, then $\lambda \cdot \mu$ is also a measure on $X$. Moreover, if $\{\mu_\alpha\}_{\alpha\in J}$ are all measures on $X$, then $\mu = \sum_{\alpha\in J} \mu_\alpha$, i.e.

$$\mu(A) = \sum_{\alpha\in J} \mu_\alpha(A) \text{ for all } A \subset X$$

is a measure on $X$. (See Section 2 for the meaning of this sum.) To prove this we must show that $\mu$ is countably additive. Suppose that $\{A_i\}_{i=1}^\infty$ is a collection of pair-wise disjoint subsets of $X$, then

$$\begin{aligned}
\mu(\cup_{i=1}^\infty A_i) &= \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \sum_{\alpha\in J} \mu_\alpha(A_i) \\
&= \sum_{\alpha\in J} \sum_{i=1}^\infty \mu_\alpha(A_i) = \sum_{\alpha\in J} \mu_\alpha(\cup_{i=1}^\infty A_i) \\
&= \mu(\cup_{i=1}^\infty A_i)
\end{aligned}$$

wherein the third equality we used Theorem 4.22 and in the fourth we used that fact that $\mu_\alpha$ is a measure.

*Example 17.4.* Suppose that $X$ is a set $\lambda : X \to [0,\infty]$ is a function. Then

$$\mu := \sum_{x\in X} \lambda(x)\delta_x$$

is a measure, explicitly

$$\mu(A) = \sum_{x\in A} \lambda(x)$$

for all $A \subset X$.

## 17.1 The problem with Lebesgue "measure"

So far all of the examples of measures given above are "counting" type measures, i.e. a weighted count of the number of points in a set. We certainly are going to want other types of measures too. In particular, it will be of great interest to have a measure on $\mathbb{R}$ (called Lebesgue measure) which measures the "length" of a subset of $\mathbb{R}$. Unfortunately as the next theorem shows, there is no such reasonable measure of length if we insist on measuring all subsets of $\mathbb{R}$.

**Theorem 17.5.** *There is no measure $\mu : 2^{\mathbb{R}} \to [0,\infty]$ such that*

1. *$\mu([a,b)) = (b-a)$ for all $a<b$ and*
2. *is translation invariant, i.e. $\mu(A+x) = \mu(A)$ for all $x \in \mathbb{R}$ and $A \in 2^{\mathbb{R}}$, where*

$$A + x := \{y + x : y \in A\} \subset \mathbb{R}.$$

*In fact the theorem is still true even if (1) is replaced by the weaker condition that $0 < \mu((0,1]) < \infty$.*

The counting measure $\mu(A) = \#(A)$ is translation invariant. However $\mu((0,1]) = \infty$ in this case and so $\mu$ does not satisfy condition 1.

*Proof.* **First proof.** Let us identify $[0,1)$ with the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ by the map

$$\phi(t) = e^{i2\pi t} = (\cos 2\pi t + i \sin 2\pi t) \in S^1$$

for $t \in [0,1)$. Using this identification we may use $\mu$ to define a function $\nu$ on $2^{S^1}$ by $\nu(\phi(A)) = \mu(A)$ for all $A \subset [0,1)$. This new function is a measure on $S^1$ with the property that $0 < \nu((0,1]) < \infty$. For $z \in S^1$ and $N \subset S^1$ let

$$zN := \{zn \in S^1 : n \in N\}, \tag{17.1}$$



that is to say $e^{i\theta}N$ is $N$ rotated counter clockwise by angle $\theta$. We now claim that $\nu$ is invariant under these rotations, i.e.

$$\nu(zN) = \nu(N) \tag{17.2}$$

for all $z \in S^1$ and $N \subset S^1$. To verify this, write $N = \phi(A)$ and $z = \phi(t)$ for some $t \in [0,1)$ and $A \subset [0,1)$. Then

$$\phi(t)\phi(A) = \phi(t + A \bmod 1)$$

where for $A \subset [0,1)$ and $\alpha \in [0,1)$,

$$\begin{aligned} t + A \bmod 1 &:= \{a + t \bmod 1 \in [0,1) : a \in N\} \\ &= (a + A \cap \{a < 1-t\}) \cup ((t-1) + A \cap \{a \geq 1-t\}). \end{aligned}$$

Thus

$$\begin{aligned} \nu(\phi(t)\phi(A)) &= \mu(t + A \bmod 1) \\ &= \mu\left((a + A \cap \{a < 1-t\}) \cup ((t-1) + A \cap \{a \geq 1-t\})\right) \\ &= \mu\left((a + A \cap \{a < 1-t\})\right) + \mu\left(((t-1) + A \cap \{a \geq 1-t\})\right) \\ &= \mu\left(A \cap \{a < 1-t\}\right) + \mu\left(A \cap \{a \geq 1-t\}\right) \\ &= \mu\left((A \cap \{a < 1-t\}) \cup (A \cap \{a \geq 1-t\})\right) \\ &= \mu(A) = \nu(\phi(A)). \end{aligned}$$

Therefore it suffices to prove that no finite non-trivial measure $\nu$ on $S^1$ such that Eq. (17.2) holds. To do this we will "construct" a non-measurable set $N = \phi(A)$ for some $A \subset [0,1)$.

Let

$$R := \{z = e^{i2\pi t} : t \in \mathbb{Q}\} = \{z = e^{i2\pi t} : t \in [0,1) \cap \mathbb{Q}\}$$

– a countable subgroup of $S^1$. As above $R$ acts on $S^1$ by rotations and divides $S^1$ up into equivalence classes, where $z, w \in S^1$ are equivalent if $z = rw$ for some $r \in R$. Choose (using the axiom of choice) one representative point $n$ from each of these equivalence classes and let $N \subset S^1$ be the set of these representative points. Then every point $z \in S^1$ may be uniquely written as $z = nr$ with $n \in N$ and $r \in R$. That is to say

$$S^1 = \coprod_{r\in R} (rN) \tag{17.3}$$

where $\coprod_\alpha A_\alpha$ is used to denote the union of pair-wise disjoint sets $\{A_\alpha\}$. By Eqs. (17.2) and (17.3),

$$\nu(S^1) = \sum_{r\in R} \nu(rN) = \sum_{r\in R} \nu(N).$$

The right member from this equation is either 0 or $\infty$, 0 if $\nu(N) = 0$ and $\infty$ if $\nu(N) > 0$. In either case it is not equal $\nu(S^1) \in (0,1)$. Thus we have reached the desired contradiction. ■

*Proof.* **Second proof of Theorem** 17.5. For $N \subset [0,1)$ and $\alpha \in [0,1)$, let

$$\begin{aligned} N^\alpha &= N + \alpha \operatorname{mod} 1 \\ &= \{a + \alpha \operatorname{mod} 1 \in [0,1) : a \in N\} \\ &= (\alpha + N \cap \{a < 1-\alpha\}) \cup ((\alpha - 1) + N \cap \{a \geq 1 - \alpha\}). \end{aligned}$$

Then

$$\begin{aligned} \mu(N^\alpha) &= \mu(\alpha + N \cap \{a < 1-\alpha\}) + \mu((\alpha - 1) + N \cap \{a \geq 1 - \alpha\}) \\ &= \mu(N \cap \{a < 1-\alpha\}) + \mu(N \cap \{a \geq 1 - \alpha\}) \\ &= \mu(N \cap \{a < 1-\alpha\} \cup (N \cap \{a \geq 1 - \alpha\})) \\ &= \mu(N). \end{aligned} \tag{17.4}$$

We will now construct a bad set $N$ which coupled with Eq. (17.4) will lead to a contradiction.

Set

$$Q_x := \{x + r \in \mathbb{R} : r \in \mathbb{Q}\} = x + \mathbb{Q}.$$

Notice that $Q_x \cap Q_y \neq \emptyset$ implies that $Q_x = Q_y$. Let $\mathcal{O} = \{Q_x : x \in \mathbb{R}\}$ – the orbit space of the $\mathbb{Q}$ action. For all $A \in \mathcal{O}$ choose $f(A) \in [0,1/3) \cap A$[1] and define $N = f(\mathcal{O})$. Then observe:

1. $f(A) = f(B)$ implies that $A \cap B \neq \emptyset$ which implies that $A = B$ so that $f$ is injective.
2. $\mathcal{O} = \{Q_n : n \in N\}$.

Let $R$ be the countable set,

$$R := \mathbb{Q} \cap [0,1).$$

We now claim that

$$N^r \cap N^s = \emptyset \text{ if } r \neq s \text{ and} \tag{17.5}$$

$$[0,1) = \cup_{r \in R} N^r. \tag{17.6}$$

Indeed, if $x \in N^r \cap N^s \neq \emptyset$ then $x = r + n \operatorname{mod} 1$ and $x = s + n' \operatorname{mod} 1$, then $n - n' \in \mathbb{Q}$, i.e. $Q_n = Q_{n'}$. That is to say, $n = f(Q_n) = f(Q_{n'}) = n'$ and hence that $s = r \operatorname{mod} 1$, but $s, r \in [0,1)$ implies that $s = r$. Furthermore, if $x \in [0,1)$ and $n := f(Q_x)$, then $x - n = r \in \mathbb{Q}$ and $x \in N^{r \operatorname{mod} 1}$.

Now that we have constructed $N$, we are ready for the contradiction. By Equations (17.4–17.6) we find

$$\begin{aligned} 1 = \mu([0,1)) &= \sum_{r \in R} \mu(N^r) = \sum_{r \in R} \mu(N) \\ &= \begin{cases} \infty & \text{if } \mu(N) > 0 \\ 0 & \text{if } \mu(N) = 0 \end{cases}. \end{aligned}$$

[1] We have used the Axiom of choice here, i.e. $\prod_{A \in \mathcal{F}} (A \cap [0,1/3]) \neq \emptyset$



which is certainly inconsistent. Incidentally we have just produced an example of so called "non – measurable" set. ■

Because of Theorem 17.5, it is necessary to modify Definition 17.1. Theorem 17.5 points out that we will have to give up the idea of trying to measure all subsets of $\mathbb{R}$ but only measure some sub-collections of "measurable" sets. This leads us to the notion of $\sigma$ – algebra discussed in the next chapter. Our revised notion of a measure will appear in Definition 19.1 of Chapter 19 below.

# 18

# Measurability

## 18.1 Algebras and $\sigma$ – Algebras

**Definition 18.1.** *A collection of subsets $\mathcal{A}$ of a set $X$ is an **algebra** if*

1. $\emptyset, X \in \mathcal{A}$
2. $A \in \mathcal{A}$ *implies that* $A^c \in \mathcal{A}$
3. $\mathcal{A}$ *is closed under finite unions, i.e. if* $A_1, \dots, A_n \in \mathcal{A}$ *then* $A_1 \cup \dots \cup A_n \in \mathcal{A}$.
   *In view of conditions 1. and 2., 3. is equivalent to*

3′. $\mathcal{A}$ *is closed under finite intersections.*

**Definition 18.2.** *A collection of subsets $\mathcal{M}$ of $X$ is a $\sigma$ – **algebra** (or sometimes called a $\sigma$ – field) if $\mathcal{M}$ is an algebra which also closed under countable unions, i.e. if $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$, then $\cup_{i=1}^\infty A_i \in \mathcal{M}$. (Notice that since $\mathcal{M}$ is also closed under taking complements, $\mathcal{M}$ is also closed under taking countable intersections.) A pair $(X, \mathcal{M})$, where $X$ is a set and $\mathcal{M}$ is a $\sigma$ – algebra on $X$, is called a **measurable space.***

The reader should compare these definitions with that of a topology in Definition 10.1. Recall that the elements of a topology are called open sets. Analogously, elements of and algebra $\mathcal{A}$ or a $\sigma$ – algebra $\mathcal{M}$ will be called **measurable** sets.

*Example 18.3.* Here are some examples of algebras.

1. $\mathcal{M} = 2^X$, then $\mathcal{M}$ is a topology, an algebra and a $\sigma$ – algebra.
2. Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on $X$ which is not an algebra.
3. $\tau = \mathcal{A} = \{\{1\}, \{2, 3\}, \emptyset, X\}$ is a topology, an algebra, and a $\sigma$ – algebra on $X$. The sets $X$, $\{1\}$, $\{2, 3\}$, $\emptyset$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor **closed** and are not measurable.

The reader should compare this example with Example 10.3.



**Proposition 18.4.** *Let $\mathcal{E}$ be any collection of subsets of $X$. Then there exists a unique smallest algebra $\mathcal{A}(\mathcal{E})$ and $\sigma$ – algebra $\sigma(\mathcal{E})$ which contains $\mathcal{E}$.*

*Proof.* The proof is the same as the analogous Proposition 10.6 for topologies, i.e.

$$\mathcal{A}(\mathcal{E}) := \bigcap \{\mathcal{A} : \mathcal{A} \text{ is an algebra such that } \mathcal{E} \subset \mathcal{A}\}$$

and

$$\sigma(\mathcal{E}) := \bigcap \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma - \text{algebra such that } \mathcal{E} \subset \mathcal{M}\}.$$

■

*Example 18.5.* Suppose $X = \{1, 2, 3\}$ and $\mathcal{E} = \{\emptyset, X, \{1, 2\}, \{1, 3\}\}$, see Figure 18.1.

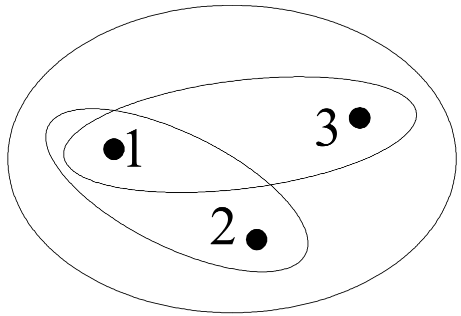



**Fig. 18.1.** A collection of subsets.

Then

$$\tau(\mathcal{E}) = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$$
$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = 2^X.$$

The next proposition is the analogue to Proposition 10.7 for topologies and enables us to give and explicit descriptions of $\mathcal{A}(\mathcal{E})$. On the other hand it should be noted that $\sigma(\mathcal{E})$ typically does not admit a simple concrete description.

**Proposition 18.6.** *Let $X$ be a set and $\mathcal{E} \subset 2^X$. Let $\mathcal{E}^c := \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c := \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$ Then*

$$\mathcal{A}(\mathcal{E}) := \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\}. \quad (18.1)$$

*Proof.* Let $\mathcal{A}$ denote the right member of Eq. (18.1). From the definition of an algebra, it is clear that $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$. Hence to finish that proof it suffices to show $\mathcal{A}$ is an algebra. The proof of these assertions are routine except for possibly showing that $\mathcal{A}$ is closed under complementation.

To check $\mathcal{A}$ is closed under complementation, let $Z \in \mathcal{A}$ be expressed as

$$Z = \bigcup_{i=1}^{N} \bigcap_{j=1}^{K} A_{ij}$$

where $A_{ij} \in \mathcal{E}_c$. Therefore, writing $B_{ij} = A_{ij}^c \in \mathcal{E}_c$, we find that

$$Z^c = \bigcap_{i=1}^{N} \bigcup_{j=1}^{K} B_{ij} = \bigcup_{j_1,\dots,j_N=1}^{K} (B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}) \in \mathcal{A}$$

wherein we have used the fact that $B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}$ is a finite intersection of sets from $\mathcal{E}_c$. ■

*Remark 18.7.* One might think that in general $\sigma(\mathcal{E})$ may be described as the countable unions of countable intersections of sets in $\mathcal{E}^c$. However this is in general **false,** since if

$$Z = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij}$$

with $A_{ij} \in \mathcal{E}_c$, then

$$Z^c = \bigcup_{j_1=1, j_2=1, \dots j_N=1, \dots}^{\infty} \left( \bigcap_{\ell=1}^{\infty} A_{\ell, j_\ell}^c \right)$$

which is now an **uncountable** union. Thus the above description is not correct. In general it is complicated to explicitly describe $\sigma(\mathcal{E})$, see Proposition 1.23 on page 39 of Folland for details. Also see Proposition 18.13 below.

**Exercise 18.1.** Let $\tau$ be a topology on a set $X$ and $\mathcal{A} = \mathcal{A}(\tau)$ be the algebra generated by $\tau$. Show $\mathcal{A}$ is the collection of subsets of $X$ which may be written as finite union of sets of the form $F \cap V$ where $F$ is closed and $V$ is open.

The following notion will be useful in the sequel and plays an analogous role for algebras as a base (Definition 10.8) does for a topology.

**Definition 18.8.** *A set $\mathcal{E} \subset 2^X$ is said to be an **elementary family or elementary class** provided that*

- $\emptyset \in \mathcal{E}$
- *$\mathcal{E}$ is closed under finite intersections*
- *if $E \in \mathcal{E}$, then $E^c$ is a finite disjoint union of sets from $\mathcal{E}$. (In particular $X = \emptyset^c$ is a finite disjoint union of elements from $\mathcal{E}$.)*



*Example 18.9.* Let $X = \mathbb{R}$, then

$$\begin{aligned}\mathcal{E} &:= \left\{(a,b] \cap \mathbb{R} : a, b \in \bar{\mathbb{R}}\right\} \\ &= \{(a,b] : a \in [-\infty, \infty) \text{ and } a < b < \infty\} \cup \{\emptyset, \mathbb{R}\}\end{aligned}$$

is an elementary family.

**Exercise 18.2.** Let $\mathcal{A} \subset 2^X$ and $\mathcal{B} \subset 2^Y$ be elementary families. Show the collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is also an elementary family.

**Proposition 18.10.** *Suppose $\mathcal{E} \subset 2^X$ is an elementary family, then $\mathcal{A} = \mathcal{A}(\mathcal{E})$ consists of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$.*

*Proof.* This could be proved making use of Proposition 18.6. However it is easier to give a direct proof.

Let $\mathcal{A}$ denote the collection of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$. Clearly $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$ so it suffices to show $\mathcal{A}$ is an algebra since $\mathcal{A}(\mathcal{E})$ is the smallest algebra containing $\mathcal{E}$.

By the properties of $\mathcal{E}$, we know that $\emptyset, X \in \mathcal{A}$. Now suppose that $A_i = \coprod_{F \in \Lambda_i} F \in \mathcal{A}$ where, for $i = 1, 2, \dots, n$, $\Lambda_i$ is a finite collection of disjoint sets from $\mathcal{E}$. Then

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \left( \coprod_{F \in \Lambda_i} F \right) = \bigcup_{(F_1,, \dots, F_n) \in \Lambda_1 \times \dots \times \Lambda_n} (F_1 \cap F_2 \cap \dots \cap F_n)$$

and this is a disjoint (you check) union of elements from $\mathcal{E}$. Therefore $\mathcal{A}$ is closed under finite intersections. Similarly, if $A = \coprod_{F \in \Lambda} F$ with $\Lambda$ being a finite collection of disjoint sets from $\mathcal{E}$, then $A^c = \bigcap_{F \in \Lambda} F^c$. Since by assumption $F^c \in \mathcal{A}$ for $F \in \Lambda \subset \mathcal{E}$ and $\mathcal{A}$ is closed under finite intersections, it follows that $A^c \in \mathcal{A}$. ■

**Definition 18.11.** *Let $X$ be a set. We say that a family of sets $\mathcal{F} \subset 2^X$ is a **partition** of $X$ if distinct members of $\mathcal{F}$ are disjoint and if $X$ is the union of the sets in $\mathcal{F}$.*

*Example 18.12.* Let $X$ be a set and $\mathcal{E} = \{A_1, \dots, A_n\}$ where $A_1, \dots, A_n$ is a partition of $X$. In this case

$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = \tau(\mathcal{E}) = \{\cup_{i \in \Lambda} A_i : \Lambda \subset \{1, 2, \dots, n\}\}$$

where $\cup_{i \in \Lambda} A_i := \emptyset$ when $\Lambda = \emptyset$. Notice that

$$\#\left(\mathcal{A}(\mathcal{E})\right) = \#(2^{\{1,2,\dots,n\}}) = 2^n.$$

**Proposition 18.13.** *Suppose that $\mathcal{M} \subset 2^X$ is a $\sigma$ – algebra and $\mathcal{M}$ is at most a countable set. Then there exists a unique **finite** partition $\mathcal{F}$ of $X$ such that $\mathcal{F} \subset \mathcal{M}$ and every element $B \in \mathcal{M}$ is of the form*

$$B = \cup \{A \in \mathcal{F} : A \subset B\}. \tag{18.2}$$

*In particular $\mathcal{M}$ is actually a finite set and $\#(\mathcal{M}) = 2^n$ for some $n \in \mathbb{N}$.*

*Proof.* For each $x \in X$ let

$$A_x = \cap \{A \in \mathcal{M} : x \in A\} \in \mathcal{M},$$

wherein we have used $\mathcal{M}$ is a countable $\sigma$ – algebra to insure $A_x \in \mathcal{M}$. Hence $A_x$ is the smallest set in $\mathcal{M}$ which contains $x$.

Let $C = A_x \cap A_y$. If $x \notin C$ then $A_x \setminus C \subset A_x$ is an element of $\mathcal{M}$ which contains $x$ and since $A_x$ is the smallest member of $\mathcal{M}$ containing $x$, we must have that $C = \emptyset$. Similarly if $y \notin C$ then $C = \emptyset$. Therefore if $C \neq \emptyset$, then $x, y \in A_x \cap A_y \in \mathcal{M}$ and $A_x \cap A_y \subset A_x$ and $A_x \cap A_y \subset A_y$ from which it follows that $A_x = A_x \cap A_y = A_y$. This shows that $\mathcal{F} = \{A_x : x \in X\} \subset \mathcal{M}$ is a (necessarily countable) partition of $X$ for which Eq. (18.2) holds for all $B \in \mathcal{M}$.

Enumerate the elements of $\mathcal{F}$ as $\mathcal{F} = \{P_n\}_{n=1}^N$ where $N \in \mathbb{N}$ or $N = \infty$. If $N = \infty$, then the correspondence

$$a \in \{0,1\}^{\mathbb{N}} \to A_a = \cup\{P_n : a_n = 1\} \in \mathcal{M}$$

is bijective and therefore, by Lemma 2.6, $\mathcal{M}$ is uncountable. Thus any countable $\sigma$ – algebra is necessarily finite. This finishes the proof modulo the uniqueness assertion which is left as an exercise to the reader. ■

*Example 18.14.* Let $X = \mathbb{R}$ and

$$\mathcal{E} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\} = \{(a,\infty) \cap \mathbb{R} : a \in \bar{\mathbb{R}}\} \subset 2^{\mathbb{R}}.$$

Notice that $\mathcal{E}_f = \mathcal{E}$ and that $\mathcal{E}$ is closed under unions, which shows that $\tau(\mathcal{E}) = \mathcal{E}$, i.e. $\mathcal{E}$ is already a topology. Since $(a,\infty)^c = (-\infty, a]$ we find that $\mathcal{E}_c = \{(a,\infty), (-\infty,a], -\infty \le a < \infty\} \cup \{\mathbb{R}, \emptyset\}$. Noting that

$$(a,\infty) \cap (-\infty, b] = (a,b]$$

it follows that $\mathcal{A}(\mathcal{E}) = \mathcal{A}(\tilde{\mathcal{E}})$ where

$$\tilde{\mathcal{E}} := \left\{(a,b] \cap \mathbb{R} : a,b \in \bar{\mathbb{R}}\right\}.$$

Since $\tilde{\mathcal{E}}$ is an elementary family of subsets of $\mathbb{R}$, Proposition 18.10 implies $\mathcal{A}(\mathcal{E})$ may be described as being those sets which are finite disjoint unions of sets from $\tilde{\mathcal{E}}$. The $\sigma$ – algebra, $\sigma(\mathcal{E})$, generated by $\mathcal{E}$ is **very complicated**. Here are some sets in $\sigma(\mathcal{E})$ – most of which are not in $\mathcal{A}(\mathcal{E})$.



