1

Introduction / User Guide

Not written as of yet. Topics to mention.

1. A better and more general integral.
   a) Convergence Theorems
   b) Integration over diverse collection of sets. (See probability theory.)
   c) Integration relative to different weights or densities including singular weights.
   d) Characterization of dual spaces.
   e) Completeness.
2. Infinite dimensional Linear algebra.
3. ODE and PDE.
4. Harmonic and Fourier Analysis.
5. Probability Theory

1.1 Topology beginnings

Recall the notion of a topology by extrapolating from the open sets on $\mathbb{R}^2$. Also recall what it means to be continuous, namely $f : X \to \mathbb{R}$ is continuous at x if for all $\varepsilon > 0$ there exists $V \in \tau_x$ such that

$$f(V) \subset f(x) + (-\varepsilon, \varepsilon).$$

1.2 A Better Integral and an Introduction to Measure Theory

Let $a, b \in \mathbb{R}$ with $a < b$ and let

$$I^0(f) := \int_a^b f(t)dt \text{ for all } f \in C([a, b])$$

denote the Riemann integral. Also let $\mathcal{H}$ denote the smallest **linear subspace** of bounded functions on $[a, b]$ which is closed under bounded convergence and contains $C([a, b])$. Such a space exists since we can take the intersection over all such spaces of functions.

Theorem 1.1. *There is an extension I of I^0 to $\mathcal{H}$ such that I is still linear and $\lim_{n\to\infty} I(f_n) = I(f)$ for all $f_n \in \mathcal{H}$ with $f_n \to f$ boundedly. Moreover this extension is unique and is **positive** in the sense that $I(f) \geq 0$ if $f \in \mathcal{H}$ and $f \geq 0$.*

Proof. We will only prove the uniqueness here. Suppose that J and I are two such extensions and let

$$\mathcal{K} := \{f \in \mathcal{H} : J(f) = I(f)\}.$$

Then $\mathcal{K}$ is a linear subspace closed under bounded convergence which contains $C([a, b])$ and hence $\mathcal{K} = \mathcal{H}$.

The existence of I is the hard part. The positivity of I can be seen from the existence construction. ■

Example 1.2. Here are some examples of functions in $\mathcal{H}$ and their integrals:

1. Suppose $[\alpha, \beta] \subset [a, b]$, then $1_{[\alpha,\beta]} \in \mathcal{H}$ and $I\left(1_{[\alpha,\beta]}\right) = \beta - \alpha$.(Draw a picture.)
2. $I\left(1_{\{\alpha\}}\right) = 0$.
3. The space $\mathcal{H}$ is an algebra, i.e. if $f, g \in \mathcal{H}$ then $fg \in \mathcal{H}$. To prove this, first assume that $f \in C([a, b])$ and let
$$\mathcal{H}_f := \{g \in \mathcal{H} : fg \in \mathcal{H}\}.$$
Then $\mathcal{H}_f$ is closed under bounded convergence and contains $C([a, b])$ and hence $\mathcal{H}_f = \mathcal{H}$, i.e. the product of a continuous function and an element in $\mathcal{H}$ is back in $\mathcal{H}$.
Now suppose that $f \in \mathcal{H}$ and again let $\mathcal{H}_f$ be as above. By the same reasoning we may show again that $\mathcal{H}_f = \mathcal{H}$ and this proves the assertion.
4. If $f \in \mathcal{H}$ and $\phi \in C(\mathbb{R})$, then $\phi \circ f \in \mathcal{H}$. This a consequence of the Weierstrass approximation Theorem 22.34. In particular $|f| \in \mathcal{H}$ and $f_\pm := \frac{|f| \pm f}{2} \in \mathcal{H}$ if $f \in \mathcal{H}$.
5. If $f_n \in \mathcal{H}$, $f_n \geq 0$ and $f = \sum_{n=1}^\infty f_n$ is a bounded function, then $f \in \mathcal{H}$ and
$$I(f) = \sum_{n=1}^{\infty} I(f_n). \tag{1.1}$$
To prove Eq. (1.1) we have
$$\sum_{n=1}^{\infty} I(f_n) = \lim_{N\to\infty} I\left(\sum_{n=1}^{N} f_n\right) = I(f).$$

