22

Approximation Theorems and Convolutions

22.1 Density Theorems

In this section, $(X, \mathcal{M}, \mu)$ will be a measure space $\mathcal{A}$ will be a subalgebra of $\mathcal{M}$.

Notation 22.1 *Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $\mathcal{A} \subset \mathcal{M}$ is a subalgebra of $\mathcal{M}$. Let $\mathbb{S}(\mathcal{A})$ denote those simple functions $\phi : X \to \mathbb{C}$ such that $\phi^{-1}(\{\lambda\}) \in \mathcal{A}$ for all $\lambda \in \mathbb{C}$ and let $\mathbb{S}_f(\mathcal{A}, \mu)$ denote those $\phi \in \mathbb{S}(\mathcal{A})$ such that $\mu(\phi \neq 0) < \infty$.*

Remark 22.2. For $\phi \in \mathbb{S}_f(\mathcal{A}, \mu)$ and $p \in [1, \infty)$, $|\phi|^p = \sum_{z \neq 0} |z|^p 1_{\{\phi = z\}}$ and hence

$$\int |\phi|^p \, d\mu = \sum_{z \neq 0} |z|^p \mu(\phi = z) < \infty \tag{22.1}$$

so that $\mathbb{S}_f(\mathcal{A}, \mu) \subset L^p(\mu)$. Conversely if $\phi \in \mathbb{S}(\mathcal{A}) \cap L^p(\mu)$, then from Eq. (22.1) it follows that $\mu(\phi = z) < \infty$ for all $z \neq 0$ and therefore $\mu(\phi \neq 0) < \infty$. Hence we have shown, for any $1 \leq p < \infty$,

$$\mathbb{S}_f(\mathcal{A}, \mu) = \mathbb{S}(\mathcal{A}) \cap L^p(\mu).$$

Lemma 22.3 (Simple Functions are Dense). *The simple functions, $\mathbb{S}_f(\mathcal{M}, \mu)$, form a dense subspace of $L^p(\mu)$ for all $1 \leq p < \infty$.*

Proof. Let $\{\phi_n\}_{n=1}^\infty$ be the simple functions in the approximation Theorem 18.42. Since $|\phi_n| \leq |f|$ for all n, $\phi_n \in \mathbb{S}_f(\mathcal{M}, \mu)$ and

$$|f - \phi_n|^p \leq (|f| + |\phi_n|)^p \leq 2^p |f|^p \in L^1(\mu).$$

Therefore, by the dominated convergence theorem,

$$\lim_{n \to \infty} \int |f - \phi_n|^p d\mu = \int \lim_{n \to \infty} |f - \phi_n|^p d\mu = 0.$$

■

The goal of this section is to find a number of other dense subspaces of $L^p(\mu)$ for $p \in [1,\infty)$. The next theorem is the key result of this section.

Theorem 22.4 (Density Theorem). *Let $p \in [1,\infty)$, $(X,\mathcal{M},\mu)$ be a measure space and M be an algebra of bounded $\mathbb{F}$ – valued ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) measurable functions such that*

1. *$M \subset L^p(\mu,\mathbb{F})$ and $\sigma(M) = \mathcal{M}$.*
2. *There exists $\psi_k \in M$ such that $\psi_k \to 1$ boundedly.*
3. *If $\mathbb{F} = \mathbb{C}$ we further assume that M is closed under complex conjugation.*

Then to every function $f \in L^p(\mu,\mathbb{F})$, there exists $\phi_n \in M$ such that $\lim_{n\to\infty} \|f - \phi_n\|_{L^p(\mu)} = 0$, i.e. M is dense in $L^p(\mu,\mathbb{F})$.

Proof. Fix $k \in \mathbb{N}$ for the moment and let $\mathcal{H}$ denote those bounded $\mathcal{M}$ – measurable functions, $f : X \to \mathbb{F}$, for which there exists $\{\phi_n\}_{n=1}^\infty \subset M$ such that $\lim_{n\to\infty} \|\psi_k f - \phi_n\|_{L^p(\mu)} = 0$. A routine check shows $\mathcal{H}$ is a subspace of $\ell^\infty(\mathcal{M},\mathbb{F})$ such that $1 \in \mathcal{H}$, $M \subset \mathcal{H}$ and $\mathcal{H}$ is closed under complex conjugation if $\mathbb{F} = \mathbb{C}$. Moreover, $\mathcal{H}$ is closed under bounded convergence. To see this suppose $f_n \in \mathcal{H}$ and $f_n \to f$ boundedly. Then, by the dominated convergence theorem, $\lim_{n\to\infty} \|\psi_k(f - f_n)\|_{L^p(\mu)} = 0$.[1] (Take the dominating function to be $g = [2C|\psi_k|]^p$ where C is a constant bounding all of the $\{|f_n|\}_{n=1}^\infty$.) We may now choose $\phi_n \in M$ such that $\|\phi_n - \psi_k f_n\|_{L^p(\mu)} \le \frac{1}{n}$ then

$$\begin{aligned}\limsup_{n\to\infty} \|\psi_k f - \phi_n\|_{L^p(\mu)} \le &\limsup_{n\to\infty} \|\psi_k(f - f_n)\|_{L^p(\mu)} \\ &+ \limsup_{n\to\infty} \|\psi_k f_n - \phi_n\|_{L^p(\mu)} = 0 \qquad (22.2)\end{aligned}$$

which implies $f \in \mathcal{H}$. An application of Dynkin's Multiplicative System Theorem 18.51 if $\mathbb{F} = \mathbb{R}$ or Theorem 18.52 if $\mathbb{F} = \mathbb{C}$ now shows $\mathcal{H}$ contains all bounded measurable functions on X.

Let $f \in L^p(\mu)$ be given. The dominated convergence theorem implies $\lim_{k\to\infty} \left\|\psi_k 1_{\{|f|\le k\}} f - f\right\|_{L^p(\mu)} = 0$. (Take the dominating function to be $g = [2C|f|]^p$ where C is a bound on all of the $|\psi_k|$.) Using this and what we have just proved, there exists $\phi_k \in M$ such that

$$\left\|\psi_k 1_{\{|f|\le k\}} f - \phi_k\right\|_{L^p(\mu)} \le \frac{1}{k}.$$

The same line of reasoning used in Eq. (22.2) now implies $\lim_{k\to\infty} \|f - \phi_k\|_{L^p(\mu)} = 0$. ■

[1] It is at this point that the proof would break down if $p = \infty$.

Definition 22.5. *Let (X, τ) be a topological space and μ be a measure on $\mathcal{B}_X = \sigma(\tau)$. A **locally integrable** function is a Borel measurable function $f : X \to \mathbb{C}$ such that $\int_K |f| \, d\mu < \infty$ for all compact subsets $K \subset X$. We will write $L^1_{loc}(\mu)$ for the space of locally integrable functions. More generally we say $f \in L^p_{loc}(\mu)$ iff $\|1_K f\|_{L^p(\mu)} < \infty$ for all compact subsets $K \subset X$.*

Definition 22.6. *Let (X, τ) be a topological space. A K**-finite measure** on X is Borel measure μ such that $\mu(K) < \infty$ for all compact subsets $K \subset X$.*

Lebesgue measure on $\mathbb{R}$ is an example of a K-finite measure while counting measure on $\mathbb{R}$ is not a K-finite measure.

Example 22.7. Suppose that μ is a K-finite measure on $\mathcal{B}_{\mathbb{R}^d}$. An application of Theorem 22.4 shows $C_c(\mathbb{R}, \mathbb{C})$ is dense in $L^p(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}, \mu; \mathbb{C})$. To apply Theorem 22.4, let $M := C_c(\mathbb{R}^d, \mathbb{C})$ and $\psi_k(x) := \psi(x/k)$ where $\psi \in C_c(\mathbb{R}^d, \mathbb{C})$ with $\psi(x) = 1$ in a neighborhood of 0. The proof is completed by showing $\sigma(M) = \sigma(C_c(\mathbb{R}^d, \mathbb{C})) = \mathcal{B}_{\mathbb{R}^d}$, which follows directly from Lemma 18.57.

We may also give a more down to earth proof as follows. Let $x_0 \in \mathbb{R}^d$, $R > 0$, $A := B(x_0, R)^c$ and $f_n(x) := d_A^{1/n}(x)$. Then $f_n \in M$ and $f_n \to 1_{B(x_0,R)}$ as $n \to \infty$ which shows $1_{B(x_0,R)}$ is $\sigma(M)$-measurable, i.e. $B(x_0, R) \in \sigma(M)$. Since $x_0 \in \mathbb{R}^d$ and $R > 0$ were arbitrary, $\sigma(M) = \mathcal{B}_{\mathbb{R}^d}$.

