23

L^2 - Hilbert Spaces Techniques and Fourier Series

This section is concerned with Hilbert spaces presented as in the following example.

Example 23.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $H := L^2(X, \mathcal{M}, \mu)$ with inner product
$$\langle f|g\rangle = \int_X f \cdot \bar{g} d\mu$$
is a Hilbert space.

It will be convenient to define
$$\langle f, g\rangle := \int_X f(x)\, g(x)\, d\mu(x) \tag{23.1}$$
for all measurable functions f, g on X such that $fg \in L^1(\mu)$. So with this notation we have $\langle f|g\rangle = \langle f, \bar{g}\rangle$ for all $f, g \in H$.

Exercise 23.1. Let $K : L^2(\nu) \to L^2(\mu)$ be the operator defined in Exercise 21.12. Show $K^* : L^2(\mu) \to L^2(\nu)$ is the operator given by
$$K^* g(y) = \int_X \bar{k}(x, y) g(x) d\mu(x).$$

23.1 L^2-Orthonoramal Basis

Example 23.2. 1. Let $H = L^2([-1, 1], dm)$ and $A := \{1, x, x^2, x^3 \dots\}$. Then A is total in H by the Stone-Weierstrass theorem and a similar argument as in the first example or directly from Exercise 22.13. The result of doing Gram-Schmidt on this set gives an orthonormal basis of H consisting of the "**Legendre Polynomials.**"

2. Let $H = L^2(\mathbb{R}, e^{-\frac{1}{2}x^2}dx)$.Exercise 22.13 implies $A := \{1, x, x^2, x^3 \dots\}$ is total in H and the result of doing Gram-Schmidt on A now gives an orthonormal basis for H consisting of "**Hermite Polynomials.**"

Remark 23.3 (An Interesting Phenomena). Let $H = L^2([-1, 1], dm)$ and $B := \{1, x^3, x^6, x^9, \dots\}$. Then again A is total in H by the same argument as in item 2. Example 23.2. This is true even though B is a proper subset of A. Notice that A is an algebraic basis for the polynomials on $[-1, 1]$ while B is not! The following computations may help relieve some of the reader's anxiety. Let $f \in L^2([-1, 1], dm)$, then, making the change of variables $x = y^{1/3}$, shows that
$$\int_{-1}^1 |f(x)|^2 dx = \int_{-1}^1 \left|f(y^{1/3})\right|^2 \frac{1}{3} y^{-2/3} dy = \int_{-1}^1 \left|f(y^{1/3})\right|^2 d\mu(y) \tag{23.2}$$
where $d\mu(y) = \frac{1}{3} y^{-2/3} dy$. Since $\mu([-1, 1]) = m([-1, 1]) = 2$, μ is a finite measure on $[-1, 1]$ and hence by Exercise 22.13 $A := \{1, x, x^2, x^3 \dots\}$ is a total (see Definition 14.25) in $L^2([-1, 1], d\mu)$. In particular for any $\varepsilon > 0$ there exists a polynomial $p(y)$ such that
$$\int_{-1}^1 \left|f(y^{1/3}) - p(y)\right|^2 d\mu(y) < \varepsilon^2.$$

However, by Eq. (23.2) we have
$$\varepsilon^2 > \int_{-1}^1 \left|f(y^{1/3}) - p(y)\right|^2 d\mu(y) = \int_{-1}^1 \left|f(x) - p(x^3)\right|^2 dx.$$

Alternatively, if $f \in C([-1, 1])$, then $g(y) = f(y^{1/3})$ is back in $C([-1, 1])$. Therefore for any $\varepsilon > 0$, there exists a polynomial $p(y)$ such that
$$\begin{aligned}
\varepsilon > \|g - p\|_\infty &= \sup\{|g(y) - p(y)| : y \in [-1, 1]\} \\
&= \sup\left\{\left|g(x^3) - p(x^3)\right| : x \in [-1, 1]\right\} \\
&= \sup\left\{\left|f(x) - p(x^3)\right| : x \in [-1, 1]\right\}.
\end{aligned}$$

This gives another proof the polynomials in x^3 are dense in $C([-1, 1])$ and hence in $L^2([-1, 1])$.

Exercise 23.2. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces such that $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_n\}_{n=1}^\infty$ and $\{g_m\}_{m=1}^\infty$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ respectively, then $\beta := \{f_n \otimes g_m : m, n \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\mu \otimes \nu)$. (Recall that $f \otimes g(x, y) := f(x) g(y)$, see Notation 20.4.) **Hint:** model your proof of the proof of Proposition 14.28.

Exercise 23.3. Suppose H is a Hilbert space and $\{H_n : n \in \mathbb{N}\}$ are closed subspaces of H such that $H_n \perp H_m$ for all $m \neq n$ and if $f \in H$ with $f \perp H_n$ for all $n \in \mathbb{N}$, then $f = 0$. Show:

1. If $f_n \in H_n$ for all $n \in \mathbb{N}$ satisfy $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$ then $\sum_{n=1}^{\infty} f_n$ exists in H.
2. Every element $f \in H$ may be uniquely written as $f = \sum_{n=1}^{\infty} f_n$ with $f_n \in H$ as in item 1.
(For this reason we will write $H = \oplus_{n=1}^{\infty} H_n$ under the hypothesis of this exercise.)

Exercise 23.4. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $X = \coprod_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{M}$ and $\mu(X_n) > 0$ for all n. Then $U : L^2(X, \mu) \to \oplus_{n=1}^{\infty} L^2(X_n, \mu)$ defined by $(Uf)_n := f1_{X_n}$ is unitary.

23.2 Hilbert Schmidt Operators

In this section H and B will be Hilbert spaces. Typically H and B will be separable, but we will not assume this until it is needed later.

Proposition 23.4. *Let H and B be a separable Hilbert spaces, $K : H \to B$ be a bounded linear operator, $\{e_n\}_{n=1}^{\infty}$ and $\{u_m\}_{m=1}^{\infty}$ be orthonormal basis for H and B respectively. Then:*

1. *$\sum_{n=1}^{\infty} \|Ke_n\|^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2$ allowing for the possibility that the sums are infinite. In particular the **Hilbert Schmidt norm** of K,*
$$\|K\|_{HS}^2 := \sum_{n=1}^{\infty} \|Ke_n\|^2,$$
*is well defined independent of the choice of orthonormal basis $\{e_n\}_{n=1}^{\infty}$. We say $K : H \to B$ is a **Hilbert Schmidt operator** if $\|K\|_{HS} < \infty$ and let $HS(H, B)$ denote the space of Hilbert Schmidt operators from H to B.*
2. *For all $K \in L(H, B)$, $\|K\|_{HS} = \|K^*\|_{HS}$ and*
$$\|K\|_{HS} \geq \|K\|_{op} := \sup\{\|Kh\| : h \in H \ni \|h\| = 1\}.$$
3. *The set $HS(H, B)$ is a subspace of $\mathcal{K}(H, B)$ (the compact operators from $H \to B$) and $\|\cdot\|_{HS}$ is a norm on $HS(H, B)$ for which $(HS(H, B), \|\cdot\|_{HS})$ is a Hilbert space. The inner product on $HS(H, B)$ is given by*
$$\langle K_1 | K_2 \rangle_{HS} = \sum_{n=1}^{\infty} \langle K_1 e_n | K_2 e_n \rangle. \tag{23.3}$$
4. *If $K : H \to B$ is a bounded finite rank operator, then K is Hilbert Schmidt.*
5. *Let $P_N x := \sum_{n=1}^{N} (x, e_n) e_n$ be orthogonal projection onto* span $\{e_i : i \leq N\} \subset$ *H and for $K \in HS(H, B)$, let $K_n := KP_n$. Then*
$$\|K - K_N\|_{op}^2 \leq \|K - K_N\|_{HS}^2 \to 0 \text{ as } N \to \infty,$$
which shows that finite rank operators are dense in $(HS(H, B), \|\cdot\|_{HS})$.

6. *If L is another Hilbert space and $A : L \to H$ and $C : B \to L$ are bounded operators, then*
$$\|KA\|_{HS} \leq \|K\|_{HS} \|A\|_{op} \text{ and } \|CK\|_{HS} \leq \|K\|_{HS} \|C\|_{op}.$$

Proof. Items 1. and 2. By Parseval's equality and Fubini's theorem for sums,
$$\begin{aligned}\sum_{n=1}^{\infty} \|Ke_n\|^2 &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |(Ke_n, u_m)|^2 \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |(e_n, K^* u_m)|^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2.\end{aligned}$$

This proves $\|K\|_{HS}$ is well defined independent of basis and that $\|K\|_{HS} = \|K^*\|_{HS}$. For $x \in H \setminus \{0\}$, $x/\|x\|$ may be taken to be the first element in an orthonormal basis for H and hence
$$\left\| K \frac{x}{\|x\|} \right\| \leq \|K\|_{HS}.$$

Multiplying this inequality by $\|x\|$ shows $\|Kx\| \leq \|K\|_{HS} \|x\|$ and hence $\|K\|_{op} \leq \|K\|_{HS}$.

