

Bruce K. Driver

Analysis Tools with Applications

October 30, 2003 *File:anal.tex*

Springer

Berlin Heidelberg New York

Hong Kong London

Milan Paris Tokyo

Contents

Part I Back Ground Material

1	Introduction / User Guide	3
1.1	Topology beginnings	3
1.2	A Better Integral and an Introduction to Measure Theory	3
2	Set Operations	7
2.1	Exercises	10
3	The Real and Complex Numbers	11
3.1	The Real Numbers	12
3.1.1	The Decimal Representation of a Real Number	16
3.2	The Complex Numbers	19
3.3	Exercises	20
4	Limits and Sums	21
4.1	Limsups, Liminfs and Extended Limits	21
4.2	Sums of positive functions	24
4.3	Sums of complex functions	28
4.4	Iterated sums and the Fubini and Tonelli Theorems	32
4.5	Exercises	34
4.5.1	Limit Problems	34
4.5.2	Dominated Convergence Theorem Problems	35
5	ℓ^p – spaces, Minkowski and Holder Inequalities	39
5.1	Exercises	44

Part II Metric and Banach Space Basics

6	Metric Spaces	49
	6.1 Continuity	51
	6.2 Completeness in Metric Spaces	53
	6.3 Supplementary Remarks	55
	6.3.1 Word of Caution	55
	6.3.2 Riemannian Metrics	56
	6.4 Exercises	57
7	Banach Spaces	61
	7.1 Examples	61
	7.2 Bounded Linear Operators Basics	64
	7.3 General Sums in Banach Spaces	68
	7.4 Inverting Elements in $L(X)$	70
	7.5 Hahn Banach Theorem	71
	7.6 Exercises	75
	7.6.1 Hahn-Banach Theorem Problems	77
8	The Riemann Integral	79
	8.1 The Fundamental Theorem of Calculus	83
	8.2 Integral Operators as Examples of Bounded Operators	85
	8.3 Linear Ordinary Differential Equations	87
	8.4 Classical Weierstrass Approximation Theorem	91
	8.5 Iterated Integrals	96
	8.6 Exercises	99
9	Hölder Spaces as Banach Spaces	103
	9.1 Exercises	107

Part III Topological Spaces: I

10	Topological Space Basics	113
	10.1 Constructing Topologies and Checking Continuity	114
	10.2 Product Spaces I	120
	10.3 Closure operations	123
	10.4 Countability Axioms	125
	10.5 Connectedness	127
	10.6 Exercises	130
	10.6.1 General Topological Space Problems	130
	10.6.2 Connectedness Problems	131
	10.6.3 Metric Spaces as Topological Spaces	132

11 Compactness 135

- 11.1 Metric Space Compactness Criteria 136
- 11.2 Compact Operators 142
- 11.3 Local and σ – Compactness 144
- 11.4 Function Space Compactness Criteria 146
- 11.5 Tychonoff’s Theorem 149
- 11.6 Exercises 152
 - 11.6.1 Ascoli-Arzelà Theorem Problems 153
 - 11.6.2 Tychonoff’s Theorem Problem 154

Part IV Topological Spaces II

12 Locally Compact Hausdorff Spaces 157

- 12.1 Locally compact form of Urysohn’s Metrization Theorem 161
- 12.2 Partitions of Unity 164
- 12.3 $C_0(X)$ and the Alexanderov Compactification 169
- 12.4 More on Separation Axioms: Normal Spaces 171
- 12.5 Stone-Weierstrass Theorem 174
- 12.6 Locally Compact Version of Stone-Weierstrass Theorem 178
- 12.7 Exercises 179

13 Baire Category Theorem 183

- 13.1 Metric Space Baire Category Theorem 183
- 13.2 Locally Compact Hausdorff Space Baire Category Theorem ... 184
- 13.3 Exercises 190

14 Topological Vector Spaces 191

- 14.1 Basic Facts 191
- 14.2 The Structure of Finite Dimensional Topological Vector Spaces 193
- 14.3 Metrizable Topological Vector Spaces 195

Part V Calculus and Ordinary Differential Equations in Banach Spaces

15 Ordinary Differential Equations in a Banach Space 203

- 15.1 Examples 203
- 15.2 Uniqueness Theorem and Continuous Dependence on Initial Data 205
- 15.3 Local Existence (Non-Linear ODE) 207
- 15.4 Global Properties 210
- 15.5 Semi-Group Properties of time independent flows 216
- 15.6 Exercises 218