More generally we have the following result.

Theorem 22.8. *Let (X, τ) be a second countable locally compact Hausdorff space and $\mu : \mathcal{B}_X \to [0, \infty]$ be a K-finite measure. Then $C_c(X)$ (the space of continuous functions with compact support) is dense in $L^p(\mu)$ for all $p \in [1, \infty)$. (See also Proposition 25.23 below.)*

Proof. Let $M := C_c(X)$ and use Item 3. of Lemma 18.57 to find functions $\psi_k \in M$ such that $\psi_k \to 1$ to boundedly as $k \to \infty$. The result now follows from an application of Theorem 22.4 along with the aid of item 4. of Lemma 18.57. ∎

Exercise 22.1. Show that $BC(\mathbb{R}, \mathbb{C})$ is not dense in $L^\infty(\mathbb{R}, \mathcal{B}_\mathbb{R}, m; \mathbb{C})$. Hence the hypothesis that $p < \infty$ in Theorem 22.4 can not be removed.

Corollary 22.9. *Suppose $X \subset \mathbb{R}^n$ is an open set, $\mathcal{B}_X$ is the Borel σ – algebra on X and μ be a K-finite measure on $(X, \mathcal{B}_X)$. Then $C_c(X)$ is dense in $L^p(\mu)$ for all $p \in [1, \infty)$.*

Corollary 22.10. *Suppose that X is a compact subset of $\mathbb{R}^n$ and μ is a finite measure on $(X, \mathcal{B}_X)$, then polynomials are dense in $L^p(X, \mu)$ for all $1 \leq p < \infty$.*

Proof. Consider X to be a metric space with usual metric induced from $\mathbb{R}^n$. Then X is a locally compact separable metric space and therefore

$C_c(X,\mathbb{C}) = C(X,\mathbb{C})$ is dense in $L^p(\mu)$ for all $p \in [1,\infty)$. Since, by the dominated convergence theorem, uniform convergence implies $L^p(\mu)$ – convergence, it follows from the Weierstrass approximation theorem (see Theorem 8.34 and Corollary 8.36 or Theorem 12.31 and Corollary 12.32) that polynomials are also dense in $L^p(\mu)$. ∎

Lemma 22.11. *Let (X,τ) be a second countable locally compact Hausdorff space and $\mu : \mathcal{B}_X \to [0,\infty]$ be a K-finite measure on X. If $h \in L^1_{loc}(\mu)$ is a function such that*

$$\int_X fhd\mu = 0 \text{ for all } f \in C_c(X) \tag{22.3}$$

then $h(x) = 0$ for μ – a.e. x. (See also Corollary 25.26 below.)

Proof. Let $d\nu(x) = |h(x)|\,dx$, then ν is a K-finite measure on X and hence $C_c(X)$ is dense in $L^1(\nu)$ by Theorem 22.8. Notice that

$$\int_X f\cdot \operatorname{sgn}(h)d\nu = \int_X fhd\mu = 0 \text{ for all } f \in C_c(X). \tag{22.4}$$

Let $\{K_k\}_{k=1}^\infty$ be a sequence of compact sets such that $K_k \uparrow X$ as in Lemma 11.23. Then $1_{K_k}\overline{\operatorname{sgn}(h)} \in L^1(\nu)$ and therefore there exists $f_m \in C_c(X)$ such that $f_m \to 1_{K_k}\overline{\operatorname{sgn}(h)}$ in $L^1(\nu)$. So by Eq. (22.4),

$$\nu(K_k) = \int_X 1_{K_k} d\nu = \lim_{m\to\infty}\int_X f_m \operatorname{sgn}(h)d\nu = 0.$$

Since $K_k \uparrow X$ as $k \to \infty$, $0 = \nu(X) = \int_X |h|\,d\mu$, i.e. $h(x) = 0$ for μ – a.e. x. ∎

As an application of Lemma 22.11 and Example 12.34, we will show that the Laplace transform is injective.

Theorem 22.12 (Injectivity of the Laplace Transform). *For $f \in L^1([0,\infty),dx)$, the Laplace transform of f is defined by*

$$\mathcal{L}f(\lambda) := \int_0^\infty e^{-\lambda x} f(x)dx \text{ for all } \lambda > 0.$$

If $\mathcal{L}f(\lambda) := 0$ then $f(x) = 0$ for m -a.e. x.

Proof. Suppose that $f \in L^1([0,\infty),dx)$ such that $\mathcal{L}f(\lambda) \equiv 0$. Let $g \in C_0([0,\infty),\mathbb{R})$ and $\varepsilon > 0$ be given. By Example 12.34 we may choose $\{a_\lambda\}_{\lambda>0}$ such that $\#(\{\lambda > 0 : a_\lambda \neq 0\}) < \infty$ and

$$|g(x) - \sum_{\lambda>0} a_\lambda e^{-\lambda x}| < \varepsilon \text{ for all } x \geq 0.$$

Then

$$\left|\int_0^\infty g(x)f(x)dx\right| = \left|\int_0^\infty \left(g(x) - \sum_{\lambda>0} a_\lambda e^{-\lambda x}\right) f(x)dx\right|$$
$$\leq \int_0^\infty \left|g(x) - \sum_{\lambda>0} a_\lambda e^{-\lambda x}\right| |f(x)|\, dx \leq \varepsilon \|f\|_1.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\int_0^\infty g(x)f(x)dx = 0$ for all $g \in C_0([0,\infty),\mathbb{R})$. The proof is finished by an application of Lemma 22.11. ■

Here is another variant of Theorem 22.8.

Theorem 22.13. *Let (X,d) be a metric space, τ_d be the topology on X generated by d and $\mathcal{B}_X = \sigma(\tau_d)$ be the Borel σ – algebra. Suppose $\mu : \mathcal{B}_X \to [0,\infty]$ is a measure which is σ – finite on τ_d and let $BC_f(X)$ denote the bounded continuous functions on X such that $\mu(f \neq 0) < \infty$. Then $BC_f(X)$ is a dense subspace of $L^p(\mu)$ for any $p \in [1,\infty)$.*

Proof. Let $X_k \in \tau_d$ be open sets such that $X_k \uparrow X$ and $\mu(X_k) < \infty$ and let

$$\psi_k(x) = \min(1, k \cdot d_{X_k^c}(x)) = \phi_k(d_{X_k^c}(x)),$$

see Figure 22.1 below. It is easily verified that $M := BC_f(X)$ is an algebra,

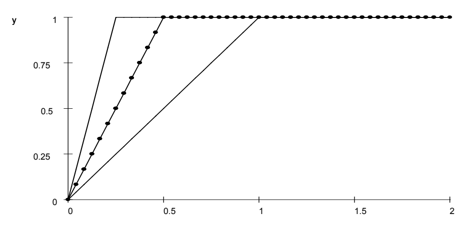

Fig. 22.1. The plot of ϕ_n for $n = 1$, 2, and 4. Notice that $\phi_n \to 1_{(0,\infty)}$.

$\psi_k \in M$ for all k and $\psi_k \to 1$ boundedly as $k \to \infty$. Given $V \in \tau$ and $k, n \in \mathbb{N}$,let

$$f_{k,n}(x) := \min(1, n \cdot d_{(V\cap X_k)^c}(x)).$$

Then $\{f_{k,n} \neq 0\} = V \cap X_k$ so $f_{k,n} \in BC_f(X)$. Moreover

$$\lim_{k\to\infty}\lim_{n\to\infty} f_{k,n} = \lim_{k\to\infty} 1_{V\cap X_k} = 1_V$$

which shows $V \in \sigma(M)$ and hence $\sigma(M) = \mathcal{B}_X$. The proof is now completed by an application of Theorem 22.4. ■

Exercise 22.2. (BRUCE: Should drop this exercise.) Suppose that (X,d) is a metric space, μ is a measure on $\mathcal{B}_X := \sigma(\tau_d)$ which is finite on bounded measurable subsets of X. Show $BC_b(X,\mathbb{R})$, defined in Eq. (19.26), is dense in $L^p(\mu)$. **Hints:** let ψ_k be as defined in Eq. (19.27) which incidentally may be used to show $\sigma(BC_b(X,\mathbb{R})) = \sigma(BC(X,\mathbb{R}))$. Then use the argument in the proof of Corollary 18.55 to show $\sigma(BC(X,\mathbb{R})) = \mathcal{B}_X$.

