5. Measures and Integration

Definition 5.1. A **measure** μ on a measurable space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$ and
2. (Finite Additivity) If $\{A_i\}_{i=1}^n \subset \mathcal{M}$ are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$ when $i \neq j$, then
$$\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$
3. (Continuity) If $A_n \in \mathcal{M}$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.

We call a triple $(X, \mathcal{M}, \mu)$, where $(X, \mathcal{M})$ is a measurable space and $\mu : \mathcal{M} \to [0, \infty]$ is a measure, a **measure space.**

Remark 5.2. Properties 2) and 3) in Definition 5.1 are equivalent to the following condition. If $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ are pairwise disjoint then

$$\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i). \tag{5.1}$$

To prove this suppose that Properties 2) and 3) in Definition 5.1 and $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ are pairwise disjoint. Let $B_n := \bigcup_{i=1}^n A_i \uparrow B := \bigcup_{i=1}^\infty A_i$, so that

$$\mu(B) \overset{(3)}{=} \lim_{n\to\infty} \mu(B_n) \overset{(2)}{=} \lim_{n\to\infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i).$$

Conversely, if Eq. (5.1) holds we may take $A_j = \emptyset$ for all $j \geq n$ to see that Property 2) of Definition 5.1 holds. Also if $A_n \uparrow A$, let $B_n := A_n \setminus A_{n-1}$. Then $\{B_n\}_{n=1}^\infty$ are pairwise disjoint, $A_n = \cup_{j=1}^n B_j$ and $A = \cup_{j=1}^\infty B_j$. So if Eq. (5.1) holds we have

$$\begin{aligned}\mu(A) &= \mu\left(\cup_{j=1}^\infty B_j\right) = \sum_{j=1}^\infty \mu(B_j) \\ &= \lim_{n\to\infty} \sum_{j=1}^n \mu(B_j) = \lim_{n\to\infty} \mu(\cup_{j=1}^n B_j) = \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

Proposition 5.3 (Basic properties of measures). *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$ and $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$, then :*

1. *$\mu(E) \leq \mu(F)$ if $E \subset F$.*
2. *$\mu(\cup E_j) \leq \sum \mu(E_j)$.*
3. *If $\mu(E_1) < \infty$ and $E_j \downarrow E$, i.e. $E_1 \supset E_2 \supset E_3 \supset \dots$ and $E = \cap_j E_j$, then $\mu(E_j) \downarrow \mu(E)$ as $j \to \infty$.*

Proof.

1. Since $F = E \cup (F \setminus E)$,
$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$
2. Let $\widetilde{E}_j = E_j \setminus (E_1 \cup \dots \cup E_{j-1})$ so that the $\tilde{E}_j$'s are pair-wise disjoint and $E = \cup \widetilde{E}_j$. Since $\tilde{E}_j \subset E_j$ it follows from Remark 5.2 and part (1), that
$$\mu(E) = \sum \mu(\widetilde{E}_j) \leq \sum \mu(E_j).$$

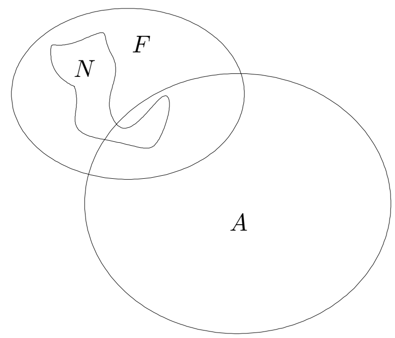

FIGURE 13. Completing a σ – algebra.

3. Define $D_i \equiv E_1 \setminus E_i$ then $D_i \uparrow E_1 \setminus E$ which implies that
$$\mu(E_1) - \mu(E) = \lim_{i\to\infty} \mu(D_i) = \mu(E_1) - \lim_{i\to\infty} \mu(E_i)$$
which shows that $\lim_{i\to\infty} \mu(E_i) = \mu(E)$.

■

Definition 5.4. A set $E \subset X$ is a **null** set if $E \in \mathcal{M}$ and $\mu(E) = 0$. If P is some "property" which is either true or false for each $x \in X$, we will use the terminology P a.e. (to be read P almost everywhere) to mean
$$E := \{x \in X : P \text{ is false for } x\}$$
is a null set. For example if f and g are two measurable functions on $(X, \mathcal{M}, \mu)$, $f = g$ a.e. means that $\mu(f \neq g) = 0$.