Item 3. For $K_1, K_2 \in L(H, B)$,
$$\begin{aligned}\|K_1 + K_2\|_{HS} &= \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n + K_2 e_n\|^2} \\ &\leq \sqrt{\sum_{n=1}^{\infty} [\|K_1 e_n\| + \|K_2 e_n\|]^2} \\ &= \|\{\|K_1 e_n\| + \|K_2 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} \\ &\leq \|\{\|K_1 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} + \|\{\|K_2 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} \\ &= \|K_1\|_{HS} + \|K_2\|_{HS}.\end{aligned}$$

From this triangle inequality and the homogeneity properties of $\|\cdot\|_{HS}$, we now easily see that $HS(H, B)$ is a subspace of $\mathcal{K}(H, B)$ and $\|\cdot\|_{HS}$ is a norm on $HS(H, B)$. Since
$$\begin{aligned}\sum_{n=1}^{\infty} |\langle K_1 e_n | K_2 e_n \rangle| &\leq \sum_{n=1}^{\infty} \|K_1 e_n\| \|K_2 e_n\| \\ &\leq \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n\|^2} \sqrt{\sum_{n=1}^{\infty} \|K_2 e_n\|^2} = \|K_1\|_{HS} \|K_2\|_{HS},\end{aligned}$$

the sum in Eq. (23.3) is well defined and is easily checked to define an inner product on $HS(H,B)$ such that $\|K\|_{HS}^2 = \langle K_1, K_2\rangle_{HS}$. To see that $HS(H,B)$ is complete in this inner product suppose $\{K_m\}_{m=1}^\infty$ is a $\|\cdot\|_{HS}$ – Cauchy sequence in $HS(H,B)$. Because $L(H,B)$ is complete, there exists $K \in L(H,B)$ such that $\|K_m - K\|_{op} \to 0$ as $m \to \infty$. Since

$$\sum_{n=1}^N \|(K-K_m)\,e_n\|^2 = \lim_{l\to\infty} \sum_{n=1}^N \|(K_l - K_m)\,e_n\|^2 \le \limsup_{l\to\infty} \|K_l - K_m\|_{HS},$$

$$\begin{aligned}\|K_m - K\|_{HS}^2 &= \sum_{n=1}^\infty \|(K-K_m)\,e_n\|^2 = \lim_{N\to\infty}\sum_{n=1}^N \|(K-K_m)\,e_n\|^2 \\ &\le \limsup_{l\to\infty} \|K_l - K_m\|_{HS} \to 0 \text{ as } m\to\infty.\end{aligned}$$

Item 4. Let $\{v_n\}_{n=1}^{N:=\dim K(H)}$ be an orthonormal basis for $\mathrm{Ran}(K) = K(H)$. Then, for all $h \in H$,

$$\|Kh\|_B^2 = \sum_{n=1}^N |\langle Kh|v_n\rangle|^2 = \sum_{n=1}^N |\langle h|K^* v_n\rangle|^2.$$

Summing this equation on $h \in \Gamma$ where an Γ is an orthonormal basis for H shows

$$\|K\|_{HS}^2 = \sum_{h\in\Gamma} \|Kh\|_B^2 = \sum_{n=1}^N \|K^* v_n\|_H^2 < \infty.$$

Item 5. Simply observe,

$$\|K - K_N\|_{op}^2 \le \|K - K_N\|_{HS}^2 = \sum_{n>N} \|Ke_n\|^2 \to 0 \text{ as } N\to\infty.$$

Item 6. For $C \in L(B,L)$ and $K \in L(H,B)$ then

$$\|CK\|_{HS}^2 = \sum_{n=1}^\infty \|CKe_n\|^2 \le \|C\|_{op}^2 \sum_{n=1}^\infty \|Ke_n\|^2 = \|C\|_{op}^2 \|K\|_{HS}^2$$

and for $A \in L(L,H)$,

$$\|KA\|_{HS} = \|A^*K^*\|_{HS} \le \|A^*\|_{op}\|K^*\|_{HS} = \|A\|_{op}\|K\|_{HS}.$$

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Remark 23.5. The separability assumptions made in Proposition 23.4 are unnecessary. In general, we define

$$\|K\|_{HS}^2 = \sum_{e\in\Gamma} \|Ke\|^2$$

where $\Gamma \subset H$ is an orthonormal basis. The same proof of Item 1. of Proposition 23.4 shows $\|K\|_{HS}$ is well defined and $\|K\|_{HS} = \|K^*\|_{HS}$. If $\|K\|_{HS}^2 < \infty$, then there exists a countable subset $\Gamma_0 \subset \Gamma$ such that $Ke = 0$ if $e \in \Gamma\setminus\Gamma_0$. Let $H_0 := \overline{\mathrm{span}(\Gamma_0)}$ and $B_0 := \overline{K(H_0)}$. Then $K(H) \subset B_0$, $K|_{H_0^\perp} = 0$ and hence by applying the results of Proposition 23.4 to $K|_{H_0} : H_0 \to B_0$ one easily sees that the separability of H and B are unnecessary in Proposition 23.4.

Example 23.6. Let (X,μ) be a measure space, $H = L^2(X,\mu)$ and

$$k(x,y) := \sum_{i=1}^n f_i(x) g_i(y)$$

where

$$f_i, g_i \in L^2(X,\mu) \text{ for } i = 1,\dots,n.$$

Define $(Kf)(x) = \int_X k(x,y) f(y) d\mu(y)$, then $K : L^2(X,\mu) \to L^2(X,\mu)$ is a finite rank operator and hence Hilbert Schmidt.

Exercise 23.5. Suppose that (X,μ) is a σ–finite measure space such that $H = L^2(X,\mu)$ is separable and $k : X\times X \to \mathbb{R}$ is a measurable function, such that

$$\|k\|_{L^2(X\times X,\mu\otimes\mu)}^2 := \int_{X\times X} |k(x,y)|^2 d\mu(x) d\mu(y) < \infty.$$

Define, for $f \in H$,

$$Kf(x) = \int_X k(x,y) f(y) d\mu(y),$$

when the integral makes sense. Show:

1. $Kf(x)$ is defined for μ–a.e. x in X.
2. The resulting function Kf is in H and $K : H \to H$ is linear.
3. $\|K\|_{HS} = \|k\|_{L^2(X\times X,\mu\otimes\mu)} < \infty$. (This implies $K \in HS(H,H)$.)

Example 23.7. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded set, $\alpha < n$, then the operator $K : L^2(\Omega, m) \to L^2(\Omega, m)$ defined by

$$Kf(x) := \int_\Omega \frac{1}{|x-y|^\alpha} f(y) dy$$

is compact.

Proof. For $\varepsilon \ge 0$, let

$$K_\varepsilon f(x) := \int_\Omega \frac{1}{|x-y|^\alpha + \varepsilon} f(y) dy = [g_\varepsilon * (1_\Omega f)](x)$$

where $g_\varepsilon(x) = \frac{1}{|x|^\alpha+\varepsilon} 1_C(x)$ with $C \subset \mathbb{R}^n$ a sufficiently large ball such that $\Omega - \Omega \subset C$. Since $\alpha < n$, it follows that

$$g_\varepsilon \le g_0 = |\cdot|^{-\alpha} 1_C \in L^1(\mathbb{R}^n, m).$$

Hence it follows by Proposition 22.23 that

$$\begin{aligned}\|(K - K_\varepsilon) f\|_{L^2(\Omega)} &\le \|(g_0 - g_\varepsilon) * (1_\Omega f)\|_{L^2(\mathbb{R}^n)} \\ &\le \|(g_0 - g_\varepsilon)\|_{L^1(\mathbb{R}^n)} \|1_\Omega f\|_{L^2(\mathbb{R}^n)} \\ &= \|(g_0 - g_\varepsilon)\|_{L^1(\mathbb{R}^n)} \|f\|_{L^2(\Omega)}\end{aligned}$$

which implies

$$\begin{aligned}\|K - K_\varepsilon\|_{B(L^2(\Omega))} &\le \|g_0 - g_\varepsilon\|_{L^1(\mathbb{R}^n)} \\ &= \int_C \left| \frac{1}{|x|^\alpha + \varepsilon} - \frac{1}{|x|^\alpha} \right| dx \to 0 \text{ as } \varepsilon \downarrow 0 \qquad (23.4)\end{aligned}$$

by the dominated convergence theorem. For any $\varepsilon > 0$,

$$\int_{\Omega \times \Omega} \left[\frac{1}{|x-y|^\alpha + \varepsilon} \right]^2 dx dy < \infty,$$

and hence K_ε is Hilbert Schmidt and hence compact. By Eq. (23.4), $K_\varepsilon \to K$ as $\varepsilon \downarrow 0$ and hence it follows that K is compact as well. ∎

Exercise 23.6. Let $H := L^2([0,1], m)$, $k(x,y) := \min(x,y)$ for $x, y \in [0,1]$ and define $K : H \to H$ by

$$Kf(x) = \int_0^1 k(x,y) f(y) dy.$$

By Exercise 23.5, K is a Hilbert Schmidt operator and it is easily seen that K is self-adjoint. Show:

1. Show $\langle Kf | g'' \rangle = -\langle f | g \rangle$ for all $g \in C_c^\infty((0,1))$ and use this to conclude that $\mathrm{Nul}(K) = \{0\}$.
2. Now suppose that $f \in H$ is an eigenvector of K with eigenvalue $\lambda \neq 0$. Show that there is a version of f in $C([0,1]) \cap C^2((0,1))$ and this version, still denoted by f, solves
$$\lambda f'' = -f \text{ with } f(0) = f'(1) = 0. \qquad (23.5)$$
where $f'(1) := \lim_{x \uparrow 1} f'(x)$.
3. Use Eq. (23.5) to find all the eigenvalues and eigenfunctions of K.
4. Use the results above along with the spectral Theorem 14.45, to show
$$\left\{ \sqrt{2} \sin\left(n \frac{\pi}{2} x\right) : n \in \mathbb{N} \right\}$$
is an orthonormal basis for $L^2([0,1], m)$.

Exercise 23.7. Suppose $a \in L^\infty(X, \mathcal{M}, \mu)$ and let A be the bounded operator on $H := L^2(X, \mathcal{M}, \mu)$ defined by $Af(x) = a(x) f(x)$ for all $f \in H$. (We will denote A by M_a in the future.) Show:

1. A is a bounded operator and $A^* = M_{\bar{a}}$.
2. $\sigma(A) = \mathrm{essran}(a)$ where $\sigma(A)$ is the spectrum of A and $\mathrm{essran}(a)$ is the essential range of a, see Definitions 14.30 and 21.40 respectively.
3. Show λ is an eigenvalue for a iff $\mu(\{a = \lambda\}) > 0$.