16 Banach Space Calculus	221
16.1 The Differential	221
16.2 Product and Chain Rules	222
16.3 Partial Derivatives	225
16.4 Smooth Dependence of ODE's on Initial Conditions	226
16.5 Higher Order Derivatives	229
16.6 Contraction Mapping Principle	233
16.7 Inverse and Implicit Function Theorems	235
16.8 More on the Inverse Function Theorem	239
16.8.1 Alternate construction of g	243
16.9 Applications	243
16.10 Exercises	245

Part VI Lebesgue Integration Theory

17 Introduction: What are measures and why “measurable” sets	251
17.1 The problem with Lebesgue “measure”	252
18 Measurability	257
18.1 Algebras and σ – Algebras	257
18.2 Measurable Functions	262
18.2.1 More general pointwise limits	269
18.3 σ – Function Algebras	269
18.4 Product σ – Algebras	276
18.4.1 Factoring of Measurable Maps	279
18.5 Exercises	279
19 Measures and Integration	281
19.1 Example of Measures	284
19.2 Integrals of Simple functions	286
19.3 Integrals of positive functions	287
19.4 Integrals of Complex Valued Functions	295
19.5 Measurability on Complete Measure Spaces	303
19.6 Comparison of the Lebesgue and the Riemann Integral	304
19.7 Determining Classes for Measures	307
19.8 Appendix: Bochner Integral	310
19.9 Bochner Integrals (NEEDS WORK)	314
19.9.1 Bochner Integral Problems From Folland	314
19.10 Exercises	316

20 Fubini’s Theorem 319

20.1 Fubini-Tonelli’s Theorem and Product Measure 320

 20.1.1 Application: More proofs of the Weierstrass
 Approximation Theorem ~~8.27~~ ^{t.8.27} 327

20.2 Lebesgue measure on \mathbb{R}^d 330

20.3 Polar Coordinates and Surface Measure 333

20.4 Exercises 337

21 L^p -spaces 339

21.1 Jensen’s Inequality 343

21.2 Modes of Convergence 346

21.3 Completeness of L^p – spaces 350

 21.3.1 Summary: 354

21.4 Converse of Hölder’s Inequality 355

21.5 Uniform Integrability 361

21.6 Exercises 367

22 Approximation Theorems and Convolutions 369

22.1 Convolution and Young’s Inequalities 374

 22.1.1 Smooth Partitions of Unity 383

22.2 Exercises 384

Part VII Construction of Measures

23 Uniqueness and Regularity of Measures 389

23.1 Monotone Class and $\pi - \lambda$ Theorems 389

 23.1.1 Some other proofs of previously proved theorems 392

23.2 Regularity of Measures 394

24 Daniell – Stone Construction of Measures 401

24.1 Finitely Additive Measures and Associated Integrals 401

 24.1.1 Integrals associated to finitely additive measures 403

24.2 The Daniell-Stone Construction Theorem 406

24.3 Extensions of premeasures to measures I 410

24.4 Riesz Representation Theorem 414

 24.4.1 The Riemann – Stieljtes – Lebesgue Integral 418

24.5 Metric space regularity results resisted 421

24.6 Measure on Products of Metric spaces 422

24.7 Measures on general infinite product spaces 424

24.8 Extensions of premeasures to measures II 426

 24.8.1 “Radon” measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ Revisited ~~t.24.35~~ 428

24.9 Supplement: Generalizations of Theorem ~~24.35~~ ^{t.24.35} to \mathbb{R}^n 429

24.10 Exercises 433

 24.10.1 The Laws of Large Number Exercises 434

25 Daniell Integral Proofs	437
25.1 Extension of Integrals	438
25.2 The Structure of $L^1(I)$	446
25.3 Relationship to Measure Theory	447
26 Carathéodory's Method of Constructing Measures	453
26.1 Outer Measures	453
26.2 Carathéodory's Construction Theorem	455
26.3 Regularity results revisited	459
27 Radon Measures and $C_0(X)^*$	463
27.1 More Regularity Results	466
27.2 The Riesz Representation Theorem	469
27.2.1 Another proof of Theorem 27.11	473
27.3 The dual of $C_0(X)$	473
27.4 Special case of Riesz Theorem on $[0, 1]$	477
27.5 Applications	478
27.6 The General Riesz Representation by Daniell Integrals	481
27.7 Regularity Results	483
27.8 Old Stuff	489
27.8.1 Construction of measures on a simple product space... ..	489