Definition 5.5. A measure space $(X, \mathcal{M}, \mu)$ is **complete** if every subset of a null set is in $\mathcal{M}$, i.e. for all $F \subset X$ such that $F \subset E \in \mathcal{M}$ with $\mu(E) = 0$ implies that $F \in \mathcal{M}$.

Proposition 5.6. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Set*
$$\mathcal{N} \equiv \{N \subset X : \exists\, F \in \mathcal{M} \ni N \subset F \text{ and } \mu(F) = 0\}$$
and
$$\bar{\mathcal{M}} = \{A \cup N : A \in \mathcal{M}, N \in \mathcal{M}\},$$
see Fig. 13. Then $\bar{\mathcal{M}}$ *is a* σ*-algebra. Define* $\bar{\mu}(A \cup N) = \mu(A)$, *then* $\bar{\mu}$ *is the unique measure on* $\bar{\mathcal{M}}$ *which extends* μ.

Proof. Clearly $X, \emptyset \in \bar{\mathcal{M}}$.

Let $A \in \mathcal{M}$ and $N \in \mathcal{N}$ and choose $F \in \mathcal{M}$ such that $N \subset F$ and $\mu(F) = 0$. Since $N^c = (F \setminus N) \cup F^c$,
$$(A \cup N)^c = A^c \cap N^c = A^c \cap (F \setminus N \cup F^c) = [A^c \cap (F \setminus N)] \cup [A^c \cap F^c]$$
where $[A^c \cap (F \setminus N)] \in \mathcal{N}$ and $[A^c \cap F^c] \in \mathcal{M}$. Thus $\bar{\mathcal{M}}$ is closed under complements.

If $A_i \in \mathcal{M}$ and $N_i \subset F_i \in \mathcal{M}$ such that $\mu(F_i) = 0$ then $\cup(A_i \cup N_i) = (\cup A_i) \cup (\cup N_i) \in \bar{\mathcal{M}}$ since $\cup A_i \in \mathcal{M}$ and $\cup N_i \subset \cup F_i$ and $\mu(\cup F_i) \leq \sum \mu(F_i) = 0$. Therefore, $\bar{\mathcal{M}}$ is a σ-algebra.

Suppose $A \cup N_1 = B \cup N_2$ with $A, B \in \mathcal{M}$ and $N_1, N_2, \in \mathcal{N}$. Then $A \subset A \cup N_1 \subset A \cup N_1 \cup F_2 = B \cup F_2$ which shows that

$$\mu(A) \leq \mu(B) + \mu(F_2) = \mu(B).$$

Similarly, we show that $\mu(B) \leq \mu(A)$ so that $\mu(A) = \mu(B)$ and hence $\bar{\mu}(A \cup N) := \mu(A)$ is well defined. It is left as an exercise to show $\bar{\mu}$ is a measure, i.e. that it is countable additive. ■

Many theorems in the sequel will require some control on the size of a measure μ. The relevant notion for our purposes (and most purposes) is that of a σ – finite measure defined next.

Definition 5.7. Suppose X is a set, $\mathcal{E} \subset \mathcal{M} \subset \mathcal{P}(X)$ and $\mu : \mathcal{M} \to [0, \infty]$ is a function. The function μ is σ – finite on $\mathcal{E}$ if there exists $E_n \in \mathcal{E}$ such that $\mu(E_n) < \infty$ and $X = \cup_{n=1} E_n$. If $\mathcal{M}$ is a σ – algebra and μ is a measure on $\mathcal{M}$ which is σ – finite on $\mathcal{M}$ we will say $(X, \mathcal{M}, \mu)$ is a σ-finite measure space.

The reader should check that if μ is a finitely additive measure on an algebra, $\mathcal{M}$, then μ is σ – finite on $\mathcal{M}$ iff there exists $X_n \in \mathcal{M}$ such that $X_n \uparrow X$ and $\mu(X_n) < \infty$.

5.1. **Example of Measures.** Most σ – algebras and σ -additive measures are somewhat difficult to describe and define. However, one special case is fairly easy to understand. Namely suppose that $\mathcal{F} \subset \mathcal{P}(X)$ is a countable or finite partition of X and $\mathcal{M} \subset \mathcal{P}(X)$ is the σ – algebra which consists of the collection of sets $A \subset X$ such that

$$A = \cup \{\alpha \in \mathcal{F} : \alpha \subset A\}. \tag{5.2}$$

It is easily seen that $\mathcal{M}$ is a σ – algebra.