6. As an example of item 4., $1_{\mathbb{Q}\cap[a,b]} = \sum_{n=1}^{\infty} 1_{\{\alpha_n\}} \in \mathcal{H}$ and $I\left(1_{\mathbb{Q}\cap[a,b]}\right) = 0$. Here $\{\alpha_n\}_{n=1}^{\infty}$ is an enumeration of the rationale numbers in the interval $[a, b]$.
7. Let $\mathcal{M} := \{A \subset [a, b] : 1_A \in \mathcal{H}\}$ and for $A \in \mathcal{M}$ let $m(A) := I(1_A)$. Then $\mathcal{M}$ and m have the following properties:
   a) $\emptyset, [a, b] \in \mathcal{M}$ and $m(\emptyset) = 0$ and $m([a, b]) = b - a$. Moreover $m(A) \geq 0$ for all $A \in \mathcal{M}$.
   b) If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$ and $m(A^c) = b - a - m(A)$. This follows from the fact that $1_{A^c} = 1 - 1_A$.
   c) If $A, B \in \mathcal{M}$, then $A \cap B \in \mathcal{M}$ since if $1_{A\cap B} = 1_A \cdot 1_B$ and $\mathcal{H}$ is an algebra.
   Definition: a collection of sets $\mathcal{M}$ satisfying a) – c) is called an **algebra** of subsets of $[a, b]$.
   d) More generally if $A_n \in \mathcal{M}$ then $\cap A_n \in \mathcal{M}$ since $1_{\cap A_n} = \lim_{N\to\infty} 1_{A_1} \cdots 1_{A_N}$ and the convergence is bounded.
   Definition: a collection of sets $\mathcal{M}$ satisfying a) – d) is called an $\sigma -$ **algebra.**
   e) If $A_n \in \mathcal{M}$, then $\cup A_n \in \mathcal{M}$. Indeed we know $\cup A_n \in \mathcal{M}$ iff $(\cup A_n)^c \in \mathcal{M}$. But
   $$(\cup A_n)^c = \cap A_n^c \in \mathcal{M}$$
   by item d. above.
   f) If $A_n \in \mathcal{M}$ are pairwise disjoint, then
   $$m(\cup A_n) = \sum_{n=1}^{\infty} m(A_n).$$
   To prove this it suffices to observe that $1_{\cup A_n} = \sum_{n=1}^{\infty} 1_{A_n}$.
   g) $\mathcal{M}$ is not $2^{[a,b]}$, i.e. $\mathcal{M}$ is not all subset of $[a, b]$. This is not obvious and it is not possible to really write down an "explicit" subset $[a, b]$ which is not in $\mathcal{M}$. We will prove the existence of such sets later.
8. **Fact:** $\mathcal{M}$ is the smallest σ – algebra on $[a, b]$ which contains all subintervals of $[a, b]$.
9. **Fact:** A bounded function $f : [a, b] \to \mathbb{R}$ is in $\mathcal{H}$ iff $\{f > \alpha\} \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$.
10. **Fact:** The integral I may be recovered from the measure m by the formula
$$I(f) = \lim_{\text{mesh}\to 0} \sum_{0<\alpha_1<\alpha_2<\alpha_3<\ldots}^{\infty} \alpha_i m(\{x \in [a, b] : \alpha_i < f(x) \leq \alpha_i\}).$$

We will prove items 8. – 10. later in the course. The proof if Items 9. and 10. is not so hard and the energetic reader may wish to give them a try.

Notation 1.3 *The collection of sets $\mathcal{M}$ is called the Borel σ – algebra on $[a, b]$ and the function $m : \mathcal{M} \to \mathbb{R}$ is called Lebesgue measure. We will usually* *write* $I(f)$ *as* $\int_{[a,b]} f dm$ *and* $I(f)$ *will be called the Lebesgue integral of* f*. This integral may be extended to all positive functions* f *such that* $f 1_{|f| \leq M} \in \mathcal{H}$ *for all* M *by*

$$I(f) = \lim_{M \to \infty} I\left(f 1_{|f| \leq M}\right).$$

Again, we will come back to all of this again later.