23.3 Fourier Series Considerations

Throughout this section we will let $d\theta$, dx, $d\alpha$, etc. denote Lebesgue measure on $\mathbb{R}^d$ normalized so that the cube, $Q := (-\pi, \pi]^d$, has measure one, i.e. $d\theta = (2\pi)^{-d} dm(\theta)$ where m is standard Lebesgue measure on $\mathbb{R}^d$. As usual, for $\alpha \in \mathbb{N}_0^d$, let

$$D_\theta^\alpha = \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial \theta_1^{\alpha_1} \dots \partial \theta_d^{\alpha_d}}.$$

Notation 23.8 *Let $C_{per}^k(\mathbb{R}^d)$ denote the 2π – periodic functions in $C^k(\mathbb{R}^d)$, that is $f \in C_{per}^k(\mathbb{R}^d)$ iff $f \in C^k(\mathbb{R}^d)$ and $f(\theta + 2\pi e_i) = f(\theta)$ for all $\theta \in \mathbb{R}^d$ and $i = 1, 2, \dots, d$. Further let $\langle \cdot | \cdot \rangle$ denote the inner product on the Hilbert space, $H := L^2([-\pi, \pi]^d)$, given by*

$$\langle f | g \rangle := \int_Q f(\theta) \bar{g}(\theta) d\theta = \left(\frac{1}{2\pi}\right)^d \int_Q f(\theta) \bar{g}(\theta) dm(\theta)$$

and define $e_k(\theta) := e^{ik\cdot\theta}$ for all $k \in \mathbb{Z}^d$. For $f \in L^1(Q)$, we will write $\tilde{f}(k)$ for the ***Fourier coefficient****,*

$$\tilde{f}(k) := \langle f | e_k \rangle = \int_Q f(\theta) e^{-ik\cdot\theta} d\theta. \qquad (23.6)$$

Since any 2π – periodic functions on $\mathbb{R}^d$ may be identified with function on the d - dimensional torus, $\mathbb{T}^d \cong \mathbb{R}^d / (2\pi\mathbb{Z})^d \cong (S^1)^d$, I may also write $C^k(\mathbb{T}^d)$ for $C_{per}^k(\mathbb{R}^d)$ and $L^p(\mathbb{T}^d)$ for $L^p(Q)$ where elements in $f \in L^p(Q)$ are to be thought of as there extensions to 2π – periodic functions on $\mathbb{R}^d$.

Theorem 23.9 (Fourier Series). *The functions $\beta := \{e_k : k \in \mathbb{Z}^d\}$ form an orthonormal basis for H, i.e. if $f \in H$ then*

$$f = \sum_{k \in \mathbb{Z}^d} \langle f | e_k \rangle e_k = \sum_{k \in \mathbb{Z}^d} \tilde{f}(k) e_k \qquad (23.7)$$

where the convergence takes place in $L^2([-\pi, \pi]^d)$.

Proof. Simple computations show $\beta := \left\{e_k : k \in \mathbb{Z}^d\right\}$ is an orthonormal set. We now claim that β is an orthonormal basis. To see this recall that $C_c((-\pi,\pi)^d)$ is dense in $L^2((-\pi,\pi)^d, dm)$. Any $f \in C_c((-\pi,\pi))$ may be extended to be a continuous 2π – periodic function on $\mathbb{R}$ and hence by Exercise 12.13 and Remark 12.44, f may uniformly (and hence in L^2) be approximated by a trigonometric polynomial. Therefore β is a total orthonormal set, i.e. β is an orthonormal basis.

This may also be proved by first proving the case $d = 1$ as above and then using Exercise 23.2 inductively to get the result for any d. ∎

Exercise 23.8. Let A be the operator defined in Lemma 14.36 and for $g \in L^2(\mathbb{T})$, let $Ug(k) := \tilde{g}(k)$ so that $U : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is unitary. Show $U^{-1}AU = M_a$ where $a \in C^\infty_{per}(\mathbb{R})$ is a function to be found. Use this representation and the results in Exercise 23.7 to give a simple proof of the results in Lemma 14.36.

23.3.1 Dirichlet, Fejér and Kernels

Although the sum in Eq. (23.7) is guaranteed to converge relative to the Hilbertian norm on H it certainly need not converge pointwise even if $f \in C_{per}\left(\mathbb{R}^d\right)$ as will be proved in Section 35.1.1 below. Nevertheless, if f is sufficiently regular, then the sum in Eq. (23.7) will converge pointwise as we will now show. In the process we will give a direct and constructive proof of the result in Exercise 12.13, see Theorem 23.11 below.

Let us restrict our attention to $d = 1$ here. Consider

$$\begin{aligned} f_n(\theta) &= \sum_{|k|\le n} \tilde{f}(k) e_k(\theta) = \sum_{|k|\le n} \frac{1}{2\pi}\left[\int_{[-\pi,\pi]} f(x) e^{-ik\cdot x} dx\right] e_k(\theta) \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x) \sum_{|k|\le n} e^{ik\cdot(\theta - x)} dx = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x) D_n(\theta - x) dx \end{aligned} \tag{23.8}$$

where

$$D_n(\theta) := \sum_{k=-n}^{n} e^{ik\theta}$$

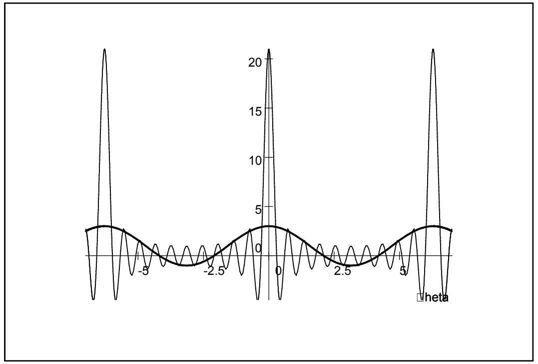
is called the **Dirichlet kernel**. Letting $\alpha = e^{i\theta/2}$, we have

$$\begin{aligned} D_n(\theta) &= \sum_{k=-n}^{n} \alpha^{2k} = \frac{\alpha^{2(n+1)} - \alpha^{-2n}}{\alpha^2 - 1} = \frac{\alpha^{2n+1} - \alpha^{-(2n+1)}}{\alpha - \alpha^{-1}} \\ &= \frac{2i\sin(n+\frac{1}{2})\theta}{2i\sin\frac{1}{2}\theta} = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \end{aligned}$$

and therefore

$$D_n(\theta) := \sum_{k=-n}^{n} e^{ik\theta} = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \tag{23.9}$$

see Figure 23.3.1.



This is a plot D_1 and D_{10}.

with the understanding that the right side of this equation is $2n+1$ whenever $\theta \in 2\pi\mathbb{Z}$.

Theorem 23.10. *Suppose $f \in L^1([-\pi,\pi], dm)$ and f is differentiable at some $\theta \in [-\pi,\pi]$, then $\lim_{n\to\infty} f_n(\theta) = f(\theta)$ where f_n is as in Eq. (23.8).*

Proof. Observe that

$$\frac{1}{2\pi}\int_{[-\pi,\pi]} D_n(\theta - x)dx = \frac{1}{2\pi}\int_{[-\pi,\pi]} \sum_{|k|\le n} e^{ik\cdot(\theta - x)}dx = 1$$

and therefore,

$$\begin{aligned} f_n(\theta) - f(\theta) &= \frac{1}{2\pi}\int_{[-\pi,\pi]} \left[f(x) - f(\theta)\right] D_n(\theta - x)dx \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} \left[f(x) - f(\theta - x)\right] D_n(x)dx \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} \left[\frac{f(\theta - x) - f(\theta)}{\sin\frac{1}{2}x}\right] \sin(n+\frac{1}{2})x\, dx. \end{aligned} \tag{23.10}$$

If f is differentiable at θ, the last expression in Eq. (23.10) tends to 0 as $n \to \infty$ by the Riemann Lebesgue Lemma (Corollary 22.17 or Lemma 22.37) and the fact that $1_{[-\pi,\pi]}(x)\frac{f(\theta - x) - f(\theta)}{\sin\frac{1}{2}x} \in L^1(dx)$. ∎

Despite the Dirichlet kernel not being positive, it still satisfies the approximate δ – sequence property, $\frac{1}{2\pi}D_n \to \delta_0$ as $n \to \infty$, when acting on C^1 – periodic functions in θ. In order to improve the convergence properties it is reasonable to try to replace $\{f_n : n \in \mathbb{N}_0\}$ by the sequence of averages (see Exercise 7.15),

$$\begin{aligned} F_N(\theta) &= \frac{1}{N+1}\sum_{n=0}^{N} f_n(\theta) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{[-\pi,\pi]} f(x)\sum_{|k|\le n} e^{ik\cdot(\theta-x)}dx \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} K_N(\theta-x)f(x)dx \end{aligned}$$

where

$$K_N(\theta) := \frac{1}{N+1}\sum_{n=0}^{N}\sum_{|k|\le n} e^{ik\cdot\theta} \tag{23.11}$$

is the **Fejér kernel.**

Theorem 23.11. *The Fejér kernel K_N in Eq. (23.11) satisfies:*

1.

$$K_N(\theta) = \sum_{n=-N}^{N}\left[1 - \frac{|n|}{N+1}\right]e^{in\theta} \tag{23.12}$$

$$= \frac{1}{N+1}\frac{\sin^2\left(\frac{N+1}{2}\theta\right)}{\sin^2\left(\frac{\theta}{2}\right)}. \tag{23.13}$$

2. $K_N(\theta) \ge 0$.
3. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1$
4. $\sup_{\varepsilon\le|\theta|\le\pi} K_N(\theta) \to 0$ *as* $N \to \infty$ *for all* $\varepsilon > 0$*, see Figure 23.1.*
5. For any continuous 2π *– periodic function* f *on* $\mathbb{R}$*,* $K_N * f(\theta) \to f(\theta)$ *uniformly in* θ *as* $N \to \infty$*, where*

$$\begin{aligned} K_N * f(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta-\alpha)f(\alpha)d\alpha \\ &= \sum_{n=-N}^{N}\left[1-\frac{|n|}{N+1}\right]\tilde{f}(n)e^{in\theta}. \end{aligned} \tag{23.14}$$