(a) $(a,b) = \bigcup\limits_{n=1}^{\infty} (a, b - \frac{1}{n}] \in \sigma(\mathcal{E})$.
(b) All of the standard open subsets of $\mathbb{R}$ are in $\sigma(\mathcal{E})$.
(c) $\{x\} = \bigcap\limits_n \left(x - \frac{1}{n}, x\right] \in \sigma(\mathcal{E})$
(d) $[a,b] = \{a\} \cup (a,b] \in \sigma(\mathcal{E})$
(e) Any countable subset of $\mathbb{R}$ is in $\sigma(\mathcal{E})$.

*Remark 18.15.* In the above example, one may replace $\mathcal{E}$ by $\mathcal{E} = \{(a,\infty) : a \in \mathbb{Q}\} \cup \{\mathbb{R}, \emptyset\}$, in which case $\mathcal{A}(\mathcal{E})$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\{(a,\infty), (-\infty, a], (a,b] : a,b \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}.$$

This shows that $\mathcal{A}(\mathcal{E})$ is a countable set – a useful fact which will be needed later.

**Notation 18.16** *For a general topological space $(X,\tau)$, the **Borel $\sigma$ – algebra** is the $\sigma$ – algebra $\mathcal{B}_X := \sigma(\tau)$ on $X$. In particular if $X = \mathbb{R}^n$, $\mathcal{B}_{\mathbb{R}^n}$ will be used to denote the Borel $\sigma$ – algebra on $\mathbb{R}^n$ when $\mathbb{R}^n$ is equipped with its standard Euclidean topology.*

**Exercise 18.3.** Verify the $\sigma$ – algebra, $\mathcal{B}_{\mathbb{R}}$, is generated by any of the following collection of sets:

$$1.\ \{(a,\infty) : a \in \mathbb{R}\},\ 2.\ \{(a,\infty) : a \in \mathbb{Q}\} \text{ or } 3.\ \{[a,\infty) : a \in \mathbb{Q}\}.$$

**Proposition 18.17.** *If $\tau$ is a second countable topology on $X$ and $\mathcal{E}$ is a countable collection of subsets of $X$ such that $\tau = \tau(\mathcal{E})$, then $\mathcal{B}_X := \sigma(\tau) = \sigma(\mathcal{E})$, i.e. $\sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E})$.*

*Proof.* Let $\mathcal{E}_f$ denote the collection of subsets of $X$ which are finite intersection of elements from $\mathcal{E}$ along with $X$ and $\emptyset$. Notice that $\mathcal{E}_f$ is still countable (you prove). A set $Z$ is in $\tau(\mathcal{E})$ iff $Z$ is an arbitrary union of sets from $\mathcal{E}_f$. Therefore $Z = \bigcup\limits_{A \in \mathcal{F}} A$ for some subset $\mathcal{F} \subset \mathcal{E}_f$ which is necessarily countable. Since $\mathcal{E}_f \subset \sigma(\mathcal{E})$ and $\sigma(\mathcal{E})$ is closed under countable unions it follows that $Z \in \sigma(\mathcal{E})$ and hence that $\tau(\mathcal{E}) \subset \sigma(\mathcal{E})$. Lastly, since $\mathcal{E} \subset \tau(\mathcal{E}) \subset \sigma(\mathcal{E})$, $\sigma(\mathcal{E}) \subset \sigma(\tau(\mathcal{E})) \subset \sigma(\mathcal{E})$. ■

## 18.2 Measurable Functions

Our notion of a "measurable" function will be analogous to that for a continuous function. For motivational purposes, suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to \mathbb{R}_+$. Roughly speaking, in the next Chapter we are going to define $\int\limits_X f d\mu$ as a certain limit of sums of the form,

$$\sum_{0<a_1<a_2<a_3<\dots}^{\infty} a_i \mu(f^{-1}(a_i, a_{i+1}]).$$

For this to make sense we will need to require $f^{-1}((a,b]) \in \mathcal{M}$ for all $a < b$. Because of Lemma 18.22 below, this last condition is equivalent to the condition $f^{-1}(\mathcal{B}_{\mathbb{R}}) \subset \mathcal{M}$.

**Definition 18.18.** *Let $(X, \mathcal{M})$ and $(Y, \mathcal{F})$ be measurable spaces. A function $f : X \to Y$ is* ***measurable*** *if $f^{-1}(\mathcal{F}) \subset \mathcal{M}$. We will also say that $f$ is $\mathcal{M}/\mathcal{F}$ – measurable or $(\mathcal{M}, \mathcal{F})$ – measurable.*

*Example 18.19 (Characteristic Functions).* Let $(X, \mathcal{M})$ be a measurable space and $A \subset X$. We define the **characteristic function** $1_A : X \to \mathbb{R}$ by

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

If $A \in \mathcal{M}$, then $1_A$ is $(\mathcal{M}, 2^{\mathbb{R}})$ – measurable because $1_A^{-1}(W)$ is either $\emptyset$, $X$, $A$ or $A^c$ for any $W \subset \mathbb{R}$. Conversely, if $\mathcal{F}$ is any $\sigma$ – algebra on $\mathbb{R}$ containing a set $W \subset \mathbb{R}$ such that $1 \in W$ and $0 \in W^c$, then $A \in \mathcal{M}$ if $1_A$ is $(\mathcal{M}, \mathcal{F})$ – measurable. This is because $A = 1_A^{-1}(W) \in \mathcal{M}$.

**Exercise 18.4.** Suppose $f : X \to Y$ is a function, $\mathcal{F} \subset 2^Y$ and $\mathcal{M} \subset 2^X$. Show $f^{-1}\mathcal{F}$ and $f_*\mathcal{M}$ (see Notation 2.7) are algebras ($\sigma$ – algebras) provided $\mathcal{F}$ and $\mathcal{M}$ are algebras ($\sigma$ – algebras).

*Remark 18.20.* Let $f : X \to Y$ be a function. Given a $\sigma$ – algebra $\mathcal{F} \subset 2^Y$, the $\sigma$ – algebra $\mathcal{M} := f^{-1}(\mathcal{F})$ is the smallest $\sigma$ – algebra on $X$ such that $f$ is $(\mathcal{M}, \mathcal{F})$ - measurable . Similarly, if $\mathcal{M}$ is a $\sigma$ - algebra on $X$ then $\mathcal{F} = f_*\mathcal{M}$ is the largest $\sigma$ – algebra on $Y$ such that $f$ is $(\mathcal{M}, \mathcal{F})$ - measurable .

Recall from Definition 2.8 that for $\mathcal{E} \subset 2^X$ and $A \subset X$ that

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}$$

where $i_A : A \to X$ is the inclusion map. Because of Exercise 10.3, when $\mathcal{E} = \mathcal{M}$ is an algebra ($\sigma$ – algebra), $\mathcal{M}_A$ is an algebra ($\sigma$ – algebra) on $A$ and we call $\mathcal{M}_A$ the relative or induced algebra ($\sigma$ – algebra) on $A$.

The next two Lemmas are direct analogues of their topological counter parts in Lemmas 10.13 and 10.14. For completeness, the proofs will be given even though they are same as those for Lemmas 10.13 and 10.14.

**Lemma 18.21.** *Suppose that $(X, \mathcal{M})$, $(Y, \mathcal{F})$ and $(Z, \mathcal{G})$ are measurable spaces. If $f : (X, \mathcal{M}) \to (Y, \mathcal{F})$ and $g : (Y, \mathcal{F}) \to (Z, \mathcal{G})$ are measurable functions then $g \circ f : (X, \mathcal{M}) \to (Z, \mathcal{G})$ is measurable as well.*



*Proof.* By assumption $g^{-1}(\mathcal{G}) \subset \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subset \mathcal{M}$ so that

$$(g \circ f)^{-1}(\mathcal{G}) = f^{-1}\left(g^{-1}(\mathcal{G})\right) \subset f^{-1}(\mathcal{F}) \subset \mathcal{M}.$$

■

**Lemma 18.22.** *Suppose that $f : X \to Y$ is a function and $\mathcal{E} \subset 2^Y$ and $A \subset Y$ then*

$$\sigma\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\sigma(\mathcal{E})) \text{ and} \tag{18.3}$$

$$(\sigma(\mathcal{E}))_A = \sigma(\mathcal{E}_A). \tag{18.4}$$

*(Similar assertion hold with $\sigma(\cdot)$ being replaced by $\mathcal{A}(\cdot)$.) Moreover, if $\mathcal{F} = \sigma(\mathcal{E})$ and $\mathcal{M}$ is a $\sigma$ – algebra on $X$, then $f$ is $(\mathcal{M}, \mathcal{F})$ – measurable iff $f^{-1}(\mathcal{E}) \subset \mathcal{M}$.*

*Proof.* By Exercise 18.4, $f^{-1}(\sigma(\mathcal{E}))$ is a $\sigma$ – algebra and since $\mathcal{E} \subset \mathcal{F}$, $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$. It now follows that $\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. For the reverse inclusion, notice that

$$f_*\sigma\left(f^{-1}(\mathcal{E})\right) = \left\{B \subset Y : f^{-1}(B) \in \sigma\left(f^{-1}(\mathcal{E})\right)\right\}$$

is a $\sigma$ – algebra which contains $\mathcal{E}$ and thus $\sigma(\mathcal{E}) \subset f_*\sigma\left(f^{-1}(\mathcal{E})\right)$. Hence if $B \in \sigma(\mathcal{E})$ we know that $f^{-1}(B) \in \sigma\left(f^{-1}(\mathcal{E})\right)$, i.e. $f^{-1}(\sigma(\mathcal{E})) \subset \sigma\left(f^{-1}(\mathcal{E})\right)$ and Eq. (18.3) has been proved.

Applying Eq. (18.3) with $X = A$ and $f = i_A$ being the inclusion map implies

$$(\sigma(\mathcal{E}))_A = i_A^{-1}(\sigma(\mathcal{E})) = \sigma(i_A^{-1}(\mathcal{E})) = \sigma(\mathcal{E}_A).$$

Lastly if $f^{-1}\mathcal{E} \subset \mathcal{M}$, then $f^{-1}\sigma(\mathcal{E}) = \sigma\left(f^{-1}\mathcal{E}\right) \subset \mathcal{M}$ which shows $f$ is $(\mathcal{M}, \mathcal{F})$ – measurable. ■

**Corollary 18.23.** *Suppose that $(X, \mathcal{M})$ is a measurable space. Then the following conditions on a function $f : X \to \mathbb{R}$ are equivalent:*

1. *$f$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable,*
2. *$f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,*
3. *$f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$,*
4. *$f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.*

*Proof.* An exercise in using Lemma 18.22 and is the content of Exercise 18.8. ■

Here is yet another way to generate $\sigma$ – algebras. (Compare with the analogous topological Definition 10.20.)

**Definition 18.24 ($\sigma$ – Algebras Generated by Functions).** *Let $X$ be a set and suppose there is a collection of measurable spaces $\{(Y_\alpha, \mathcal{F}_\alpha) : \alpha \in A\}$ and functions $f_\alpha : X \to Y_\alpha$ for all $\alpha \in A$. Let $\sigma(f_\alpha : \alpha \in A)$ denote the smallest $\sigma$ – algebra on $X$ such that each $f_\alpha$ is measurable, i.e.*

$$\sigma(f_\alpha : \alpha \in A) = \sigma(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)).$$

**Proposition 18.25.** *Assuming the notation in Definition 18.24 and additionally let $(Z, \mathcal{M})$ be a measurable space and $g : Z \to X$ be a function. Then $g$ is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable iff $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$–measurable for all $\alpha \in A$.*

*Proof.* This proof is essentially the same as the proof of the topological analogue in Proposition 10.21.

($\Rightarrow$) If $g$ is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable, then the composition $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable by Lemma 18.21.

($\Leftarrow$) Let

$$\mathcal{G} = \sigma(f_\alpha : \alpha \in A) = \sigma\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right).$$

If $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable for all $\alpha$, then

$$g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{M} \, \forall \, \alpha \in A$$

and therefore

$$g^{-1}\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right) = \cup_{\alpha \in A} g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{M}.$$

Hence

$$g^{-1}(\mathcal{G}) = g^{-1}\left(\sigma\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right)\right) = \sigma(g^{-1}\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right) \subset \mathcal{M}$$

which shows that $g$ is $(\mathcal{M}, \mathcal{G})$ – measurable. ■

**Definition 18.26.** *A function $f : X \to Y$ between two topological spaces is* ***Borel measurable*** *if $f^{-1}(\mathcal{B}_Y) \subset \mathcal{B}_X$.*

**Proposition 18.27.** *Let $X$ and $Y$ be two topological spaces and $f : X \to Y$ be a continuous function. Then $f$ is Borel measurable.*

*Proof.* Using Lemma 18.22 and $\mathcal{B}_Y = \sigma(\tau_Y)$,

$$f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\tau_Y)) = \sigma(f^{-1}(\tau_Y)) \subset \sigma(\tau_X) = \mathcal{B}_X.$$

■

**Definition 18.28.** *Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{F})$ and a subset $A \subset X$. We say a function $f : A \to Y$ is measurable iff $f$ if $\mathcal{M}_A/\mathcal{F}$ – measurable.*

**Proposition 18.29 (Localizing Measurability).** *Let $(X, \mathcal{M})$ and $(Y, \mathcal{F})$ be measurable spaces and $f : X \to Y$ be a function.*

1. *If $f$ is measurable and $A \subset X$ then $f|_A : A \to Y$ is measurable.*
2. *Suppose there exist $A_n \in \mathcal{M}$ such that $X = \cup_{n=1}^\infty A_n$ and $f|A_n$ is $\mathcal{M}_{A_n}$ measurable for all $n$, then $f$ is $\mathcal{M}$ – measurable.*



*Proof.* As the reader will notice, the proof given below is essentially identical to the proof of Proposition 10.19 which is the topological analogue of this proposition.

1. If $f : X \to Y$ is measurable, $f^{-1}(B) \in \mathcal{M}$ for all $B \in \mathcal{F}$ and therefore

$$f|_A^{-1}(B) = A \cap f^{-1}(B) \in \mathcal{M}_A \text{ for all } B \in \mathcal{F}.$$

2. If $B \in \mathcal{F}$, then

$$f^{-1}(B) = \cup_{n=1}^\infty \left(f^{-1}(B) \cap A_n\right) = \cup_{n=1}^\infty f|_{A_n}^{-1}(B).$$

Since each $A_n \in \mathcal{M}$, $\mathcal{M}_{A_n} \subset \mathcal{M}$ and so the previous displayed equation shows $f^{-1}(B) \in \mathcal{M}$. ■

**Proposition 18.30.** *If $(X, \mathcal{M})$ is a measurable space, then*

$$f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$$

*is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^n})$ – measurable iff $f_i : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable for each $i$. In particular, a function $f : X \to \mathbb{C}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable iff* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable.*

*Proof.* This is formally a consequence of Corollary 18.65 and Proposition 18.60 below. Nevertheless it is instructive to give a direct proof now.

Let $\tau = \tau_{\mathbb{R}^n}$ denote the usual topology on $\mathbb{R}^n$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be projection onto the $i^{\text{th}}$ – factor. Since $\pi_i$ is continuous, $\pi_i$ is $\mathcal{B}_{\mathbb{R}^n}/\mathcal{B}_{\mathbb{R}}$ – measurable and therefore if $f : X \to \mathbb{R}^n$ is measurable then so is $f_i = \pi_i \circ f$.

Now suppose $f_i : X \to \mathbb{R}$ is measurable for all $i = 1, 2, \ldots, n$. Let

$$\mathcal{E} := \{(a, b) : a, b \in \mathbb{Q}^n \ni a < b\},$$

where, for $a, b \in \mathbb{R}^n$, we write $a < b$ iff $a_i < b_i$ for $i = 1, 2, \ldots, n$ and let

$$(a, b) = (a_1, b_1) \times \cdots \times (a_n, b_n).$$

Since $\mathcal{E} \subset \tau$ and every element $V \in \tau$ may be written as a (necessarily) countable union of elements from $\mathcal{E}$, we have $\sigma(\mathcal{E}) \subset \mathcal{B}_{\mathbb{R}^n} = \sigma(\tau) \subset \sigma(\mathcal{E})$, i.e. $\sigma(\mathcal{E}) = \mathcal{B}_{\mathbb{R}^n}$. (This part of the proof is essentially a direct proof of Corollary 18.65 below.)

Because

$$f^{-1}((a, b)) = f_1^{-1}((a_1, b_1)) \cap f_2^{-1}((a_2, b_2)) \cap \cdots \cap f_n^{-1}((a_n, b_n)) \in \mathcal{M}$$

for all $a, b \in \mathbb{Q}$ with $a < b$, it follows that $f^{-1}\mathcal{E} \subset \mathcal{M}$ and therefore

$$f^{-1}\mathcal{B}_{\mathbb{R}^n} = f^{-1}\sigma(\mathcal{E}) = \sigma\left(f^{-1}\mathcal{E}\right) \subset \mathcal{M}.$$

■

**Corollary 18.31.** *Let $(X, \mathcal{M})$ be a measurable space and $f, g : X \to \mathbb{C}$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable functions. Then $f \pm g$ and $f \cdot g$ are also $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable.*

*Proof.* Define $F : X \to \mathbb{C} \times \mathbb{C}$, $A_{\pm} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ by $F(x) = (f(x), g(x))$, $A_{\pm}(w, z) = w \pm z$ and $M(w, z) = wz$. Then $A_{\pm}$ and $M$ are continuous and hence $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$ – measurable. Also $F$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}}) = (\mathcal{M}, \mathcal{B}_{\mathbb{C}^2})$ – measurable since $\pi_1 \circ F = f$ and $\pi_2 \circ F = g$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable. Therefore $A_{\pm} \circ F = f \pm g$ and $M \circ F = f \cdot g$, being the composition of measurable functions, are also measurable. ■

**Lemma 18.32.** *Let $\alpha \in \mathbb{C}$, $(X, \mathcal{M})$ be a measurable space and $f : X \to \mathbb{C}$ be a $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable function. Then*

$$F(x) := \begin{cases} \frac{1}{f(x)} & \text{if} \quad f(x) \neq 0 \\ \alpha & \text{if} \quad f(x) = 0 \end{cases}$$

*is measurable.*

*Proof.* Define $i : \mathbb{C} \to \mathbb{C}$ by

$$i(z) = \begin{cases} \frac{1}{z} \text{ if } & z \neq 0 \\ \alpha \text{ if } & z = 0. \end{cases}$$

For any open set $V \subset \mathbb{C}$ we have

$$i^{-1}(V) = i^{-1}(V \setminus \{0\}) \cup i^{-1}(V \cap \{0\})$$

Because $i$ is continuous except at $z = 0$, $i^{-1}(V \setminus \{0\})$ is an open set and hence in $\mathcal{B}_{\mathbb{C}}$. Moreover, $i^{-1}(V \cap \{0\}) \in \mathcal{B}_{\mathbb{C}}$ since $i^{-1}(V \cap \{0\})$ is either the empty set or the one point set $\{\alpha\}$. Therefore $i^{-1}(\tau_{\mathbb{C}}) \subset \mathcal{B}_{\mathbb{C}}$ and hence $i^{-1}(\mathcal{B}_{\mathbb{C}}) = i^{-1}(\sigma(\tau_{\mathbb{C}})) = \sigma(i^{-1}(\tau_{\mathbb{C}})) \subset \mathcal{B}_{\mathbb{C}}$ which shows that $i$ is Borel measurable. Since $F = i \circ f$ is the composition of measurable functions, $F$ is also measurable. ■

We will often deal with functions $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. When talking about measurability in this context we will refer to the $\sigma$ – algebra on $\bar{\mathbb{R}}$ defined by

$$\mathcal{B}_{\bar{\mathbb{R}}} := \sigma\left(\{[a, \infty] : a \in \mathbb{R}\}\right). \tag{18.5}$$

**Proposition 18.33 (The Structure of $\mathcal{B}_{\bar{\mathbb{R}}}$).** *Let $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\bar{\mathbb{R}}}$ be as above, then*

$$\mathcal{B}_{\bar{\mathbb{R}}} = \{A \subset \bar{\mathbb{R}} : A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}. \tag{18.6}$$

*In particular $\{\infty\}, \{-\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}$ and $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\bar{\mathbb{R}}}$.*

*Proof.* Let us first observe that

$$\{-\infty\} = \cap_{n=1}^{\infty}[-\infty, -n) = \cap_{n=1}^{\infty}[-n, \infty]^c \in \mathcal{B}_{\bar{\mathbb{R}}},$$
$$\{\infty\} = \cap_{n=1}^{\infty}[n, \infty] \in \mathcal{B}_{\bar{\mathbb{R}}} \text{ and } \mathbb{R} = \bar{\mathbb{R}} \setminus \{\pm\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}.$$



Letting $i : \mathbb{R} \to \bar{\mathbb{R}}$ be the inclusion map,

$$\begin{aligned} i^{-1}\left(\mathcal{B}_{\bar{\mathbb{R}}}\right) &= \sigma\left(i^{-1}\left(\left\{[a, \infty] : a \in \bar{\mathbb{R}}\right\}\right)\right) = \sigma\left(\left\{i^{-1}\left([a, \infty]\right) : a \in \bar{\mathbb{R}}\right\}\right) \\ &= \sigma\left(\left\{[a, \infty] \cap \mathbb{R} : a \in \bar{\mathbb{R}}\right\}\right) = \sigma\left(\{[a, \infty) : a \in \mathbb{R}\}\right) = \mathcal{B}_{\mathbb{R}}. \end{aligned}$$

Thus we have shown

$$\mathcal{B}_{\mathbb{R}} = i^{-1}\left(\mathcal{B}_{\bar{\mathbb{R}}}\right) = \{A \cap \mathbb{R} : A \in \mathcal{B}_{\bar{\mathbb{R}}}\}.$$

This implies:

1. $A \in \mathcal{B}_{\bar{\mathbb{R}}} \Longrightarrow A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ and
2. if $A \subset \bar{\mathbb{R}}$ is such that $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ there exists $B \in \mathcal{B}_{\bar{\mathbb{R}}}$ such that $A \cap \mathbb{R} = B \cap \mathbb{R}$. Because $A \Delta B \subset \{\pm\infty\}$ and $\{\infty\}, \{-\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}$ we may conclude that $A \in \mathcal{B}_{\bar{\mathbb{R}}}$ as well.

