F.1 Study Guide For Math 240A: Fall 2003

F.1.1 Basic things you should know about numbers and limits

1. I am taking for granted that you know the basic properties of $\mathbb{R}$ and $\mathbb{C}$ and that they are complete.
2. Should know how to compute $\lim a_n$, $\limsup a_n$ and $\liminf a_n$ and their basic properties. See Lemma 4.2 and Proposition 4.5 for example.

F.1.2 Basic things you should know about topological and measurable spaces:

1. You should know the basic definitions, Definition 10.1 and Definition 18.1.
2. It would be good to understand the notion of generating a topology or a σ – algebra by either a collection of sets or functions. This is key to understanding product topologies and product σ – algebras. See Propositions 10.7, 10.21 and 18.4 and Definition 18.24 and Proposition 18.25.
3. You should be able to check whether a given function is continuous or measurable. **Hints:**
   a) If possible avoid going back to the definition of continuity or measurability. Do this by using the stability properties of continuous (measurable) functions. For example continuous (measurable) functions are stable under compositions and algebraic operations, under uniform (pointwise) limits and sums. Measurable functions are also stable under Closure under sups, infs and $\liminf$ and $\limsup$, see Proposition 18.36. Also recall if we are using the Borel σ – algebras, then continuous functions are automatically measurable.
   b) It is also possible to check continuity and measurability by splitting the space up and checking continuity and measurability on the individual pieces. See Proposition 10.19 and Exercise 10.7 and Proposition 18.29.
   c) If you must go back to first principles, then the fact that $\sigma\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\sigma(\mathcal{E}))$ and $\tau\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\tau(\mathcal{E}))$ is key, see Lemma 18.22 and 10.14 respectively.
4. Dynkin's multiplicative system Theorems 18.51 and 18.52 are extremely useful for understanding the structure of measurable functions. They are also very useful for proving general theorems which are to hold for all bounded measurable functions. For examples, see the examples following Theorem 18.52 and the examples in Section 19.7.

F.1.3 Basic things you should know about Metric Spaces

1. The associated topology, see Example 10.3.
2. How to find the closure of a set. I typically would use the sequential definition of closure here.
3. Continuity is equivalent to the sequential notion of continuity, see Section 6.1.
4. The continuity properties of the metric, see Lemma 6.6.
5. The notions of Cauchy sequences and completeness.

F.1.4 Basic things you should know about Banach spaces

1. They are complete normed spaces.
2. $\ell^p(\mu)$ – spaces are Banach spaces, see Theorems 5.6, 5.8, and 7.5. Later we will see that all of these theorem hold for more general $L^p(\mu)$ – spaces as well.
3. $BC(X)$ is a closed subset of the Banach space $\ell^\infty(X)$ and hence is a Banach space, see Lemma 7.3.
4. The space of operators $L(X, Y)$ between two Banach spaces is a Banach space. In particular the dual space X^* is a Banach space, see Proposition 7.12.
5. How to find the norm of an operator and the basic properties of the operator norm, Lemma 7.10.
6. Boundedness of an operator is equivalent to continuity, Proposition 7.8.
7. Small perturbations of an invertible operator is still invertible, see Proposition 7.19 and Corollary 7.20.

F.1.5 The Riemann integral

The material on Riemann integral in Chapter 8 served as an illustration of much of the general Banach space theory described above. We also saw interesting applications to linear ODE.

However the **most important** result from Chapter 8 is the Weierstrass Approximation Theorem 8.31 and its complex version in Corollary 8.33.

F.1.6 Basic things you should know about Lebesgue integration theory and infinite sums

Recall that the Lebesgue integral relative to a counting type measure corresponds to an infinite sum, see Lemma 19.15. As a rule one does not need to go back to the definitions of integrals to work with them. The key points to working with integrals (and hence sums as well) are the following facts.

1. The integral is linear and satisfies the monotonicity properties: $\int f \leq \int g$ if $f \leq g$ a.e. and $\left|\int f\right| \leq \int |f|$.
2. The monotone convergence Theorem 19.16 and its Corollary 19.18 about interchanging sums and integrals.
3. The dominated convergence Theorem 19.38 and its Corollary 19.39 about interchanging sums and integrals.
4. Fatou's Lemma 19.28 is used to a lesser extent.
5. Fubini and Tonelli theorems for computing multiple integrals. We have not done this yet for integrals, but the result for sums is in Theorems 4.22 and 4.23.
6. To compute integrals involving Lebesgue measure you will need to know the basic properties of Lebesgue measure, Theorem 19.10 and the fundamental theorem of calculus, Theorem 19.40.
7. You should understand when it is permissible to differentiate past the integral, see Corollary 19.43.

Remark F.1. Again let me stress that the above properties are typically all that are needed to work with integrals (sums). In particular to understand $\int_X f d\mu$ for a general measurable f it suffices to understand:

1. If $A \in \mathcal{M}$, then $\int_X 1_A d\mu = \mu(A)$. By linearity of the integral this determines $\int_X f d\mu$ on simple functions f.
2. Using either the monotone or dominated convergence theorem along with the approximation Theorem 18.42, $\int_X f d\mu$ may be written as a limit of integrals of simple functions.