Proof. 1. Equation (23.12) is a consequence of the identity,

$$\sum_{n=0}^{N}\sum_{|k|\le n} e^{ik\cdot\theta} = \sum_{|k|\le n\le N} e^{ik\cdot\theta} = \sum_{|k|\le N}(N+1-|k|)e^{ik\cdot\theta}.$$

Moreover, letting $\alpha = e^{i\theta/2}$ and using Eq. (3.3) shows

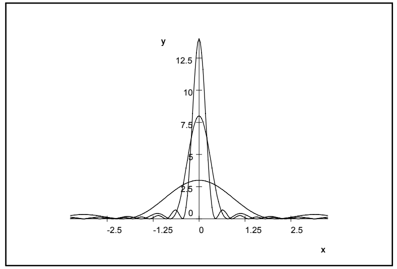


Fig. 23.1. Plots of $K_N(\theta)$ for $N = 2$, 7 and 13.

$$\begin{aligned} K_N(\theta) &= \frac{1}{N+1}\sum_{n=0}^{N}\sum_{|k|\le n}\alpha^{2k} = \frac{1}{N+1}\sum_{n=0}^{N}\frac{\alpha^{2n+2}-\alpha^{-2n}}{\alpha^2-1} \\ &= \frac{1}{(N+1)(\alpha-\alpha^{-1})}\sum_{n=0}^{N}\left[\alpha^{2n+1}-\alpha^{-2n-1}\right] \\ &= \frac{1}{(N+1)(\alpha-\alpha^{-1})}\sum_{n=0}^{N}\left[\alpha\alpha^{2n}-\alpha^{-1}\alpha^{-2n}\right] \\ &= \frac{1}{(N+1)(\alpha-\alpha^{-1})}\left[\alpha\frac{\alpha^{2N+2}-1}{\alpha^2-1}-\alpha^{-1}\frac{\alpha^{-2N-2}-1}{\alpha^{-2}-1}\right] \\ &= \frac{1}{(N+1)(\alpha-\alpha^{-1})^2}\left[\alpha^{2(N+1)}-1+\alpha^{-2(N+1)}-1\right] \\ &= \frac{1}{(N+1)(\alpha-\alpha^{-1})^2}\left[\alpha^{(N+1)}-\alpha^{-(N+1)}\right]^2 \\ &= \frac{1}{N+1}\frac{\sin^2((N+1)\theta/2)}{\sin^2(\theta/2)}. \end{aligned}$$

Items 2. and 3. follow easily from Eqs. (23.13) and (23.12) respectively. Item 4. is a consequence of the elementary estimate;

$$\sup_{\varepsilon\le|\theta|\le\pi} K_N(\theta) \le \frac{1}{N+1}\frac{1}{\sin^2\left(\frac{\varepsilon}{2}\right)}$$

and is clearly indicated in Figure 23.1. Item 5. now follows by the standard approximate δ – function arguments, namely,

$$\begin{aligned}|K_N * f(\theta) - f(\theta)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} K_N(\theta-\alpha)\left[f(\alpha) - f(\theta)\right] d\alpha\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\alpha)\left|f(\theta-\alpha) - f(\theta)\right| d\alpha \\ &\leq \frac{1}{\pi}\frac{1}{N+1}\frac{1}{\sin^2\left(\frac{\varepsilon}{2}\right)}\|f\|_\infty + \frac{1}{2\pi}\int_{|\alpha|\leq\varepsilon} K_N(\alpha)\left|f(\theta-\alpha) - f(\theta)\right| d\alpha \\ &\leq \frac{1}{\pi}\frac{1}{N+1}\frac{1}{\sin^2\left(\frac{\varepsilon}{2}\right)}\|f\|_\infty + \sup_{|\alpha|\leq\varepsilon}\left|f(\theta-\alpha) - f(\theta)\right|.\end{aligned}$$

Therefore,

$$\lim_{N\to\infty}\sup \|K_N * f - f\|_\infty \leq \sup_\theta \sup_{|\alpha|\leq\varepsilon} |f(\theta-\alpha) - f(\theta)| \to 0 \text{ as } \varepsilon \downarrow 0.$$

■

23.3.2 The Dirichlet Problems on D and the Poisson Kernel

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C} \cong \mathbb{R}^2$, write $z \in \mathbb{C}$ as $z = x + iy$ or $z = re^{i\theta}$, and let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ be the **Laplacian** acting on $C^2(D)$.

Theorem 23.12 (Dirichlet problem for D). *To every continuous function $g \in C(\mathrm{bd}(D))$ there exists a unique function $u \in C(\bar{D}) \cap C^2(D)$ solving*

$$\Delta u(z) = 0 \text{ for } z \in D \text{ and } u|_{\partial D} = g. \tag{23.15}$$

Moreover for $r < 1$, u is given by,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\alpha) u(e^{i\alpha}) d\alpha =: P_r * u(e^{i\theta}) \tag{23.16}$$

$$= \frac{1}{2\pi}\mathrm{Re}\int_{-\pi}^{\pi} \frac{1 + re^{i(\theta-\alpha)}}{1 - re^{i(\theta-\alpha)}} u(e^{i\alpha}) d\alpha \tag{23.17}$$

*where P_r is the **Poisson kernel** defined by*

$$P_r(\delta) := \frac{1 - r^2}{1 - 2r\cos\delta + r^2}.$$

*(The problem posed in Eq. (23.15) is called the **Dirichlet problem for** D.)*

Proof. In this proof, we are going to be identifying $S^1 = \mathrm{bd}(D) := \left\{z \in \bar{D} : |z| = 1\right\}$ with $[-\pi,\pi]/(\pi \sim -\pi)$ by the map $\theta \in [-\pi,\pi] \to e^{i\theta} \in S^1$. Also recall that the Laplacian Δ may be expressed in polar coordinates as,

$$\Delta u = r^{-1}\partial_r\left(r^{-1}\partial_r u\right) + \frac{1}{r^2}\partial_\theta^2 u,$$

where

$$\left(\partial_r u\right)\left(re^{i\theta}\right) = \frac{\partial}{\partial r}u\left(re^{i\theta}\right) \text{ and } \left(\partial_\theta u\right)\left(re^{i\theta}\right) = \frac{\partial}{\partial\theta}u\left(re^{i\theta}\right).$$

Uniqueness. Suppose u is a solution to Eq. (23.15) and let

$$\tilde{g}(k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{ik\theta})e^{-ik\theta}d\theta$$

and

$$\tilde{u}(r,k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})e^{-ik\theta}d\theta \tag{23.18}$$

be the Fourier coefficients of $g(\theta)$ and $\theta \to u\left(re^{i\theta}\right)$ respectively. Then for $r \in (0,1)$,

$$\begin{aligned}r^{-1}\partial_r\left(r\partial_r\tilde{u}(r,k)\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{-1}\partial_r\left(r^{-1}\partial_r u\right)(re^{i\theta})e^{-ik\theta}d\theta \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{r^2}\partial_\theta^2 u(re^{i\theta})e^{-ik\theta}d\theta \\ &= -\frac{1}{r^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})\partial_\theta^2 e^{-ik\theta}d\theta \\ &= \frac{1}{r^2}k^2\tilde{u}(r,k)\end{aligned}$$

or equivalently

$$r\partial_r\left(r\partial_r\tilde{u}(r,k)\right) = k^2\tilde{u}(r,k). \tag{23.19}$$

Recall the general solution to

$$r\partial_r\left(r\partial_r y(r)\right) = k^2 y(r) \tag{23.20}$$

may be found by trying solutions of the form $y(r) = r^\alpha$ which then implies $\alpha^2 = k^2$ or $\alpha = \pm k$. From this one sees that $\tilde{u}(r,k)$ solving Eq. (23.19) may be written as $\tilde{u}(r,k) = A_k r^{|k|} + B_k r^{-|k|}$ for some constants A_k and B_k when $k \neq 0$. If $k = 0$, the solution to Eq. (23.20) is gotten by simple integration and the result is $\tilde{u}(r,0) = A_0 + B_0 \ln r$. Since $\tilde{u}(r,k)$ is bounded near the origin for each k it must be that $B_k = 0$ for all $k \in \mathbb{Z}$. Hence we have shown there exists $A_k \in \mathbb{C}$ such that, for all $r \in (0,1)$,

$$A_k r^{|k|} = \tilde{u}(r,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})e^{-ik\theta}d\theta. \tag{23.21}$$

Since all terms of this equation are continuous for $r \in [0,1]$, Eq. (23.21) remains valid for all $r \in [0,1]$ and in particular we have, at $r = 1$, that

$$A_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})e^{-ik\theta}d\theta = \tilde{g}(k).$$

Hence if u is a solution to Eq. (23.15) then u must be given by

$$u(re^{i\theta}) = \sum_{k\in\mathbb{Z}} \tilde{g}(k) r^{|k|} e^{ik\theta} \text{ for } r < 1. \tag{23.22}$$

or equivalently,

$$u(z) = \sum_{k\in\mathbb{N}_0} \tilde{g}(k) z^k + \sum_{k\in\mathbb{N}} \tilde{g}(-k) \bar{z}^k.$$

Notice that the theory of the Fourier series implies Eq. (23.22) is valid in the $L^2(d\theta)$ - sense. However more is true, since for $r < 1$, the series in Eq. (23.22) is absolutely convergent and in fact defines a C^∞ – function (see Exercise 4.11 or Corollary 19.43) which must agree with the continuous function, $\theta \to u(re^{i\theta})$, for almost every θ and hence for all θ. This completes the proof of uniqueness.