This proves Eq. (18.6). ■

The proofs of the next two corollaries are left to the reader, see Exercises 18.5 and 18.6.

**Corollary 18.34.** *Let $(X, \mathcal{M})$ be a measurable space and $f : X \to \bar{\mathbb{R}}$ be a function. Then the following are equivalent*

1. *$f$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ - measurable,*
2. *$f^{-1}((a, \infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$,*
3. *$f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$,*
4. *$f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$ and $f^0 : X \to \mathbb{R}$ is measurable where*

$$f^0(x) := \begin{cases} f(x) \text{ if } & x \in \mathbb{R} \\ 0 & \text{if } x \in \{\pm\infty\}. \end{cases}$$

**Corollary 18.35.** *Let $(X, \mathcal{M})$ be a measurable space, $f, g : X \to \bar{\mathbb{R}}$ be functions and define $f \cdot g : X \to \bar{\mathbb{R}}$ and $(f + g) : X \to \bar{\mathbb{R}}$ using the conventions, $0 \cdot \infty = 0$ and $(f + g)(x) = 0$ if $f(x) = \infty$ and $g(x) = -\infty$ or $f(x) = -\infty$ and $g(x) = \infty$. Then $f \cdot g$ and $f + g$ are measurable functions on $X$ if both $f$ and $g$ are measurable.*

**Exercise 18.5.** Prove Corollary 18.34 noting that the equivalence of items 1. – 3. is a direct analogue of Corollary 18.23. Use Proposition 18.33 to handle item 4.

**Exercise 18.6.** Prove Corollary 18.35.

**Proposition 18.36 (Closure under sups, infs and limits).** *Suppose that $(X, \mathcal{M})$ is a measurable space and $f_j : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ for $j \in \mathbb{N}$ is a sequence of $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. Then*

$$\sup{}_j f_j, \quad \inf{}_j f_j, \quad \limsup_{j \to \infty} f_j \text{ and } \liminf_{j \to \infty} f_j$$

*are all $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. (Note that this result is in generally false when $(X, \mathcal{M})$ is a topological space and measurable is replaced by continuous in the statement.)*

*Proof.* Define $g_+(x) := \sup_j f_j(x)$, then

$$\begin{aligned}\{x : g_+(x) \le a\} &= \{x : f_j(x) \le a\ \forall j\} \\ &= \cap_j \{x : f_j(x) \le a\} \in \mathcal{M}\end{aligned}$$

so that $g_+$ is measurable. Similarly if $g_-(x) = \inf_j f_j(x)$ then

$$\{x : g_-(x) \ge a\} = \cap_j \{x : f_j(x) \ge a\} \in \mathcal{M}.$$

Since

$$\begin{aligned}\limsup_{j\to\infty} f_j &= \inf_n \sup\{f_j : j \ge n\} \text{ and} \\ \liminf_{j\to\infty} f_j &= \sup_n \inf\{f_j : j \ge n\}\end{aligned}$$

we are done by what we have already proved. ■

**Definition 18.37.** *Given a function $f : X \to \bar{\mathbb{R}}$ let $f_+(x) := \max\{f(x), 0\}$ and $f_-(x) := \max(-f(x), 0) = -\min(f(x), 0)$. Notice that $f = f_+ - f_-$.*

**Corollary 18.38.** *Suppose $(X, \mathcal{M})$ is a measurable space and $f : X \to \bar{\mathbb{R}}$ is a function. Then $f$ is measurable iff $f_\pm$ are measurable.*

*Proof.* If $f$ is measurable, then Proposition 18.36 implies $f_\pm$ are measurable. Conversely if $f_\pm$ are measurable then so is $f = f_+ - f_-$. ■

### 18.2.1 More general pointwise limits

**Lemma 18.39.** *Suppose that $(X, \mathcal{M})$ is a measurable space, $(Y, d)$ is a metric space and $f_j : X \to Y$ is $(\mathcal{M}, \mathcal{B}_Y)$ – measurable for all $j$. Also assume that for each $x \in X$, $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Then $f : X \to Y$ is also $(\mathcal{M}, \mathcal{B}_Y)$ – measurable.*

*Proof.* Let $V \in \tau_d$ and $W_m := \{y \in Y : d_{V^c}(y) > 1/m\}$ for $m = 1, 2, \dots$. Then $W_m \in \tau_d$,

$$W_m \subset \bar{W}_m \subset \{y \in Y : d_{V^c}(y) \ge 1/m\} \subset V$$

for all $m$ and $W_m \uparrow V$ as $m \to \infty$. The proof will be completed by verifying the identity,

$$f^{-1}(V) = \cup_{m=1}^\infty \cup_{N=1}^\infty \cap_{n\ge N} f_n^{-1}(W_m) \in \mathcal{M}.$$

If $x \in f^{-1}(V)$ then $f(x) \in V$ and hence $f(x) \in W_m$ for some $m$. Since $f_n(x) \to f(x)$, $f_n(x) \in W_m$ for almost all $n$. That is $x \in \cup_{m=1}^\infty \cup_{N=1}^\infty \cap_{n\ge N} f_n^{-1}(W_m)$. Conversely when $x \in \cup_{m=1}^\infty \cup_{N=1}^\infty \cap_{n\ge N} f_n^{-1}(W_m)$ there exists an $m$ such that $f_n(x) \in W_m \subset \bar{W}_m$ for almost all $n$. Since $f_n(x) \to f(x) \in \bar{W}_m \subset V$, it follows that $x \in f^{-1}(V)$. ■



*Remark 18.40.* In the previous Lemma 18.39 it is possible to let $(Y, \tau)$ be any topological space which has the "regularity" property that if $V \in \tau$ there exists $W_m \in \tau$ such that $W_m \subset \bar{W}_m \subset V$ and $V = \cup_{m=1}^\infty W_m$. Moreover, some extra condition is necessary on the topology $\tau$ in order for Lemma 18.39 to be correct. For example if $Y = \{1, 2, 3\}$ and $\tau = \{Y, \emptyset, \{1, 2\}, \{2, 3\}, \{2\}\}$ as in Example 10.36 and $X = \{a, b\}$ with the trivial $\sigma$ – algebra. Let $f_j(a) = f_j(b) = 2$ for all $j$, then $f_j$ is constant and hence measurable. Let $f(a) = 1$ and $f(b) = 2$, then $f_j \to f$ as $j \to \infty$ with $f$ being non-measurable. Notice that the Borel $\sigma$ – algebra on $Y$ is $2^Y$.

## 18.3 $\sigma$ – Function Algebras

In this subsection, we are going to relate $\sigma$ – algebras of subsets of a set $X$ to certain algebras of functions on $X$. We will begin this endeavor after proving the simple but very useful approximation Theorem 18.42 below.

**Definition 18.41.** *Let $(X, \mathcal{M})$ be a measurable space. A function $\phi : X \to \mathbb{F}$ ($\mathbb{F}$ denotes either $\mathbb{R}$, $\mathbb{C}$ or $[0, \infty] \subset \bar{\mathbb{R}}$) is a* ***simple function*** *if $\phi$ is $\mathcal{M}$ – $\mathcal{B}_\mathbb{F}$ measurable and $\phi(X)$ contains only finitely many elements.*

Any such simple functions can be written as

$$\phi = \sum_{i=1}^n \lambda_i 1_{A_i} \text{ with } A_i \in \mathcal{M} \text{ and } \lambda_i \in \mathbb{F}. \tag{18.7}$$

Indeed, take $\lambda_1, \lambda_2, \dots, \lambda_n$ to be an enumeration of the range of $\phi$ and $A_i = \phi^{-1}(\{\lambda_i\})$. Note that this argument shows that any simple function may be written intrinsically as

$$\phi = \sum_{y\in\mathbb{F}} y 1_{\phi^{-1}(\{y\})}. \tag{18.8}$$

The next theorem shows that simple functions are "pointwise dense" in the space of measurable functions.

**Theorem 18.42 (Approximation Theorem).** *Let $f : X \to [0, \infty]$ be measurable and define*

$$\begin{aligned}\phi_n(x) &:= \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)}(x) + 2^n 1_{f^{-1}((2^n,\infty])}(x) \\ &= \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{\left\{\frac{k}{2^n} < f \le \frac{k+1}{2^n}\right\}}(x) + 2^n 1_{\{f > 2^n\}}(x)\end{aligned}$$

*then $\phi_n \le f$ for all $n$, $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on the sets $X_M := \{x \in X : f(x) \le M\}$ with $M < \infty$. Moreover, if $f : X \to \mathbb{C}$ is a measurable function, then there exists simple functions $\phi_n$ such that $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all $x$ and $|\phi_n| \uparrow |f|$ as $n \to \infty$.*

*Proof.* Since

$$\left(\frac{k}{2^n},\frac{k+1}{2^n}\right] = \left(\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}}\right] \cup \left(\frac{2k+1}{2^{n+1}},\frac{2k+2}{2^{n+1}}\right],$$

if $x \in f^{-1}\left(\left(\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}}\right]\right)$ then $\phi_n(x) = \phi_{n+1}(x) = \frac{2k}{2^{n+1}}$ and if $x \in f^{-1}\left(\left(\frac{2k+1}{2^{n+1}},\frac{2k+2}{2^{n+1}}\right]\right)$ then $\phi_n(x) = \frac{2k}{2^{n+1}} < \frac{2k+1}{2^{n+1}} = \phi_{n+1}(x)$. Similarly

$$(2^n,\infty] = (2^n,2^{n+1}] \cup (2^{n+1},\infty],$$

and so for $x \in f^{-1}((2^{n+1},\infty])$, $\phi_n(x) = 2^n < 2^{n+1} = \phi_{n+1}(x)$ and for $x \in f^{-1}((2^n,2^{n+1}])$, $\phi_{n+1}(x) \geq 2^n = \phi_n(x)$. Therefore $\phi_n \leq \phi_{n+1}$ for all $n$. It is clear by construction that $\phi_n(x) \leq f(x)$ for all $x$ and that $0 \leq f(x)-\phi_n(x) \leq 2^{-n}$ if $x \in X_{2^n}$. Hence we have shown that $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on bounded sets.

For the second assertion, first assume that $f : X \to \mathbb{R}$ is a measurable function and choose $\phi_n^{\pm}$ to be simple functions such that $\phi_n^{\pm} \uparrow f_{\pm}$ as $n \to \infty$ and define $\phi_n = \phi_n^+ - \phi_n^-$. Then

$$|\phi_n| = \phi_n^+ + \phi_n^- \leq \phi_{n+1}^+ + \phi_{n+1}^- = |\phi_{n+1}|$$

and clearly $|\phi_n| = \phi_n^+ + \phi_n^- \uparrow f_+ + f_- = |f|$ and $\phi_n = \phi_n^+ - \phi_n^- \to f_+ - f_- = f$ as $n \to \infty$.

Now suppose that $f : X \to \mathbb{C}$ is measurable. We may now choose simple function $u_n$ and $v_n$ such that $|u_n| \uparrow |\operatorname{Re} f|$, $|v_n| \uparrow |\operatorname{Im} f|$, $u_n \to \operatorname{Re} f$ and $v_n \to \operatorname{Im} f$ as $n \to \infty$. Let $\phi_n = u_n + iv_n$, then

$$|\phi_n|^2 = u_n^2 + v_n^2 \uparrow |\operatorname{Re} f|^2 + |\operatorname{Im} f|^2 = |f|^2$$

and $\phi_n = u_n + iv_n \to \operatorname{Re} f + i \operatorname{Im} f = f$ as $n \to \infty$. ■

For the rest of this section let $X$ be a given set.

**Definition 18.43 (Bounded Convergence).** *We say that a sequence of functions $f_n$ from $X$ to $\mathbb{R}$ or $\mathbb{C}$ **converges boundedly** to a function $f$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$ and*

$$\sup\{|f_n(x)| : x \in X \text{ and } n = 1,2,\dots\} < \infty.$$

**Definition 18.44.** *A **function algebra** $\mathcal{H}$ on $X$ is a linear subspace of $\ell^\infty(X,\mathbb{R})$ which contains 1 and is closed under pointwise multiplication, i.e. $\mathcal{H}$ is a subalgebra of $\ell^\infty(X,\mathbb{R})$ which contains 1. If $\mathcal{H}$ is further closed under bounded convergence then $\mathcal{H}$ is said to be a $\sigma$ – **function algebra.***

*Example 18.45.* Suppose $\mathcal{M}$ is a $\sigma$ – algebra on $X$, then

$$\ell^\infty(\mathcal{M},\mathbb{R}) := \{f \in \ell^\infty(X,\mathbb{R}) : \ f \text{ is } \mathcal{M}/\mathcal{B}_{\mathbb{R}} \text{ – measurable}\} \tag{18.9}$$

is a $\sigma$ – function algebra. The next theorem will show that these are the only example of $\sigma$ – function algebras. (See Exercise 18.7 below for examples of function algebras on $X$.)



**Notation 18.46** *If $\mathcal{H} \subset \ell^\infty(X,\mathbb{R})$ be a function algebra, let*

$$\mathcal{M}(\mathcal{H}) := \{A \subset X : 1_A \in \mathcal{H}\}. \tag{18.10}$$

**Theorem 18.47.** *Let $\mathcal{H}$ be a $\sigma$ – function algebra on a set $X$. Then*

1. *$\mathcal{M}(\mathcal{H})$ is a $\sigma$ – algebra on $X$.*
2. *$\mathcal{H} = \ell^\infty(\mathcal{M}(\mathcal{H}),\mathbb{R})$.*
3. *The map*

$$\mathcal{M} \in \{\sigma\text{ – algebras on } X\} \to \ell^\infty(\mathcal{M},\mathbb{R}) \in \{\sigma\text{ – function algebras on } X\} \tag{18.11}$$

*is bijective with inverse given by $\mathcal{H} \to \mathcal{M}(\mathcal{H})$.*

*Proof.* Let $\mathcal{M} := \mathcal{M}(\mathcal{H})$.

1. Since $0,1 \in \mathcal{H}$, $\emptyset, X \in \mathcal{M}$. If $A \in \mathcal{M}$ then, since $\mathcal{H}$ is a linear subspace of $\ell^\infty(X,\mathbb{R})$, $1_{A^c} = 1 - 1_A \in \mathcal{H}$ which shows $A^c \in \mathcal{M}$. If $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$, then since $\mathcal{H}$ is an algebra,
$$1_{\cap_{n=1}^N A_n} = \prod_{n=1}^N 1_{A_n} =: f_N \in \mathcal{H}$$
for all $N \in \mathbb{N}$. Because $\mathcal{H}$ is closed under bounded convergence it follows that
$$1_{\cap_{n=1}^\infty A_n} = \lim_{N\to\infty} f_N \in \mathcal{H}$$
and this implies $\cap_{n=1}^\infty A_n \in \mathcal{M}$. Hence we have shown $\mathcal{M}$ is a $\sigma$ – algebra.
2. Since $\mathcal{H}$ is an algebra, $p(f) \in \mathcal{H}$ for any $f \in \mathcal{H}$ and any polynomial $p$ on $\mathbb{R}$. The Weierstrass approximation Theorem 8.31, asserts that polynomials on $\mathbb{R}$ are uniformly dense in the space of continuos functions on any compact subinterval of $\mathbb{R}$. Hence if $f \in \mathcal{H}$ and $\phi \in C(\mathbb{R})$, there exists polynomials $p_n$ on $\mathbb{R}$ such that $p_n \circ f(x)$ converges to $\phi \circ f(x)$ uniformly (and hence boundedly) in $x \in X$ as $n \to \infty$. Therefore $\phi \circ f \in \mathcal{H}$ for all $f \in \mathcal{H}$ and $\phi \in C(\mathbb{R})$ and in particular $|f| \in \mathcal{H}$ and $f_\pm := \frac{|f| \pm f}{2} \in \mathcal{H}$ if $f \in \mathcal{H}$.
Fix an $\alpha \in \mathbb{R}$ and for $n \in \mathbb{N}$ let $\phi_n(t) := (t-\alpha)_+^{1/n}$, where $(t-\alpha)_+ := \max\{t-\alpha, 0\}$. Then $\phi_n \in C(\mathbb{R})$ and $\phi_n(t) \to 1_{t>\alpha}$ as $n \to \infty$ and the convergence is bounded when $t$ is restricted to any compact subset of $\mathbb{R}$. Hence if $f \in \mathcal{H}$ it follows that $1_{f>\alpha} = \lim_{n\to\infty} \phi_n(f) \in \mathcal{H}$ for all $\alpha \in \mathbb{R}$, i.e. $\{f > \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$. Therefore if $f \in \mathcal{H}$ then $f \in \ell^\infty(\mathcal{M},\mathbb{R})$ and we have shown $\mathcal{H} \subset \ell^\infty(\mathcal{M},\mathbb{R})$. Conversely if $f \in \ell^\infty(\mathcal{M},\mathbb{R})$, then for any $\alpha < \beta$, $\{\alpha < f \leq \beta\} \in \mathcal{M} = \mathcal{M}(\mathcal{H})$ and so by assumption $1_{\{\alpha < f \leq \beta\}} \in \mathcal{H}$. Combining this remark with the approximation Theorem 18.42 and the fact that $\mathcal{H}$ is closed under bounded convergence shows that $f \in \mathcal{H}$. Hence we have shown $\ell^\infty(\mathcal{M},\mathbb{R}) \subset \mathcal{H}$ which combined with $\mathcal{H} \subset \ell^\infty(\mathcal{M},\mathbb{R})$ already proved shows $\mathcal{H} = \ell^\infty(\mathcal{M}(\mathcal{H}),\mathbb{R})$.

3. Items 1. and 2. shows the map in Eq. (18.11) is surjective. To see the map is injective suppose $\mathcal{M}$ and $\mathcal{F}$ are two $\sigma$ – algebras on $X$ such that $\ell^\infty(\mathcal{M},\mathbb{R}) = \ell^\infty(\mathcal{F},\mathbb{R})$, then

$$\begin{aligned}\mathcal{M} &= \{A \subset X : 1_A \in \ell^\infty(\mathcal{M},\mathbb{R})\} \\ &= \{A \subset X : 1_A \in \ell^\infty(\mathcal{F},\mathbb{R})\} = \mathcal{F}.\end{aligned}$$

■

**Notation 18.48** *Suppose $M$ is a subset of $\ell^\infty(X,\mathbb{R})$.*

1. *Let $\mathcal{H}(M)$ denote the smallest subspace of $\ell^\infty(X,\mathbb{R})$ which contains $M$ and the constant functions and is closed under bounded convergence.*
2. *Let $\mathcal{H}_\sigma(M)$ denote the smallest $\sigma$ – function algebra containing $M$.*

**Theorem 18.49.** *Suppose $M$ is a subset of $\ell^\infty(X,\mathbb{R})$, then $\mathcal{H}_\sigma(M) = \ell^\infty(\sigma(M),\mathbb{R})$ or in other words the following diagram commutes:*

$$\begin{array}{ccccc} & M & \longrightarrow & \sigma(M) & \\ M & \{\text{Multiplicative Subsets}\} & \longrightarrow & \{\sigma - \text{algebras}\} & \mathcal{M} \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \mathcal{H}_\sigma(M) & \{\sigma\text{– function algebras}\} & = & \{\sigma\text{– function algebras}\} & \ell^\infty(\mathcal{M},\mathbb{R}). \end{array}$$

*Proof.* Since $\ell^\infty(\sigma(M),\mathbb{R})$ is $\sigma$ – function algebra which contains $M$ it follows that

$$\mathcal{H}_\sigma(M) \subset \ell^\infty(\sigma(M),\mathbb{R}).$$

For the opposite inclusion, let

$$\mathcal{M} = \mathcal{M}(\mathcal{H}_\sigma(M)) := \{A \subset X : 1_A \in \mathcal{H}_\sigma(M)\}.$$

By Theorem 18.47, $M \subset \mathcal{H}_\sigma(M) = \ell^\infty(\mathcal{M},\mathbb{R})$ which implies that every $f \in M$ is $\mathcal{M}$ – measurable. This then implies $\sigma(M) \subset \mathcal{M}$ and therefore

$$\ell^\infty(\sigma(M),\mathbb{R}) \subset \ell^\infty(\mathcal{M},\mathbb{R}) = \mathcal{H}_\sigma(M).$$

■

**Definition 18.50 (Multiplicative System).** *A collection of bounded real or complex valued functions, $M$, on a set $X$ is called a **multiplicative system** if $f \cdot g \in M$ whenever $f$ and $g$ are in $M$.*

**Theorem 18.51 (Dynkin's Multiplicative System Theorem).** *Suppose $M \subset \ell^\infty(X,\mathbb{R})$ is a multiplicative system, then*

$$\mathcal{H}(M) = \mathcal{H}_\sigma(M) = \ell^\infty(\sigma(M),\mathbb{R}). \tag{18.12}$$

*In words, the smallest subspace of bounded real valued functions on $X$ which contains $M$ that is closed under bounded convergence is the same as the space of bounded real valued $\sigma(M)$ – measurable functions on $X$.*



*Proof.* We begin by proving $\mathcal{H} := \mathcal{H}(M)$ is a $\sigma$ – function algebra. To do this, for any $f \in \mathcal{H}$ let

$$\mathcal{H}_f := \{g \in \mathcal{H} : fg \in \mathcal{H}\} \subset \mathcal{H}$$

and notice that $\mathcal{H}_f$ is a linear subspace of $\ell^\infty(X,\mathbb{R})$ which is closed under bounded convergence. Moreover if $f \in M$, $M \subset \mathcal{H}_f$ since $M$ is multiplicative. Therefore $\mathcal{H}_f = \mathcal{H}$ and we have shown that $fg \in \mathcal{H}$ whenever $f \in M$ and $g \in \mathcal{H}$. Given this it now follows that $M \subset \mathcal{H}_f$ for any $f \in \mathcal{H}$ and by the same reasoning just used, $\mathcal{H}_f = \mathcal{H}$. Since $f \in \mathcal{H}$ is arbitrary, we have shown $fg \in \mathcal{H}$ for all $f, g \in \mathcal{H}$, i.e. $\mathcal{H}$ is an algebra.