Any measure $\mu : \mathcal{M} \to [0, \infty]$ is determined uniquely by its values on $\mathcal{F}$. Conversely, if we are given any function $\lambda : \mathcal{F} \to [0, \infty]$ we may define, for $A \in \mathcal{M}$,

$$\mu(A) = \sum_{\alpha \in \mathcal{F} \ni \alpha \subset A} \lambda(\alpha) = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A}$$

where $1_{\alpha \subset A}$ is one if $\alpha \subset A$ and zero otherwise. We may check that μ is a measure on $\mathcal{M}$. Indeed, if $A = \coprod_{i=1}^{\infty} A_i$ and $\alpha \in \mathcal{F}$, then $\alpha \subset A$ iff $\alpha \subset A_i$ for one and hence exactly one A_i. Therefore $1_{\alpha \subset A} = \sum_{i=1}^{\infty} 1_{\alpha \subset A_i}$ and hence

$$\begin{aligned}
\mu(A) &= \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A} = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) \sum_{i=1}^{\infty} 1_{\alpha \subset A_i} \\
&= \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A_i} = \sum_{i=1}^{\infty} \mu(A_i)
\end{aligned}$$

as desired. Thus we have shown that there is a one to one correspondence between measures μ on $\mathcal{M}$ and functions $\lambda : \mathcal{F} \to [0, \infty]$.

We will leave the issue of constructing measures until Sections 10 and 11. However, let us point out that interesting measures do exist. The following theorem may be found in Theorem 10.32 or see Section 10.8.1.

Theorem 5.8. *To every right continuous non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ there exists a unique measure μ_F on $\mathcal{B}_{\mathbb{R}}$ such that*

$$\mu_F((a, b]) = F(b) - F(a) \ \forall \ -\infty < a \leq b < \infty \tag{5.3}$$

Moreover, if $A \in \mathcal{B}_{\mathbb{R}}$ then

$$\mu_F(A) = \inf\left\{\sum_{i=1}^{\infty}(F(b_i) - F(a_i)) : A \subset \cup_{i=1}^{\infty}(a_i, b_i]\right\} \tag{5.4}$$

$$= \inf\left\{\sum_{i=1}^{\infty}(F(b_i) - F(a_i)) : A \subset \coprod_{i=1}^{\infty}(a_i, b_i]\right\}. \tag{5.5}$$

In fact the map $F \to \mu_F$ is a one to one correspondence between right continuous functions F with $F(0) = 0$ on one hand and measures μ on $\mathcal{B}_{\mathbb{R}}$ such that $\mu(J) < \infty$ on any bounded set $J \in \mathcal{B}_{\mathbb{R}}$ on the other.

Example 5.9. The most important special case of Theorem 5.8 is when $F(x) = x$, in which case we write m for μ_F. The measure m is called Lebesgue measure.

Theorem 5.10. *Lebesgue measure m is invariant under translations, i.e. for $B \in \mathcal{B}_{\mathbb{R}}$ and $x \in \mathbb{R}$,*

$$m(x + B) = m(B). \tag{5.6}$$

Moreover, m is the unique measure on $\mathcal{B}_{\mathbb{R}}$ such that $m((0, 1]) = 1$ and Eq. (5.6) holds for $B \in \mathcal{B}_{\mathbb{R}}$ and $x \in \mathbb{R}$. Moreover, m has the scaling property

$$m(\lambda B) = |\lambda|\, m(B) \tag{5.7}$$

where $\lambda \in \mathbb{R}$, $B \in \mathcal{B}_{\mathbb{R}}$ and $\lambda B := \{\lambda x : x \in B\}$.

Proof. Let $m_x(B) := m(x+B)$, then one easily shows that m_x is a measure on $\mathcal{B}_{\mathbb{R}}$ such that $m_x((a, b]) = b - a$ for all $a < b$. Therefore, $m_x = m$ by the uniqueness assertion in Theorem 5.8.

For the converse, suppose that m is translation invariant and $m((0, 1]) = 1$. Given $n \in \mathbb{N}$, we have

$$(0, 1] = \cup_{k=1}^{n}(\frac{k-1}{n}, \frac{k}{n}] = \cup_{k=1}^{n}\left(\frac{k-1}{n} + (0, \frac{1}{n}]\right).$$

Therefore,

$$1 = m((0, 1]) = \sum_{k=1}^{n} m\left(\frac{k-1}{n} + (0, \frac{1}{n}]\right)$$
$$= \sum_{k=1}^{n} m((0, \frac{1}{n}]) = n \cdot m((0, \frac{1}{n}]).$$