Existence. Given $g \in C(\mathrm{bd}(D))$, let u be defined as in Eq. (23.22). Then, again by Exercise 4.11 or Corollary 19.43, $u \in C^\infty(D)$. So to finish the proof it suffices to show $\lim_{x\to y} u(x) = g(y)$ for all $y \in \mathrm{bd}(D)$. Inserting the formula for $\tilde{g}(k)$ into Eq. (23.22) gives

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\alpha) u(e^{i\alpha}) d\alpha \text{ for all } r < 1$$

where

$$\begin{aligned} P_r(\delta) &= \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\delta} = \sum_{k=0}^{\infty} r^k e^{ik\delta} + \sum_{k=0}^{\infty} r^k e^{-ik\delta} - 1 = \\ &= \mathrm{Re}\left[2\frac{1}{1-re^{i\delta}} - 1\right] = \mathrm{Re}\left[\frac{1+re^{i\delta}}{1-re^{i\delta}}\right] \\ &= \mathrm{Re}\left[\frac{(1+re^{i\delta})(1-re^{-i\delta})}{|1-re^{i\delta}|^2}\right] = \mathrm{Re}\left[\frac{1-r^2+2ir\sin\delta}{1-2r\cos\delta+r^2}\right] \\ &= \frac{1-r^2}{1-2r\cos\delta+r^2}. \end{aligned} \tag{23.23}$$

The Poisson kernel again solves the usual approximate δ – function properties (see Figure 2), namely:

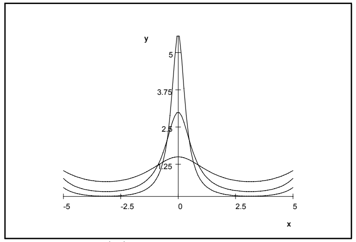
1. $P_r(\delta) > 0$ and

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\alpha) d\alpha &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik(\theta-\alpha)} d\alpha \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} r^{|k|} \int_{-\pi}^{\pi} e^{ik(\theta-\alpha)} d\alpha = 1 \end{aligned}$$

and

2.

$$\sup_{\varepsilon\le|\theta|\le\pi} P_r(\theta) \le \frac{1-r^2}{1-2r\cos\varepsilon+r^2} \to 0 \text{ as } r \uparrow 1.$$



A plot of $P_r(\delta)$ for $r = 0.2$, 0.5 and 0.7.

Therefore by the same argument used in the proof of Theorem 23.11,

$$\lim_{r\uparrow 1}\sup_\theta \left|u(re^{i\theta}) - g(e^{i\theta})\right| = \lim_{r\uparrow 1}\sup_\theta \left|(P_r * g)(e^{i\theta}) - g(e^{i\theta})\right| = 0$$

which certainly implies $\lim_{x\to y} u(x) = g(y)$ for all $y \in \mathrm{bd}(D)$. ■

Remark 23.13 (Harmonic Conjugate). Writing $z = re^{i\theta}$, Eq. (23.17) may be rewritten as

$$u(z) = \frac{1}{2\pi}\mathrm{Re}\int_{-\pi}^{\pi} \frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}} u(e^{i\alpha}) d\alpha$$

which shows $u = \mathrm{Re}\, F$ where

$$F(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}} u(e^{i\alpha}) d\alpha.$$

Moreover it follows from Eq. (23.23) that

$$\begin{aligned} \mathrm{Im}\, F(re^{i\theta}) &= \frac{1}{\pi}\mathrm{Im}\int_{-\pi}^{\pi} \frac{r\sin(\theta-\alpha)}{1-2r\cos(\theta-\alpha)+r^2} g(e^{i\alpha}) d\alpha \\ &=: (Q_r * u)(e^{i\theta}) \end{aligned}$$

where

$$Q_r(\delta) := \frac{r\sin(\delta)}{1-2r\cos(\delta)+r^2}.$$

From these remarks it follows that $v =: (Q_r * g)(e^{i\theta})$ is the harmonic conjugate of u and $\tilde{P}_r = Q_r$. For more on this point see Section 49.7 below.

23.4 Weak L^2-Derivatives

Theorem 23.14 (Weak and Strong Differentiability). *Suppose that $f \in L^2(\mathbb{R}^n)$ and $v \in \mathbb{R}^n \setminus \{0\}$. Then the following are equivalent:*

1. *There exists* $\{t_n\}_{n=1}^\infty \subset \mathbb{R}\setminus\{0\}$ *such that* $\lim_{n\to\infty} t_n = 0$ *and*
$$\sup_n \left\| \frac{f(\cdot + t_n v) - f(\cdot)}{t_n} \right\|_2 < \infty.$$
2. *There exists* $g \in L^2(\mathbb{R}^n)$ *such that* $\langle f, \partial_v \phi \rangle = -\langle g, \phi \rangle$ *for all* $\phi \in C_c^\infty(\mathbb{R}^n)$.
3. *There exists* $g \in L^2(\mathbb{R}^n)$ *and* $f_n \in C_c^\infty(\mathbb{R}^n)$ *such that* $f_n \overset{L^2}{\to} f$ *and* $\partial_v f_n \overset{L^2}{\to} g$ *as* $n \to \infty$.
4. *There exists* $g \in L^2$ *such that*
$$\frac{f(\cdot + tv) - f(\cdot)}{t} \overset{L^2}{\to} g \text{ as } t \to 0.$$

(See Theorem 36.18 for the L^p *generalization of this theorem.)*

Proof. 1. $\Longrightarrow$ 2. We may assume, using Theorem 14.52 and passing to a subsequence if necessary, that $\frac{f(\cdot+t_n v)-f(\cdot)}{t_n} \overset{w}{\to} g$ for some $g \in L^2(\mathbb{R}^n)$. Now for $\phi \in C_c^\infty(\mathbb{R}^n)$,
$$\begin{aligned}\langle g|\phi\rangle &= \lim_{n\to\infty} \left\langle \frac{f(\cdot + t_n v) - f(\cdot)}{t_n}, \phi \right\rangle = \lim_{n\to\infty} \left\langle f, \frac{\phi(\cdot - t_n v) - \phi(\cdot)}{t_n} \right\rangle \\ &= \left\langle f, \lim_{n\to\infty} \frac{\phi(\cdot - t_n v) - \phi(\cdot)}{t_n} \right\rangle = -\langle f, \partial_v \phi \rangle,\end{aligned}$$
wherein we have used the translation invariance of Lebesgue measure and the dominated convergence theorem. 2. $\Longrightarrow$ 3. Let $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$ and let $\phi_m(x) = m^n \phi(mx)$, then by Proposition 22.34, $h_m := \phi_m * f \in C^\infty(\mathbb{R}^n)$ for all m and
$$\begin{aligned}\partial_v h_m(x) &= \partial_v \phi_m * f(x) = \int_{\mathbb{R}^n} \partial_v \phi_m(x - y) f(y) dy = \langle f, -\partial_v [\phi_m(x - \cdot)] \rangle \\ &= \langle g, \phi_m(x - \cdot) \rangle = \phi_m * g(x).\end{aligned}$$
By Theorem 22.32, $h_m \to f \in L^2(\mathbb{R}^n)$ and $\partial_v h_m = \phi_m * g \to g$ in $L^2(\mathbb{R}^n)$ as $m \to \infty$. This shows 3. holds except for the fact that h_m need not have compact support. To fix this let $\psi \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $\psi = 1$ in a neighborhood of 0 and let $\psi_\varepsilon(x) = \psi(\varepsilon x)$ and $(\partial_v \psi)_\varepsilon(x) := (\partial_v \psi)(\varepsilon x)$. Then
$$\partial_v (\psi_\varepsilon h_m) = \partial_v \psi_\varepsilon h_m + \psi_\varepsilon \partial_v h_m = \varepsilon (\partial_v \psi)_\varepsilon h_m + \psi_\varepsilon \partial_v h_m$$
so that $\psi_\varepsilon h_m \to h_m$ in L^2 and $\partial_v(\psi_\varepsilon h_m) \to \partial_v h_m$ in L^2 as $\varepsilon \downarrow 0$. Let $f_m = \psi_{\varepsilon_m} h_m$ where ε_m is chosen to be greater than zero but small enough so that
$$\|\psi_{\varepsilon_m} h_m - h_m\|_2 + \|\partial_v(\psi_{\varepsilon_m} h_m) \to \partial_v h_m\|_2 < 1/m.$$
Then $f_m \in C_c^\infty(\mathbb{R}^n)$, $f_m \to f$ and $\partial_v f_m \to g$ in L^2 as $m \to \infty$. 3. $\Longrightarrow$ 4. By the fundamental theorem of calculus

$$\begin{aligned}\frac{\tau_{-tv} f_m(x) - f_m(x)}{t} &= \frac{f_m(x + tv) - f_m(x)}{t} \\ &= \frac{1}{t}\int_0^1 \frac{d}{ds} f_m(x + stv) ds = \int_0^1 (\partial_v f_m)(x + stv) ds.\end{aligned} \tag{23.24}$$

Let
$$G_t(x) := \int_0^1 \tau_{-stv} g(x) ds = \int_0^1 g(x + stv) ds$$
which is defined for almost every x and is in $L^2(\mathbb{R}^n)$ by Minkowski's inequality for integrals, Theorem 21.27. Therefore
$$\frac{\tau_{-tv} f_m(x) - f_m(x)}{t} - G_t(x) = \int_0^1 [(\partial_v f_m)(x + stv) - g(x + stv)] ds$$
and hence again by Minkowski's inequality for integrals,
$$\begin{aligned}\left\| \frac{\tau_{-tv} f_m - f_m}{t} - G_t \right\|_2 &\le \int_0^1 \|\tau_{-stv}(\partial_v f_m) - \tau_{-stv} g\|_2 ds \\ &= \int_0^1 \|\partial_v f_m - g\|_2 ds.\end{aligned}$$
Letting $m \to \infty$ in this equation implies $(\tau_{-tv} f - f)/t = G_t$ a.e. Finally one more application of Minkowski's inequality for integrals implies,
$$\begin{aligned}\left\| \frac{\tau_{-tv} f - f}{t} - g \right\|_2 &= \|G_t - g\|_2 = \left\| \int_0^1 (\tau_{-stv} g - g) ds \right\|_2 \\ &\le \int_0^1 \|\tau_{-stv} g - g\|_2 ds.\end{aligned}$$
By the dominated convergence theorem and Proposition 22.24, the latter term tends to 0 as $t \to 0$ and this proves 4. The proof is now complete since 4. $\Longrightarrow$ 1. is trivial. ■

23.5 *Conditional Expectation

In this section let $(\Omega, \mathcal{F}, P)$ be a probability space, i.e. $(\Omega, \mathcal{F}, P)$ is a measure space and $P(\Omega) = 1$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub – sigma algebra of $\mathcal{F}$ and write $f \in \mathcal{G}_b$ if $f : \Omega \to \mathbb{C}$ is bounded and f is $(\mathcal{G}, \mathcal{B}_\mathbb{C})$ – measurable. In this section we will write
$$Ef := \int_\Omega f dP.$$

Definition 23.15 (Conditional Expectation). *Let* $E_\mathcal{G} : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{G}, P)$ *denote orthogonal projection of* $L^2(\Omega, \mathcal{F}, P)$ *onto the closed subspace* $L^2(\Omega, \mathcal{G}, P)$*. For* $f \in L^2(\Omega, \mathcal{G}, P)$*, we say that* $E_\mathcal{G} f \in L^2(\Omega, \mathcal{F}, P)$ *is the* ***conditional expectation*** *of* f.