Part VIII Hilbert Spaces and Spectral Theory of Compact Operators

28 Hilbert Spaces	495
28.1 Hilbert Spaces Basics	495
28.2 Hilbert Space Basis	504
28.3 Fourier Series Considerations	507
28.4 Weak Convergence	510
28.5 Supplement 1: Converse of the Parallelogram Law	514
28.6 Supplement 2. Non-complete inner product spaces	516
28.7 Supplement 3: Conditional Expectation	518
28.8 Exercises	521
28.9 Fourier Series Exercises	524
28.10 Dirichlet Problems on D	528
29 Polar Decomposition of an Operator	533
30 Compact Operators	541
30.1 Hilbert Schmidt and Trace Class Operators	543
30.2 The Spectral Theorem for Self Adjoint Compact Operators ...	548
30.3 Structure of Compact Operators	551
30.4 Trace Class Operators	551

30.5 Fredholm Operators 554

30.6 Tensor Product Spaces 560

31 Spectral Theorem for Self-Adjoint Operators 565

31.1 $*$ -Algebras (over complexes) 572

 31.1.1 Exercises 576

31.2 The Spectral Theorem 578

 31.2.1 Problems on the Spectral Theorem (Multiplication Operator Form) 585

31.3 Spectral Theory in Hilbert Space 588

 31.3.1 Multiplication operators 591

 31.3.2 Spectrum 593

 31.3.3 Spectral Theorem and the Functional Calculus 595

 31.3.4 Weak integrals of operator valued functions 596

 31.3.5 Fourier Transform, Proof of Theorem 31.83 598

 31.3.6 Proof of Theorem 31.83 602

 31.3.7 Extensions to commuting self-adjoint operators 603

 31.3.8 Exercises 611

31.4 Unbounded Self-Adjoint Operators 612

 31.4.1 Affiliated operators 622

31.5 Bounded Self - Adjoint Operators 623

31.6 The Structure of Abelian Banach Algebras 628

Part IX Synthesis of Integral and Differential Calculus

32 Complex Measures, Radon-Nikodym Theorem and the Dual of L^p 637

32.1 Radon-Nikodym Theorem I 638

32.2 Signed Measures 644

 32.2.1 Hahn Decomposition Theorem 645

 32.2.2 Jordan Decomposition 646

32.3 Complex Measures II 650

32.4 Absolute Continuity on an Algebra 654

32.5 Dual Spaces and the Complex Riesz Theorem 656

 32.5.1 Reflexivity 660

32.6 Exercises 661

33 Lebesgue Differentiation and the Fundamental Theorem of Calculus 663

33.1 A Covering Lemma and Averaging Operators 664

33.2 Maximal Functions 665

33.3 Lebesgue Set 667

33.4 The Fundamental Theorem of Calculus 670

33.5 Alternative method to the Fundamental Theorem of Calculus 681

33.5.1 Proof of Theorem <u>t:31:29</u>	683
33.6 Examples:	683
33.7 Exercises	684
34 The Change of Variable Theorem	687
34.1 Appendix: Other Approaches to proving Theorem <u>t:32:1</u>	692
34.2 Sard's Theorem	694
35 Surfaces, Surface Integrals and Integration by Parts	699
35.1 Surface Integrals	701
35.2 More spherical coordinates	711
35.3 n – dimensional manifolds with boundaries	715
35.4 Divergence Theorem	718
35.5 The proof of the Divergence Theorem	722
35.5.1 The Proof of the Divergence Theorem <u>t:33:23</u>	724
35.5.2 Extensions of the Divergence Theorem to Lipschitz domains	726
35.6 Application to Holomorphic functions	727
35.7 Dirichlet Problems on D	730
35.7.1 Appendix: More Proofs of Proposition <u>p:33:32</u>	734
35.8 Exercises	736