Theorem 22.14. *Suppose $p \in [1,\infty)$, $\mathcal{A} \subset \mathcal{M}$ is an algebra such that $\sigma(\mathcal{A}) = \mathcal{M}$ and μ is σ – finite on $\mathcal{A}$. Then $\mathbb{S}_f(\mathcal{A},\mu)$ is dense in $L^p(\mu)$. (See also Remark 25.7 below.)*

Proof. Let $M := \mathbb{S}_f(\mathcal{A},\mu)$. By assumption there exits $X_k \in \mathcal{A}$ such that $\mu(X_k) < \infty$ and $X_k \uparrow X$ as $k \to \infty$. If $A \in \mathcal{A}$, then $X_k \cap A \in \mathcal{A}$ and $\mu(X_k \cap A) < \infty$ so that $1_{X_k \cap A} \in M$. Therefore $1_A = \lim_{k\to\infty} 1_{X_k \cap A}$ is $\sigma(M)$ – measurable for every $A \in \mathcal{A}$. So we have shown that $\mathcal{A} \subset \sigma(M) \subset \mathcal{M}$ and therefore $\mathcal{M} = \sigma(\mathcal{A}) \subset \sigma(M) \subset \mathcal{M}$, i.e. $\sigma(M) = \mathcal{M}$. The theorem now follows from Theorem 22.4 after observing $\psi_k := 1_{X_k} \in M$ and $\psi_k \to 1$ boundedly. ■

Theorem 22.15 (Separability of L^p – Spaces). *Suppose, $p \in [1,\infty)$, $\mathcal{A} \subset \mathcal{M}$ is a countable algebra such that $\sigma(\mathcal{A}) = \mathcal{M}$ and μ is σ – finite on $\mathcal{A}$. Then $L^p(\mu)$ is separable and*

$$\mathbb{D} = \{\sum a_j 1_{A_j} : a_j \in \mathbb{Q} + i\mathbb{Q},\ A_j \in \mathcal{A} \text{ with } \mu(A_j) < \infty\}$$

is a countable dense subset.

Proof. It is left to reader to check $\mathbb{D}$ is dense in $\mathbb{S}_f(\mathcal{A},\mu)$ relative to the $L^p(\mu)$ – norm. The proof is then complete since $\mathbb{S}_f(\mathcal{A},\mu)$ is a dense subspace of $L^p(\mu)$ by Theorem 22.14. ■

Example 22.16. The collection of functions of the form $\phi = \sum_{k=1}^n c_k 1_{(a_k,b_k]}$ with $a_k, b_k \in \mathbb{Q}$ and $a_k < b_k$ are dense in $L^p(\mathbb{R},\mathcal{B}_\mathbb{R},m;\mathbb{C})$ and $L^p(\mathbb{R},\mathcal{B}_\mathbb{R},m;\mathbb{C})$ is separable for any $p \in [1,\infty)$. To prove this simply apply Theorem 22.14 with $\mathcal{A}$ being the algebra on $\mathbb{R}$ generated by the half open intervals $(a,b] \cap \mathbb{R}$ with $a < b$ and $a, b \in \mathbb{Q} \cup \{\pm\infty\}$, i.e. $\mathcal{A}$ consists of sets of the form $\coprod_{k=1}^n (a_k,b_k] \cap \mathbb{R}$, where $a_k, b_k \in \mathbb{Q} \cup \{\pm\infty\}$.

Exercise 22.3. Show $L^\infty([0,1],\mathcal{B}_\mathbb{R},m;\mathbb{C})$ is not separable. **Hint:** Suppose Γ is a dense subset of $L^\infty([0,1],\mathcal{B}_\mathbb{R},m;\mathbb{C})$ and for $\lambda \in (0,1)$, let $f_\lambda(x) := 1_{[0,\lambda]}(x)$. For each $\lambda \in (0,1)$, choose $g_\lambda \in \Gamma$ such that $\|f_\lambda - g_\lambda\|_\infty < 1/2$ and then show the map $\lambda \in (0,1) \to g_\lambda \in \Gamma$ is injective. Use this to conclude that Γ must be uncountable.

Corollary 22.17 (Riemann Lebesgue Lemma). *Suppose that $f \in L^1(\mathbb{R},m)$, then*

$$\lim_{\lambda\to\pm\infty} \int_\mathbb{R} f(x) e^{i\lambda x} dm(x) = 0.$$

Proof. By Example 22.16, given $\varepsilon > 0$ there exists $\phi = \sum_{k=1}^n c_k 1_{(a_k,b_k]}$ with $a_k, b_k \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} |f - \phi| dm < \varepsilon.$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}} \phi(x) e^{i\lambda x} dm(x) &= \int_{\mathbb{R}} \sum_{k=1}^n c_k 1_{(a_k,b_k]}(x) e^{i\lambda x} dm(x) \\ &= \sum_{k=1}^n c_k \int_{a_k}^{b_k} e^{i\lambda x} dm(x) = \sum_{k=1}^n c_k \lambda^{-1} e^{i\lambda x}|_{a_k}^{b_k} \\ &= \lambda^{-1} \sum_{k=1}^n c_k \left(e^{i\lambda b_k} - e^{i\lambda a_k}\right) \to 0 \text{ as } |\lambda| \to \infty.\end{aligned}$$

Combining these two equations with

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x) e^{i\lambda x} dm(x)\right| &\le \left|\int_{\mathbb{R}} (f(x) - \phi(x)) e^{i\lambda x} dm(x)\right| + \left|\int_{\mathbb{R}} \phi(x) e^{i\lambda x} dm(x)\right| \\ &\le \int_{\mathbb{R}} |f - \phi| dm + \left|\int_{\mathbb{R}} \phi(x) e^{i\lambda x} dm(x)\right| \\ &\le \varepsilon + \left|\int_{\mathbb{R}} \phi(x) e^{i\lambda x} dm(x)\right|\end{aligned}$$

we learn that

$$\lim \sup_{|\lambda|\to\infty} \left|\int_{\mathbb{R}} f(x) e^{i\lambda x} dm(x)\right| \le \varepsilon + \lim \sup_{|\lambda|\to\infty} \left|\int_{\mathbb{R}} \phi(x) e^{i\lambda x} dm(x)\right| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of the Riemann Lebesgue lemma. ■

Corollary 22.18. *Suppose $\mathcal{A} \subset \mathcal{M}$ is an algebra such that $\sigma(\mathcal{A}) = \mathcal{M}$ and μ is σ – finite on $\mathcal{A}$. Then for every $B \in \mathcal{M}$ such that $\mu(B) < \infty$ and $\varepsilon > 0$ there exists $D \in \mathcal{A}$ such that $\mu(B \triangle D) < \varepsilon$. (See also Remark 25.7 below.)*

Proof. By Theorem 22.14, there exists a collection, $\{A_i\}_{i=1}^n$, of pairwise disjoint subsets of $\mathcal{A}$ and $\lambda_i \in \mathbb{R}$ such that $\int_X |1_B - f| \, d\mu < \varepsilon$ where $f = \sum_{i=1}^n \lambda_i 1_{A_i}$. Let $A_0 := X \setminus \cup_{i=1}^n A_i \in \mathcal{A}$ then

$$\begin{aligned}
\int_X |1_B - f|\, d\mu &= \sum_{i=0}^{n} \int_{A_i} |1_B - f|\, d\mu \\
&= \mu(A_0 \cap B) + \sum_{i=1}^{n} \left[\int_{A_i \cap B} |1_B - \lambda_i|\, d\mu + \int_{A_i \setminus B} |1_B - \lambda_i|\, d\mu \right] \\
&= \mu(A_0 \cap B) + \sum_{i=1}^{n} \left[|1 - \lambda_i|\, \mu(B \cap A_i) + |\lambda_i|\, \mu(A_i \setminus B) \right] \qquad (22.5) \\
&\geq \mu(A_0 \cap B) + \sum_{i=1}^{n} \min\{\mu(B \cap A_i), \mu(A_i \setminus B)\} \qquad (22.6)
\end{aligned}$$

where the last equality is a consequence of the fact that $1 \leq |\lambda_i| + |1 - \lambda_i|$. Let

$$\alpha_i = \begin{cases} 0 \text{ if } \mu(B \cap A_i) < \mu(A_i \setminus B) \\ 1 \text{ if } \mu(B \cap A_i) \geq \mu(A_i \setminus B) \end{cases}$$

and $g = \sum_{i=1}^{n} \alpha_i 1_{A_i} = 1_D$ where

$$D := \cup \{A_i : i > 0 \ \&\ \alpha_i = 1\} \in \mathcal{A}.$$

Equation (22.5) with λ_i replaced by α_i and f by g implies

$$\int_X |1_B - 1_D|\, d\mu = \mu(A_0 \cap B) + \sum_{i=1}^{n} \min\{\mu(B \cap A_i), \mu(A_i \setminus B)\}.$$

The latter expression, by Eq. (22.6), is bounded by $\int_X |1_B - f|\, d\mu < \varepsilon$ and therefore,

$$\mu(B \triangle D) = \int_X |1_B - 1_D|\, d\mu < \varepsilon.$$

■

Remark 22.19. We have to assume that $\mu(B) < \infty$ as the following example shows. Let $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}$, $\mu = m$, $\mathcal{A}$ be the algebra generated by half open intervals of the form $(a, b]$, and $B = \cup_{n=1}^{\infty}(2n, 2n+1]$. It is easily checked that for every $D \in \mathcal{A}$, that $m(B \Delta D) = \infty$.