That is to say

$$m((0, \frac{1}{n}]) = 1/n.$$

Similarly, $m((0, \frac{l}{n}]) = l/n$ for all $l, n \in \mathbb{N}$ and therefore by the translation invariance of m,

$$m((a, b]) = b - a \text{ for all } a, b \in \mathbb{Q} \text{ with } a < b.$$

Finally for $a, b \in \mathbb{R}$ such that $a < b$, choose $a_n, b_n \in \mathbb{Q}$ such that $b_n \downarrow b$ and $a_n \uparrow a$, then $(a_n, b_n] \downarrow (a, b]$ and thus

$$m((a, b]) = \lim_{n\to\infty} m((a_n, b_n]) = \lim_{n\to\infty}(b_n - a_n) = b - a,$$

i.e. m is Lebesgue measure.

To prove Eq. (5.7) we may assume that $\lambda \neq 0$ since this case is trivial to prove. Now let $m_\lambda(B) := |\lambda|^{-1} m(\lambda B)$. It is easily checked that m_λ is again a measure on $\mathcal{B}_{\mathbb{R}}$ which satisfies

$$m_\lambda((a,b]) = \lambda^{-1} m((\lambda a, \lambda b]) = \lambda^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda > 0$ and

$$m_\lambda((a,b]) = |\lambda|^{-1} m([\lambda b, \lambda a)) = -|\lambda|^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda < 0$. Hence $m_\lambda = m$. ■

We are now going to develope integration theory relative to a measure. The integral defined in the case for Lebesgue measure, m, will be an extension of the standard Riemann integral on $\mathbb{R}$.

5.2. **Integrals of Simple functions.** Let $(X, \mathcal{M}, \mu)$ be a fixed measure space in this section.

Definition 5.11. A function $\phi : X \to \mathbb{F}$ is a **simple function** if ϕ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable and $\phi(X)$ is a finite set. Any such simple functions can be written as

$$\phi = \sum_{i=1}^{n} \lambda_i 1_{A_i} \text{ with } A_i \in \mathcal{M} \text{ and } \lambda_i \in \mathbb{F}. \tag{5.8}$$

Indeed, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be an enumeration of the range of ϕ and $A_i = \phi^{-1}(\{\lambda_i\})$. Also note that Eq. (5.8) may be written more intrinsically as

$$\phi = \sum_{y \in \mathbb{F}} y 1_{\phi^{-1}(\{y\})}.$$

The next theorem shows that simple functions are "pointwise dense" in the space of measurable functions.

Theorem 5.12 (Approximation Theorem)**.** *Let $f : X \to [0,\infty]$ be measurable and define*

$$\begin{aligned}\phi_n(x) &\equiv \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)}(x) + 2^n 1_{f^{-1}((2^n,\infty])}(x) \\ &= \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{\left\{\frac{k}{2^n} < f \le \frac{k+1}{2^n}\right\}}(x) + 2^n 1_{\{f > 2^n\}}(x)\end{aligned}$$

then $\phi_n \le f$ for all n, $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on the sets $X_M := \{x \in X : f(x) \le M\}$ with $M < \infty$. Moreover, if $f : X \to \mathbb{C}$ is a measurable function, then there exists simple functions ϕ_n such that $\lim_{n\to\infty} \phi_n(x) = f(x)$ *for all x and $|\phi_n| \uparrow |f|$ as $n \to \infty$.*

Proof. It is clear by construction that $\phi_n(x) \le f(x)$ for all x and that $0 \le f(x) - \phi_n(x) \le 2^{-n}$ if $x \in X_{2^n}$. From this it follows that $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on bounded sets.

Also notice that

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right] = \left(\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right] = \left(\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right] \cup \left(\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]$$

and for $x \in f^{-1}\left(\left(\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right]\right)$, $\phi_n(x) = \phi_{n+1}(x) = \frac{2k}{2^{n+1}}$ and for $x \in f^{-1}\left(\left(\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]\right)$, $\phi_n(x) = \frac{2k}{2^{n+1}} < \frac{2k+1}{2^{n+1}} = \phi_{n+1}(x)$. Similarly

$$(2^n, \infty] = (2^n, 2^{n+1}] \cup (2^{n+1}, \infty],$$

so for $x \in f^{-1}((2^{n+1}, \infty])$ $\phi_n(x) = 2^n < 2^{n+1} = \phi_{n+1}(x)$ and for $x \in f^{-1}((2^n, 2^{n+1}])$, $\phi_{n+1}(x) \geq 2^n = \phi_n(x)$. Therefore $\phi_n \leq \phi_{n+1}$ for all n and we have completed the proof of the first assertion.