Since it is harder to be an algebra of functions containing $M$ (see Exercise 18.13) than it is to be a subspace of functions containing $M$ it follows that $\mathcal{H}(M) \subset \mathcal{H}_\sigma(M)$. But as we have just seen $\mathcal{H}(M)$ is a $\sigma$ – function algebra which contains $M$ so we must have $\mathcal{H}_\sigma(M) \subset \mathcal{H}(M)$ because $\mathcal{H}_\sigma(M)$ is by definition the smallest such $\sigma$ – function algebra. Hence $\mathcal{H}_\sigma(M) = \mathcal{H}(M)$. The assertion that $\mathcal{H}_\sigma(M) = \ell^\infty(\sigma(M),\mathbb{R})$ has already been proved in Theorem 18.49. ■

**Theorem 18.52 (Complex Multiplicative System Theorem).** *Suppose $\mathcal{H}$ is a complex linear subspace of $\ell^\infty(X,\mathbb{C})$ such that: $1 \in \mathcal{H}$, $\mathcal{H}$ is closed under complex conjugation, and $\mathcal{H}$ is closed under bounded convergence. If $M \subset \mathcal{H}$ is multiplicative system which is closed under conjugation, then $\mathcal{H}$ contains all bounded complex valued $\sigma(M)$-measurable functions, i.e. $\ell^\infty(\sigma(M),\mathbb{C}) \subset \mathcal{H}$.*

*Proof.* Let $M_0 = \operatorname{span}_{\mathbb{C}}(M \cup \{1\})$ be the complex span of $M$. As the reader should verify, $M_0$ is an algebra, $M_0 \subset \mathcal{H}$, $M_0$ is closed under complex conjugation and that $\sigma(M_0) = \sigma(M)$. Let $\mathcal{H}^{\mathbb{R}} := \mathcal{H} \cap \ell^\infty(X,\mathbb{R})$ and $M_0^{\mathbb{R}} = M \cap \ell^\infty(X,\mathbb{R})$. Then (you verify) $M_0^{\mathbb{R}}$ is a multiplicative system, $M_0^{\mathbb{R}} \subset \mathcal{H}^{\mathbb{R}}$ and $\mathcal{H}^{\mathbb{R}}$ is a linear space containing 1 which is closed under bounded convergence. Therefore by Theorem 18.51, $\ell^\infty(\sigma(M_0^{\mathbb{R}}),\mathbb{R}) \subset \mathcal{H}^{\mathbb{R}}$.

Since $\mathcal{H}$ and $M_0$ are complex linear spaces closed under complex conjugation, for any $f \in \mathcal{H}$ or $f \in M_0$, the functions $\operatorname{Re} f = \frac{1}{2}(f + \bar{f})$ and $\operatorname{Im} f = \frac{1}{2i}(f - \bar{f})$ are in $\mathcal{H}$ $(M_0)$ or $M_0$ respectively. Therefore $\mathcal{H} = \mathcal{H}^{\mathbb{R}} + i\mathcal{H}^{\mathbb{R}}$, $M_0 = M_0^{\mathbb{R}} + iM_0^{\mathbb{R}}$, $\sigma(M_0^{\mathbb{R}}) = \sigma(M_0) = \sigma(M)$ and

$$\begin{aligned}\ell^\infty(\sigma(M),\mathbb{C}) &= \ell^\infty(\sigma(M_0^{\mathbb{R}}),\mathbb{R}) + i\ell^\infty(\sigma(M_0^{\mathbb{R}}),\mathbb{R}) \\ &\subset \mathcal{H}^{\mathbb{R}} + i\mathcal{H}^{\mathbb{R}} = \mathcal{H}.\end{aligned}$$

■

**Exercise 18.7 (Algebra analogue of Theorem 18.47).** Call a function algebra $\mathcal{H} \subset \ell^\infty(X,\mathbb{R})$ a **simple function algebra** if the range of each function $f \in \mathcal{H}$ is a finite subset of $\mathbb{R}$. Prove there is a one to one correspondence between algebras $\mathcal{A}$ on a set $X$ and simple function algebras $\mathcal{H}$ on $X$.

**Definition 18.53.** *A collection of subsets, $\mathcal{C}$, of $X$ is a **multiplicative class**(or a $\pi$ – **class**) if $\mathcal{C}$ is closed under finite intersections.*

**Corollary 18.54.** *Suppose $\mathcal{H}$ is a subspace of $\ell^\infty(X,\mathbb{R})$ which is closed under bounded convergence and $1 \in \mathcal{H}$. If $\mathcal{C} \subset 2^X$ is a multiplicative class such that $1_A \in \mathcal{H}$ for all $A \in \mathcal{C}$, then $\mathcal{H}$ contains all bounded $\sigma(\mathcal{C})$ – measurable functions.*

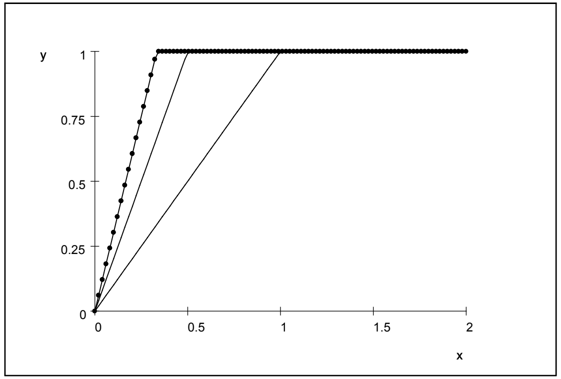
*Proof.* Let $M = \{1\} \cup \{1_A : A \in \mathcal{C}\}$. Then $M \subset \mathcal{H}$ is a multiplicative system and the proof is completed with an application of Theorem 18.51. ■

**Corollary 18.55.** *Suppose that $(X,d)$ is a metric space and $\mathcal{B}_X = \sigma(\tau_d)$ is the Borel $\sigma$ – algebra on $X$ and $\mathcal{H}$ is a subspace of $\ell^\infty(X,\mathbb{R})$ such that $BC(X,\mathbb{R}) \subset \mathcal{H}$ and $\mathcal{H}$ is closed under bounded convergence*[1]*. Then $\mathcal{H}$ contains all bounded $\mathcal{B}_X$ – measurable real valued functions on $X$. (This may be stated as follows: the smallest vector space of bounded functions which is closed under bounded convergence and contains $BC(X,\mathbb{R})$ is the space of bounded $\mathcal{B}_X$ – measurable real valued functions on $X$.)*

*Proof.* Let $V \in \tau_d$ be an open subset of $X$ and for $n \in \mathbb{N}$ let

$$f_n(x) := \min(n \cdot d_{V^c}(x), 1) \text{ for all } x \in X.$$

Notice that $f_n = \phi_n \circ d_{V^c}$ where $\phi_n(t) = \min(nt, 1)$ (see Figure 18.3) which is continuous and hence $f_n \in BC(X,\mathbb{R})$ for all $n$. Furthermore, $f_n$ converges boundedly to $1_{d_{V^c}>0} = 1_V$ as $n \to \infty$ and therefore $1_V \in \mathcal{H}$ for all $V \in \tau$. Since $\tau$ is a $\pi$ – class, the result now follows by an application of Corollary 18.54.




Plots of $\phi_1$, $\phi_2$ and $\phi_3$.

■

Here are some more variants of Corollary 18.55.

**Proposition 18.56.** *Let $(X,d)$ be a metric space, $\mathcal{B}_X = \sigma(\tau_d)$ be the Borel $\sigma$ – algebra and assume there exists compact sets $K_k \subset X$ such that $K_k^o \uparrow X$.*

[1] Recall that $BC(X,\mathbb{R})$ are the bounded continuous functions on $X$.



*Suppose that $\mathcal{H}$ is a subspace of $\ell^\infty(X,\mathbb{R})$ such that $C_c(X,\mathbb{R}) \subset \mathcal{H}$ ($C_c(X,\mathbb{R})$ is the space of continuous functions with compact support) and $\mathcal{H}$ is closed under bounded convergence. Then $\mathcal{H}$ contains all bounded $\mathcal{B}_X$ – measurable real valued functions on $X$.*

*Proof.* Let $k$ and $n$ be positive integers and set $\psi_{n,k}(x) = \min(1, n \cdot d_{(K_k^o)^c}(x))$. Then $\psi_{n,k} \in C_c(X,\mathbb{R})$ and $\{\psi_{n,k} \neq 0\} \subset K_k^o$. Let $\mathcal{H}_{n,k}$ denote those bounded $\mathcal{B}_X$ – measurable functions, $f : X \to \mathbb{R}$, such that $\psi_{n,k} f \in \mathcal{H}$. It is easily seen that $\mathcal{H}_{n,k}$ is closed under bounded convergence and that $\mathcal{H}_{n,k}$ contains $BC(X,\mathbb{R})$ and therefore by Corollary 18.55, $\psi_{n,k} f \in \mathcal{H}$ for all bounded measurable functions $f : X \to \mathbb{R}$. Since $\psi_{n,k} f \to 1_{K_k^o} f$ boundedly as $n \to \infty$, $1_{K_k^o} f \in \mathcal{H}$ for all $k$ and similarly $1_{K_k^o} f \to f$ boundedly as $k \to \infty$ and therefore $f \in \mathcal{H}$. ■

**Lemma 18.57.** *Suppose that $(X,\tau)$ is a locally compact second countable Hausdorff space.*[2] *Then:*

1. *every open subset $U \subset X$ is $\sigma$ – compact.*
2. *If $F \subset X$ is a closed set, there exist open sets $V_n \subset X$ such that $V_n \downarrow F$ as $n \to \infty$.*
3. *To each open set $U \subset X$ there exists $f_n \prec U$ (i.e. $f_n \in C_c(U,[0,1])$) such that $\lim_{n\to\infty} f_n = 1_U$.*
4. *$\mathcal{B}_X = \sigma(C_c(X,\mathbb{R}))$, i.e. the $\sigma$ – algebra generated by $C_c(X)$ is the Borel $\sigma$ – algebra on $X$.*

*Proof.*
1. Let $U$ be an open subset of $X$, $\mathcal{V}$ be a countable base for $\tau$ and
$$\mathcal{V}^U := \{W \in \mathcal{V} : \bar{W} \subset U \text{ and } \bar{W} \text{ is compact}\}.$$
For each $x \in U$, by Proposition 12.7, there exists an open neighborhood $V$ of $x$ such that $\bar{V} \subset U$ and $\bar{V}$ is compact. Since $\mathcal{V}$ is a base for the topology $\tau$, there exists $W \in \mathcal{V}$ such that $x \in W \subset V$. Because $\bar{W} \subset \bar{V}$, it follows that $\bar{W}$ is compact and hence $W \in \mathcal{V}^U$. As $x \in U$ was arbitrary, $U = \cup \mathcal{V}^U$.
Let $\{W_n\}_{n=1}^\infty$ be an enumeration of $\mathcal{V}^U$ and set $K_n := \cup_{k=1}^n \bar{W}_k$. Then $K_n \uparrow U$ as $n \to \infty$ and $K_n$ is compact for each $n$.
2. Let $\{K_n\}_{n=1}^\infty$ be compact subsets of $F^c$ such that $K_n \uparrow F^c$ as $n \to \infty$ and set $V_n := K_n^c = X \setminus K_n$. Then $V_n \downarrow F$ and by Proposition 12.5, $V_n$ is open for each $n$.
3. Let $U \subset X$ be an open set and $\{K_n\}_{n=1}^\infty$ be compact subsets of $U$ such that $K_n \uparrow U$. By Lemma 12.8, there exist $f_n \prec U$ such that $f_n = 1$ on $K_n$. These functions satisfy, $1_U = \lim_{n\to\infty} f_n$.

[2] For example any separable locally compact metric space and in particular any open subset of $\mathbb{R}^n$.

4. By item 3., $1_U$ is $\sigma(C_c(X,\mathbb{R}))$ – measurable for all $U \in \tau$ and hence $\tau \subset \sigma(C_c(X,\mathbb{R}))$. Therefore $\mathcal{B}_X = \sigma(\tau) \subset \sigma(C_c(X,\mathbb{R}))$. The converse inclusion holds because continuous functions are always Borel measurable. ■

Here is a variant of Corollary 18.55.

**Corollary 18.58.** *Suppose that $(X,\tau)$ is a second countable locally compact Hausdorff space and $\mathcal{B}_X = \sigma(\tau)$ is the Borel $\sigma$ – algebra on $X$. If $\mathcal{H}$ is a subspace of $\ell^\infty(X,\mathbb{R})$ which is closed under bounded convergence and contains $C_c(X,\mathbb{R})$, then $\mathcal{H}$ contains all bounded $\mathcal{B}_X$ – measurable real valued functions on $X$.*

*Proof.* By Item 3. of Lemma 18.57, for every $U \in \tau$ the characteristic function, $1_U$, may be written as a bounded pointwise limit of functions from $C_c(X,\mathbb{R})$. Therefore $1_U \in \mathcal{H}$ for all $U \in \tau$. Since $\tau$ is a $\pi$ – class, the proof is finished with an application of Corollary 18.54 ■

## 18.4 Product $\sigma$ – Algebras

Let $\{(X_\alpha,\mathcal{M}_\alpha)\}_{\alpha\in A}$ be a collection of measurable spaces $X = X_A = \prod_{\alpha\in A} X_\alpha$ and $\pi_\alpha : X_A \to X_\alpha$ be the canonical projection map as in Notation 2.2.

**Definition 18.59 (Product $\sigma$ – Algebra).** *The **product $\sigma$ – algebra,** $\otimes_{\alpha\in A}\mathcal{M}_\alpha$, is the smallest $\sigma$ – algebra on $X$ such that each $\pi_\alpha$ for $\alpha \in A$ is measurable, i.e.*

$$\otimes_{\alpha\in A}\mathcal{M}_\alpha := \sigma(\pi_\alpha : \alpha \in A) = \sigma\left(\cup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

Applying Proposition 18.25 in this setting implies the following proposition.

**Proposition 18.60.** *Suppose $Y$ is a measurable space and $f : Y \to X = X_A$ is a map. Then $f$ is measurable iff $\pi_\alpha \circ f : Y \to X_\alpha$ is measurable for all $\alpha \in A$. In particular if $A = \{1,2,\dots,n\}$ so that $X = X_1 \times X_2 \times \cdots \times X_n$ and $f(y) = (f_1(y), f_2(y), \dots, f_n(y)) \in X_1 \times X_2 \times \cdots \times X_n$, then $f : Y \to X_A$ is measurable iff $f_i : Y \to X_i$ is measurable for all $i$.*

**Proposition 18.61.** *Suppose that $(X_\alpha,\mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of measurable spaces and $\mathcal{E}_\alpha \subset \mathcal{M}_\alpha$ generates $\mathcal{M}_\alpha$ for each $\alpha \in A$, then*

$$\otimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \tag{18.13}$$

*Moreover, suppose that $A$ is either finite or countably infinite, $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, and $\mathcal{M}_\alpha = \sigma(\mathcal{E}_\alpha)$ for each $\alpha \in A$. Then the product $\sigma$ – algebra satisfies*



$$\otimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right). \tag{18.14}$$

*In particular if $A = \{1,2,\dots,n\}$, then $X = X_1 \times X_2 \times \cdots \times X_n$ and*

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n = \sigma(\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n),$$

*where $\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$ is as defined in Notation 10.26.*

*Proof.* Since $\cup_\alpha\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \cup_\alpha\pi_\alpha^{-1}(\mathcal{M}_\alpha)$, it follows that

$$\mathcal{F} := \sigma\left(\cup_\alpha\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\cup_\alpha\pi_\alpha^{-1}(\mathcal{M}_\alpha)\right) = \otimes_{\alpha\in A}\mathcal{M}_\alpha.$$

Conversely,

$$\mathcal{F} \supset \sigma(\pi_\alpha^{-1}(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\sigma(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\mathcal{M}_\alpha)$$

holds for all $\alpha$ implies that

$$\cup_\alpha\pi_\alpha^{-1}(\mathcal{M}_\alpha) \subset \mathcal{F}$$

and hence that $\otimes_{\alpha\in A}\mathcal{M}_\alpha \subset \mathcal{F}$.

We now prove Eq. (18.14). Since we are assuming that $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, we see that

$$\cup_\alpha\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}$$

and therefore by Eq. (18.13)

$$\otimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\cup_\alpha\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right).$$

This last statement is true independent as to whether $A$ is countable or not. For the reverse inclusion it suffices to notice that since $A$ is countable,

$$\prod_{\alpha\in A} E_\alpha = \cap_{\alpha\in A}\pi_\alpha^{-1}(E_\alpha) \in \otimes_{\alpha\in A}\mathcal{M}_\alpha$$

and hence

$$\sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right) \subset \otimes_{\alpha\in A}\mathcal{M}_\alpha.$$

■

*Remark 18.62.* One can not relax the assumption that $X_\alpha \in \mathcal{E}_\alpha$ in the second part of Proposition 18.61. For example, if $X_1 = X_2 = \{1,2\}$ and $\mathcal{E}_1 = \mathcal{E}_2 = \{\{1\}\}$, then $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \{\emptyset, X_1 \times X_2, \{(1,1)\}\}$ while $\sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = 2^{X_1\times X_2}$.

**Theorem 18.63.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a sequence of sets where $A$ is at most countable. Suppose for each $\alpha \in A$ we are given a countable set $\mathcal{E}_\alpha \subset 2^{X_\alpha}$. Let $\tau_\alpha = \tau(\mathcal{E}_\alpha)$ be the topology on $X_\alpha$ generated by $\mathcal{E}_\alpha$ and $X$ be the product space $\prod_{\alpha\in A} X_\alpha$ with equipped with the product topology $\tau := \otimes_{\alpha\in A}\tau(\mathcal{E}_\alpha)$. Then the Borel $\sigma$ – algebra $\mathcal{B}_X = \sigma(\tau)$ is the same as the product $\sigma$ – algebra:*

$$\mathcal{B}_X = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha},$$

*where $\mathcal{B}_{X_\alpha} = \sigma(\tau(\mathcal{E}_\alpha)) = \sigma(\mathcal{E}_\alpha)$ for all $\alpha \in A$.*

*In particular if $A = \{1, 2, \ldots, n\}$ and each $(X_i, \tau_i)$ is a second countable topological space, then*

$$\mathcal{B}_X := \sigma(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n) = \sigma(\mathcal{B}_{X_1} \times \cdots \times \mathcal{B}_{X_n}) =: \mathcal{B}_{X_1} \otimes \cdots \otimes \mathcal{B}_{X_n}.$$

*Proof.* By Proposition 10.25, the topology $\tau$ may be described as the smallest topology containing $\mathcal{E} = \cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)$. Now $\mathcal{E}$ is the countable union of countable sets so is still countable. Therefore by Proposition 18.17 and Proposition 18.61,

$$\begin{aligned}\mathcal{B}_X &= \sigma(\tau) = \sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E}) = \otimes_{\alpha\in A}\sigma(\mathcal{E}_\alpha)\\ &= \otimes_{\alpha\in A}\sigma(\tau_\alpha) = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha}.\end{aligned}$$

■

**Corollary 18.64.** *If $(X_i, d_i)$ are separable metric spaces for $i = 1, \ldots, n$, then*

$$\mathcal{B}_{X_1} \otimes \cdots \otimes \mathcal{B}_{X_n} = \mathcal{B}_{(X_1\times\cdots\times X_n)}$$

*where $\mathcal{B}_{X_i}$ is the Borel $\sigma$ – algebra on $X_i$ and $\mathcal{B}_{(X_1\times\cdots\times X_n)}$ is the Borel $\sigma$ – algebra on $X_1 \times \cdots \times X_n$ equipped with the metric topology associated to the metric $d(x, y) = \sum_{i=1}^n d_i(x_i, y_i)$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.*

*Proof.* This is a combination of the results in Lemma 10.28, Exercise 10.9 and Theorem 18.63. ■

Because all norms on finite dimensional spaces are equivalent, the usual Euclidean norm on $\mathbb{R}^m \times \mathbb{R}^n$ is equivalent to the "product" norm defined by

$$\|(x, y)\|_{\mathbb{R}^m\times\mathbb{R}^n} = \|x\|_{\mathbb{R}^m} + \|y\|_{\mathbb{R}^n}.$$

Hence by Lemma 10.28, the Euclidean topology on $\mathbb{R}^{m+n}$ is the same as the product topology on $\mathbb{R}^{m+n} \cong \mathbb{R}^m \times \mathbb{R}^n$. Here we are identifying $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ by the map

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \to (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^{m+n}.$$

These comments along with Corollary 18.64 proves the following result.

**Corollary 18.65.** *After identifying $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ as above and letting $\mathcal{B}_{\mathbb{R}^n}$ denote the Borel $\sigma$ –algebra on $\mathbb{R}^n$, we have*

$$\mathcal{B}_{\mathbb{R}^{m+n}} = \mathcal{B}_{\mathbb{R}^n} \otimes \mathcal{B}_{\mathbb{R}^m} \text{ and } \mathcal{B}_{\mathbb{R}^n} = \overbrace{\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}}^{n\text{–times}}.$$



### 18.4.1 Factoring of Measurable Maps

**Lemma 18.66.** *Suppose that $(Y, \mathcal{F})$ is a measurable space and $F : X \to Y$ is a map. Then to every $(\sigma(F), \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function, $H : X \to \bar{\mathbb{R}}$, there is a $(\mathcal{F}, \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function $h : Y \to \bar{\mathbb{R}}$ such that $H = h \circ F$.*

*Proof.* First suppose that $H = 1_A$ where $A \in \sigma(F) = F^{-1}(\mathcal{F})$. Let $B \in \mathcal{F}$ such that $A = F^{-1}(B)$ then $1_A = 1_{F^{-1}(B)} = 1_B \circ F$ and hence the Lemma is valid in this case with $h = 1_B$. More generally if $H = \sum a_i 1_{A_i}$ is a simple function, then there exists $B_i \in \mathcal{F}$ such that $1_{A_i} = 1_{B_i} \circ F$ and hence $H = h \circ F$ with $h := \sum a_i 1_{B_i}$ – a simple function on $\bar{\mathbb{R}}$.

For general $(\sigma(F), \mathcal{F})$ – measurable function, $H$, from $X \to \bar{\mathbb{R}}$, choose simple functions $H_n$ converging to $H$. Let $h_n$ be simple functions on $\bar{\mathbb{R}}$ such that $H_n = h_n \circ F$. Then it follows that

$$H = \lim_{n\to\infty} H_n = \limsup_{n\to\infty} H_n = \limsup_{n\to\infty} h_n \circ F = h \circ F$$

where $h := \limsup_{n\to\infty} h_n$ – a measurable function from $Y$ to $\bar{\mathbb{R}}$. ■

The following is an immediate corollary of Proposition 18.25 and Lemma 18.66.

**Corollary 18.67.** *Let $X$ and $A$ be sets, and suppose for $\alpha \in A$ we are give a measurable space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. Let $Y := \prod_{\alpha\in A} Y_\alpha$, $\mathcal{F} := \otimes_{\alpha\in A}\mathcal{F}_\alpha$ be the product $\sigma$ – algebra on $Y$ and $\mathcal{M} := \sigma(f_\alpha : \alpha \in A)$ be the smallest $\sigma$ – algebra on $X$ such that each $f_\alpha$ is measurable. Then the function $F : X \to Y$ defined by $[F(x)]_\alpha := f_\alpha(x)$ for each $\alpha \in A$ is $(\mathcal{M}, \mathcal{F})$ – measurable and a function $H : X \to \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable iff there exists a $(\mathcal{F}, \mathcal{B}_{\bar{\mathbb{R}}})$ – measurable function $h$ from $Y$ to $\bar{\mathbb{R}}$ such that $H = h \circ F$.*

## 18.5 Exercises

**Exercise 18.8.** Prove Corollary 18.23. **Hint:** See Exercise 18.3.

**Exercise 18.9.** If $\mathcal{M}$ is the $\sigma$ – algebra generated by $\mathcal{E} \subset 2^X$, then $\mathcal{M}$ is the union of the $\sigma$ – algebras generated by countable subsets $\mathcal{F} \subset \mathcal{E}$. (Folland, Problem 1.5 on p.24.)