22.2 Convolution and Young's Inequalities

Throughout this section we will be solely concerned with d – dimensional Lebesgue measure, m, and we will simply write L^p for $L^p\left(\mathbb{R}^d, m\right)$.

Definition 22.20 (Convolution). *Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be measurable functions. We define*

$$f * g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) dy \qquad (22.7)$$

whenever the integral is defined, i.e. either $f(x-\cdot)\,g(\cdot) \in L^1(\mathbb{R}^d, m)$ *or* $f(x-\cdot)\,g(\cdot) \geq 0$. *Notice that the condition that* $f(x-\cdot)\,g(\cdot) \in L^1(\mathbb{R}^d, m)$ *is equivalent to writing* $|f| * |g|(x) < \infty$. *By convention, if the integral in Eq. (22.7) is not defined, let* $f * g(x) := 0$.

Notation 22.21 *Given a multi-index* $\alpha \in \mathbb{Z}_+^d$, *let* $|\alpha| = \alpha_1 + \cdots + \alpha_d$,

$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \text{ and } \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^d \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}.$$

For $z \in \mathbb{R}^d$ *and* $f : \mathbb{R}^d \to \mathbb{C}$, *let* $\tau_z f : \mathbb{R}^d \to \mathbb{C}$ *be defined by* $\tau_z f(x) = f(x-z)$.

Remark 22.22 (The Significance of Convolution).

1. Suppose that $f, g \in L^1(m)$ are positive functions and let μ be the measure on $\left(\mathbb{R}^d\right)^2$ defined by
$$d\mu(x,y) := f(x)\,g(y)\,dm(x)\,dm(y).$$
Then if $h : \mathbb{R} \to [0,\infty]$ is a measurable function we have
$$\begin{aligned}\int_{(\mathbb{R}^d)^2} h(x+y)\,d\mu(x,y) &= \int_{(\mathbb{R}^d)^2} h(x+y)\,f(x)\,g(y)\,dm(x)\,dm(y) \\ &= \int_{(\mathbb{R}^d)^2} h(x)\,f(x-y)\,g(y)\,dm(x)\,dm(y) \\ &= \int_{\mathbb{R}^d} h(x)\,f*g(x)\,dm(x).\end{aligned}$$
In other words, this shows the measure $(f*g)\,m$ is the same as $S_*\mu$ where $S(x,y) := x+y$. In probability lingo, the distribution of a sum of two "independent" (i.e. product measure) random variables is the the convolution of the individual distributions.
2. Suppose that $L = \sum_{|\alpha|\leq k} a_\alpha \partial^\alpha$ is a constant coefficient differential operator and suppose that we can solve (uniquely) the equation $Lu = g$ in the form
$$u(x) = Kg(x) := \int_{\mathbb{R}^d} k(x,y)g(y)dy$$
where $k(x,y)$ is an "integral kernel." (This is a natural sort of assumption since, in view of the fundamental theorem of calculus, integration is the inverse operation to differentiation.) Since $\tau_z L = L\tau_z$ for all $z \in \mathbb{R}^d$, (this is another way to characterize constant coefficient differential operators) and $L^{-1} = K$ we should have $\tau_z K = K\tau_z$. Writing out this equation then says
$$\begin{aligned}\int_{\mathbb{R}^d} k(x-z,y)g(y)dy &= (Kg)(x-z) = \tau_z Kg(x) = (K\tau_z g)(x) \\ &= \int_{\mathbb{R}^d} k(x,y)g(y-z)dy = \int_{\mathbb{R}^d} k(x,y+z)g(y)dy.\end{aligned}$$

Since g is arbitrary we conclude that $k(x-z,y)=k(x,y+z)$. Taking $y=0$ then gives

$$k(x,z)=k(x-z,0)=:\rho(x-z).$$

We thus find that $Kg=\rho*g$. Hence we expect the convolution operation to appear naturally when solving constant coefficient partial differential equations. More about this point later.

Proposition 22.23. *Suppose $p\in[1,\infty]$, $f\in L^1$ and $g\in L^p$, then $f*g(x)$ exists for almost every x, $f*g\in L^p$ and*

$$\|f*g\|_p\le\|f\|_1\|g\|_p.$$

Proof. This follows directly from Minkowski's inequality for integrals, Theorem 21.27. ■

Proposition 22.24. *Suppose that $p\in[1,\infty)$, then $\tau_z:L^p\to L^p$ is an isometric isomorphism and for $f\in L^p$, $z\in\mathbb{R}^d\to\tau_zf\in L^p$ is continuous.*

Proof. The assertion that $\tau_z:L^p\to L^p$ is an isometric isomorphism follows from translation invariance of Lebesgue measure and the fact that $\tau_{-z}\circ\tau_z=id$. For the continuity assertion, observe that

$$\|\tau_zf-\tau_yf\|_p=\|\tau_{-y}(\tau_zf-\tau_yf)\|_p=\|\tau_{z-y}f-f\|_p$$

from which it follows that it is enough to show $\tau_zf\to f$ in L^p as $z\to0\in\mathbb{R}^d$. When $f\in C_c(\mathbb{R}^d)$, $\tau_zf\to f$ uniformly and since the $K:=\cup_{|z|\le1}\operatorname{supp}(\tau_zf)$ is compact, it follows by the dominated convergence theorem that $\tau_zf\to f$ in L^p as $z\to0\in\mathbb{R}^d$. For general $g\in L^p$ and $f\in C_c(\mathbb{R}^d)$,

$$\begin{aligned}\|\tau_zg-g\|_p&\le\|\tau_zg-\tau_zf\|_p+\|\tau_zf-f\|_p+\|f-g\|_p\\&=\|\tau_zf-f\|_p+2\|f-g\|_p\end{aligned}$$

and thus

$$\limsup_{z\to0}\|\tau_zg-g\|_p\le\limsup_{z\to0}\|\tau_zf-f\|_p+2\|f-g\|_p=2\|f-g\|_p.$$

Because $C_c(\mathbb{R}^d)$ is dense in L^p, the term $\|f-g\|_p$ may be made as small as we please. ■

Exercise 22.4. Compute the operator norm, $\|\tau_z-I\|_{L(L^p(m))}$, of τ_z-I and use this to show $z\in\mathbb{R}^d\to\tau_z\in L(L^p(m))$ is **not** continuous.

Definition 22.25. *Suppose that (X,τ) is a topological space and μ is a measure on $\mathcal{B}_X=\sigma(\tau)$. For a measurable function $f:X\to\mathbb{C}$ we define the essential support of f by*

$$\operatorname{supp}_\mu(f)=\{x\in X:\mu(\{y\in V:f(y)\neq0\})>0\ \forall\text{ neighborhoods }V\text{ of }x\}.\tag{22.8}$$

Equivalently, $x\notin\operatorname{supp}_\mu(f)$ iff there exists an open neighborhood V of x such that $1_Vf=0$ a.e.

It is not hard to show that if $\operatorname{supp}(\mu) = X$ (see Definition 21.41) and $f \in C(X)$ then $\operatorname{supp}_\mu(f) = \operatorname{supp}(f) := \overline{\{f \neq 0\}}$, see Exercise 22.7.