Part X Miracle Properties of Banach Spaces

36 Two Fundamental Principles of Banach Spaces	741
36.1 The Open Mapping Theorem	741
36.1.1 Applications to Fourier Series	747
36.2 Banach – Alaoglu's Theorem	750
36.2.1 Weak and Strong Topologies	750
36.2.2 Weak Convergence Results	752
36.3 Supplement: Quotient spaces, adjoints, and more reflexivity	756
36.4 Exercises	761
36.4.1 More Examples of Banach Spaces	761
36.4.2 Hahn-Banach Theorem Problems	761
36.4.3 Baire Category Result Problems	761
36.4.4 Weak Topology and Convergence Problems	762
37 Weak and Strong Derivatives	763
37.1 Basic Definitions and Properties	763
37.2 The connection of Weak and pointwise derivatives	777
37.3 Exercises	783

Part XI Complex Variable Theory

38 Complex Differentiable Functions 789

 38.1 Basic Facts About Complex Numbers 789

 38.2 The complex derivative 790

 38.3 Contour integrals 796

 38.4 Weak characterizations of $H(\Omega)$ 803

 38.5 Summary of Results 807

 38.6 Exercises 808

 38.7 Problems from Rudin 811

39 More Complex Variables: The Index 813

 39.1 Unique Lifting Theorem 814

 39.2 Path Lifting Property 814

 39.2.1 **Alternate Method** 816

40 Residue Theorem 825

 40.1 Residue Theorem 825

 40.2 Open Mapping Theorem 828

 40.3 Applications of Residue Theorem 829

 40.3.1 Applications; fundamental theory of Algebra. 830

 40.4 Isolated Singularity Theory 831

41 Conformal Equivalence 835

42 Find All Conformal Homeomorphisms of $V \rightarrow U$ 839

 42.1 “Sketch of Proof of Riemann Mapping” Theorem 840

 42.1.1 Normal Families 840

43 Littlewood Payley Theory 849

 43.0.2 Applications 852

Part XII The Fourier Transform and Generalized Functions

44 Fourier Transform 859

 44.1 Fourier Transform 860

 44.2 Schwartz Test Functions 863

 44.3 Fourier Inversion Formula 865

 44.4 Summary of Basic Properties of \mathcal{F} and \mathcal{F}^{-1} 868

 44.5 Fourier Transforms of Measures and Bochner’s Theorem 869

 44.6 Supplement: Heisenberg Uncertainty Principle 873

 44.6.1 Exercises 875

 44.6.2 More Proofs of the Fourier Inversion Theorem 876

45 Constant Coefficient partial differential equations 879

45.1 Elliptic examples 880

45.2 Poisson Semi-Group 882

45.3 Heat Equation on \mathbb{R}^n 883

45.4 Wave Equation on \mathbb{R}^n 887

45.5 Elliptic Regularity 893

45.6 Exercises 898

46 Elementary Generalized Functions / Distribution Theory . . 899

46.1 Distributions on $U \subset \mathbb{R}^n$ 899

46.2 Examples of distributions and related computations 900

46.3 Other classes of test functions 908

46.4 Compactly supported distributions 914

46.5 Tempered Distributions and the Fourier Transform 916

46.6 Wave Equation 924

46.7 Appendix: Topology on $C_c^\infty(U)$ 929

47 Convolutions involving distributions 933

47.1 Tensor Product of Distributions 933

47.2 Elliptic Regularity 943

47.3 Appendix: Old Proof of Theorem 47.4 947

Part XIII An Introduction to Differentiable Manifolds

48 Inverse Function Theorem and Embedded Submanifolds . . 955

48.1 Embedded Submanifolds 955

48.2 Exercises 956

48.3 Construction of Embedded Submanifolds 957

49 The Flow of a Vector Fields on Manifolds 959

50 Co-Area Formula in Riemannian Geometry 963

50.0.1 Formal Proof of Theorem 50.3 969

50.1 Special case of the Co-area formula when $X = \mathbb{R}$ 972

50.2 Differential Geometric Version of Co-Area Formula 974

51 Application of the Co-Area Formulas 977

51.1 Existence of Densities for Push Forwards of Measures 977

51.2 Sobolev Inequalities and Isoperimetric Inequalities 980

Part XIV L^2 – Sobolev spaces and Pseudo Differential Operators

52 L^2 – Sobolev spaces on T^n 987

53 L^2 – Sobolev spaces on \mathbb{R}^n 989
 53.1 Sobolev Spaces 989
 53.2 Examples 999
 53.3 **Summary of operations on $H_{-\infty}$** 1001
 53.4 Application to Differential Equations 1004
 53.4.1 Dirichlet problem 1004