**Exercise 18.10.** Let $(X, \mathcal{M})$ be a measure space and $f_n : X \to \mathbb{F}$ be a sequence of measurable functions on $X$. Show that $\{x : \lim_{n\to\infty} f_n(x)$ exists **in** $\mathbb{F}\} \in \mathcal{M}$.

**Exercise 18.11.** Show that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$ – measurable.

**Exercise 18.12.** Show by example that the supremum of an uncountable family of measurable functions need not be measurable. (Folland problem 2.6 on p. 48.)

**Exercise 18.13.** Let $X = \{1,2,3,4\}$, $A = \{1,2\}$, $B = \{2,3\}$ and $M := \{1_A, 1_B\}$. Show $\mathcal{H}_\sigma(M) \neq \mathcal{H}(M)$ in this case.

# 19

# Measures and Integration

**Definition 19.1.** *A* ***measure*** $\mu$ *on a measurable space* $(X, \mathcal{M})$ *is a function* $\mu : \mathcal{M} \to [0, \infty]$ *such that*

1. $\mu(\emptyset) = 0$ *and*
2. *(Finite Additivity) If* $\{A_i\}_{i=1}^n \subset \mathcal{M}$ *are pairwise disjoint, i.e.* $A_i \cap A_j = \emptyset$ *when* $i \neq j$*, then*
$$\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$
3. *(Continuity) If* $A_n \in \mathcal{M}$ *and* $A_n \uparrow A$*, then* $\mu(A_n) \uparrow \mu(A)$*.*

*We call a triple* $(X, \mathcal{M}, \mu)$*, where* $(X, \mathcal{M})$ *is a measurable space and* $\mu : \mathcal{M} \to [0, \infty]$ *is a measure, a* ***measure space.***

*Remark 19.2.* Properties 2) and 3) in Definition 19.1 are equivalent to the following condition. If $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ are pairwise disjoint then

$$\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i). \tag{19.1}$$

To prove this assume that Properties 2) and 3) in Definition 19.1 hold and $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ are pairwise disjoint. Letting $B_n := \bigcup_{i=1}^n A_i \uparrow B := \bigcup_{i=1}^\infty A_i$, we have

$$\mu(\bigcup_{i=1}^\infty A_i) = \mu(B) \overset{(3)}{=} \lim_{n\to\infty} \mu(B_n) \overset{(2)}{=} \lim_{n\to\infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i).$$

Conversely, if Eq. (19.1) holds we may take $A_j = \emptyset$ for all $j > n$ to see that Property 2) of Definition 19.1 holds. Also if $A_n \uparrow A$, let $B_n := A_n \setminus A_{n-1}$ with $A_0 := \emptyset$. Then $\{B_n\}_{n=1}^\infty$ are pairwise disjoint, $A_n = \cup_{j=1}^n B_j$ and $A = \cup_{j=1}^\infty B_j$. So if Eq. (19.1) holds we have

$$\mu(A) = \mu\left(\cup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \mu(B_j)$$
$$= \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) = \lim_{n\to\infty} \mu(\cup_{j=1}^{n} B_j) = \lim_{n\to\infty} \mu(A_n).$$

**Proposition 19.3 (Basic properties of measures).** *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$ and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, then :*

1. *$\mu(E) \leq \mu(F)$ if $E \subset F$.*
2. *$\mu(\cup E_j) \leq \sum \mu(E_j)$.*
3. *If $\mu(E_1) < \infty$ and $E_j \downarrow E$, i.e. $E_1 \supset E_2 \supset E_3 \supset \ldots$ and $E = \cap_j E_j$, then $\mu(E_j) \downarrow \mu(E)$ as $j \to \infty$.*

*Proof.* 1. Since $F = E \cup (F \setminus E)$,

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

2. Let $\widetilde{E}_j = E_j \setminus (E_1 \cup \cdots \cup E_{j-1})$ so that the $\tilde{E}_j$ 's are pair-wise disjoint and $E = \cup \tilde{E}_j$. Since $\tilde{E}_j \subset E_j$ it follows from Remark 19.2 and part (1), that
$$\mu(E) = \sum \mu(\widetilde{E}_j) \leq \sum \mu(E_j).$$
3. Define $D_i := E_1 \setminus E_i$ then $D_i \uparrow E_1 \setminus E$ which implies that
$$\mu(E_1) - \mu(E) = \lim_{i\to\infty} \mu(D_i) = \mu(E_1) - \lim_{i\to\infty} \mu(E_i)$$
which shows that $\lim_{i\to\infty} \mu(E_i) = \mu(E)$.
■

**Definition 19.4.** *A set $E \subset X$ is a **null** set if $E \in \mathcal{M}$ and $\mu(E) = 0$. If $P$ is some "property" which is either true or false for each $x \in X$, we will use the terminology $P$ a.e. (to be read $P$ almost everywhere) to mean*

$$E := \{x \in X : P \text{ is false for } x\}$$

*is a null set. For example if $f$ and $g$ are two measurable functions on $(X, \mathcal{M}, \mu)$, $f = g$ a.e. means that $\mu(f \neq g) = 0$.*

**Definition 19.5.** *A measure space $(X, \mathcal{M}, \mu)$ is **complete** if every subset of a null set is in $\mathcal{M}$, i.e. for all $F \subset X$ such that $F \subset E \in \mathcal{M}$ with $\mu(E) = 0$ implies that $F \in \mathcal{M}$.*

**Proposition 19.6.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Set*

$$\mathcal{N} := \{N \subset X : \exists\, F \in \mathcal{M} \ni N \subset F \text{ and } \mu(F) = 0\}$$

*and*

$$\bar{\mathcal{M}} = \{A \cup N : A \in \mathcal{M}, N \in \mathcal{M}\},$$

*see Fig. 19.1. Then $\bar{\mathcal{M}}$ is a $\sigma$ – algebra. Define $\bar{\mu}(A \cup N) = \mu(A)$, then $\bar{\mu}$ is the unique measure on $\bar{\mathcal{M}}$ which extends $\mu$.*



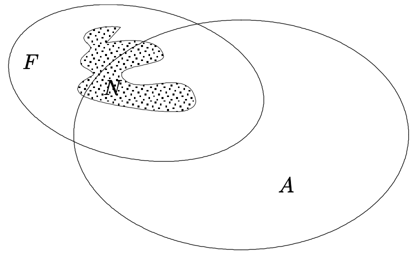



**Fig. 19.1.** Completing a $\sigma$ – algebra.

*Proof.* Clearly $X, \emptyset \in \bar{\mathcal{M}}$.

Let $A \in \mathcal{M}$ and $N \in \mathcal{N}$ and choose $F \in \mathcal{M}$ such that $N \subset F$ and $\mu(F) = 0$. Since $N^c = (F \setminus N) \cup F^c$,

$$(A \cup N)^c = A^c \cap N^c = A^c \cap (F \setminus N \cup F^c)$$
$$= [A^c \cap (F \setminus N)] \cup [A^c \cap F^c]$$

where $[A^c \cap (F \setminus N)] \in \mathcal{N}$ and $[A^c \cap F^c] \in \mathcal{M}$. Thus $\bar{\mathcal{M}}$ is closed under complements.

If $A_i \in \mathcal{M}$ and $N_i \subset F_i \in \mathcal{M}$ such that $\mu(F_i) = 0$ then $\cup(A_i \cup N_i) = (\cup A_i) \cup (\cup N_i) \in \bar{\mathcal{M}}$ since $\cup A_i \in \mathcal{M}$ and $\cup N_i \subset \cup F_i$ and $\mu(\cup F_i) \leq \sum \mu(F_i) = 0$. Therefore, $\bar{\mathcal{M}}$ is a $\sigma$ – algebra.

Suppose $A \cup N_1 = B \cup N_2$ with $A, B \in \mathcal{M}$ and $N_1, N_2, \in \mathcal{N}$. Then $A \subset A \cup N_1 \subset A \cup N_1 \cup F_2 = B \cup F_2$ which shows that

$$\mu(A) \leq \mu(B) + \mu(F_2) = \mu(B).$$

Similarly, we show that $\mu(B) \leq \mu(A)$ so that $\mu(A) = \mu(B)$ and hence $\bar{\mu}(A \cup N) := \mu(A)$ is well defined. It is left as an exercise to show $\bar{\mu}$ is a measure, i.e. that it is countable additive. ■

Many theorems in the sequel will require some control on the size of a measure $\mu$. The relevant notion for our purposes (and most purposes) is that of a $\sigma$ – finite measure defined next.

**Definition 19.7.** *Suppose $X$ is a set, $\mathcal{E} \subset \mathcal{M} \subset 2^X$ and $\mu : \mathcal{M} \to [0, \infty]$ is a function. The function $\mu$ is $\sigma$ – finite on $\mathcal{E}$ if there exists $E_n \in \mathcal{E}$ such that $\mu(E_n) < \infty$ and $X = \cup_{n=1} E_n$. If $\mathcal{M}$ is a $\sigma$ – algebra and $\mu$ is a measure on $\mathcal{M}$ which is $\sigma$ – finite on $\mathcal{M}$ we will say $(X, \mathcal{M}, \mu)$ is a $\sigma$ – finite measure space.*

The reader should check that if $\mu$ is a finitely additive measure on an algebra, $\mathcal{M}$, then $\mu$ is $\sigma$ – finite on $\mathcal{M}$ iff there exists $X_n \in \mathcal{M}$ such that $X_n \uparrow X$ and $\mu(X_n) < \infty$.

## 19.1 Example of Measures

Most $\sigma$ – algebras and $\sigma$ -additive measures are somewhat difficult to describe and define. However, one special case is fairly easy to understand. Namely suppose that $\mathcal{F} \subset 2^X$ is a countable or finite partition of $X$ and $\mathcal{M} \subset 2^X$ is the $\sigma$ – algebra which consists of the collection of sets $A \subset X$ such that

$$A = \cup \{\alpha \in \mathcal{F} : \alpha \subset A\}. \tag{19.2}$$

It is easily seen that $\mathcal{M}$ is a $\sigma$ – algebra.

Any measure $\mu : \mathcal{M} \to [0, \infty]$ is determined uniquely by its values on $\mathcal{F}$. Conversely, if we are given any function $\lambda : \mathcal{F} \to [0, \infty]$ we may define, for $A \in \mathcal{M}$,

$$\mu(A) = \sum_{\alpha \in \mathcal{F} \ni \alpha \subset A} \lambda(\alpha) = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A}$$

where $1_{\alpha \subset A}$ is one if $\alpha \subset A$ and zero otherwise. We may check that $\mu$ is a measure on $\mathcal{M}$. Indeed, if $A = \coprod_{i=1}^\infty A_i$ and $\alpha \in \mathcal{F}$, then $\alpha \subset A$ iff $\alpha \subset A_i$ for one and hence exactly one $A_i$. Therefore $1_{\alpha \subset A} = \sum_{i=1}^\infty 1_{\alpha \subset A_i}$ and hence

$$\begin{aligned}
\mu(A) &= \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A} = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) \sum_{i=1}^\infty 1_{\alpha \subset A_i} \\
&= \sum_{i=1}^\infty \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A_i} = \sum_{i=1}^\infty \mu(A_i)
\end{aligned}$$

as desired. Thus we have shown that there is a one to one correspondence between measures $\mu$ on $\mathcal{M}$ and functions $\lambda : \mathcal{F} \to [0, \infty]$.

We will leave the issue of constructing measures until Sections 25 and 26. However, let us point out that interesting measures do exist. The following theorem may be found in Theorem 25.35 or see Section 25.8.1.

**Theorem 19.8.** *To every right continuous non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ there exists a unique measure $\mu_F$ on $\mathcal{B}_\mathbb{R}$ such that*

$$\mu_F((a, b]) = F(b) - F(a) \ \forall \ -\infty < a \le b < \infty \tag{19.3}$$

*Moreover, if $A \in \mathcal{B}_\mathbb{R}$ then*

$$\begin{aligned}
\mu_F(A) &= \inf \left\{ \sum_{i=1}^\infty (F(b_i) - F(a_i)) : A \subset \cup_{i=1}^\infty (a_i, b_i] \right\} &(19.4) \\
&= \inf \left\{ \sum_{i=1}^\infty (F(b_i) - F(a_i)) : A \subset \coprod_{i=1}^\infty (a_i, b_i] \right\}. &(19.5)
\end{aligned}$$

*In fact the map $F \to \mu_F$ is a one to one correspondence between right continuous functions $F$ with $F(0) = 0$ on one hand and measures $\mu$ on $\mathcal{B}_\mathbb{R}$ such that $\mu(J) < \infty$ on any bounded set $J \in \mathcal{B}_\mathbb{R}$ on the other.*



*Example 19.9.* The most important special case of Theorem 19.8 is when $F(x) = x$, in which case we write $m$ for $\mu_F$. The measure $m$ is called Lebesgue measure.

**Theorem 19.10.** *Lebesgue measure $m$ is invariant under translations, i.e. for $B \in \mathcal{B}_\mathbb{R}$ and $x \in \mathbb{R}$,*

$$m(x + B) = m(B). \tag{19.6}$$

*Moreover, $m$ is the unique measure on $\mathcal{B}_\mathbb{R}$ such that $m((0,1]) = 1$ and Eq. (19.6) holds for $B \in \mathcal{B}_\mathbb{R}$ and $x \in \mathbb{R}$. Moreover, $m$ has the scaling property*

$$m(\lambda B) = |\lambda| \, m(B) \tag{19.7}$$

*where $\lambda \in \mathbb{R}$, $B \in \mathcal{B}_\mathbb{R}$ and $\lambda B := \{\lambda x : x \in B\}$.*

*Proof.* Let $m_x(B) := m(x + B)$, then one easily shows that $m_x$ is a measure on $\mathcal{B}_\mathbb{R}$ such that $m_x((a, b]) = b - a$ for all $a < b$. Therefore, $m_x = m$ by the uniqueness assertion in Theorem 19.8.

For the converse, suppose that $m$ is translation invariant and $m((0, 1]) = 1$. Given $n \in \mathbb{N}$, we have

$$(0, 1] = \cup_{k=1}^n (\frac{k-1}{n}, \frac{k}{n}] = \cup_{k=1}^n \left( \frac{k-1}{n} + (0, \frac{1}{n}] \right).$$

Therefore,

$$\begin{aligned}
1 = m((0, 1]) &= \sum_{k=1}^n m \left( \frac{k-1}{n} + (0, \frac{1}{n}] \right) \\
&= \sum_{k=1}^n m((0, \frac{1}{n}]) = n \cdot m((0, \frac{1}{n}]).
\end{aligned}$$

That is to say

$$m((0, \frac{1}{n}]) = 1/n.$$

Similarly, $m((0, \frac{l}{n}]) = l/n$ for all $l, n \in \mathbb{N}$ and therefore by the translation invariance of $m$,

$$m((a, b]) = b - a \text{ for all } a, b \in \mathbb{Q} \text{ with } a < b.$$

Finally for $a, b \in \mathbb{R}$ such that $a < b$, choose $a_n, b_n \in \mathbb{Q}$ such that $b_n \downarrow b$ and $a_n \uparrow a$, then $(a_n, b_n] \downarrow (a, b]$ and thus

$$m((a, b]) = \lim_{n\to\infty} m((a_n, b_n]) = \lim_{n\to\infty} (b_n - a_n) = b - a,$$

i.e. $m$ is Lebesgue measure.

To prove Eq. (19.7) we may assume that $\lambda \neq 0$ since this case is trivial to prove. Now let $m_\lambda(B) := |\lambda|^{-1} m(\lambda B)$. It is easily checked that $m_\lambda$ is again a measure on $\mathcal{B}_\mathbb{R}$ which satisfies

$$m_\lambda((a,b]) = \lambda^{-1} m\left((\lambda a, \lambda b]\right) = \lambda^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda > 0$ and

$$m_\lambda((a,b]) = |\lambda|^{-1} m\left([\lambda b, \lambda a)\right) = -|\lambda|^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda < 0$. Hence $m_\lambda = m$. ■

We are now going to develop integration theory relative to a measure. The integral defined in the case for Lebesgue measure, $m$, will be an extension of the standard Riemann integral on $\mathbb{R}$.

#### 19.1.1 ADD: Examples of Measures

BRUCE: ADD details.

1. Product measure for the flipping of a coin.
2. Haar Measure
3. Measure on embedded submanifolds, i.e. Hausdorff measure.
4. Wiener measure.
5. Gibbs states.
6. Measure associated to self-adjoint operators and classifying them.

## 19.2 Integrals of Simple functions

Let $(X, \mathcal{M}, \mu)$ be a fixed measure space in this section.

**Definition 19.11.** *Let $\mathbb{F} = \mathbb{C}$ or $[0,\infty)$ and suppose that $\phi : X \to \mathbb{F}$ is a simple function as in Definition 18.41. If $\mathbb{F} = \mathbb{C}$ assume further that $\mu(\phi^{-1}(\{y\})) < \infty$ for all $y \neq 0$ in $\mathbb{C}$. For such functions $\phi$, define $I_\mu(\phi)$ by*

$$I_\mu(\phi) = \sum_{y\in\mathbb{F}} y\mu(\phi^{-1}(\{y\})).$$

**Proposition 19.12.** *Let $\lambda \in \mathbb{F}$ and $\phi$ and $\psi$ be two simple functions, then $I_\mu$ satisfies:*

*1.*
$$I_\mu(\lambda\phi) = \lambda I_\mu(\phi). \tag{19.8}$$

*2.*
$$I_\mu(\phi + \psi) = I_\mu(\psi) + I_\mu(\phi).$$

*3. If $\phi$ and $\psi$ are non-negative simple functions such that $\phi \le \psi$ then*
$$I_\mu(\phi) \le I_\mu(\psi).$$



*Proof.* Let us write $\{\phi = y\}$ for the set $\phi^{-1}(\{y\}) \subset X$ and $\mu(\phi = y)$ for $\mu(\{\phi = y\}) = \mu(\phi^{-1}(\{y\}))$ so that

$$I_\mu(\phi) = \sum_{y\in\mathbb{C}} y\mu(\phi = y).$$

We will also write $\{\phi = a, \psi = b\}$ for $\phi^{-1}(\{a\}) \cap \psi^{-1}(\{b\})$. This notation is more intuitive for the purposes of this proof. Suppose that $\lambda \in \mathbb{F}$ then

$$\begin{aligned} I_\mu(\lambda\phi) &= \sum_{y\in\mathbb{F}} y\ \mu(\lambda\phi = y) = \sum_{y\in\mathbb{F}} y\ \mu(\phi = y/\lambda) \\ &= \sum_{z\in\mathbb{F}} \lambda z\ \mu(\phi = z) = \lambda I_\mu(\phi) \end{aligned}$$

provided that $\lambda \neq 0$. The case $\lambda = 0$ is clear, so we have proved 1.

Suppose that $\phi$ and $\psi$ are two simple functions, then

$$\begin{aligned} I_\mu(\phi + \psi) &= \sum_{z\in\mathbb{F}} z\ \mu(\phi + \psi = z) \\ &= \sum_{z\in\mathbb{F}} z\ \mu\left(\cup_{w\in\mathbb{F}}\{\phi = w,\ \psi = z - w\}\right) \\ &= \sum_{z\in\mathbb{F}} z \sum_{w\in\mathbb{F}} \mu(\phi = w,\ \psi = z - w) \\ &= \sum_{z,w\in\mathbb{F}} (z + w)\mu(\phi = w,\ \psi = z) \\ &= \sum_{z\in\mathbb{F}} z\ \mu(\psi = z) + \sum_{w\in\mathbb{F}} w\ \mu(\phi = w) \\ &= I_\mu(\psi) + I_\mu(\phi). \end{aligned}$$

which proves 2.

For 3. if $\phi$ and $\psi$ are non-negative simple functions such that $\phi \le \psi$

$$\begin{aligned} I_\mu(\phi) &= \sum_{a\ge 0} a\mu(\phi = a) = \sum_{a,b\ge 0} a\mu(\phi = a, \psi = b) \\ &\le \sum_{a,b\ge 0} b\mu(\phi = a, \psi = b) = \sum_{b\ge 0} b\mu(\psi = b) = I_\mu(\psi), \end{aligned}$$

wherein the third inequality we have used $\{\phi = a, \psi = b\} = \emptyset$ if $a > b$. ■

## 19.3 Integrals of positive functions

**Definition 19.13.** *Let $L^+ = \{f : X \to [0,\infty] : f$ is measurable$\}$. Define*

$$\int_X f d\mu = \sup\{I_\mu(\phi) : \phi \text{ is simple and } \phi \le f\}.$$

*We say the* $f \in L^+$ *is* ***integrable*** *if* $\int_X f d\mu < \infty$. *If* $A \in \mathcal{M}$, *let*

$$\int_A f d\mu := \int_X 1_A f \, d\mu.$$

*Remark 19.14.* Because of item 3. of Proposition 19.12, if $\phi$ is a non-negative simple function, $\int_X \phi d\mu = I_\mu(\phi)$ so that $\int_X$ is an extension of $I_\mu$. This extension still has the monotonicity property if $I_\mu$ : namely if $0 \le f \le g$ then

$$\int_X f d\mu = \sup\{I_\mu(\phi) : \phi \text{ is simple and } \phi \le f\}$$
$$\le \sup\{I_\mu(\phi) : \phi \text{ is simple and } \phi \le g\} \le \int_X g.$$

Similarly if $c > 0$,

$$\int_X c f d\mu = c \int_X f d\mu.$$

Also notice that if $f$ is integrable, then $\mu(\{f = \infty\}) = 0$.