Lemma 22.26. *Suppose (X, τ) is second countable and $f : X \to \mathbb{C}$ is a measurable function and μ is a measure on $\mathcal{B}_X$. Then $X := U \setminus \operatorname{supp}_\mu(f)$ may be described as the largest open set W such that $f1_W(x) = 0$ for μ – a.e. x. Equivalently put, $C := \operatorname{supp}_\mu(f)$ is the smallest closed subset of X such that $f = f1_C$ a.e.*

Proof. To verify that the two descriptions of $\operatorname{supp}_\mu(f)$ are equivalent, suppose $\operatorname{supp}_\mu(f)$ is defined as in Eq. (22.8) and $W := X \setminus \operatorname{supp}_\mu(f)$. Then

$$\begin{aligned} W &= \{x \in X : \exists\, \tau \ni V \ni x \text{ such that } \mu(\{y \in V : f(y) \neq 0\}) = 0\} \\ &= \cup \{V \subset_o X : \mu(f1_V \neq 0) = 0\} \\ &= \cup \{V \subset_o X : f1_V = 0 \text{ for } \mu - \text{a.e.}\}. \end{aligned}$$

So to finish the argument it suffices to show $\mu(f1_W \neq 0) = 0$. To to this let $\mathcal{U}$ be a countable base for τ and set

$$\mathcal{U}_f := \{V \in \mathcal{U} : f1_V = 0 \text{ a.e.}\}.$$

Then it is easily seen that $W = \cup \mathcal{U}_f$ and since $\mathcal{U}_f$ is countable

$$\mu(f1_W \neq 0) \leq \sum_{V \in \mathcal{U}_f} \mu(f1_V \neq 0) = 0.$$

■

Lemma 22.27. *Suppose $f, g, h : \mathbb{R}^d \to \mathbb{C}$ are measurable functions and assume that x is a point in $\mathbb{R}^d$ such that $|f| * |g|(x) < \infty$ and $|f| * (|g| * |h|)(x) < \infty$, then*

1. *$f * g(x) = g * f(x)$*
2. *$f * (g * h)(x) = (f * g) * h(x)$*
3. *If $z \in \mathbb{R}^d$ and $\tau_z(|f| * |g|)(x) = |f| * |g|(x - z) < \infty$, then*
$$\tau_z(f * g)(x) = \tau_z f * g(x) = f * \tau_z g(x)$$
4. *If $x \notin \operatorname{supp}_m(f) + \operatorname{supp}_m(g)$ then $f * g(x) = 0$ and in particular,*
$$\operatorname{supp}_m(f * g) \subset \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$$
where in defining $\operatorname{supp}_m(f * g)$ we will use the convention that "$f * g(x) \neq 0$" when $|f| * |g|(x) = \infty$.

Proof. For item 1.,

$$|f| * |g|(x) = \int_{\mathbb{R}^d} |f|(x - y)\, |g|(y) dy = \int_{\mathbb{R}^d} |f|(y)\, |g|(y - x) dy = |g| * |f|(x)$$

where in the second equality we made use of the fact that Lebesgue measure invariant under the transformation $y \to x - y$. Similar computations prove all of the remaining assertions of the first three items of the lemma. Item 4. Since $f * g(x) = \tilde{f} * \tilde{g}(x)$ if $f = \tilde{f}$ and $g = \tilde{g}$ a.e. we may, by replacing f by $f 1_{\mathrm{supp}_m(f)}$ and g by $g 1_{\mathrm{supp}_m(g)}$ if necessary, assume that $\{f \neq 0\} \subset \mathrm{supp}_m(f)$ and $\{g \neq 0\} \subset \mathrm{supp}_m(g)$. So if $x \notin (\mathrm{supp}_m(f) + \mathrm{supp}_m(g))$ then $x \notin (\{f \neq 0\} + \{g \neq 0\})$ and for all $y \in \mathbb{R}^d$, either $x - y \notin \{f \neq 0\}$ or $y \notin \{g \neq 0\}$. That is to say either $x - y \in \{f = 0\}$ or $y \in \{g = 0\}$ and hence $f(x-y)g(y) = 0$ for all y and therefore $f * g(x) = 0$. This shows that $f * g = 0$ on $\mathbb{R}^d \setminus \left(\overline{\mathrm{supp}_m(f) + \mathrm{supp}_m(g)}\right)$ and therefore

$$\mathbb{R}^d \setminus \left(\overline{\mathrm{supp}_m(f) + \mathrm{supp}_m(g)}\right) \subset \mathbb{R}^d \setminus \mathrm{supp}_m(f * g),$$

i.e. $\mathrm{supp}_m(f * g) \subset \mathrm{supp}_m(f) + \mathrm{supp}_m(g)$. ■

Remark 22.28. Let A, B be closed sets of $\mathbb{R}^d$, it is not necessarily true that $A + B$ is still closed. For example, take

$$A = \{(x, y) : x > 0 \text{ and } y \geq 1/x\} \text{ and } B = \{(x, y) : x < 0 \text{ and } y \geq 1/|x|\},$$

then every point of $A + B$ has a positive y - component and hence is not zero. On the other hand, for $x > 0$ we have $(x, 1/x) + (-x, 1/x) = (0, 2/x) \in A + B$ for all x and hence $0 \in \overline{A + B}$ showing $A + B$ is not closed. Nevertheless if one of the sets A or B is compact, then $A + B$ is closed again. Indeed, if A is compact and $x_n = a_n + b_n \in A + B$ and $x_n \to x \in \mathbb{R}^d$, then by passing to a subsequence if necessary we may assume $\lim_{n\to\infty} a_n = a \in A$ exists. In this case

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (x_n - a_n) = x - a \in B$$

exists as well, showing $x = a + b \in A + B$.

Proposition 22.29. *Suppose that $p, q \in [1, \infty]$ and p and q are conjugate exponents, $f \in L^p$ and $g \in L^q$, then $f * g \in BC(\mathbb{R}^d)$, $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$ and if $p, q \in (1, \infty)$ then $f * g \in C_0(\mathbb{R}^d)$.*

Proof. The existence of $f*g(x)$ and the estimate $|f * g|(x) \leq \|f\|_p \|g\|_q$ for all $x \in \mathbb{R}^d$ is a simple consequence of Holders inequality and the translation invariance of Lebesgue measure. In particular this shows $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. By relabeling p and q if necessary we may assume that $p \in [1, \infty)$. Since

$$\begin{aligned} \|\tau_z (f * g) - f * g\|_u &= \|\tau_z f * g - f * g\|_u \\ &\leq \|\tau_z f - f\|_p \|g\|_q \to 0 \text{ as } z \to 0 \end{aligned}$$

it follows that $f * g$ is uniformly continuous. Finally if $p, q \in (1, \infty)$, we learn from Lemma 22.27 and what we have just proved that $f_m * g_m \in C_c(\mathbb{R}^d)$ where $f_m = f 1_{|f| \leq m}$ and $g_m = g 1_{|g| \leq m}$. Moreover,

$$\begin{aligned}\|f*g-f_m*g_m\|_\infty &\le \|f*g-f_m*g\|_\infty+\|f_m*g-f_m*g_m\|_\infty\\ &\le \|f-f_m\|_p\|g\|_q+\|f_m\|_p\|g-g_m\|_q\\ &\le \|f-f_m\|_p\|g\|_q+\|f\|_p\|g-g_m\|_q\to 0 \text{ as } m\to\infty\end{aligned}$$

showing, with the aid of Proposition 12.23, $f*g\in C_0(\mathbb{R}^d)$. ■

Theorem 22.30 (Young's Inequality). *Let $p,q,r\in[1,\infty]$ satisfy*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}. \tag{22.9}$$

If $f\in L^p$ and $g\in L^q$ then $|f|*|g|(x)<\infty$ for m – a.e. x and

$$\|f*g\|_r\le\|f\|_p\|g\|_q. \tag{22.10}$$

In particular L^1 is closed under convolution. (The space $(L^1,*)$ is an example of a "Banach algebra" without unit.)

Remark 22.31. Before going to the formal proof, let us first understand Eq. (22.9) by the following scaling argument. For $\lambda>0$, let $f_\lambda(x):=f(\lambda x)$, then after a few simple change of variables we find

$$\|f_\lambda\|_p=\lambda^{-d/p}\|f\| \text{ and } (f*g)_\lambda=\lambda^d f_\lambda*g_\lambda.$$

Therefore if Eq. (22.10) holds for some $p,q,r\in[1,\infty]$, we would also have

$$\|f*g\|_r=\lambda^{d/r}\|(f*g)_\lambda\|_r\le\lambda^{d/r}\lambda\|f_\lambda\|_p\|g_\lambda\|_q=\lambda^{(d+d/r-d/p-d/q)}\|f\|_p\|g\|_q$$

for all $\lambda>0$. This is only possible if Eq. (22.9) holds.