54 Pseudo-Differential Operators on Euclidean space 1007
 54.0.2 On the decay of $\partial^\alpha \langle x \rangle^s$ and $\partial_x^\alpha |x|^{2m}$ 1007
 54.1 Symbols and their operators 1009
 54.1.1 A remark on the Fourier Inversion Formula 1009
 54.2 A more general symbol class 1011
 54.2.1 Heuristics 1012
 54.2.2 The proofs 1014
 54.3 Schwartz Kernel Approach 1021
 54.4 Pseudo Differential Operators 1027

55 Elliptic Pseudo Differential Operators on \mathbb{R}^d 1041

56 Pseudo differential operators on Compact Manifolds 1047

57 Sobolev Spaces on M 1051
 57.1 Alternate Definition of H_k for k -integer 1056
 57.2 Scaled Spaces 1058
 57.3 General Properties of “Scaled space” 1059

Part XV PDE Examples

58 Some Examples of PDE’s 1065
 58.1 Some More Geometric Examples 1070

Part XVI First Order Scalar Equations

59 First Order Quasi-Linear Scalar PDE 1073
 59.1 Linear Evolution Equations 1073
 59.1.1 A 1-dimensional wave equation with non-constant
 coefficients 1080
 59.2 General Linear First Order PDE 1082
 59.3 Quasi-Linear Equations 1088
 59.4 Distribution Solutions for Conservation Laws 1094
 59.5 Exercises 1099

60 Fully nonlinear first order PDE	1105
60.1 An Introduction to Hamilton Jacobi Equations	1110
60.1.1 Solving the Hamilton Jacobi Equation (60.17) by characteristics	1110
60.1.2 The connection with the Euler Lagrange Equations	1111
60.2 Geometric meaning of the Legendre Transform	1117
61 Cauchy – Kovalevskaya Theorem	1119
61.1 PDE Cauchy Kovalevskaya Theorem	1124
61.2 Proof of Theorem 61.7	1129
61.3 Examples	1130

Part XVII Elliptic ODE

62 A very short introduction to generalized functions	1135
63 Elliptic Ordinary Differential Operators	1139
63.1 Symmetric Elliptic ODE	1140
63.2 General Regular 2nd order elliptic ODE	1143
63.3 Elementary Sobolev Inequalities	1153
63.4 Associated Heat and Wave Equations	1157
63.5 Extensions to Other Boundary Conditions	1159
63.5.1 Dirichlet Forms Associated to $(L, D(L))$	1161

Part XVIII Constant Coefficient Equations

64 Convolutions, Test Functions and Partitions of Unity	1169
64.1 Convolution and Young's Inequalities	1169
64.2 Smooth Partitions of Unity	1180
65 Poisson and Laplace's Equation	1183
65.1 Harmonic and Subharmonic Functions	1189
65.2 Green's Functions	1199
65.3 Explicit Green's Functions and Poisson Kernels	1203
65.4 Green's function for Ball	1207
65.5 Perron's Method for solving the Dirichlet Problem	1212
65.6 Solving the Dirichlet Problem by Integral Equations	1217
66 Introduction to the Spectral Theorem	1219
66.1 Du Hammel's principle again	1226

67 Heat Equation 1235
 67.1 Extensions of Theorem ¶:68:1 ~~67.1~~ 1237
 67.2 Representation Theorem and Regularity 1241
 67.3 Weak Max Principles 1243
 67.4 Non-Uniqueness of solutions to the Heat Equation 1249
 67.5 The Heat Equation on the Circle and \mathbb{R} 1251

68 Abstract Wave Equation 1253
 68.1 Corresponding first order O.D.E. 1254
 68.2 Du Hamel's Principle 1256

69 Wave Equation on \mathbb{R}^n 1259
 69.1 $n = 1$ Case 1260
 69.1.1 Factorization method for $n = 1$ 1262
 69.2 Solution for $n = 3$ 1263
 69.2.1 Alternate Proof of Theorem ¶:70:4 ~~69.4~~ 1265
 69.3 Du Hamel's Principle 1266
 69.4 Spherical Means 1266
 69.5 Energy methods 1269
 69.6 Wave Equation in Higher Dimensions 1271
 69.6.1 Solution derived from the heat kernel 1271
 69.6.2 Solution derived from the Poisson kernel 1272
 69.7 Explain Method of descent $n = 2$ 1276