**Lemma 19.15.** *Let* $X$ *be a set and* $\rho : X \to [0, \infty]$ *be a function, let* $\mu = \sum_{x \in X} \rho(x)\delta_x$ *on* $\mathcal{M} = 2^X$, *i.e.*

$$\mu(A) = \sum_{x \in A} \rho(x).$$

*If* $f : X \to [0, \infty]$ *is a function (which is necessarily measurable), then*

$$\int_X f d\mu = \sum_X f\rho.$$

*Proof.* Suppose that $\phi : X \to [0, \infty]$ is a simple function, then $\phi = \sum_{z \in [0,\infty]} z 1_{\{\phi = z\}}$ and

$$\sum_X \phi\rho = \sum_{x \in X} \rho(x) \sum_{z \in [0,\infty]} z 1_{\{\phi = z\}}(x) = \sum_{z \in [0,\infty]} z \sum_{x \in X} \rho(x) 1_{\{\phi = z\}}(x)$$
$$= \sum_{z \in [0,\infty]} z\mu(\{\phi = z\}) = \int_X \phi d\mu.$$

So if $\phi : X \to [0, \infty)$ is a simple function such that $\phi \le f$, then

$$\int_X \phi d\mu = \sum_X \phi\rho \le \sum_X f\rho.$$

Taking the sup over $\phi$ in this last equation then shows that



$$\int_X f d\mu \le \sum_X f\rho.$$

For the reverse inequality, let $\Lambda \subset\subset X$ be a finite set and $N \in (0, \infty)$. Set $f^N(x) = \min\{N, f(x)\}$ and let $\phi_{N,\Lambda}$ be the simple function given by $\phi_{N,\Lambda}(x) := 1_\Lambda(x) f^N(x)$. Because $\phi_{N,\Lambda}(x) \le f(x)$,

$$\sum_\Lambda f^N \rho = \sum_X \phi_{N,\Lambda}\rho = \int_X \phi_{N,\Lambda} d\mu \le \int_X f d\mu.$$

Since $f^N \uparrow f$ as $N \to \infty$, we may let $N \to \infty$ in this last equation to concluded

$$\sum_\Lambda f\rho \le \int_X f d\mu.$$

Since $\Lambda$ is arbitrary, this implies

$$\sum_X f\rho \le \int_X f d\mu.$$

■

**Theorem 19.16 (Monotone Convergence Theorem).** *Suppose* $f_n \in L^+$ *is a sequence of functions such that* $f_n \uparrow f$ *(*$f$ *is necessarily in* $L^+$*) then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

*Proof.* Since $f_n \le f_m \le f$, for all $n \le m < \infty$,

$$\int f_n \le \int f_m \le \int f$$

from which if follows $\int f_n$ is increasing in $n$ and

$$\lim_{n\to\infty} \int f_n \le \int f. \tag{19.9}$$

For the opposite inequality, let $\phi$ be a simple function such that $0 \le \phi \le f$, $\alpha \in (0, 1)$ and $E_n := \{f_n \ge \alpha\phi\}$. Notice that $E_n \uparrow X$ and by Proposition 19.12,

$$\int f_n \ge \int f_n \ge \int \alpha 1_{E_n}\phi = \alpha \int 1_{E_n}\phi. \tag{19.10}$$

Then using the continuity property of $\mu$,

$$\lim_{n\to\infty} \int 1_{E_n}\phi = \lim_{n\to\infty} \int 1_{E_n} \sum_{y>0} y 1_{\{\phi = y\}}$$
$$= \lim_{n\to\infty} \sum_{y>0} y\mu(E_n \cap \{\phi = y\}) = \sum_{y>0} y \lim_{n\to\infty} \mu(E_n \cap \{\phi = y\})$$
$$= \sum_{y>0} y \lim_{n\to\infty} \mu(\{\phi = y\}) = \int \phi.$$

This identity allows us to let $n \to \infty$ in Eq. (19.10) to conclude

$$\int_X \phi \le \frac{1}{\alpha} \lim_{n\to\infty} \int f_n.$$

Since this is true for all non-negative simple functions $\phi$ with $\phi \le f$;

$$\int f = \sup \left\{ \int_X \phi : \phi \text{ is simple and } \phi \le f \right\} \le \frac{1}{\alpha} \lim_{n\to\infty} \int f_n.$$

Because $\alpha \in (0,1)$ was arbitrary, it follows that $\int f \le \lim_{n\to\infty} \int f_n$ which combined with Eq. (19.9) proves the theorem. ■

The following simple lemma will be use often in the sequel.

**Lemma 19.17 (Chebyshev's Inequality).** *Suppose that $f \ge 0$ is a measurable function, then for any $\varepsilon > 0$,*

$$\mu(f \ge \varepsilon) \le \frac{1}{\varepsilon} \int_X f d\mu. \tag{19.11}$$

*In particular if $\int_X f d\mu < \infty$ then $\mu(f = \infty) = 0$ (i.e. $f < \infty$ a.e.) and the set $\{f > 0\}$ is $\sigma$ – finite.*

*Proof.* Since $1_{\{f\ge\varepsilon\}} \le 1_{\{f\ge\varepsilon\}} \frac{1}{\varepsilon} f \le \frac{1}{\varepsilon} f$,

$$\mu(f \ge \varepsilon) = \int_X 1_{\{f\ge\varepsilon\}} d\mu \le \int_X 1_{\{f\ge\varepsilon\}} \frac{1}{\varepsilon} f d\mu \le \frac{1}{\varepsilon} \int_X f d\mu.$$

If $M := \int_X f d\mu < \infty$, then

$$\mu(f = \infty) \le \mu(f \ge n) \le \frac{M}{n} \to 0 \text{ as } n \to \infty$$

and $\{f \ge 1/n\} \uparrow \{f > 0\}$ with $\mu(f \ge 1/n) \le nM < \infty$ for all $n$. ■

**Corollary 19.18.** *If $f_n \in L^+$ is a sequence of functions then*

$$\int \sum_n f_n = \sum_n \int f_n.$$

*In particular, if $\sum_n \int f_n < \infty$ then $\sum_n f_n < \infty$ a.e.*

*Proof.* First off we show that

$$\int (f_1 + f_2) = \int f_1 + \int f_2$$

by choosing non-negative simple function $\phi_n$ and $\psi_n$ such that $\phi_n \uparrow f_1$ and $\psi_n \uparrow f_2$. Then $(\phi_n + \psi_n)$ is simple as well and $(\phi_n + \psi_n) \uparrow (f_1 + f_2)$ so by the monotone convergence theorem,



$$\int (f_1 + f_2) = \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \left( \int \phi_n + \int \psi_n \right)$$
$$= \lim_{n\to\infty} \int \phi_n + \lim_{n\to\infty} \int \psi_n = \int f_1 + \int f_2.$$

Now to the general case. Let $g_N := \sum_{n=1}^N f_n$ and $g = \sum_1^\infty f_n$, then $g_N \uparrow g$ and so again by monotone convergence theorem and the additivity just proved,

$$\sum_{n=1}^\infty \int f_n := \lim_{N\to\infty} \sum_{n=1}^N \int f_n = \lim_{N\to\infty} \int \sum_{n=1}^N f_n$$
$$= \lim_{N\to\infty} \int g_N = \int g = \sum_{n=1}^\infty \int f_n.$$

■

*Remark 19.19.* It is in the proof of this corollary (i.e. the linearity of the integral) that we really make use of the assumption that all of our functions are measurable. In fact the definition $\int f d\mu$ makes sense for **all** functions $f : X \to [0,\infty]$ not just measurable functions. Moreover the monotone convergence theorem holds in this generality with no change in the proof. However, in the proof of Corollary 19.18, we use the approximation Theorem 18.42 which relies heavily on the measurability of the functions to be approximated.

The following Lemma and the next Corollary are simple applications of Corollary 19.18.

**Lemma 19.20 (The First Borell – Carntelli Lemma).** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $A_n \in \mathcal{M}$, and set*

$$\{A_n \text{ i.o.}\} = \{x \in X : x \in A_n \text{ for infinitely many } n\text{'s}\} = \bigcap_{N=1}^\infty \bigcup_{n\ge N} A_n.$$

*If $\sum_{n=1}^\infty \mu(A_n) < \infty$ then $\mu(\{A_n \text{ i.o.}\}) = 0$.*

*Proof.* (First Proof.) Let us first observe that

$$\{A_n \text{ i.o.}\} = \left\{ x \in X : \sum_{n=1}^\infty 1_{A_n}(x) = \infty \right\}.$$

Hence if $\sum_{n=1}^\infty \mu(A_n) < \infty$ then

$$\infty > \sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \int_X 1_{A_n} \, d\mu = \int_X \sum_{n=1}^\infty 1_{A_n} \, d\mu$$

implies that $\sum_{n=1}^{\infty} 1_{A_n}(x) < \infty$ for $\mu$ - a.e. $x$. That is to say $\mu(\{A_n \text{ i.o.}\}) = 0$.

(Second Proof.) Of course we may give a strictly measure theoretic proof of this fact:

$$\mu(A_n \text{ i.o.}) = \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} A_n\right) \\ \leq \lim_{N\to\infty} \sum_{n\geq N} \mu(A_n)$$

and the last limit is zero since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. ■

**Corollary 19.21.** *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ is a collection of sets such that $\mu(A_i \cap A_j) = 0$ for all $i \neq j$, then*

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

*Proof.* Since

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \int_X 1_{\cup_{n=1}^{\infty} A_n} d\mu \text{ and}$$
$$\sum_{n=1}^{\infty} \mu(A_n) = \int_X \sum_{n=1}^{\infty} 1_{A_n} d\mu$$

it suffices to show

$$\sum_{n=1}^{\infty} 1_{A_n} = 1_{\cup_{n=1}^{\infty} A_n} \ \mu - \text{a.e.} \tag{19.12}$$

Now $\sum_{n=1}^{\infty} 1_{A_n} \geq 1_{\cup_{n=1}^{\infty} A_n}$ and $\sum_{n=1}^{\infty} 1_{A_n}(x) \neq 1_{\cup_{n=1}^{\infty} A_n}(x)$ iff $x \in A_i \cap A_j$ for some $i \neq j$, that is

$$\left\{x : \sum_{n=1}^{\infty} 1_{A_n}(x) \neq 1_{\cup_{n=1}^{\infty} A_n}(x)\right\} = \cup_{i<j} A_i \cap A_j$$

and the later set has measure 0 being the countable union of sets of measure zero. This proves Eq. (19.12) and hence the corollary. ■

*Example 19.22.* Suppose $-\infty < a < b < \infty$, $f \in C([a,b],[0,\infty))$ and $m$ be Lebesgue measure on $\mathbb{R}$. Also let $\pi_k = \{a = a_0^k < a_1^k < \cdots < a_{n_k}^k = b\}$ be a sequence of refining partitions (i.e. $\pi_k \subset \pi_{k+1}$ for all $k$) such that

$$\text{mesh}(\pi_k) := \max\{\left|a_j^k - a_{j-1}^{k+1}\right| : j = 1, \ldots, n_k\} \to 0 \text{ as } k \to \infty.$$

For each $k$, let



$$f_k(x) = f(a)1_{\{a\}} + \sum_{l=0}^{n_k-1} \min\left\{f(x) : a_l^k \leq x \leq a_{l+1}^k\right\} 1_{(a_l^k, a_{l+1}^k]}(x)$$

then $f_k \uparrow f$ as $k \to \infty$ and so by the monotone convergence theorem,

$$\int_a^b f dm := \int_{[a,b]} f dm = \lim_{k\to\infty} \int_a^b f_k \, dm \\ = \lim_{k\to\infty} \sum_{l=0}^{n_k} \min\left\{f(x) : a_l^k \leq x \leq a_{l+1}^k\right\} m\left((a_l^k, a_{l+1}^k]\right) \\ = \int_a^b f(x) dx.$$

The latter integral being the Riemann integral.

We can use the above result to integrate some non-Riemann integrable functions:

*Example 19.23.* For all $\lambda > 0$,

$$\int_0^{\infty} e^{-\lambda x} dm(x) = \lambda^{-1} \text{ and } \int_{\mathbb{R}} \frac{1}{1+x^2} dm(x) = \pi.$$

The proof of these identities are similar. By the monotone convergence theorem, Example 19.22 and the fundamental theorem of calculus for Riemann integrals (or see Theorem 8.13 above or Theorem 19.39 below),

$$\int_0^{\infty} e^{-\lambda x} dm(x) = \lim_{N\to\infty} \int_0^N e^{-\lambda x} dm(x) = \lim_{N\to\infty} \int_0^N e^{-\lambda x} dx \\ = -\lim_{N\to\infty} \frac{1}{\lambda} e^{-\lambda x}|_0^N = \lambda^{-1}$$

and

$$\int_{\mathbb{R}} \frac{1}{1+x^2} dm(x) = \lim_{N\to\infty} \int_{-N}^N \frac{1}{1+x^2} dm(x) = \lim_{N\to\infty} \int_{-N}^N \frac{1}{1+x^2} dx \\ = \tan^{-1}(N) - \tan^{-1}(-N) = \pi.$$

Let us also consider the functions $x^{-p}$,

$$\int_{(0,1]} \frac{1}{x^p} \, dm(x) = \lim_{n\to\infty} \int_0^1 1_{(\frac{1}{n},1]}(x) \frac{1}{x^p} dm(x) \\ = \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \frac{1}{x^p} dx = \lim_{n\to\infty} \left.\frac{x^{-p+1}}{1-p}\right|_{1/n}^1 \\ = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}$$

If $p = 1$ we find

$$\int_{(0,1]} \frac{1}{x^p}\, dm(x) = \lim_{n\to\infty} \int_{\frac{1}{n}}^{1} \frac{1}{x} dx = \lim_{n\to\infty} \ln(x)|_{1/n}^{1} = \infty.$$

*Example 19.24.* Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the points in $\mathbb{Q} \cap [0,1]$ and define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{1}{\sqrt{|x - r_n|}}$$

with the convention that

$$\frac{1}{\sqrt{|x - r_n|}} = 5 \text{ if } x = r_n.$$

Since, By Theorem 19.39,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{|x - r_n|}}\, dx &= \int_{r_n}^{1} \frac{1}{\sqrt{x - r_n}} dx + \int_0^{r_n} \frac{1}{\sqrt{r_n - x}} dx \\ &= 2\sqrt{x - r_n}|_{r_n}^{1} - 2\sqrt{r_n - x}|_0^{r_n} = 2\left(\sqrt{1 - r_n} - \sqrt{r_n}\right) \\ &\le 4,\end{aligned}$$

we find

$$\int_{[0,1]} f(x) dm(x) = \sum_{n=1}^{\infty} 2^{-n} \int_{[0,1]} \frac{1}{\sqrt{|x - r_n|}} dx \le \sum_{n=1}^{\infty} 2^{-n} 4 = 4 < \infty.$$

In particular, $m(f = \infty) = 0$, i.e. that $f < \infty$ for almost every $x \in [0,1]$ and this implies that

$$\sum_{n=1}^{\infty} 2^{-n} \frac{1}{\sqrt{|x - r_n|}} < \infty \text{ for a.e. } x \in [0,1].$$

This result is somewhat surprising since the singularities of the summands form a dense subset of $[0,1]$.

**Proposition 19.25.** *Suppose that $f \ge 0$ is a measurable function. Then $\int_X f d\mu = 0$ iff $f = 0$ a.e. Also if $f, g \ge 0$ are measurable functions such that $f \le g$ a.e. then $\int f d\mu \le \int g d\mu$. In particular if $f = g$ a.e. then $\int f d\mu = \int g d\mu$.*

*Proof.* If $f = 0$ a.e. and $\phi \le f$ is a simple function then $\phi = 0$ a.e. This implies that $\mu(\phi^{-1}(\{y\})) = 0$ for all $y > 0$ and hence $\int_X \phi d\mu = 0$ and therefore $\int_X f d\mu = 0$.

Conversely, if $\int f d\mu = 0$, then by Chebyshev's Inequality (Lemma 19.17),

$$\mu(f \ge 1/n) \le n \int f d\mu = 0 \text{ for all } n.$$



Therefore, $\mu(f > 0) \le \sum_{n=1}^{\infty} \mu(f \ge 1/n) = 0$, i.e. $f = 0$ a.e.

For the second assertion let $E$ be the exceptional set where $g > f$, i.e. $E := \{x \in X : g(x) > f(x)\}$. By assumption $E$ is a null set and $1_{E^c} f \le 1_{E^c} g$ everywhere. Because $g = 1_{E^c} g + 1_E g$ and $1_E g = 0$ a.e.,

$$\int g d\mu = \int 1_{E^c} g d\mu + \int 1_E g d\mu = \int 1_{E^c} g d\mu$$

and similarly $\int f d\mu = \int 1_{E^c} f d\mu$. Since $1_{E^c} f \le 1_{E^c} g$ everywhere,

$$\int f d\mu = \int 1_{E^c} f d\mu \le \int 1_{E^c} g d\mu = \int g d\mu.$$

■

**Corollary 19.26.** *Suppose that $\{f_n\}$ is a sequence of non-negative functions and $f$ is a measurable function such that $f_n \uparrow f$ off a null set, then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

*Proof.* Let $E \subset X$ be a null set such that $f_n 1_{E^c} \uparrow f 1_{E^c}$ as $n \to \infty$. Then by the monotone convergence theorem and Proposition 19.25,

$$\int f_n = \int f_n 1_{E^c} \uparrow \int f 1_{E^c} = \int f \text{ as } n \to \infty.$$

■

**Lemma 19.27 (Fatou's Lemma).** *If $f_n : X \to [0,\infty]$ is a sequence of measurable functions then*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

*Proof.* Define $g_k := \inf_{n \ge k} f_n$ so that $g_k \uparrow \liminf_{n\to\infty} f_n$ as $k \to \infty$. Since $g_k \le f_n$ for all $k \le n$,

$$\int g_k \le \int f_n \text{ for all } n \ge k$$

and therefore

$$\int g_k \le \lim\inf_{n\to\infty} \int f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k \to \infty$ to find

$$\int \lim\inf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \int g_k \le \lim\inf_{n\to\infty} \int f_n.$$

■

## 19.4 Integrals of Complex Valued Functions

**Definition 19.28.** *A measurable function $f : X \to \bar{\mathbb{R}}$ is **integrable** if $f_+ := f 1_{\{f\geq 0\}}$ and $f_- = -f\ 1_{\{f\leq 0\}}$ are **integrable**. We write $\mathrm{L}^1(\mu;\mathbb{R})$ for the space of real valued integrable functions. For $f \in \mathrm{L}^1(\mu;\mathbb{R})$, let*

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

**Convention:** If $f, g : X \to \bar{\mathbb{R}}$ are two measurable functions, let $f+g$ denote the collection of measurable functions $h : X \to \bar{\mathbb{R}}$ such that $h(x) = f(x)+g(x)$ whenever $f(x)+g(x)$ is well defined, i.e. is not of the form $\infty-\infty$ or $-\infty+\infty$. We use a similar convention for $f-g$. Notice that if $f, g \in \mathrm{L}^1(\mu;\mathbb{R})$ and $h_1, h_2 \in f+g$, then $h_1 = h_2$ a.e. because $|f| < \infty$ and $|g| < \infty$ a.e.

**Notation 19.29 (Abuse of notation)** *We will sometimes denote the integral $\int_X f d\mu$ by $\mu(f)$. With this notation we have $\mu(A) = \mu(1_A)$ for all $A \in \mathcal{M}$.*

*Remark 19.30.* Since

$$f_\pm \leq |f| \leq f_+ + f_-,$$

a measurable function $f$ is **integrable** iff $\int |f|\ d\mu < \infty$. Hence

$$\mathrm{L}^1(\mu;\mathbb{R}) := \left\{ f : X \to \bar{\mathbb{R}} :\ f \text{ is measurable and } \int_X |f|\ d\mu < \infty \right\}.$$

If $f, g \in \mathrm{L}^1(\mu;\mathbb{R})$ and $f = g$ a.e. then $f_\pm = g_\pm$ a.e. and so it follows from Proposition 19.25 that $\int f d\mu = \int g d\mu$. In particular if $f, g \in \mathrm{L}^1(\mu;\mathbb{R})$ we may define

$$\int_X (f+g)\, d\mu = \int_X h d\mu$$

where $h$ is any element of $f+g$.

**Proposition 19.31.** *The map*

$$f \in \mathrm{L}^1(\mu;\mathbb{R}) \to \int_X f d\mu \in \mathbb{R}$$

*is linear and has the monotonicity property: $\int f d\mu \leq \int g d\mu$ for all $f, g \in \mathrm{L}^1(\mu;\mathbb{R})$ such that $f \leq g$ a.e.*

*Proof.* Let $f, g \in \mathrm{L}^1(\mu;\mathbb{R})$ and $a, b \in \mathbb{R}$. By modifying $f$ and $g$ on a null set, we may assume that $f, g$ are real valued functions. We have $af+bg \in \mathrm{L}^1(\mu;\mathbb{R})$ because

$$|af+bg| \leq |a|\,|f| + |b|\,|g| \in \mathrm{L}^1(\mu;\mathbb{R}).$$

If $a < 0$, then

$$(af)_+ = -af_- \text{ and } (af)_- = -af_+$$



so that

$$\int af = -a\int f_- + a\int f_+ = a(\int f_+ - \int f_-) = a\int f.$$

A similar calculation works for $a > 0$ and the case $a = 0$ is trivial so we have shown that

$$\int af = a\int f.$$

Now set $h = f+g$. Since $h = h_+ - h_-$,

$$h_+ - h_- = f_+ - f_- + g_+ - g_-$$

or

$$h_+ + f_- + g_- = h_- + f_+ + g_+.$$

Therefore,

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+$$

and hence

$$\int h = \int h_+ - \int h_- = \int f_+ + \int g_+ - \int f_- - \int g_- = \int f + \int g.$$

Finally if $f_+ - f_- = f \leq g = g_+ - g_-$ then $f_+ + g_- \leq g_+ + f_-$ which implies that

$$\int f_+ + \int g_- \leq \int g_+ + \int f_-$$

or equivalently that

$$\int f = \int f_+ - \int f_- \leq \int g_+ - \int g_- = \int g.$$

The monotonicity property is also a consequence of the linearity of the integral, the fact that $f \leq g$ a.e. implies $0 \leq g-f$ a.e. and Proposition 19.25. ■

**Definition 19.32.** *A measurable function $f : X \to \mathbb{C}$ is **integrable** if $\int_X |f|\ d\mu < \infty$. Analogously to the real case, let*

$$\mathrm{L}^1(\mu;\mathbb{C}) := \left\{ f : X \to \mathbb{C} :\ f \text{ is measurable and } \int_X |f|\ d\mu < \infty \right\}.$$

*denote the complex valued integrable functions. Because, $\max(|\mathrm{Re}\, f|, |\mathrm{Im}\, f|) \leq |f| \leq \sqrt{2}\max(|\mathrm{Re}\, f|, |\mathrm{Im}\, f|)$, $\int |f|\ d\mu < \infty$ iff*

$$\int |\mathrm{Re}\, f|\, d\mu + \int |\mathrm{Im}\, f|\, d\mu < \infty.$$

*For $f \in \mathrm{L}^1(\mu;\mathbb{C})$ define*

$$\int f\ d\mu = \int \mathrm{Re}\, f\ d\mu + i\int \mathrm{Im}\, f\ d\mu.$$

It is routine to show the integral is still linear on $\mathrm{L}^1(\mu;\mathbb{C})$ (prove!). In the remainder of this section, let $\mathrm{L}^1(\mu)$ be either $\mathrm{L}^1(\mu;\mathbb{C})$ or $\mathrm{L}^1(\mu;\mathbb{R})$. If $A\in\mathcal{M}$ and $f\in\mathrm{L}^1(\mu;\mathbb{C})$ or $f:X\to[0,\infty]$ is a measurable function, let

$$\int_A f d\mu := \int_X 1_A f d\mu.$$

**Proposition 19.33.** *Suppose that $f\in\mathrm{L}^1(\mu;\mathbb{C})$, then*

$$\left|\int_X f d\mu\right| \le \int_X |f|\, d\mu.$$

*Proof.* Start by writing $\int_X f\ d\mu = Re^{i\theta}$. Then using the monotonicity in Proposition 19.25,

$$\begin{aligned}\left|\int_X f d\mu\right| &= R = e^{-i\theta}\int_X f d\mu = \int_X e^{-i\theta} f d\mu \\ &= \int_X \mathrm{Re}\left(e^{-i\theta} f\right) d\mu \le \int_X \left|\mathrm{Re}\left(e^{-i\theta} f\right)\right| d\mu \le \int_X |f|\, d\mu.\end{aligned}$$

■

**Proposition 19.34.** *Let $f,g\in\mathrm{L}^1(\mu)$, then*

1. *The set $\{f\neq 0\}$ is $\sigma$ – finite, in fact $\{|f|\ge\frac{1}{n}\}\uparrow\{f\neq 0\}$ and $\mu(|f|\ge\frac{1}{n})<\infty$ for all $n$.*
2. *The following are equivalent*
   a) *$\int_E f=\int_E g$ for all $E\in\mathcal{M}$*
   b) *$\int\limits_X |f-g|=0$*
   c) *$f=g$ a.e.*

*Proof.* 1. By Chebyshev's inequality, Lemma 19.17,

$$\mu(|f|\ge\frac{1}{n})\le n\int_X |f|\, d\mu<\infty$$

for all $n$.