Proof. By the usual sorts of arguments, we may assume f and g are positive functions. Let $\alpha,\beta\in[0,1]$ and $p_1,p_2\in(0,\infty]$ satisfy $p_1^{-1}+p_2^{-1}+r^{-1}=1$. Then by Hölder's inequality, Corollary 21.3,

$$\begin{aligned}f*g(x)&=\int_{\mathbb{R}^d}\left[f(x-y)^{(1-\alpha)}g(y)^{(1-\beta)}\right]f(x-y)^\alpha g(y)^\beta dy\\ &\le\left(\int_{\mathbb{R}^d}f(x-y)^{(1-\alpha)r}g(y)^{(1-\beta)r}dy\right)^{1/r}\left(\int_{\mathbb{R}^d}f(x-y)^{\alpha p_1}dy\right)^{1/p_1}\times\\ &\qquad\times\left(\int_{\mathbb{R}^d}g(y)^{\beta p_2}dy\right)^{1/p_2}\\ &=\left(\int_{\mathbb{R}^d}f(x-y)^{(1-\alpha)r}g(y)^{(1-\beta)r}dy\right)^{1/r}\|f\|_{\alpha p_1}^\alpha\|g\|_{\beta p_2}^\beta.\end{aligned}$$

Taking the r^{th} power of this equation and integrating on x gives

$$\|f*g\|_r^r \le \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)^{(1-\alpha)r} g(y)^{(1-\beta)r} dy\right) dx \cdot \|f\|_{\alpha p_1}^{\alpha} \|g\|_{\beta p_2}^{\beta}$$
$$= \|f\|_{(1-\alpha)r}^{(1-\alpha)r} \|g\|_{(1-\beta)r}^{(1-\beta)r} \|f\|_{\alpha p_1}^{\alpha r} \|g\|_{\beta p_2}^{\beta r}. \tag{22.11}$$

Let us now suppose, $(1-\alpha)r = \alpha p_1$ and $(1-\beta)r = \beta p_2$, in which case Eq. (22.11) becomes,

$$\|f*g\|_r^r \le \|f\|_{\alpha p_1}^r \|g\|_{\beta p_2}^r$$

which is Eq. (22.10) with

$$p := (1-\alpha)r = \alpha p_1 \text{ and } q := (1-\beta)r = \beta p_2. \tag{22.12}$$

So to finish the proof, it suffices to show p and q are arbitrary indices in $[1,\infty]$ satisfying $p^{-1}+q^{-1} = 1+r^{-1}$. If α, β, p_1, p_2 satisfy the relations above, then

$$\alpha = \frac{r}{r+p_1} \text{ and } \beta = \frac{r}{r+p_2}$$

and

$$\frac{1}{p}+\frac{1}{q} = \frac{1}{\alpha p_1}+\frac{1}{\alpha p_2} = \frac{1}{p_1}\frac{r+p_1}{r}+\frac{1}{p_2}\frac{r+p_2}{r}$$
$$= \frac{1}{p_1}+\frac{1}{p_2}+\frac{2}{r} = 1+\frac{1}{r}.$$

Conversely, if p, q, r satisfy Eq. (22.9), then let α and β satisfy $p = (1-\alpha)r$ and $q = (1-\beta)r$, i.e.

$$\alpha := \frac{r-p}{r} = 1-\frac{p}{r} \le 1 \text{ and } \beta = \frac{r-q}{r} = 1-\frac{q}{r} \le 1.$$

Using Eq. (22.9) we may also express α and β as

$$\alpha = p(1-\frac{1}{q}) \ge 0 \text{ and } \beta = q(1-\frac{1}{p}) \ge 0$$

and in particular we have shown $\alpha, \beta \in [0,1]$. If we now define $p_1 := p/\alpha \in (0,\infty]$ and $p_2 := q/\beta \in (0,\infty]$, then

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{r} = \beta\frac{1}{q}+\alpha\frac{1}{p}+\frac{1}{r}$$
$$= (1-\frac{1}{q})+(1-\frac{1}{p})+\frac{1}{r}$$
$$= 2-\left(1+\frac{1}{r}\right)+\frac{1}{r} = 1$$

as desired. ∎

Theorem 22.32 (Approximate δ − functions). *Let $p \in [1,\infty]$, $\phi \in L^1(\mathbb{R}^d)$, $a := \int_{\mathbb{R}^d} f(x)dx$, and for $t>0$ let $\phi_t(x) = t^{-d}\phi(x/t)$. Then*

1. *If* $f \in L^p$ *with* $p < \infty$ *then* $\phi_t * f \to af$ *in* L^p *as* $t \downarrow 0$.
2. *If* $f \in BC(\mathbb{R}^d)$ *and* f *is uniformly continuous then* $\|\phi_t * f - af\|_\infty \to 0$ *as* $t \downarrow 0$.
3. *If* $f \in L^\infty$ *and* f *is continuous on* $U \subset_o \mathbb{R}^d$ *then* $\phi_t * f \to af$ *uniformly on compact subsets of* U *as* $t \downarrow 0$.

Proof. Making the change of variables $y = tz$ implies

$$\phi_t * f(x) = \int_{\mathbb{R}^d} f(x-y)\phi_t(y)dy = \int_{\mathbb{R}^d} f(x-tz)\phi(z)dz$$

so that

$$\begin{aligned}\phi_t * f(x) - af(x) &= \int_{\mathbb{R}^d} [f(x-tz) - f(x)]\,\phi(z)dz \\ &= \int_{\mathbb{R}^d} [\tau_{tz}f(x) - f(x)]\,\phi(z)dz. \end{aligned} \tag{22.13}$$

Hence by Minkowski's inequality for integrals (Theorem 21.27), Proposition 22.24 and the dominated convergence theorem,

$$\|\phi_t * f - af\|_p \le \int_{\mathbb{R}^d} \|\tau_{tz}f - f\|_p\,|\phi(z)|\,dz \to 0 \text{ as } t \downarrow 0.$$

Item 2. is proved similarly. Indeed, form Eq. (22.13)

$$\|\phi_t * f - af\|_\infty \le \int_{\mathbb{R}^d} \|\tau_{tz}f - f\|_\infty\,|\phi(z)|\,dz$$

which again tends to zero by the dominated convergence theorem because $\lim_{t\downarrow 0}\|\tau_{tz}f - f\|_\infty = 0$ uniformly in z by the uniform continuity of f.

Item 3. Let $B_R = B(0,R)$ be a large ball in $\mathbb{R}^d$ and $K \sqsubset\sqsubset U$, then

$$\begin{aligned}&\sup_{x\in K}|\phi_t * f(x) - af(x)| \\ &\quad\le \left|\int_{B_R}[f(x-tz) - f(x)]\,\phi(z)dz\right| + \left|\int_{B_R^c}[f(x-tz) - f(x)]\,\phi(z)dz\right| \\ &\quad\le \int_{B_R}|\phi(z)|\,dz \cdot \sup_{x\in K, z\in B_R}|f(x-tz) - f(x)| + 2\|f\|_\infty \int_{B_R^c}|\phi(z)|\,dz \\ &\quad\le \|\phi\|_1 \cdot \sup_{x\in K, z\in B_R}|f(x-tz) - f(x)| + 2\|f\|_\infty \int_{|z|>R}|\phi(z)|\,dz\end{aligned}$$

so that using the uniform continuity of f on compact subsets of U,

$$\limsup_{t\downarrow 0}\sup_{x\in K}|\phi_t * f(x) - af(x)| \le 2\|f\|_\infty \int_{|z|>R}|\phi(z)|\,dz \to 0 \text{ as } R \to \infty.$$

■

See Theorem 8.15 if Folland for a statement about almost everywhere convergence.

Exercise 22.5. Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

Show $f \in C^\infty(\mathbb{R}, [0,1])$.

Lemma 22.33. *There exists $\phi \in C_c^\infty(\mathbb{R}^d, [0,\infty))$ such that $\phi(0) > 0$, $\operatorname{supp}(\phi) \subset \bar{B}(0,1)$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$.*

Proof. Define $h(t) = f(1-t)f(t+1)$ where f is as in Exercise 22.5. Then $h \in C_c^\infty(\mathbb{R}, [0,1])$, $\operatorname{supp}(h) \subset [-1,1]$ and $h(0) = e^{-2} > 0$. Define $c = \int_{\mathbb{R}^d} h(|x|^2)dx$. Then $\phi(x) = c^{-1}h(|x|^2)$ is the desired function. ■

The reader asked to prove the following proposition in Exercise 22.9 below.