Theorem 23.16. *Let $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ be as above and $f, g \in L^2(\Omega, \mathcal{F}, P)$.*

1. *If $f \geq 0$, P – a.e. then $E_{\mathcal{G}} f \geq 0$, P – a.e.*
2. *If $f \geq g$, P – a.e. there $E_{\mathcal{G}} f \geq E_{\mathcal{G}} g$, P – a.e.*
3. *$|E_{\mathcal{G}} f| \leq E_{\mathcal{G}} |f|$, P – a.e.*
4. *$\|E_{\mathcal{G}} f\|_{L^1} \leq \|f\|_{L^1}$ for all $f \in L^2$. So by the B.L.T. Theorem 8.4, $E_{\mathcal{G}}$ extends uniquely to a bounded linear map from $L^1(\Omega, \mathcal{F}, P)$ to $L^1(\Omega, \mathcal{G}, P)$ which we will still denote by $E_{\mathcal{G}}$.*
5. *If $f \in L^1(\Omega, \mathcal{F}, P)$ then $F = E_{\mathcal{G}} f \in L^1(\Omega, \mathcal{G}, P)$ iff*
$$E(Fh) = E(fh) \text{ for all } h \in \mathcal{G}_b.$$
6. *If $g \in \mathcal{G}_b$ and $f \in L^1(\Omega, \mathcal{F}, P)$, then $E_{\mathcal{G}}(gf) = g \cdot E_{\mathcal{G}} f$, P – a.e.*

Proof. By the definition of orthogonal projection for $h \in \mathcal{G}_b$,

$$E(fh) = E(f \cdot E_{\mathcal{G}} h) = E(E_{\mathcal{G}} f \cdot h).$$

So if $f, h \geq 0$ then $0 \leq E(fh) \leq E(E_{\mathcal{G}} f \cdot h)$ and since this holds for all $h \geq 0$ in $\mathcal{G}_b$, $E_{\mathcal{G}} f \geq 0$, P – a.e. This proves (1). Item (2) follows by applying item (1). to $f - g$. If f is real, $\pm f \leq |f|$ and so by Item (2), $\pm E_{\mathcal{G}} f \leq E_{\mathcal{G}} |f|$, i.e. $|E_{\mathcal{G}} f| \leq E_{\mathcal{G}} |f|$, P – a.e. For complex f, let $h \geq 0$ be a bounded and $\mathcal{G}$ – measurable function. Then

$$\begin{aligned} E[|E_{\mathcal{G}} f| h] &= E\left[E_{\mathcal{G}} f \cdot \overline{\operatorname{sgn}(E_{\mathcal{G}} f)} h\right] = E\left[f \cdot \overline{\operatorname{sgn}(E_{\mathcal{G}} f)} h\right] \\ &\leq E[|f| h] = E[E_{\mathcal{G}} |f| \cdot h]. \end{aligned}$$

Since h is arbitrary, it follows that $|E_{\mathcal{G}} f| \leq E_{\mathcal{G}} |f|$, P – a.e. Integrating this inequality implies

$$\|E_{\mathcal{G}} f\|_{L^1} \leq E|E_{\mathcal{G}} f| \leq E[E_{\mathcal{G}} |f| \cdot 1] = E[|f|] = \|f\|_{L^1}.$$

Item (5). Suppose $f \in L^1(\Omega, \mathcal{F}, P)$ and $h \in \mathcal{G}_b$. Let $f_n \in L^2(\Omega, \mathcal{F}, P)$ be a sequence of functions such that $f_n \to f$ in $L^1(\Omega, \mathcal{F}, P)$. Then

$$\begin{aligned} E(E_{\mathcal{G}} f \cdot h) &= E(\lim_{n\to\infty} E_{\mathcal{G}} f_n \cdot h) = \lim_{n\to\infty} E(E_{\mathcal{G}} f_n \cdot h) \\ &= \lim_{n\to\infty} E(f_n \cdot h) = E(f \cdot h). \end{aligned} \tag{23.25}$$

This equation uniquely determines $E_{\mathcal{G}}$, for if $F \in L^1(\Omega, \mathcal{G}, P)$ also satisfies $E(F \cdot h) = E(f \cdot h)$ for all $h \in \mathcal{G}_b$, then taking $h = \overline{\operatorname{sgn}(F - E_{\mathcal{G}} f)}$ in Eq. (23.25) gives

$$0 = E((F - E_{\mathcal{G}} f) h) = E(|F - E_{\mathcal{G}} f|).$$

This shows $F = E_{\mathcal{G}} f$, P – a.e. Item (6) is now an easy consequence of this characterization, since if $h \in \mathcal{G}_b$,

$$E[(g E_{\mathcal{G}} f) h] = E[E_{\mathcal{G}} f \cdot hg] = E[f \cdot hg] = E[gf \cdot h] = E[E_{\mathcal{G}}(gf) \cdot h].$$

Thus $E_{\mathcal{G}}(gf) = g \cdot E_{\mathcal{G}} f$, P – a.e. ■

Proposition 23.17. *If $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{F}$. Then*

$$E_{\mathcal{G}_0} E_{\mathcal{G}_1} = E_{\mathcal{G}_1} E_{\mathcal{G}_0} = E_{\mathcal{G}_0}. \tag{23.26}$$

Proof. Equation (23.26) holds on $L^2(\Omega, \mathcal{F}, P)$ by the basic properties of orthogonal projections. It then hold on $L^1(\Omega, \mathcal{F}, P)$ by continuity and the density of $L^2(\Omega, \mathcal{F}, P)$ in $L^1(\Omega, \mathcal{F}, P)$. ■

Example 23.18. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are two σ – finite measure spaces. Let $\Omega = X \times Y$, $\mathcal{F} = \mathcal{M} \otimes \mathcal{N}$ and $P(dx, dy) = \rho(x, y)\mu(dx)\nu(dy)$ where $\rho \in L^1(\Omega, \mathcal{F}, \mu \otimes \nu)$ is a positive function such that $\int_{X \times Y} \rho d(\mu \otimes \nu) = 1$. Let $\pi_X : \Omega \to X$ be the projection map, $\pi_X(x, y) = x$, and

$$\mathcal{G} := \sigma(\pi_X) = \pi_X^{-1}(\mathcal{M}) = \{A \times Y : A \in \mathcal{M}\}.$$

Then $f : \Omega \to \mathbb{R}$ is $\mathcal{G}$ – measurable iff $f = F \circ \pi_X$ for some function $F : X \to \mathbb{R}$ which is $\mathcal{N}$ – measurable, see Lemma 18.66. For $f \in L^1(\Omega, \mathcal{F}, P)$, we will now show $E_{\mathcal{G}} f = F \circ \pi_X$ where

$$F(x) = \frac{1}{\bar{\rho}(x)} 1_{(0,\infty)}(\bar{\rho}(x)) \cdot \int_Y f(x, y)\rho(x, y)\nu(dy),$$

$\bar{\rho}(x) := \int_Y \rho(x, y)\nu(dy)$. (By convention, $\int_Y f(x, y)\rho(x, y)\nu(dy) := 0$ if $\int_Y |f(x, y)| \rho(x, y)\nu(dy) = \infty$.)

By Tonelli's theorem, the set

$$E := \{x \in X : \bar{\rho}(x) = \infty\} \cup \left\{x \in X : \int_Y |f(x, y)| \rho(x, y)\nu(dy) = \infty\right\}$$

is a μ – null set. Since

$$\begin{aligned} E[|F \circ \pi_X|] &= \int_X d\mu(x) \int_Y d\nu(y) |F(x)| \rho(x, y) = \int_X d\mu(x) |F(x)| \bar{\rho}(x) \\ &= \int_X d\mu(x) \left| \int_Y \nu(dy) f(x, y)\rho(x, y) \right| \\ &\leq \int_X d\mu(x) \int_Y \nu(dy) |f(x, y)| \rho(x, y) < \infty, \end{aligned}$$

$F \circ \pi_X \in L^1(\Omega, \mathcal{G}, P)$. Let $h = H \circ \pi_X$ be a bounded $\mathcal{G}$ – measurable function, then

$$\begin{aligned} E[F \circ \pi_X \cdot h] &= \int_X d\mu(x) \int_Y d\nu(y) F(x) H(x) \rho(x, y) \\ &= \int_X d\mu(x) F(x) H(x) \bar{\rho}(x) \\ &= \int_X d\mu(x) H(x) \int_Y \nu(dy) f(x, y) \rho(x, y) \\ &= E[hf] \end{aligned}$$

and hence $E_{\mathcal{G}} f = F \circ \pi_X$ as claimed.

This example shows that conditional expectation is a generalization of the notion of performing integration over a partial subset of the variables in the integrand. Whereas to compute the expectation, one should integrate over all of the variables. See also Exercise 23.25 to gain more intuition about conditional expectations.