Part XIX Sobolev Theory

70 Sobolev Spaces 1279
 70.1 Mollifications 1281
 70.1.1 Proof of Theorem ¶:71:10 ~~70.10~~ 1285
 70.2 Difference quotients 1287
 70.3 Sobolev Spaces on Compact Manifolds 1289
 70.4 Trace Theorems 1293
 70.5 Extension Theorems 1298
 70.6 Exercises 1302

71 Sobolev Inequalities 1303
 71.1 Morrey's Inequality 1303
 71.2 Rademacher's Theorem 1309
 71.3 Gagliardo-Nirenberg-Sobolev Inequality 1310
 71.4 Sobolev Embedding Theorems Summary 1316
 71.5 Compactness Theorems 1318
 71.6 Fourier Transform Method 1322
 71.7 Other theorems along these lines 1323
 71.8 Exercises 1325

Part XX Variable Coefficient Equations

72	2nd order differential operators	1329
	72.1 Outline of future results	1333
73	Dirichlet Forms	1335
	73.1 Basics	1335
74	Unbounded operators and quadratic forms	1339
	74.1 Unbounded operator basics	1339
	74.2 Lax-Milgram Methods	1341
	74.3 Close, symmetric, semi-bounded quadratic forms and self-adjoint operators	1344
	74.4 Construction of positive self-adjoint operators	1349
	74.5 Applications to partial differential equations	1350
75	Weak Solutions for Elliptic Operators	1353
76	Elliptic Regularity	1357
	76.1 Interior Regularity	1357
	76.2 Boundary Regularity Theorem	1360
77	L^2 – operators associated to \mathcal{E}	1375
	77.1 Compact perturbations of the identity and the Fredholm Alternative	1376
	77.2 Solvability of $Lu = f$ and properties of the solution	1378
	77.3 Interior Regularity Revisited	1382
	77.4 Classical Dirichlet Problem	1383
	77.5 Some Non-Compact Considerations	1384
	77.5.1 Heat Equation	1386
	77.5.2 Wave Equation	1386
78	Spectral Considerations	1387
	78.1 Growth of Eigenvalues I	1388

Part XXI Heat Kernel Properties

79	Construction of Heat Kernels by Spectral Methods	1395
	79.1 Positivity of Dirichlet Heat Kernel by Beurling Deny Methods	1399
80	Nash Type Inequalities and Their Consequences	1401

81 T. Coulhon Lecture Notes 1409
 81.1 Weighted Riemannian Manifolds 1409
 81.2 Graph Setting 1411
 81.3 Basic Inequalities 1412
 81.4 A Scale of Inequalities 1415
 81.5 Semi-Group Theory 1419

Part XXII Heat Kernels on Vector Bundles

82 Heat Equation on \mathbb{R}^n 1425

83 An Abstract Version of E. Levi’s Argument 1427

84 Statement of the Main Results 1431
 84.1 The General Setup: the Heat Eq. for a Vector Bundle 1431
 84.2 The Jacobian (J – function) 1432
 84.3 The Approximate Heat Kernels 1433
 84.4 The Heat Kernel and its Asymptotic Expansion 1434

85 Proof of Theorems ^{t.85.7}84.7 and ^{t.85.10}84.10 1437
 85.1 Proof of Theorem ^{t.85.7}84.7 1437
 85.2 Proof of Theorem ^{t.85.10}84.10 1439

86 Properties of ρ 1441
 86.0.1 Proof of Proposition ^{p.87:1}86.1 1442
 86.0.2 On the Operator Associated to the Kernel ρ 1443

87 Proof of Theorem ^{t.85.4}84.4 and Corollary ^{c.85.6}84.6 1447
 87.1 Proof of Corollary ^{c.85.6}84.6 1447
 87.2 Proof of Theorem ^{t.85.4}84.4 1448

88 Appendix: Gauss’ Lemma & Polar Coordinates 1453
 88.1 The Laplacian of Radial Functions 1454

89 The Dirac Equation a la Roe’s Book 1457
 89.1 Kernel Construction 1460
 89.2 Asymptotics by Sobolev Theory 1463

90 Appendix: VanVleck Determinant Properties 1465
 90.1 Proof of Lemma ^{t.85.3}84.3 1465
 90.2 Another Proof of Remark ^{r.85.2}84.2: The Symmetry of $J(x, y)$ 1467
 90.3 Normal Coordinates 1468