2. (a) $\Longrightarrow$ (c) Notice that

$$\int_E f=\int_E g \Leftrightarrow \int_E (f-g)=0$$

for all $E\in\mathcal{M}$. Taking $E=\{\mathrm{Re}(f-g)>0\}$ and using $1_E\,\mathrm{Re}(f-g)\ge 0$, we learn that

$$0=\mathrm{Re}\int_E (f-g)d\mu=\int 1_E\,\mathrm{Re}(f-g)\Longrightarrow 1_E\,\mathrm{Re}(f-g)=0\text{ a.e.}$$

This implies that $1_E=0$ a.e. which happens iff



$$\mu\left(\{\mathrm{Re}(f-g)>0\}\right)=\mu(E)=0.$$

Similar $\mu(\mathrm{Re}(f-g)<0)=0$ so that $\mathrm{Re}(f-g)=0$ a.e. Similarly, $\mathrm{Im}(f-g)=0$ a.e and hence $f-g=0$ a.e., i.e. $f=g$ a.e.

(c) $\Longrightarrow$ (b) is clear and so is (b) $\Longrightarrow$ (a) since

$$\left|\int_E f-\int_E g\right|\le\int|f-g|=0.$$

■

**Definition 19.35.** *Let $(X,\mathcal{M},\mu)$ be a measure space and $L^1(\mu)=L^1(X,\mathcal{M},\mu)$ denote the set of $\mathrm{L}^1(\mu)$ functions modulo the equivalence relation; $f\sim g$ iff $f=g$ a.e. We make this into a normed space using the norm*

$$\|f-g\|_{L^1}=\int|f-g|\, d\mu$$

*and into a metric space using $\rho_1(f,g)=\|f-g\|_{L^1}$.*

**Warning:** in the future we will often not make much of a distinction between $L^1(\mu)$ and $\mathrm{L}^1(\mu)$. On occasion this can be dangerous and this danger will be pointed out when necessary.

*Remark 19.36.* More generally we may define $L^p(\mu)=L^p(X,\mathcal{M},\mu)$ for $p\in[1,\infty)$ as the set of measurable functions $f$ such that

$$\int_X |f|^p\, d\mu<\infty$$

modulo the equivalence relation; $f\sim g$ iff $f=g$ a.e.

We will see in Chapter 21 that

$$\|f\|_{L^p}=\left(\int|f|^p\, d\mu\right)^{1/p}\quad\text{for } f\in L^p(\mu)$$

is a norm and $(L^p(\mu),\|\cdot\|_{L^p})$ is a Banach space in this norm.

**Theorem 19.37 (Dominated Convergence Theorem).** *Suppose $f_n,g_n,g\in\mathrm{L}^1(\mu)$, $f_n\to f$ a.e., $|f_n|\le g_n\in\mathrm{L}^1(\mu)$, $g_n\to g$ a.e. and $\int_X g_n d\mu\to\int_X g d\mu$. Then $f\in\mathrm{L}^1(\mu)$ and*

$$\int_X f d\mu=\lim_{h\to\infty}\int_X f_n d\mu.$$

*(In most typical applications of this theorem $g_n=g\in\mathrm{L}^1(\mu)$ for all $n$.)*

*Proof.* Notice that $|f| = \lim_{n\to\infty} |f_n| \le \lim_{n\to\infty} |g_n| \le g$ a.e. so that $f \in L^1(\mu)$. By considering the real and imaginary parts of $f$ separately, it suffices to prove the theorem in the case where $f$ is real. By Fatou's Lemma,

$$\begin{aligned}\int_X (g \pm f) d\mu &= \int_X \liminf_{n\to\infty} (g_n \pm f_n)\, d\mu \le \liminf_{n\to\infty} \int_X (g_n \pm f_n)\, d\mu \\ &= \lim_{n\to\infty} \int_X g_n d\mu + \liminf_{n\to\infty} \left( \pm \int_X f_n d\mu \right) \\ &= \int_X g d\mu + \liminf_{n\to\infty} \left( \pm \int_X f_n d\mu \right)\end{aligned}$$

Since $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$, we have shown,

$$\int_X g d\mu \pm \int_X f d\mu \le \int_X g d\mu + \begin{cases} \liminf_{n\to\infty} \int_X f_n d\mu \\ -\limsup_{n\to\infty} \int_X f_n d\mu \end{cases}$$

and therefore

$$\limsup_{n\to\infty} \int_X f_n d\mu \le \int_X f d\mu \le \liminf_{n\to\infty} \int_X f_n d\mu.$$

This shows that $\lim\limits_{n\to\infty} \int_X f_n d\mu$ exists and is equal to $\int_X f d\mu$. ∎

**Corollary 19.38.** *Let $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ be a sequence such that $\sum_{n=1}^\infty \|f_n\|_{L^1(\mu)}$ $\infty$, then $\sum_{n=1}^\infty f_n$ is convergent a.e. and*

$$\int_X \left( \sum_{n=1}^\infty f_n \right) d\mu = \sum_{n=1}^\infty \int_X f_n d\mu.$$

*Proof.* The condition $\sum_{n=1}^\infty \|f_n\|_{L^1(\mu)} < \infty$ is equivalent to $\sum_{n=1}^\infty |f_n| \in L^1(\mu)$. Hence $\sum_{n=1}^\infty f_n$ is almost everywhere convergent and if $S_N := \sum_{n=1}^N f_n$, then

$$|S_N| \le \sum_{n=1}^N |f_n| \le \sum_{n=1}^\infty |f_n| \in L^1(\mu).$$

So by the dominated convergence theorem,

$$\begin{aligned}\int_X \left( \sum_{n=1}^\infty f_n \right) d\mu &= \int_X \lim_{N\to\infty} S_N d\mu = \lim_{N\to\infty} \int_X S_N d\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^N \int_X f_n d\mu = \sum_{n=1}^\infty \int_X f_n d\mu.\end{aligned}$$

∎



**Theorem 19.39 (The Fundamental Theorem of Calculus).** *Suppose $-\infty < a < b < \infty$, $f \in C((a,b),\mathbb{R}) \cap L^1((a,b),m)$ and $F(x) := \int_a^x f(y)dm(y)$. Then*

1. *$F \in C([a,b],\mathbb{R}) \cap C^1((a,b),\mathbb{R})$.*
2. *$F'(x) = f(x)$ for all $x \in (a,b)$.*
3. *If $G \in C([a,b],\mathbb{R}) \cap C^1((a,b),\mathbb{R})$ is an anti-derivative of $f$ on $(a,b)$ (i.e. $f = G'|_{(a,b)}$) then*

$$\int_a^b f(x)dm(x) = G(b) - G(a).$$

*Proof.* Since $F(x) := \int_{\mathbb{R}} 1_{(a,x)}(y) f(y) dm(y)$, $\lim_{x\to z} 1_{(a,x)}(y) = 1_{(a,z)}(y)$ for $m$ – a.e. $y$ and $\left|1_{(a,x)}(y) f(y)\right| \le 1_{(a,b)}(y) |f(y)|$ is an $L^1$ – function, it follows from the dominated convergence Theorem 19.37 that $F$ is continuous on $[a,b]$. Simple manipulations show,

$$\begin{aligned}\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &= \frac{1}{|h|} \begin{cases} \left| \int_x^{x+h} [f(y) - f(x)]\, dm(y) \right| & \text{if } h > 0 \\ \left| \int_{x+h}^x [f(y) - f(x)]\, dm(y) \right| & \text{if } h < 0 \end{cases} \\ &\le \frac{1}{|h|} \begin{cases} \int_x^{x+h} |f(y) - f(x)|\, dm(y) \text{ if } h > 0 \\ \int_{x+h}^x |f(y) - f(x)|\, dm(y) \text{ if } h < 0 \end{cases} \\ &\le \sup \{ |f(y) - f(x)| : y \in [x - |h|, x + |h|] \}\end{aligned}$$

and the latter expression, by the continuity of $f$, goes to zero as $h \to 0$. This shows $F' = f$ on $(a,b)$.

For the converse direction, we have by assumption that $G'(x) = F'(x)$ for $x \in (a,b)$. Therefore by the mean value theorem, $F - G = C$ for some constant $C$. Hence

$$\begin{aligned}\int_a^b f(x)dm(x) &= F(b) = F(b) - F(a) \\ &= (G(b) + C) - (G(a) + C) = G(b) - G(a).\end{aligned}$$

∎

*Example 19.40.* The following limit holds,

$$\lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n dm(x) = 1.$$

Let $f_n(x) = (1 - \frac{x}{n})^n 1_{[0,n]}(x)$ and notice that $\lim_{n\to\infty} f_n(x) = e^{-x}$. We will now show

$$0 \le f_n(x) \le e^{-x} \text{ for all } x \ge 0.$$

It suffices to consider $x \in [0,n]$. Let $g(x) = e^x f_n(x)$, then for $x \in (0,n)$,

$$\frac{d}{dx}\ln g(x) = 1 + n\frac{1}{(1-\frac{x}{n})}(-\frac{1}{n}) = 1 - \frac{1}{(1-\frac{x}{n})} \le 0$$

which shows that $\ln g(x)$ and hence $g(x)$ is decreasing on $[0,n]$. Therefore $g(x) \le g(0) = 1$, i.e.

$$0 \le f_n(x) \le e^{-x}.$$

From Example 19.23, we know

$$\int_0^\infty e^{-x} dm(x) = 1 < \infty,$$

so that $e^{-x}$ is an integrable function on $[0,\infty)$. Hence by the dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n dm(x) &= \lim_{n\to\infty}\int_0^\infty f_n(x)dm(x) \\ &= \int_0^\infty \lim_{n\to\infty} f_n(x)dm(x) = \int_0^\infty e^{-x}dm(x) = 1.\end{aligned}$$

*Example 19.41 (Integration of Power Series).* Suppose $R>0$ and $\{a_n\}_{n=0}^\infty$ is a sequence of complex numbers such that $\sum_{n=0}^\infty |a_n| r^n < \infty$ for all $r \in (0,R)$. Then

$$\int_\alpha^\beta \left(\sum_{n=0}^\infty a_n x^n\right) dm(x) = \sum_{n=0}^\infty a_n \int_\alpha^\beta x^n dm(x) = \sum_{n=0}^\infty a_n \frac{\beta^{n+1}-\alpha^{n+1}}{n+1}$$

for all $-R<\alpha<\beta<R$. Indeed this follows from Corollary 19.38 since

$$\begin{aligned}\sum_{n=0}^\infty \int_\alpha^\beta |a_n|\,|x|^n\,dm(x) &\le \sum_{n=0}^\infty \left(\int_0^{|\beta|} |a_n|\,|x|^n\,dm(x) + \int_0^{|\alpha|} |a_n|\,|x|^n\,dm(x)\right) \\ &\le \sum_{n=0}^\infty |a_n| \frac{|\beta|^{n+1}+|\alpha|^{n+1}}{n+1} \le 2r\sum_{n=0}^\infty |a_n|\,r^n < \infty\end{aligned}$$

where $r = \max(|\beta|,|\alpha|)$.

**Corollary 19.42 (Differentiation Under the Integral).** *Suppose that $J \subset \mathbb{R}$ is an open interval and $f : J \times X \to \mathbb{C}$ is a function such that*

1. *$x \to f(t,x)$ is measurable for each $t \in J$.*
2. *$f(t_0,\cdot) \in L^1(\mu)$ for some $t_0 \in J$.*
3. *$\frac{\partial f}{\partial t}(t,x)$ exists for all $(t,x)$.*
4. *There is a function $g \in \mathrm{L}^1(\mu)$ such that $\left|\frac{\partial f}{\partial t}(t,\cdot)\right| \le g \in \mathrm{L}^1(\mu)$ for each $t \in J$.*
   *Then $f(t,\cdot) \in \mathrm{L}^1(\mu)(\mu)$ for all $t \in J$ (i.e. $\int |f(t,x)|\,d\mu(x) < \infty$), $t \to \int_X f(t,x)d\mu(x)$ is a differentiable function on $J$ and*

$$\frac{d}{dt}\int_X f(t,x)d\mu(x) = \int_X \frac{\partial f}{\partial t}(t,x)d\mu(x).$$



*Proof.* (The proof is essentially the same as for sums.) By considering the real and imaginary parts of $f$ separately, we may assume that $f$ is real. Also notice that

$$\frac{\partial f}{\partial t}(t,x) = \lim_{n\to\infty} n(f(t+n^{-1},x) - f(t,x))$$

and therefore, for $x \to \frac{\partial f}{\partial t}(t,x)$ is a sequential limit of measurable functions and hence is measurable for all $t \in J$. By the mean value theorem,

$$|f(t,x) - f(t_0,x)| \le g(x)\,|t-t_0| \text{ for all } t \in J \tag{19.13}$$

and hence

$$|f(t,x)| \le |f(t,x)-f(t_0,x)| + |f(t_0,x)| \le g(x)\,|t-t_0| + |f(t_0,x)|\,.$$

This shows $f(t,\cdot) \in \mathrm{L}^1(\mu)$ for all $t \in J$. Let $G(t) := \int_X f(t,x)d\mu(x)$, then

$$\frac{G(t)-G(t_0)}{t-t_0} = \int_X \frac{f(t,x)-f(t_0,x)}{t-t_0}d\mu(x).$$

By assumption,

$$\lim_{t\to t_0}\frac{f(t,x)-f(t_0,x)}{t-t_0} = \frac{\partial f}{\partial t}(t,x) \text{ for all } x \in X$$

and by Eq. (19.13),

$$\left|\frac{f(t,x)-f(t_0,x)}{t-t_0}\right| \le g(x) \text{ for all } t \in J \text{ and } x \in X.$$

Therefore, we may apply the dominated convergence theorem to conclude

$$\begin{aligned}\lim_{n\to\infty}\frac{G(t_n)-G(t_0)}{t_n-t_0} &= \lim_{n\to\infty}\int_X \frac{f(t_n,x)-f(t_0,x)}{t_n-t_0}d\mu(x) \\ &= \int_X \lim_{n\to\infty}\frac{f(t_n,x)-f(t_0,x)}{t_n-t_0}d\mu(x) \\ &= \int_X \frac{\partial f}{\partial t}(t_0,x)d\mu(x)\end{aligned}$$

for **all** sequences $t_n \in J\setminus\{t_0\}$ such that $t_n \to t_0$. Therefore, $\dot{G}(t_0) = \lim_{t\to t_0}\frac{G(t)-G(t_0)}{t-t_0}$ exists and

$$\dot{G}(t_0) = \int_X \frac{\partial f}{\partial t}(t_0,x)d\mu(x).$$

■

*Example 19.43.* Recall from Example 19.23 that

$$\lambda^{-1} = \int_{[0,\infty)} e^{-\lambda x} dm(x) \text{ for all } \lambda > 0.$$

Let $\varepsilon > 0$. For $\lambda \geq 2\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $C_n(\varepsilon) < \infty$ such that

$$0 \leq \left(-\frac{d}{d\lambda}\right)^n e^{-\lambda x} = x^n e^{-\lambda x} \leq C(\varepsilon) e^{-\varepsilon x}.$$

Using this fact, Corollary 19.42 and induction gives

$$n!\lambda^{-n-1} = \left(-\frac{d}{d\lambda}\right)^n \lambda^{-1} = \int_{[0,\infty)} \left(-\frac{d}{d\lambda}\right)^n e^{-\lambda x} dm(x)$$
$$= \int_{[0,\infty)} x^n e^{-\lambda x} dm(x).$$

That is $n! = \lambda^n \int_{[0,\infty)} x^n e^{-\lambda x} dm(x)$. Recall that

$$\Gamma(t) := \int_{[0,\infty)} x^{t-1} e^{-x} dx \text{ for } t > 0.$$

(The reader should check that $\Gamma(t) < \infty$ for all $t > 0$.) We have just shown that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

*Remark 19.44.* Corollary 19.42 may be generalized by allowing the hypothesis to hold for $x \in X \setminus E$ where $E \in \mathcal{M}$ is a **fixed** null set, i.e. $E$ must be independent of $t$. Consider what happens if we formally apply Corollary 19.42 to $g(t) := \int_0^\infty 1_{x \leq t} dm(x)$,

$$\dot{g}(t) = \frac{d}{dt} \int_0^\infty 1_{x \leq t} dm(x) \stackrel{?}{=} \int_0^\infty \frac{\partial}{\partial t} 1_{x \leq t} dm(x).$$

The last integral is zero since $\frac{\partial}{\partial t} 1_{x \leq t} = 0$ unless $t = x$ in which case it is not defined. On the other hand $g(t) = t$ so that $\dot{g}(t) = 1$. (The reader should decide which hypothesis of Corollary 19.42 has been violated in this example.)

## 19.5 Measurability on Complete Measure Spaces

In this subsection we will discuss a couple of measurability results concerning completions of measure spaces.

**Proposition 19.45.** *Suppose that $(X, \mathcal{M}, \mu)$ is a complete measure space*[1] *and $f : X \to \mathbb{R}$ is measurable.*

[1] Recall this means that if $N \subset X$ is a set such that $N \subset A \in \mathcal{M}$ and $\mu(A) = 0$, then $N \in \mathcal{M}$ as well.



1. *If $g : X \to \mathbb{R}$ is a function such that $f(x) = g(x)$ for $\mu$ – a.e. $x$, then $g$ is measurable.*
2. *If $f_n : X \to \mathbb{R}$ are measurable and $f : X \to \mathbb{R}$ is a function such that $\lim_{n\to\infty} f_n = f$, $\mu$ - a.e., then $f$ is measurable as well.*

*Proof.* 1. Let $E = \{x : f(x) \neq g(x)\}$ which is assumed to be in $\mathcal{M}$ and $\mu(E) = 0$. Then $g = 1_{E^c} f + 1_E g$ since $f = g$ on $E^c$. Now $1_{E^c} f$ is measurable so $g$ will be measurable if we show $1_E g$ is measurable. For this consider,

$$(1_E g)^{-1}(A) = \begin{cases} E^c \cup (1_E g)^{-1}(A \setminus \{0\}) & \text{if } 0 \in A \\ (1_E g)^{-1}(A) & \text{if } 0 \notin A \end{cases} \tag{19.14}$$

Since $(1_E g)^{-1}(B) \subset E$ if $0 \notin B$ and $\mu(E) = 0$, it follow by completeness of $\mathcal{M}$ that $(1_E g)^{-1}(B) \in \mathcal{M}$ if $0 \notin B$. Therefore Eq. (19.14) shows that $1_E g$ is measurable.

2. Let $E = \{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}$ by assumption $E \in \mathcal{M}$ and $\mu(E) = 0$. Since $g := 1_E f = \lim_{n\to\infty} 1_{E^c} f_n$, $g$ is measurable. Because $f = g$ on $E^c$ and $\mu(E) = 0$, $f = g$ a.e. so by part 1. $f$ is also measurable. ■

The above results are in general false if $(X, \mathcal{M}, \mu)$ is not complete. For example, let $X = \{0, 1, 2\}$, $\mathcal{M} = \{\{0\}, \{1, 2\}, X, \phi\}$ and $\mu = \delta_0$. Take $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, then $g = 0$ a.e. yet $g$ is not measurable.

**Lemma 19.46.** *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $\bar{\mathcal{M}}$ is the completion of $\mathcal{M}$ relative to $\mu$ and $\bar{\mu}$ is the extension of $\mu$ to $\bar{\mathcal{M}}$. Then a function $f : X \to \mathbb{R}$ is $(\bar{\mathcal{M}}, \mathcal{B} = \mathcal{B}_{\mathbb{R}})$ – measurable iff there exists a function $g : X \to \mathbb{R}$ that is $(\mathcal{M}, \mathcal{B})$ – measurable such $E = \{x : f(x) \neq g(x)\} \in \bar{\mathcal{M}}$ and $\bar{\mu}(E) = 0$, i.e. $f(x) = g(x)$ for $\bar{\mu}$ – a.e. $x$. Moreover for such a pair $f$ and $g$, $f \in L^1(\bar{\mu})$ iff $g \in L^1(\mu)$ and in which case*

$$\int_X f d\bar{\mu} = \int_X g d\mu.$$

*Proof.* Suppose first that such a function $g$ exists so that $\bar{\mu}(E) = 0$. Since $g$ is also $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable, we see from Proposition 19.45 that $f$ is $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable.

Conversely if $f$ is $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable, by considering $f_\pm$ we may assume that $f \geq 0$. Choose $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable simple function $\phi_n \geq 0$ such that $\phi_n \uparrow f$ as $n \to \infty$. Writing

$$\phi_n = \sum a_k 1_{A_k}$$

with $A_k \in \bar{\mathcal{M}}$, we may choose $B_k \in \mathcal{M}$ such that $B_k \subset A_k$ and $\bar{\mu}(A_k \setminus B_k) = 0$. Letting

$$\tilde{\phi}_n := \sum a_k 1_{B_k}$$

we have produced a $(\mathcal{M}, \mathcal{B})$ – measurable simple function $\tilde{\phi}_n \geq 0$ such that $E_n := \{\phi_n \neq \tilde{\phi}_n\}$ has zero $\bar{\mu}$ – measure. Since $\bar{\mu}(\cup_n E_n) \leq \sum_n \bar{\mu}(E_n)$, there exists $F \in \mathcal{M}$ such that $\cup_n E_n \subset F$ and $\mu(F) = 0$. It now follows that

$$1_F\tilde{\phi}_n = 1_F\phi_n \uparrow g := 1_F f \text{ as } n \to \infty.$$

This shows that $g = 1_F f$ is $(\mathcal{M}, \mathcal{B})$ – measurable and that $\{f \neq g\} \subset F$ has $\bar{\mu}$ – measure zero.