Proposition 22.34. *Suppose that $f \in L^1_{loc}(\mathbb{R}^d, m)$ and $\phi \in C_c^1(\mathbb{R}^d)$, then $f * \phi \in C^1(\mathbb{R}^d)$ and $\partial_i(f * \phi) = f * \partial_i\phi$. Moreover if $\phi \in C_c^\infty(\mathbb{R}^d)$ then $f * \phi \in C^\infty(\mathbb{R}^d)$.*

Corollary 22.35 (C^∞ – Uryhson's Lemma). *Given $K \sqsubset\sqsubset U \subset_o \mathbb{R}^d$, there exists $f \in C_c^\infty(\mathbb{R}^d, [0,1])$ such that $\operatorname{supp}(f) \subset U$ and $f = 1$ on K.*

Proof. Let ϕ be as in Lemma 22.33, $\phi_t(x) = t^{-d}\phi(x/t)$ be as in Theorem 22.32, d be the standard metric on $\mathbb{R}^d$ and $\varepsilon = d(K, U^c)$. Since K is compact and U^c is closed, $\varepsilon > 0$. Let $V_\delta = \left\{x \in \mathbb{R}^d : d(x,K) < \delta\right\}$ and $f = \phi_{\varepsilon/3} * 1_{V_{\varepsilon/3}}$, then

$$\operatorname{supp}(f) \subset \overline{\operatorname{supp}(\phi_{\varepsilon/3}) + V_{\varepsilon/3}} \subset \bar{V}_{2\varepsilon/3} \subset U.$$

Since $\bar{V}_{2\varepsilon/3}$ is closed and bounded, $f \in C_c^\infty(U)$ and for $x \in K$,

$$f(x) = \int_{\mathbb{R}^d} 1_{d(y,K)<\varepsilon/3} \cdot \phi_{\varepsilon/3}(x-y)dy = \int_{\mathbb{R}^d} \phi_{\varepsilon/3}(x-y)dy = 1.$$

The proof will be finished after the reader (easily) verifies $0 \le f \le 1$. ■

Here is an application of this corollary whose proof is left to the reader, Exercise 22.10.

Lemma 22.36 (Integration by Parts). *Suppose f and g are measurable functions on $\mathbb{R}^d$ such that $t \to f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)$ and $t \to g(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)$ are continuously differentiable functions on $\mathbb{R}$ for each fixed $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Moreover assume $f \cdot g$, $\frac{\partial f}{\partial x_i} \cdot g$ and $f \cdot \frac{\partial g}{\partial x_i}$ are in $L^1(\mathbb{R}^d, m)$. Then*

$$\int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \cdot g dm = -\int_{\mathbb{R}^d} f \cdot \frac{\partial g}{\partial x_i} dm.$$

With this result we may give another proof of the Riemann Lebesgue Lemma.

Lemma 22.37 (Riemann Lebesgue Lemma). *For $f \in L^1(\mathbb{R}^d, m)$ let*

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dm(x)$$

be the Fourier transform of f. Then $\hat{f} \in C_0(\mathbb{R}^d)$ and $\left\|\hat{f}\right\|_\infty \leq (2\pi)^{-d/2} \|f\|_1$. (The choice of the normalization factor, $(2\pi)^{-d/2}$, in $\hat{f}$ is for later convenience.)

Proof. The fact that $\hat{f}$ is continuous is a simple application of the dominated convergence theorem. Moreover,

$$\left|\hat{f}(\xi)\right| \leq \int_{\mathbb{R}^d} |f(x)| \, dm(x) \leq (2\pi)^{-d/2} \|f\|_1$$

so it only remains to see that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. First suppose that $f \in C_c^\infty(\mathbb{R}^d)$ and let $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ be the Laplacian on $\mathbb{R}^d$. Notice that $\frac{\partial}{\partial x_j} e^{-i\xi\cdot x} = -i\xi_j e^{-i\xi\cdot x}$ and $\Delta e^{-i\xi\cdot x} = -|\xi|^2 e^{-i\xi\cdot x}$. Using Lemma 22.36 repeatedly,

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta^k f(x) e^{-i\xi\cdot x} dm(x) &= \int_{\mathbb{R}^d} f(x) \Delta_x^k e^{-i\xi\cdot x} dm(x) = -|\xi|^{2k} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dm(x) \\ &= -(2\pi)^{d/2} |\xi|^{2k} \hat{f}(\xi)\end{aligned}$$

for any $k \in \mathbb{N}$. Hence

$$(2\pi)^{d/2} \left|\hat{f}(\xi)\right| \leq |\xi|^{-2k} \left\|\Delta^k f\right\|_1 \to 0$$

as $|\xi| \to \infty$ and $\hat{f} \in C_0(\mathbb{R}^d)$. Suppose that $f \in L^1(m)$ and $f_k \in C_c^\infty(\mathbb{R}^d)$ is a sequence such that $\lim_{k\to\infty} \|f - f_k\|_1 = 0$, then $\lim_{k\to\infty} \left\|\hat{f} - \hat{f}_k\right\|_\infty = 0$. Hence $\hat{f} \in C_0(\mathbb{R}^d)$ by an application of Proposition 12.23. ■

Corollary 22.38. *Let $X \subset \mathbb{R}^d$ be an open set and μ be a Radon measure on $\mathcal{B}_X$.*

1. *Then $C_c^\infty(X)$ is dense in $L^p(\mu)$ for all $1 \leq p < \infty$.*
2. *If $h \in L^1_{loc}(\mu)$ satisfies*

$$\int_X f h d\mu = 0 \text{ for all } f \in C_c^\infty(X) \tag{22.14}$$

then $h(x) = 0$ for μ – a.e. x.

Proof. Let $f \in C_c(X)$, ϕ be as in Lemma 22.33, ϕ_t be as in Theorem 22.32 and set $\psi_t := \phi_t * (f 1_X)$. Then by Proposition 22.34 $\psi_t \in C^\infty(X)$ and by Lemma 22.27 there exists a compact set $K \subset X$ such that $\text{supp}(\psi_t) \subset K$ for all t sufficiently small. By Theorem 22.32, $\psi_t \to f$ uniformly on X as $t \downarrow 0$

1. The dominated convergence theorem (with dominating function being $\|f\|_\infty 1_K$), shows $\psi_t \to f$ in $L^p(\mu)$ as $t \downarrow 0$. This proves Item 1., since Theorem 22.8 guarantees that $C_c(X)$ is dense in $L^p(\mu)$.
2. Keeping the same notation as above, the dominated convergence theorem (with dominating function being $\|f\|_\infty |h| 1_K$) implies

$$0 = \lim_{t\downarrow 0} \int_X \psi_t h d\mu = \int_X \lim_{t\downarrow 0} \psi_t h d\mu = \int_X f h d\mu.$$

The proof is now finished by an application of Lemma 22.11.

■

22.2.1 Smooth Partitions of Unity

We have the following smooth variants of Proposition 12.16, Theorem 12.18 and Corollary 12.20. The proofs of these results are the same as their continuous counterparts. One simply uses the smooth version of Urysohn's Lemma of Corollary 22.35 in place of Lemma 12.8.

Proposition 22.39 (Smooth Partitions of Unity for Compacts). *Suppose that X is an open subset of $\mathbb{R}^d$, $K \subset X$ is a compact set and $\mathcal{U} = \{U_j\}_{j=1}^n$ is an open cover of K. Then there exists a smooth (i.e. $h_j \in C^\infty(X,[0,1])$) partition of unity $\{h_j\}_{j=1}^n$ of K such that $h_j \prec U_j$ for all $j = 1, 2, \dots, n$.*

Theorem 22.40 (Locally Compact Partitions of Unity). *Suppose that X is an open subset of $\mathbb{R}^d$ and $\mathcal{U}$ is an open cover of X. Then there exists a smooth partition of unity of $\{h_i\}_{i=1}^N$ ($N = \infty$ is allowed here) subordinate to the cover $\mathcal{U}$ such that* supp(h_i) *is compact for all i.*

Corollary 22.41. *Suppose that X is an open subset of $\mathbb{R}^d$ and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A} \subset \tau$ is an open cover of X. Then there exists a smooth partition of unity of $\{h_\alpha\}_{\alpha\in A}$ subordinate to the cover $\mathcal{U}$ such that* supp$(h_\alpha) \subset U_\alpha$ *for all $\alpha \in A$. Moreover if $\bar{U}_\alpha$ is compact for each $\alpha \in A$ we may choose h_α so that $h_\alpha \prec U_\alpha$.*

22.3 Exercises

Exercise 22.6. Let (X, τ) be a topological space, μ a measure on $\mathcal{B}_X = \sigma(\tau)$ and $f : X \to \mathbb{C}$ be a measurable function. Letting ν be the measure, $d\nu = |f| d\mu$, show $\operatorname{supp}(\nu) = \operatorname{supp}_\mu(f)$, where $\operatorname{supp}(\nu)$ is defined in Definition 21.41).