91 Miscellaneous 1473

91.1 Jazzed up version of Proposition p:92.1 1473

91.1.1 Proof of Eq. (e:92.3) 1475

91.1.2 Old proof of Proposition p:84.1 1475

91.1.3 Old Stuff related to Theorem t:85.7 1479

92 Remarks on Covariant Derivatives on Vector Bundles 1481

93 Spin Bundle Stuff 1485

94 The Case where $M = \mathbb{R}^n$ 1487

94.1 Formula involving p 1487

94.2 Asymptotics of a perturbed Heat Eq. on \mathbb{R}^n 1488

Part XXIII PDE Extras

95 Higher Order Elliptic Equations 1495

96 Abstract Evolution Equations 1499

96.1 Basic Definitions and Examples 1499

96.2 General Theory of Contraction Semigroups 1502

97 Solutions to Exercises 1513

97.1 Section s:60 Solutions 1513

97.1.1 Old Exercises Solutions 1514

97.2 Section s:33 Solutions 1517

97.3 Section s:68 Solutions 1519

Part XXIV Old PDE Stuff

98 Old Section Stuff 1523

98.1 Section s:60 1523

98.2 Section s:61 1525

98.3 Section s:62 1527

98.4 Old Section s:64 Stuff 1528

98.5 Old Section s:66 stuff 1528

98.6 Old Section s:70 Now to the $n = 3$ Case: formally compute: 1531

98.7 Probably delete the following stuff 1535

99 A Little Distribution Theory 1537

99.1 Old Section s:67 1538

99.1.1 Old Proof of Proposition p:67.8 1538

99.1.2 Special case of Theorem t:67.21 1540

99.2 Old Section s:68 1540

99.2.1	more section ^{s.68} 67 stuff	1542
99.3	Section ^{s.74} 73	1545
99.4	Boundary value problems	1549
99.5	Section ^{s.77} 76	1551
99.5.1	Summary of the Proof of Proposition ^{p.77:10} 76.10	1552
99.5.2	Old Proofs of Lemma ^{p.77:12} 76.12	1553
99.6	Old Section ^{s.78} 77	1555
99.7	More domain comments	1555
99.8	OLD Compactness	1557
99.9	Fourier Transform Method	1557
99.10	Old Lax-Milgram Theorem	1560
99.11	Old Local regularity	1561

Part XXV Gaussian Measures

100	Infinite Dimensional Gaussian Measures	1567
100.1	Finite Dimensional Examples and Results	1569
100.2	Density Theorems	1572
100.3	Product measures on \mathbb{R}^N	1575
100.4	Basic Infinite Dimensional Results	1577
100.5	Gaussian Measure for ℓ^2	1585
100.6	Classical Wiener Measure	1589
100.7	Basic Properties Wiener Measure	1594
100.8	The Cameron-Martin Space and Theorem	1596
100.9	Cameron-Martin Theorem	1599
100.10	Exercises	1603
100.11	Gross' Abstract Wiener Spaces	1604

Part XXVI Old Unused Analysis Material

101	Old Stuff	1615
101.1	Chapter ^{c.4} 4	1615
101.1.1	Old proofs of Tonelli's Theorem ^{t.4:22} 4.22	1615
101.1.2	Old Proof of theorem ^{c.4:23} 4.23	1617
101.1.3	Old Proof of Proposition ^{p.4:17} 4.17	1618
101.1.4	Old Fubini – sum	1620
101.2	Chapter ^{c.6} 6	1621
101.3	Compactness on metric spaces	1621
101.4	Compact Sets in \mathbb{R}^n	1624
101.5	Section ^{c.8} 8	1626
101.6	Chapter ^{c.11} II	1627
101.6.1	Old Proof of Theorem ^{t.8:27} 8.27 for $d = 1$	1627
101.7	Old ^{c.10} II Stuff	1628