Theorem 23.19 (Jensen's inequality). *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Assume $f \in L^1(\Omega, \mathcal{F}, P; \mathbb{R})$ is a function such that (for simplicity) $\varphi(f) \in L^1(\Omega, \mathcal{F}, P; \mathbb{R})$, then $\varphi(E_\mathcal{G} f) \leq E_\mathcal{G}[\varphi(f)]$, P – a.e.*

Proof. Let us first assume that ϕ is C^1 and f is bounded. In this case

$$\varphi(x) - \varphi(x_0) \geq \varphi'(x_0)(x - x_0) \text{ for all } x_0, x \in \mathbb{R}. \tag{23.27}$$

Taking $x_0 = E_\mathcal{G} f$ and $x = f$ in this inequality implies

$$\varphi(f) - \varphi(E_\mathcal{G} f) \geq \varphi'(E_\mathcal{G} f)(f - E_\mathcal{G} f)$$

and then applying $E_\mathcal{G}$ to this inequality gives

$$\begin{aligned} E_\mathcal{G}[\varphi(f)] - \varphi(E_\mathcal{G} f) &= E_\mathcal{G}[\varphi(f) - \varphi(E_\mathcal{G} f)] \\ &\geq \varphi'(E_\mathcal{G} f)(E_\mathcal{G} f - E_\mathcal{G} E_\mathcal{G} f) = 0 \end{aligned}$$

The same proof works for general ϕ, one need only use Proposition 21.8 to replace Eq. (23.27) by

$$\varphi(x) - \varphi(x_0) \geq \varphi'_-(x_0)(x - x_0) \text{ for all } x_0, x \in \mathbb{R}$$

where $\varphi'_-(x_0)$ is the left hand derivative of ϕ at x_0. If f is not bounded, apply what we have just proved to $f^M = f 1_{|f| \leq M}$, to find

$$E_\mathcal{G}\left[\varphi(f^M)\right] \geq \varphi(E_\mathcal{G} f^M). \tag{23.28}$$

Since $E_\mathcal{G} : L^1(\Omega, \mathcal{F}, P; \mathbb{R}) \to L^1(\Omega, \mathcal{F}, P; \mathbb{R})$ is a bounded operator and $f^M \to f$ and $\varphi(f^M) \to \phi(f)$ in $L^1(\Omega, \mathcal{F}, P; \mathbb{R})$ as $M \to \infty$, there exists $\{M_k\}_{k=1}^\infty$ such that $M_k \uparrow \infty$ and $f^{M_k} \to f$ and $\varphi(f^{M_k}) \to \phi(f)$, P – a.e. So passing to the limit in Eq. (23.28) shows $E_\mathcal{G}[\varphi(f)] \geq \varphi(E_\mathcal{G} f)$, P – a.e. ■

23.6 Exercises

Exercise 23.9. Let $(X, \mathcal{M}, \mu)$ be a measure space and $H := L^2(X, \mathcal{M}, \mu)$. Given $f \in L^\infty(\mu)$ let $M_f : H \to H$ be the multiplication operator defined by $M_f g = fg$. Show $M_f^2 = M_f$ iff there exists $A \in \mathcal{M}$ such that $f = 1_A$ a.e.

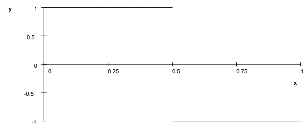
Exercise 23.10 (Haar Basis). In this problem, let L^2 denote $L^2([0,1], m)$ with the standard inner product,

$$\psi(x) = 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x)$$

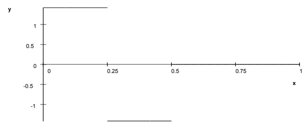
and for $k, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with $0 \leq j < 2^k$ let

$$\psi_{kj}(x) := 2^{k/2}\psi(2^k x - j).$$

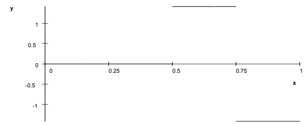
The following pictures shows the graphs of ψ_{00}, $\psi_{1,0}, \psi_{1,1}, \psi_{2,1}, \psi_{2,2}$ and $\psi_{2,3}$ respectively.



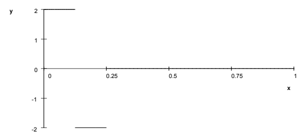
Plot of $\psi_0, 0$.



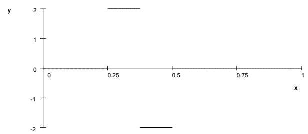
Plot of $\psi_1 0$.



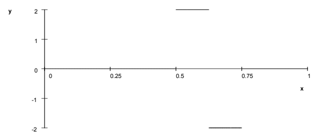
Plot of $\psi_1 1$.



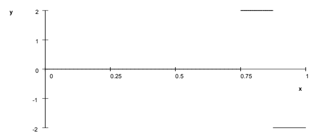
Plot of $\psi_2 0$.



Plot of $\psi_2 1$.



Plot of $\psi_2 2$.



Plot of $\psi_2 3$.

1. Show $\beta := \{\mathbf{1}\} \cup \{\psi_{kj} : 0 \le k \text{ and } 0 \le j < 2^k\}$ is an orthonormal set, $\mathbf{1}$ denotes the constant function 1.
2. For $n \in \mathbb{N}$, let $M_n := \operatorname{span}\left(\{\mathbf{1}\} \cup \{\psi_{kj} : 0 \le k < n \text{ and } 0 \le j < 2^k\}\right)$. Show
$$M_n = \operatorname{span}\left(\{1_{[j2^{-n},(j+1)2^{-n})} : \text{and } 0 \le j < 2^n\right).$$
3. Show $\cup_{n=1}^\infty M_n$ is a dense subspace of L^2 and therefore β is an orthonormal basis for L^2. **Hint:** see Theorem 22.15.
4. For $f \in L^2$, let
$$H_n f := \langle f|\mathbf{1}\rangle\mathbf{1} + \sum_{k=0}^{n-1}\sum_{j=0}^{2^k-1}\langle f|\psi_{kj}\rangle\psi_{kj}.$$
Show (compare with Exercise 23.25)
$$H_n f = \sum_{j=0}^{2^n-1}\left(2^n\int_{j2^{-n}}^{(j+1)2^{-n}} f(x)dx\right)1_{[j2^{-n},(j+1)2^{-n})}$$
and use this to show $\|f - H_n f\|_\infty \to 0$ as $n \to \infty$ for all $f \in C([0,1])$.

Exercise 23.11. Let $O(n)$ be the orthogonal groups consisting of $n \times n$ real orthogonal matrices O, i.e. $O^{tr}O = I$. For $O \in O(n)$ and $f \in L^2(\mathbb{R}^n)$ let $U_O f(x) = f(O^{-1}x)$. Show

1. $U_O f$ is well defined, namely if $f = g$ a.e. then $U_O f = U_O g$ a.e.
2. $U_O : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitary and satisfies $U_{O_1}U_{O_2} = U_{O_1O_2}$ for all $O_1, O_2 \in O(n)$. That is to say the map $O \in O(n) \to \mathcal{U}(L^2(\mathbb{R}^n))$ – the unitary operators on $L^2(\mathbb{R}^n)$ is a group homomorphism, i.e. a "unitary representation" of $O(n)$.
3. For each $f \in L^2(\mathbb{R}^n)$, the map $O \in O(n) \to U_O f \in L^2(\mathbb{R}^n)$ is continuous. Take the topology on $O(n)$ to be that inherited from the Euclidean topology on the vector space of all $n \times n$ matrices. **Hint**: see the proof of Proposition 22.24.

Exercise 23.12. Euclidean group representation and its infinitesimal generators including momentum and angular momentum operators.

Exercise 23.13. Spherical Harmonics.

Exercise 23.14. The gradient and the Laplacian in spherical coordinates.

Exercise 23.15. Legendre polynomials.

23.7 Fourier Series Exercises

Exercise 23.16. Show $\sum_{k=1}^\infty k^{-2} = \pi^2/6$, by taking $f(x) = x$ on $[-\pi,\pi]$ and computing $\|f\|_2^2$ directly and then in terms of the Fourier Coefficients $\tilde f$ of f.

Exercise 23.17 (Riemann Lebesgue Lemma for Fourier Series). Show for $f \in L^1([-\pi,\pi]^d)$ that $\tilde f \in c_0(\mathbb{Z}^d)$, i.e. $\tilde f : \mathbb{Z}^d \to \mathbb{C}$ and $\lim_{k\to\infty}\tilde f(k) = 0$. **Hint:** If $f \in H$, this follows form Bessel's inequality. Now use a density argument.

Exercise 23.18. Suppose $f \in L^1([-\pi,\pi]^d)$ is a function such that $\tilde f \in \ell^1(\mathbb{Z}^d)$ and set
$$g(x) := \sum_{k\in\mathbb{Z}^d}\tilde f(k)e^{ik\cdot x} \text{ (pointwise)}.$$

1. Show $g \in C_{per}(\mathbb{R}^d)$.
2. Show $g(x) = f(x)$ for m – a.e. x in $[-\pi,\pi]^d$. **Hint**: Show $\tilde g(k) = \tilde f(k)$ and then use approximation arguments to show
$$\int_{[-\pi,\pi]^d} f(x)h(x)dx = \int_{[-\pi,\pi]^d} g(x)h(x)dx \;\forall\; h \in C([-\pi,\pi]^d).$$
3. Conclude that $f \in L^1([-\pi,\pi]^d) \cap L^\infty([-\pi,\pi]^d)$ and in particular $f \in L^p([-\pi,\pi]^d)$ for all $p \in [1,\infty]$.

Exercise 23.19. Suppose $m \in \mathbb{N}_0$, α is a multi-index such that $|\alpha| \le 2m$ and $f \in C^{2m}_{per}(\mathbb{R}^d)$[1].