Exercise 22.7. Let (X, τ) be a topological space, μ a measure on $\mathcal{B}_X = \sigma(\tau)$ such that $\operatorname{supp}(\mu) = X$ (see Definition 21.41). Show $\operatorname{supp}_\mu(f) = \operatorname{supp}(f) = \overline{\{f \neq 0\}}$ for all $f \in C(X)$.

Exercise 22.8. Prove the following strong version of item 3. of Proposition 10.52, namely to every pair of points, $x_0,\ x_1$, in a connected open subset V of $\mathbb{R}^d$ there exists $\sigma \in C^\infty(\mathbb{R}, V)$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. **Hint**: First choose a continuous path $\gamma : [0,1] \to V$ such that $\gamma(t) = x_0$ for t near 0 and $\gamma(t) = x_1$ for t near 1 and then use a convolution argument to smooth γ.

Exercise 22.9. Prove Proposition 22.34 by appealing to Corollary 19.43.

Exercise 22.10 (Integration by Parts). Suppose that $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to f(x,y) \in \mathbb{C}$ and $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to g(x,y) \in \mathbb{C}$ are measurable functions such that for each fixed $y \in \mathbb{R}^d$, $x \to f(x,y)$ and $x \to g(x,y)$ are continuously differentiable. Also assume $f \cdot g$, $\partial_x f \cdot g$ and $f \cdot \partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R} \times \mathbb{R}^{d-1}$, where $\partial_x f(x,y) := \frac{d}{dt} f(x+t,y)|_{t=0}$. Show

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} \partial_x f(x,y) \cdot g(x,y) dx dy = - \int_{\mathbb{R}\times\mathbb{R}^{d-1}} f(x,y) \cdot \partial_x g(x,y) dx dy. \quad (22.15)$$

(Note: this result and Fubini's theorem proves Lemma 22.36.)

Hints: Let $\psi \in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of $0 \in \mathbb{R}$ and set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. First verify Eq. (22.15) with $f(x,y)$ replaced by $\psi_\varepsilon(x) f(x,y)$ by doing the x – integral first. Then use the dominated convergence theorem to prove Eq. (22.15) by passing to the limit, $\varepsilon \downarrow 0$.

Exercise 22.11. Let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$, then $\mathbb{D} := \operatorname{span}\{e^{i\lambda\cdot x} : \lambda \in \mathbb{R}^d\}$ is a dense subspace of $L^p(\mu)$ for all $1 \le p < \infty$. **Hints:** By Theorem 22.8, $C_c(\mathbb{R}^d)$ is a dense subspace of $L^p(\mu)$. For $f \in C_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$, let

$$f_N(x) := \sum_{n\in\mathbb{Z}^d} f(x + 2\pi N n).$$

Show $f_N \in BC(\mathbb{R}^d)$ and $x \to f_N(Nx)$ is 2π – periodic, so by Exercise 12.13, $x \to f_N(Nx)$ can be approximated uniformly by trigonometric polynomials. Use this fact to conclude that $f_N \in \bar{\mathbb{D}}^{L^p(\mu)}$. After this show $f_N \to f$ in $L^p(\mu)$.

Exercise 22.12. Suppose that μ and ν are two finite measures on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\mu(x) = \int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\nu(x) \quad (22.16)$$

for all $\lambda \in \mathbb{R}^d$. Show $\mu = \nu$.

Hint: Perhaps the easiest way to do this is to use Exercise 22.11 with the measure μ being replaced by $\mu + \nu$. Alternatively, use the method of proof of Exercise 22.11 to show Eq. (22.16) implies $\int_{\mathbb{R}^d} f d\mu(x) = \int_{\mathbb{R}^d} f d\nu(x)$ for all $f \in C_c(\mathbb{R}^d)$ and then apply Corollary 18.58.

Exercise 22.13. Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$. Further assume that $C_M := \int_{\mathbb{R}^d} e^{M|x|} d\mu(x) < \infty$ for all $M \in (0,\infty)$. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of polynomials, $\rho(x) = \sum_{|\alpha|\le N} \rho_\alpha x^\alpha$ with $\rho_\alpha \in \mathbb{C}$, on $\mathbb{R}^d$. (Notice that $|\rho(x)|^p \le$

$Ce^{M|x|}$ for some constant $C = C(\rho, p, M)$, so that $\mathcal{P}(\mathbb{R}^d) \subset L^p(\mu)$ for all $1 \le p < \infty$.) Show $\mathcal{P}(\mathbb{R}^d)$ is dense in $L^p(\mu)$ for all $1 \le p < \infty$. Here is a possible outline.

Outline: Fix a $\lambda \in \mathbb{R}^d$ and let $f_n(x) = (\lambda \cdot x)^n / n!$ for all $n \in \mathbb{N}$.

1. Use calculus to verify $\sup_{t \ge 0} t^\alpha e^{-Mt} = (\alpha/M)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/M)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-M|x|}\right) |\lambda|^{pn} e^{M|x|}$$
to find an estimate on $\|f_n\|_p$.
2. Use your estimate on $\|f_n\|_p$ to show $\sum_{n=0}^\infty \|f_n\|_p < \infty$ and conclude
$$\lim_{N \to \infty} \left\| e^{i\lambda \cdot (\cdot)} - \sum_{n=0}^N i^n f_n \right\|_p = 0.$$
3. Now finish by appealing to Exercise 22.11.

Exercise 22.14. Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$ but now assume there exists an $\varepsilon > 0$ such that $C := \int_{\mathbb{R}^d} e^{\varepsilon|x|} d\mu(x) < \infty$. Also let $q > 1$ and $h \in L^q(\mu)$ be a function such that $\int_{\mathbb{R}^d} h(x) x^\alpha d\mu(x) = 0$ for all $\alpha \in \mathbb{N}_0^d$. (As mentioned in Exercise 22.14, $\mathcal{P}(\mathbb{R}^d) \subset L^p(\mu)$ for all $1 \le p < \infty$, so $x \to h(x)x^\alpha$ is in $L^1(\mu)$.) Show $h(x) = 0$ for μ– a.e. x using the following outline.

Outline: Fix a $\lambda \in \mathbb{R}^d$, let $f_n(x) = (\lambda \cdot x)^n / n!$ for all $n \in \mathbb{N}$, and let $p = q/(q-1)$ be the conjugate exponent to q.

1. Use calculus to verify $\sup_{t \ge 0} t^\alpha e^{-\varepsilon t} = (\alpha/\varepsilon)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/\varepsilon)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-\varepsilon|x|}\right) |\lambda|^{pn} e^{\varepsilon|x|}$$
to find an estimate on $\|f_n\|_p$.
2. Use your estimate on $\|f_n\|_p$ to show there exists $\delta > 0$ such that $\sum_{n=0}^\infty \|f_n\|_p < \infty$ when $|\lambda| \le \delta$ and conclude for $|\lambda| \le \delta$ that $e^{i\lambda \cdot x} = L^p(\mu)\text{-}\sum_{n=0}^\infty i^n f_n(x)$. Conclude from this that
$$\int_{\mathbb{R}^d} h(x) e^{i\lambda \cdot x} d\mu(x) = 0 \text{ when } |\lambda| \le \delta.$$
3. Let $\lambda \in \mathbb{R}^d$ ($|\lambda|$ not necessarily small) and set $g(t) := \int_{\mathbb{R}^d} e^{it\lambda \cdot x} h(x) d\mu(x)$ for $t \in \mathbb{R}$. Show $g \in C^\infty(\mathbb{R})$ and
$$g^{(n)}(t) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{it\lambda \cdot x} h(x) d\mu(x) \text{ for all } n \in \mathbb{N}.$$

4. Let $T = \sup\{\tau \geq 0 : g|_{[0,\tau]} \equiv 0\}$. By Step 2., $T \geq \delta$. If $T < \infty$, then

$$0 = g^{(n)}(T) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{iT\lambda \cdot x} h(x) d\mu(x) \text{ for all } n \in \mathbb{N}.$$

Use Step 3. with h replaced by $e^{iT\lambda \cdot x} h(x)$ to conclude

$$g(T+t) = \int_{\mathbb{R}^d} e^{i(T+t)\lambda \cdot x} h(x) d\mu(x) = 0 \text{ for all } t \leq \delta / |\lambda|.$$

This violates the definition of T and therefore $T = \infty$ and in particular we may take $T = 1$ to learn

$$\int_{\mathbb{R}^d} h(x) e^{i\lambda \cdot x} d\mu(x) = 0 \text{ for all } \lambda \in \mathbb{R}^d.$$

5. Use Exercise 22.11 to conclude that

$$\int_{\mathbb{R}^d} h(x) g(x) d\mu(x) = 0$$

for all $g \in L^p(\mu)$. Now choose g judiciously to finish the proof.