101.8	Chapter 12 ^{c.12}	1628
101.8.1	Old Urysohn's metrization Theorem	1628
101.9	Section 15 ^{s.15}	1634
101.10	Section 18 ^{s.18}	1636
101.10	Miscellaneous measurability results	1636
101.11	Section 20 ^{s.20}	1638
101.11	Alternate proof of comment after Notation 20.2 ^{h.20.2}	1639
101.11	Old Theorem 20.3 ^{t.20.3}	1639
101.11	Z's Regularity of measures on metric spaces	1641
101.11	Old Regularity and Density Results	1644
101.11	A general regularity result	1644
101.11	B regularity of measures on metric spaces	1645
101.11	Low tech. proof of Theorem 23.9 ^{t.23.9}	1649
101.12	Section 21 ^{s.21}	1651
101.13	Section 22 ^{s.22}	1652
101.13	Section 26 ^{s.26}	1653
101.14	Proofs for Section 28 ^{s.26} via orthonormal bases	1655
101.15	Section 30 ^{s.28}	1656
101.15	Old Proof of Theorem 30.18 ^{t.28.18}	1656
101.16	Section 24 ^{s.24}	1660
101.16	Additive measures on \mathbb{R}	1665
101.16	More exercises.	1666
101.17	Section 25 ^{s.25}	1667
101.18	Section 32 ^{s.30} old Stuff	1668
101.19	Signed measures	1669
101.19	Old Radon-Nikodym proof.	1671
101.20	The Total Variation on an Algebra by B.	1673
101.21	The Total Variation an Algebra by Z.	1675
101.22	Section 33 ^{s.31}	1677
101.23	Old Absolute Continuity	1678
101.24	Appendix: Absolute Continuity on an algebra by Z. (Delete?) ..	1678
101.25	Other Hahn Decomposition Proofs	1678
101.26	Old Dual to L^p spaces	1679
101.27	Section 101.18 ^{s.101.18}	1682
101.28	Section 33.4 ^{s.31.4}	1683
101.29	Section ?? ^{??}	1685
101.30	Section 36 ^{s.34}	1689
101.31	Section 37 ^{s.35}	1693
101.31	Old Proof of Proposition 37.12 ^{p.35.12} in the a special case. ...	1693
101.31	Old Proof of Lemma 70.23 ^{l.71.23}	1695
101.31	Parts of old proof of Theorem 37.27 ^{p.35.27}	1695
101.32	Section 44 ^{s.42}	1697
101.33	Old Section 45 ^{s.43}	1698
101.33	Old proof of Proposition 45.5 ^{p.43.5}	1698
101.34	1699

101.3	Section ^{s.71} 70	1699
101.3	Application to regularity	1699
101.3	Old Section ^{s.32} 34	1701
101.3	Old Section ^{s.75} 74	1702
101.38	An alternate proof of part of Theorem ^{t.75:10} 74.10	1702
101.38	Old Proof of Theorem ^{t.75:23} 74.23	1703
101.38	Proof of Theorem ^{t.75:23} 74.23	1704
101.3	Old Section ^{s.A} A	1707
101.4	Old Section ^{s.54:6} 100.6	1709
101.4	Miscellaneous Old Stuff	1710
101.41	Products with a Finite Number of Factors	1710

Part XXVII Solutions to Selected Exercises

102	Section ^{s.13} 13	Solutions	1739
102.1	Section ^{s.15} 15	Solutions	1739
102.2	Section ^{s.16} 16	Solutions	1740
103	Section ^{s.18} 18	Solutions	1743
104	Section ^{s.19} 19	Solutions	1745
104.1	Section ^{s.20} 20	Solutions	1750
104.2	Section ^{s.21} 21	Solutions	1758
104.3	Section ^{s.22} 22	Solutions	1768
104.4	Section ^{s.24} 24	Solutions	1770
104.5	Section ^{s.26} 28	Solutions	1774
104.6	Section ^{s.30} 32	Solutions	1780
104.7	Section ^{s.31} 33	Solutions	1789
104.8	Section ^{s.34} 36	Solutions	1793
	104.8.1	Other Folland Chapter 5 problems	1809
104.9	Size of ℓ^2 - spaces		1812
104.1	Bochner Integral Problems form chapter 5 of first edition.		1813
104.1	Section ^{s.35} 37	Solutions	1816
104.1	Section ^{s.42} 44	Solutions	1818
104.1	Section ^{s.43} 45	Solutions	1823
104.1	Section ^{s.53} 27	Solutions	1824
104.1	Problems from Folland Sec. 7		1825
104.1	Folland Chapter 2 problems		1828
104.1	Folland Chapter 4 problems		1829

Part XXVIII Appendices

A	Multinomial Theorems and Calculus Results	5
A.1	Multinomial Theorems and Product Rules	5
A.2	Taylor's Theorem.....	7
B	Zorn's Lemma and the Hausdorff Maximal Principle	11
C	Nets	15
	References	19
	Index	21