Since $f = g$, $\bar{\mu}$ – a.e., $\int_X f d\bar{\mu} = \int_X g d\bar{\mu}$ so to prove Eq. (19.15) it suffices to prove

$$\int_X g d\bar{\mu} = \int_X g d\mu. \tag{19.15}$$

Because $\bar{\mu} = \mu$ on $\mathcal{M}$, Eq. (19.15) is easily verified for non-negative $\mathcal{M}$ – measurable simple functions. Then by the monotone convergence theorem and the approximation Theorem 18.42 it holds for all $\mathcal{M}$ – measurable functions $g : X \to [0, \infty]$. The rest of the assertions follow in the standard way by considering $(\operatorname{Re} g)_\pm$ and $(\operatorname{Im} g)_\pm$. ■

## 19.6 Comparison of the Lebesgue and the Riemann Integral

For the rest of this chapter, let $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. A partition of $[a, b]$ is a finite subset $\pi \subset [a, b]$ containing $\{a, b\}$. To each partition

$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\} \tag{19.16}$$

of $[a, b]$ let

$$\operatorname{mesh}(\pi) := \max\{|t_j - t_{j-1}| : j = 1, \ldots, n\},$$

$$M_j = \sup\{f(x) : t_j \le x \le t_{j-1}\}, \quad m_j = \inf\{f(x) : t_j \le x \le t_{j-1}\}$$

$$G_\pi = f(a)1_{\{a\}} + \sum_1^n M_j 1_{(t_{j-1}, t_j]}, \quad g_\pi = f(a)1_{\{a\}} + \sum_1^n m_j 1_{(t_{j-1}, t_j]} \text{ and}$$

$$S_\pi f = \sum M_j (t_j - t_{j-1}) \text{ and } s_\pi f = \sum m_j (t_j - t_{j-1}).$$

Notice that

$$S_\pi f = \int_a^b G_\pi dm \text{ and } s_\pi f = \int_a^b g_\pi dm.$$

The upper and lower Riemann integrals are defined respectively by

$$\overline{\int_a^b} f(x)dx = \inf_\pi S_\pi f \text{ and } \underline{\int_b^a} f(x)dx = \sup_\pi \; s_\pi f.$$

**Definition 19.47.** *The function $f$ is **Riemann integrable** iff $\overline{\int_a^b} f = \underline{\int_a^b} f$ and which case the Riemann integral $\int_a^b f$ is defined to be the common value:*

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$



The proof of the following Lemma is left as an exercise to the reader.

**Lemma 19.48.** *If $\pi'$ and $\pi$ are two partitions of $[a, b]$ and $\pi \subset \pi'$ then*

$$G_\pi \ge G_{\pi'} \ge f \ge g_{\pi'} \ge g_\pi \text{ and}$$
$$S_\pi f \ge S_{\pi'} f \ge s_{\pi'} f \ge s_\pi f.$$

*There exists an increasing sequence of partitions $\{\pi_k\}_{k=1}^\infty$ such that $\operatorname{mesh}(\pi_k) \downarrow 0$ and*

$$S_{\pi_k} f \downarrow \overline{\int_a^b} f \text{ and } s_{\pi_k} f \uparrow \underline{\int_a^b} f \text{ as } k \to \infty.$$

If we let

$$G := \lim_{k\to\infty} G_{\pi_k} \text{ and } g := \lim_{k\to\infty} g_{\pi_k} \tag{19.17}$$

then by the dominated convergence theorem,

$$\int_{[a,b]} g dm = \lim_{k\to\infty} \int_{[a,b]} g_{\pi_k} = \lim_{k\to\infty} s_{\pi_k} f = \underline{\int_a^b} f(x)dx \tag{19.18}$$

and

$$\int_{[a,b]} G dm = \lim_{k\to\infty} \int_{[a,b]} G_{\pi_k} = \lim_{k\to\infty} S_{\pi_k} f = \overline{\int_a^b} f(x)dx. \tag{19.19}$$

**Notation 19.49** *For $x \in [a, b]$, let*

$$H(x) = \limsup_{y\to x} f(y) := \lim_{\varepsilon\downarrow 0} \; \sup\{f(y) : |y - x| \le \varepsilon, \; y \in [a, b]\} \text{ and}$$
$$h(x) = \liminf_{y\to x} f(y) := \lim_{\varepsilon\downarrow 0} \; \inf \; \{f(y) : |y - x| \le \varepsilon, \; y \in [a, b]\}.$$

**Lemma 19.50.** *The functions $H, h : [a, b] \to \mathbb{R}$ satisfy:*

1. *$h(x) \le f(x) \le H(x)$ for all $x \in [a, b]$ and $h(x) = H(x)$ iff $f$ is continuous at $x$.*
2. *If $\{\pi_k\}_{k=1}^\infty$ is any increasing sequence of partitions such that $\operatorname{mesh}(\pi_k) \downarrow 0$ and $G$ and $g$ are defined as in Eq. (19.17), then*

$$G(x) = H(x) \ge f(x) \ge h(x) = g(x) \quad \forall \; x \notin \pi := \cup_{k=1}^\infty \pi_k. \tag{19.20}$$

*(Note $\pi$ is a countable set.)*

3. *$H$ and $h$ are Borel measurable.*

*Proof.* Let $G_k := G_{\pi_k} \downarrow G$ and $g_k := g_{\pi_k} \uparrow g$.

1. It is clear that $h(x) \le f(x) \le H(x)$ for all $x$ and $H(x) = h(x)$ iff $\lim_{y\to x} f(y)$ exists and is equal to $f(x)$. That is $H(x) = h(x)$ iff $f$ is continuous at $x$.

2. For $x \notin \pi$,
$$G_k(x) \geq H(x) \geq f(x) \geq h(x) \geq g_k(x) \; \forall \; k$$
and letting $k \to \infty$ in this equation implies
$$G(x) \geq H(x) \geq f(x) \geq h(x) \geq g(x) \; \forall \; x \notin \pi. \tag{19.21}$$
Moreover, given $\varepsilon > 0$ and $x \notin \pi$,
$$\sup\{f(y) : |y - x| \leq \varepsilon, \; y \in [a, b]\} \geq G_k(x)$$
for all $k$ large enough, since eventually $G_k(x)$ is the supremum of $f(y)$ over some interval contained in $[x-\varepsilon, x+\varepsilon]$. Again letting $k \to \infty$ implies $\sup_{|y-x|\leq\varepsilon} f(y) \geq G(x)$ and therefore, that
$$H(x) = \limsup_{y \to x} f(y) \geq G(x)$$
for all $x \notin \pi$. Combining this equation with Eq. (19.21) then implies $H(x) = G(x)$ if $x \notin \pi$. A similar argument shows that $h(x) = g(x)$ if $x \notin \pi$ and hence Eq. (19.20) is proved.
3. The functions $G$ and $g$ are limits of measurable functions and hence measurable. Since $H = G$ and $h = g$ except possibly on the countable set $\pi$, both $H$ and $h$ are also Borel measurable. (You justify this statement.)

■

**Theorem 19.51.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then*
$$\overline{\int_a^b} f = \int_{[a,b]} H dm \text{ and } \underline{\int_a^b} f = \int_{[a,b]} h dm \tag{19.22}$$
*and the following statements are equivalent:*

1. *$H(x) = h(x)$ for $m$ -a.e. $x$,*
2. *the set*
$$E := \{x \in [a, b] : f \text{ is discontinuous at } x\}$$
*is an $\bar{m}$ – null set.*
3. *$f$ is Riemann integrable.*

*If $f$ is Riemann integrable then $f$ is Lebesgue measurable*[2]*, i.e. $f$ is $\mathcal{L}/\mathcal{B}$ – measurable where $\mathcal{L}$ is the Lebesgue $\sigma$ – algebra and $\mathcal{B}$ is the Borel $\sigma$ – algebra on $[a, b]$. Moreover if we let $\bar{m}$ denote the completion of $m$, then*
$$\int_{[a,b]} H dm = \int_a^b f(x)dx = \int_{[a,b]} f d\bar{m} = \int_{[a,b]} h dm. \tag{19.23}$$

[2] $f$ need not be Borel measurable.



*Proof.* Let $\{\pi_k\}_{k=1}^\infty$ be an increasing sequence of partitions of $[a, b]$ as described in Lemma 19.48 and let $G$ and $g$ be defined as in Lemma 19.50. Since $m(\pi) = 0$, $H = G$ a.e., Eq. (19.22) is a consequence of Eqs. (19.18) and (19.19). From Eq. (19.22), $f$ is Riemann integrable iff
$$\int_{[a,b]} H dm = \int_{[a,b]} h dm$$
and because $h \leq f \leq H$ this happens iff $h(x) = H(x)$ for $m$ - a.e. $x$. Since $E = \{x : H(x) \neq h(x)\}$, this last condition is equivalent to $E$ being a $m$ – null set. In light of these results and Eq. (19.20), the remaining assertions including Eq. (19.23) are now consequences of Lemma 19.46. ■

**Notation 19.52** *In view of this theorem we will often write $\int_a^b f(x)dx$ for $\int_a^b f dm$.*

## 19.7 Determining Classes for Measures

**Theorem 19.53 (Uniqueness).** *Suppose that $\mathcal{C} \subset 2^X$ is a $\pi$ – class such that $\mathcal{M} = \sigma(\mathcal{C})$. If $\mu$ and $\nu$ are two measures on $\mathcal{M}$ and there exists $X_n \in \mathcal{C}$ such that $X_n \uparrow X$ and $\mu(X_n) = \nu(X_n) < \infty$ for each $n$, then $\mu = \nu$ on $\mathcal{M}$.*

*Proof.* We begin first with the special case where $\mu(X) < \infty$ and therefore also
$$\nu(X) = \lim_{n\to\infty} \nu(X_n) = \lim_{n\to\infty} \mu(X_n) = \mu(X) < \infty.$$
Let
$$\mathcal{H} := \{f \in \ell^\infty(\mathcal{M}, \mathbb{R}) : \mu(f) = \nu(f)\}.$$
Then $\mathcal{H}$ is a linear subspace which is closed under bounded convergence, contains 1 and contains the multiplicative system, $M := \{1_C : C \in \mathcal{C}\}$. Therefore, by Theorem 18.51 or Corollary 18.54, $\mathcal{H} = \ell^\infty(\mathcal{M}, \mathbb{R})$ and hence $\mu = \nu$.

For the general $\sigma$ – finite case, let $X_n \in \mathcal{C}$ be as in the statement and define two measures $\mu_n$ and $\nu_n$ on $\mathcal{M}$ for each $n$ by
$$\mu_n(A) := \mu(A \cap X_n) \text{ and } \nu_n(A) = \nu(A \cap X_n).$$
Then, as the reader should verify, $\mu_n$ and $\nu_n$ are finite measure on $\mathcal{M}$ such that $\mu_n = \nu_n$ on $\mathcal{C}$. Therefore, by the special case just proved, $\mu_n = \nu_n$ on $\mathcal{M}$. Finally, using the continuity of the measures, $\mu$ and $\nu$,
$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) = \lim_{n\to\infty} \nu(A \cap X_n) = \nu(A)$$
for all $A \in \mathcal{M}$. ■

As an immediate consequence we have the following corollaries.

**Corollary 19.54.** *Suppose that $(X,\tau)$ is a topological space, $\mathcal{B}_X = \sigma(\tau)$ is the Borel $\sigma$ – algebra on $X$ and $\mu$ and $\nu$ are two measures on $\mathcal{B}_X$ which are $\sigma$ – finite on $\tau$. If $\mu = \nu$ on $\tau$ then $\mu = \nu$ on $\mathcal{B}_X$, i.e. $\mu := \nu$.*

**Corollary 19.55.** *Suppose that $\mu$ and $\nu$ are two measures on $\mathcal{B}_{\mathbb{R}^n}$ which are finite on bounded sets and such that $\mu(A) = \nu(A)$ for all sets $A$ of the form*

$$A = (a,b] = (a_1,b_1] \times \cdots \times (a_n,b_n]$$

*with $a,b \in \mathbb{R}^n$ and $a \le b$, i.e. $a_i \le b_i$ for all $i$. Then $\mu = \nu$ on $\mathcal{B}_{\mathbb{R}^n}$.*

**Proposition 19.56.** *Suppose that $(X,d)$ is a metric space, $\mu$ and $\nu$ are two measures on $\mathcal{B}_X = \sigma(\tau_d)$ which are finite on bounded measurable subsets of $X$ and*

$$\int_X f d\mu = \int_X f d\nu \tag{19.24}$$

*for all $f \in BC_b(X,\mathbb{R})$ where*

$$BC_b(X,\mathbb{R}) = \{f \in BC(X,\mathbb{R}) : \operatorname{supp}(f) \text{ is bounded}\}.$$

*Then $\mu \equiv \nu$.*

*Proof.* To prove this fix a $o \in X$ and let

$$\psi_R(x) = ([R+1-d(x,o)] \wedge 1) \vee 0$$

so that $\psi_R \in BC_b(X,[0,1])$, $\operatorname{supp}(\psi_R) \subset B(o,R+2)$ and $\psi_R \uparrow 1$ as $R \to \infty$. Let $\mathcal{H}_R$ denote the space of bounded real valued $\mathcal{B}_X$ – measurable functions $f$ such that

$$\int_X \psi_R f d\mu = \int_X \psi_R f d\nu. \tag{19.25}$$

Then $\mathcal{H}_R$ is closed under bounded convergence and because of Eq. (19.24) contains $BC(X,\mathbb{R})$. Therefore by Corollary 18.55, $\mathcal{H}_R$ contains all bounded measurable functions on $X$. Take $f = 1_A$ in Eq. (19.25) with $A \in \mathcal{B}_X$, and then use the monotone convergence theorem to let $R \to \infty$. The result is $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}_X$. ■

Here is another version of Proposition 19.56.

**Proposition 19.57.** *Suppose that $(X,d)$ is a metric space, $\mu$ and $\nu$ are two measures on $\mathcal{B}_X = \sigma(\tau_d)$ which are both finite on compact sets. Further assume there exists compact sets $K_k \subset X$ such that $K_k^o \uparrow X$. If*

$$\int_X f d\mu = \int_X f d\nu \tag{19.26}$$

*for all $f \in C_c(X,\mathbb{R})$ then $\mu \equiv \nu$.*



*Proof.* Let $\psi_{n,k}$ be defined as in the proof of Proposition 18.56 and let $\mathcal{H}_{n,k}$ denote those bounded $\mathcal{B}_X$ – measurable functions, $f : X \to \mathbb{R}$ such that

$$\int_X f\psi_{n,k} d\mu = \int_X f\psi_{n,k} d\nu.$$

By assumption $BC(X,\mathbb{R}) \subset \mathcal{H}_{n,k}$ and one easily checks that $\mathcal{H}_{n,k}$ is closed under bounded convergence. Therefore, by Corollary 18.55, $\mathcal{H}_{n,k}$ contains all bounded measurable function. In particular for $A \in \mathcal{B}_X$,

$$\int_X 1_A \psi_{n,k} d\mu = \int_X 1_A \psi_{n,k} d\nu.$$

Letting $n \to \infty$ in this equation, using the dominated convergence theorem, one shows

$$\int_X 1_A 1_{K_k^o} d\mu = \int_X 1_A 1_{K_k^o} d\nu$$

holds for $k$. Finally using the monotone convergence theorem we may let $k \to \infty$ to conclude

$$\mu(A) = \int_X 1_A d\mu = \int_X 1_A d\nu = \nu(A)$$

for all $A \in \mathcal{B}_X$. ■

## 19.8 Exercises

**Exercise 19.1.** Let $\mu$ be a measure on an algebra $\mathcal{A} \subset 2^X$, then $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$ for all $A, B \in \mathcal{A}$.

**Exercise 19.2 (From problem 12 on p. 27 of Folland.).** Let $(X,\mathcal{M},\mu)$ be a finite measure space and for $A, B \in \mathcal{M}$ let $\rho(A,B) = \mu(A\Delta B)$ where $A\Delta B = (A \setminus B) \cup (B \setminus A)$. It is clear that $\rho(A,B) = \rho(B,A)$. Show:

1. $\rho$ satisfies the triangle inequality:
$$\rho(A,C) \le \rho(A,B) + \rho(B,C) \text{ for all } A,B,C \in \mathcal{M}.$$
2. Define $A \sim B$ iff $\mu(A\Delta B) = 0$ and notice that $\rho(A,B) = 0$ iff $A \sim B$. Show "$\sim$" is an equivalence relation.
3. Let $\mathcal{M}/\sim$ denote $\mathcal{M}$ modulo the equivalence relation, $\sim$, and let $[A] := \{B \in \mathcal{M} : B \sim A\}$. Show that $\bar\rho([A],[B]) := \rho(A,B)$ is gives a well defined metric on $\mathcal{M}/\sim$.
4. Similarly show $\tilde\mu([A]) = \mu(A)$ is a well defined function on $\mathcal{M}/\sim$ and show $\tilde\mu : (\mathcal{M}/\sim) \to \mathbb{R}_+$ is $\bar\rho$ – continuous.

**Exercise 19.3.** Suppose that $\mu_n : \mathcal{M} \to [0,\infty]$ are measures on $\mathcal{M}$ for $n \in \mathbb{N}$. Also suppose that $\mu_n(A)$ is increasing in $n$ for all $A \in \mathcal{M}$. Prove that $\mu : \mathcal{M} \to [0,\infty]$ defined by $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ is also a measure.

**Exercise 19.4.** Now suppose that $\Lambda$ is some index set and for each $\lambda \in \Lambda$, $\mu_\lambda : \mathcal{M} \to [0,\infty]$ is a measure on $\mathcal{M}$. Define $\mu : \mathcal{M} \to [0,\infty]$ by $\mu(A) = \sum_{\lambda\in\Lambda} \mu_\lambda(A)$ for each $A \in \mathcal{M}$. Show that $\mu$ is also a measure.

**Exercise 19.5.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $\rho : X \to [0,\infty]$ be a measurable function. For $A \in \mathcal{M}$, set $\nu(A) := \int_A \rho d\mu$.

1. Show $\nu : \mathcal{M} \to [0,\infty]$ is a measure.
2. Let $f : X \to [0,\infty]$ be a measurable function, show
$$\int_X f d\nu = \int_X f \rho d\mu. \tag{19.27}$$
**Hint:** first prove the relationship for characteristic functions, then for simple functions, and then for general positive measurable functions.
3. Show that $f \in L^1(\nu)$ iff $f\rho \in L^1(\mu)$ and if $f \in L^1(\nu)$ then Eq. (19.27) still holds.

**Notation 19.58** *It is customary to informally describe $\nu$ defined in Exercise 19.5 by writing $d\nu = \rho d\mu$.*

**Exercise 19.6.** Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{F})$ be a measurable space and $f : X \to Y$ be a measurable map. Define a function $\nu : \mathcal{F} \to [0,\infty]$ by $\nu(A) := \mu(f^{-1}(A))$ for all $A \in \mathcal{F}$.

1. Show $\nu$ is a measure. (We will write $\nu = f_*\mu$ or $\nu = \mu \circ f^{-1}$.)
2. Show
$$\int_Y g d\nu = \int_X (g \circ f)\, d\mu \tag{19.28}$$
for all measurable functions $g : Y \to [0,\infty]$. **Hint:** see the hint from Exercise 19.5.
3. Show $g \in L^1(\nu)$ iff $g \circ f \in L^1(\mu)$ and that Eq. (19.28) holds for all $g \in L^1(\nu)$.

**Exercise 19.7.** Let $F : \mathbb{R} \to \mathbb{R}$ be a $C^1$-function such that $F'(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to\pm\infty} F(x) = \pm\infty$. (Notice that $F$ is strictly increasing so that $F^{-1} : \mathbb{R} \to \mathbb{R}$ exists and moreover, by the implicit function theorem that $F^{-1}$ is a $C^1$ – function.) Let $m$ be Lebesgue measure on $\mathcal{B}_\mathbb{R}$ and

$$\nu(A) = m(F(A)) = m(\left(F^{-1}\right)^{-1}(A)) = \left(F^{-1}_* m\right)(A)$$

for all $A \in \mathcal{B}_\mathbb{R}$. Show $d\nu = F'dm$. Use this result to prove the change of variable formula,



$$\int_\mathbb{R} h \circ F \cdot F' dm = \int_\mathbb{R} h dm \tag{19.29}$$

which is valid for all Borel measurable functions $h : \mathbb{R} \to [0,\infty]$.

**Hint:** Start by showing $d\nu = F'dm$ on sets of the form $A = (a,b]$ with $a, b \in \mathbb{R}$ and $a < b$. Then use the uniqueness assertions in Theorem 19.8 (or see Corollary 19.55) to conclude $d\nu = F'dm$ on all of $\mathcal{B}_\mathbb{R}$. To prove Eq. (19.29) apply Exercise 19.6 with $g = h \circ F$ and $f = F^{-1}$.

**Exercise 19.8.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$, show

$$\mu(\{A_n \text{ a.a.}\}) \le \liminf_{n\to\infty} \mu(A_n)$$

and if $\mu(\cup_{m\ge n} A_m) < \infty$ for some $n$, then

$$\mu(\{A_n \text{ i.o.}\}) \ge \limsup_{n\to\infty} \mu(A_n).$$

**Exercise 19.9.** Suppose $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0\,\infty]$ be a measurable function such that $\int_X f d\mu < \infty$. Show $\mu(\{f = \infty\}) = 0$ and the set $\{f > 0\}$ is $\sigma$ – finite.

**Exercise 19.10.** Folland 2.13 on p. 52. **Hint:** "Fatou times two."

**Exercise 19.11.** Folland 2.14 on p. 52. BRUCE: delete this exercise

**Exercise 19.12.** Give examples of measurable functions $\{f_n\}$ on $\mathbb{R}$ such that $f_n$ decreases to 0 uniformly yet $\int f_n dm = \infty$ for all $n$. Also give an example of a sequence of measurable functions $\{g_n\}$ on $[0,1]$ such that $g_n \to 0$ while $\int g_n dm = 1$ for all $n$.

**Exercise 19.13.** Folland 2.19 on p. 59. (This problem is essentially covered in the previous exercise.)

**Exercise 19.14.** Suppose $\{a_n\}_{n=-\infty}^\infty \subset \mathbb{C}$ is a summable sequence (i.e. $\sum_{n=-\infty}^\infty |a_n| < \infty$), then $f(\theta) := \sum_{n=-\infty}^\infty a_n e^{in\theta}$ is a continuous function for $\theta \in \mathbb{R}$ and

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

**Exercise 19.15.** For any function $f \in L^1(m)$, show $x \in \mathbb{R} \to \int_{(-\infty,x]} f(t)\, dm(t)$ is continuous in $x$. Also find a finite measure, $\mu$, on $\mathcal{B}_\mathbb{R}$ such that $x \to \int_{(-\infty,x]} f(t)\, d\mu(t)$ is not continuous.

**Exercise 19.16.** Folland 2.28 on p. 60.

**Exercise 19.17.** Folland 2.31b on p. 60.

**Exercise 19.18.** There exists a meager (see Definition 13.4 and Proposition 13.3) subsets of $\mathbb{R}$ which have full Lebesgue measure, i.e. whose complement is a Lebesgue null set. (This is Folland 5.27. **Hint:** Consider the generalized Cantor sets discussed on p. 39 of Folland.)