1. Using integration by parts, show (using Notation 22.21) that
$$(ik)^\alpha\tilde f(k) = \langle\partial^\alpha f|e_k\rangle \text{ for all } k \in \mathbb{Z}^d.$$
Note: This equality implies
$$\left|\tilde f(k)\right| \le \frac{1}{k^\alpha}\|\partial^\alpha f\|_H \le \frac{1}{k^\alpha}\|\partial^\alpha f\|_\infty.$$
2. Now let $\Delta f = \sum_{i=1}^d \partial^2 f/\partial x_i^2$, Working as in part 1) show
$$\langle(1-\Delta)^m f|e_k\rangle = (1+|k|^2)^m\tilde f(k). \tag{23.29}$$

Remark 23.20. Suppose that m is an even integer, α is a multi-index and $f \in C^{m+|\alpha|}_{per}(\mathbb{R}^d)$, then

[1] We view $C_{per}(\mathbb{R})$ as a subspace of $H = L^2([-\pi,\pi])$ by identifying $f \in C_{per}(\mathbb{R})$ with $f|_{[-\pi,\pi]} \in H$.

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^d}|k^\alpha|\left|\tilde{f}(k)\right|\right)^2 &= \left(\sum_{k\in\mathbb{Z}^d}|\langle\partial^\alpha f|e_k\rangle|\,(1+|k|^2)^{m/2}(1+|k|^2)^{-m/2}\right)^2\\ &= \left(\sum_{k\in\mathbb{Z}^d}\left|\langle(1-\Delta)^{m/2}\partial^\alpha f|e_k\rangle\right|(1+|k|^2)^{-m/2}\right)^2\\ &\le \sum_{k\in\mathbb{Z}^d}\left|\langle(1-\Delta)^{m/2}\partial^\alpha f|e_k\rangle\right|^2\cdot\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-m}\\ &= C_m\left\|(1-\Delta)^{m/2}\partial^\alpha f\right\|_H^2\end{aligned}$$

where $C_m := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-m} < \infty$ iff $m > d/2$. So the smoother f is the faster $\tilde{f}$ decays at infinity. The next problem is the converse of this assertion and hence smoothness of f corresponds to decay of $\tilde{f}$ at infinity and visa-versa.

Exercise 23.20 (A Sobolev Imbedding Theorem). Suppose $s\in\mathbb{R}$ and $\{c_k\in\mathbb{C} : k\in\mathbb{Z}^d\}$ are coefficients such that

$$\sum_{k\in\mathbb{Z}^d}|c_k|^2(1+|k|^2)^s < \infty.$$

Show if $s > \frac{d}{2}+m$, the function f defined by

$$f(x) = \sum_{k\in\mathbb{Z}^d} c_k e^{ik\cdot x}$$

is in $C^m_{per}(\mathbb{R}^d)$. **Hint**: Work as in the above remark to show

$$\sum_{k\in\mathbb{Z}^d}|c_k|\,|k^\alpha| < \infty \text{ for all } |\alpha|\le m.$$

Exercise 23.21 (Poisson Summation Formula). Let $F\in L^1(\mathbb{R}^d)$,

$$E := \left\{x\in\mathbb{R}^d : \sum_{k\in\mathbb{Z}^d}|F(x+2\pi k)| = \infty\right\}$$

and set

$$\hat{F}(k) := (2\pi)^{-d/2}\int_{\mathbb{R}^d}F(x)e^{-ik\cdot x}dx.$$

Further assume $\hat{F}\in\ell^1(\mathbb{Z}^d)$.

1. Show $m(E)=0$ and $E+2\pi k = E$ for all $k\in\mathbb{Z}^d$. **Hint**: Compute $\int_{[-\pi,\pi]^d}\sum_{k\in\mathbb{Z}^d}|F(x+2\pi k)|\,dx$.

2. Let
$$f(x) := \begin{cases}\sum_{k\in\mathbb{Z}^d}F(x+2\pi k) & \text{for } x\notin E\\ 0 & \text{if } x\in E.\end{cases}$$
Show $f\in L^1([-\pi,\pi]^d)$ and $\tilde{f}(k) = (2\pi)^{-d/2}\hat{F}(k)$.
3. Using item 2) and the assumptions on F, show $f\in L^1([-\pi,\pi]^d)\cap L^\infty([-\pi,\pi]^d)$ and
$$f(x) = \sum_{k\in\mathbb{Z}^d}\tilde{f}(k)e^{ik\cdot x} = \sum_{k\in\mathbb{Z}^d}(2\pi)^{-d/2}\hat{F}(k)e^{ik\cdot x} \text{ for } m-\text{a.e. } x,$$
i.e.
$$\sum_{k\in\mathbb{Z}^d}F(x+2\pi k) = (2\pi)^{-d/2}\sum_{k\in\mathbb{Z}^d}\hat{F}(k)e^{ik\cdot x} \text{ for } m-\text{a.e. } x. \tag{23.30}$$
4. Suppose we now assume that $F\in C(\mathbb{R}^d)$ and F satisfies 1) $|F(x)|\le C(1+|x|)^{-s}$ for some $s>d$ and $C<\infty$ and 2) $\hat{F}\in\ell^1(\mathbb{Z}^d)$, then show Eq. (23.30) holds for **all** $x\in\mathbb{R}^d$ and in particular
$$\sum_{k\in\mathbb{Z}^d}F(2\pi k) = (2\pi)^{-d/2}\sum_{k\in\mathbb{Z}^d}\hat{F}(k).$$

For notational simplicity, in the remaining problems we will assume that $d=1$.

Exercise 23.22 (Heat Equation 1.). Let $(t,x)\in[0,\infty)\times\mathbb{R}\to u(t,x)$ be a continuous function such that $u(t,\cdot)\in C_{per}(\mathbb{R})$ for all $t\ge 0$, $\dot{u} := u_t$, u_x, and u_{xx} exists and are continuous when $t>0$. Further assume that u satisfies the heat equation $\dot{u} = \frac{1}{2}u_{xx}$. Let $\tilde{u}(t,k) := \langle u(t,\cdot)|e_k\rangle$ for $k\in\mathbb{Z}$. Show for $t>0$ and $k\in\mathbb{Z}$ that $\tilde{u}(t,k)$ is differentiable in t and $\frac{d}{dt}\tilde{u}(t,k) = -k^2\tilde{u}(t,k)/2$. Use this result to show

$$u(t,x) = \sum_{k\in\mathbb{Z}}e^{-\frac{t}{2}k^2}\tilde{f}(k)e^{ikx} \tag{23.31}$$

where $f(x) := u(0,x)$ and as above

$$\tilde{f}(k) = \langle f|e_k\rangle = \int_{-\pi}^{\pi}f(y)e^{-iky}dy = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iky}dm(y).$$

Notice from Eq. (23.31) that $(t,x)\to u(t,x)$ is C^∞ for $t>0$.

Exercise 23.23 (Heat Equation 2.). Let $q_t(x) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}e^{-\frac{t}{2}k^2}e^{ikx}$. Show that Eq. (23.31) may be rewritten as

$$u(t,x) = \int_{-\pi}^{\pi}q_t(x-y)f(y)dy$$

and

$$q_t(x) = \sum_{k\in\mathbb{Z}} p_t(x + k2\pi)$$

where $p_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2}$. Also show $u(t,x)$ may be written as

$$u(t,x) = p_t * f(x) := \int_{\mathbb{R}^d} p_t(x-y) f(y) dy.$$

Hint: To show $q_t(x) = \sum_{k\in\mathbb{Z}} p_t(x+k2\pi)$, use the Poisson summation formula and the Gaussian integration identity,

$$\hat{p}_t(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} p_t(x) e^{i\omega x} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}\omega^2}. \tag{23.32}$$

Equation (23.32) will be discussed in Example 31.4 below.

Exercise 23.24 (Wave Equation). Let $u \in C^2(\mathbb{R}\times\mathbb{R})$ be such that $u(t,\cdot) \in C_{per}(\mathbb{R})$ for all $t \in \mathbb{R}$. Further assume that u solves the wave equation, $u_{tt} = u_{xx}$. Let $f(x) := u(0,x)$ and $g(x) = \dot{u}(0,x)$. Show $\tilde{u}(t,k) := \langle u(t,\cdot), e_k\rangle$ for $k \in \mathbb{Z}$ is twice continuously differentiable in t and $\frac{d^2}{dt^2}\tilde{u}(t,k) = -k^2\tilde{u}(t,k)$. Use this result to show

$$u(t,x) = \sum_{k\in\mathbb{Z}} \left(\tilde{f}(k)\cos(kt) + \tilde{g}(k)\frac{\sin kt}{k} \right) e^{ikx} \tag{23.33}$$

with the sum converging absolutely. Also show that $u(t,x)$ may be written as

$$u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{-t}^{t} g(x+\tau) d\tau. \tag{23.34}$$

Hint: To show Eq. (23.33) implies (23.34) use

$$\cos kt = \frac{e^{ikt} + e^{-ikt}}{2},$$
$$\sin kt = \frac{e^{ikt} - e^{-ikt}}{2i}, \text{ and}$$
$$\frac{e^{ik(x+t)} - e^{ik(x-t)}}{ik} = \int_{-t}^{t} e^{ik(x+\tau)} d\tau.$$

23.8 Conditional Expectation Exercises

Exercise 23.25. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{A} := \{A_i\}_{i=1}^\infty \subset \mathcal{F}$ is a partition of Ω. (Recall this means $\Omega = \coprod_{i=1}^\infty A_i$.) Let $\mathcal{G}$ be the σ – algebra generated by $\mathcal{A}$. Show:

1. $B \in \mathcal{G}$ iff $B = \cup_{i\in\Lambda} A_i$ for some $\Lambda \subset \mathbb{N}$.

2. $g : \Omega \to \mathbb{R}$ is $\mathcal{G}$ – measurable iff $g = \sum_{i=1}^\infty \lambda_i 1_{A_i}$ for some $\lambda_i \in \mathbb{R}$.
3. For $f \in L^1(\Omega, \mathcal{F}, P)$, let $E(f|A_i) := E[1_{A_i} f]/P(A_i)$ if $P(A_i) \neq 0$ and $E(f|A_i) = 0$ otherwise. Show

$$E_{\mathcal{G}} f = \sum_{i=1}^\infty E(f|A_i) 1_{A_i}.$$