For the second assertion, first assume that $f : X \to \mathbb{R}$ is a measurable function and choose $\phi_n^{\pm}$ to be simple functions such that $\phi_n^{\pm} \uparrow f_{\pm}$ as $n \to \infty$ and define $\phi_n = \phi_n^+ - \phi_n^-$. Then

$$|\phi_n| = \phi_n^+ + \phi_n^- \leq \phi_{n+1}^+ + \phi_{n+1}^- = |\phi_{n+1}|$$

and clearly $|\phi_n| = \phi_n^+ + \phi_n^- \uparrow f_+ + f_- = |f|$ and $\phi_n = \phi_n^+ - \phi_n^- \to f_+ - f_- = f$ as $n \to \infty$.

Now suppose that $f : X \to \mathbb{C}$ is measurable. We may now choose simple function u_n and v_n such that $|u_n| \uparrow |\operatorname{Re} f|$, $|v_n| \uparrow |\operatorname{Im} f|$, $u_n \to \operatorname{Re} f$ and $v_n \to \operatorname{Im} f$ as $n \to \infty$. Let $\phi_n = u_n + iv_n$, then

$$|\phi_n|^2 = u_n^2 + v_n^2 \uparrow |\operatorname{Re} f|^2 + |\operatorname{Im} f|^2 = |f|^2$$

and $\phi_n = u_n + iv_n \to \operatorname{Re} f + i \operatorname{Im} f = f$ as $n \to \infty$. ■

We are now ready to define the Lebesgue integral. We will start by integrating simple functions and then proceed to general measurable functions.

Definition 5.13. Let $\mathbb{F} = \mathbb{C}$ or $[0, \infty)$ and suppose that $\phi : X \to \mathbb{F}$ is a simple function. If $\mathbb{F} = \mathbb{C}$ assume further that $\mu(\phi^{-1}(\{y\})) < \infty$ for all $y \neq 0$ in $\mathbb{C}$. For such functions ϕ, define $I_\mu(\phi)$ by

$$I_\mu(\phi) = \sum_{y \in \mathbb{F}} y \mu(\phi^{-1}(\{y\})).$$

Proposition 5.14. *Let $\lambda \in \mathbb{F}$ and ϕ and ψ be two simple functions, then I_μ satisfies:*

1.

$$I_\mu(\lambda\phi) = \lambda I_\mu(\phi). \tag{5.9}$$

2.

$$I_\mu(\phi + \psi) = I_\mu(\psi) + I_\mu(\phi).$$

3. *If ϕ and ψ are non-negative simple functions such that $\phi \leq \psi$ then*

$$I_\mu(\phi) \leq I_\mu(\psi).$$

Proof. Let us write $\{\phi = y\}$ for the set $\phi^{-1}(\{y\}) \subset X$ and $\mu(\phi = y)$ for $\mu(\{\phi = y\}) = \mu(\phi^{-1}(\{y\}))$ so that

$$I_\mu(\phi) = \sum_{y \in \mathbb{C}} y \mu(\phi = y).$$

We will also write $\{\phi = a, \psi = b\}$ for $\phi^{-1}(\{a\}) \cap \psi^{-1}(\{b\})$. This notation is more intuitive for the purposes of this proof. Suppose that $\lambda \in \mathbb{F}$ then

$$\begin{aligned}I_\mu(\lambda\phi) &= \sum_{y\in\mathbb{F}} y\ \mu(\lambda\phi = y) = \sum_{y\in\mathbb{F}} y\ \mu(\phi = y/\lambda) \\ &= \sum_{z\in\mathbb{F}} \lambda z\ \mu(\phi = z) = \lambda I_\mu(\phi)\end{aligned}$$

provided that $\lambda \neq 0$. The case $\lambda = 0$ is clear, so we have proved 1.

Suppose that ϕ and ψ are two simple functions, then

$$\begin{aligned}I_\mu(\phi + \psi) &= \sum_{z\in\mathbb{F}} z\ \mu(\phi + \psi = z) \\ &= \sum_{z\in\mathbb{F}} z\ \mu\left(\cup_{w\in\mathbb{F}} \{\phi = w,\ \psi = z - w\}\right) \\ &= \sum_{z\in\mathbb{F}} z \sum_{w\in\mathbb{F}} \mu(\phi = w,\ \psi = z - w) \\ &= \sum_{z,w\in\mathbb{F}} (z + w)\mu(\phi = w,\ \psi = z) \\ &= \sum_{z\in\mathbb{F}} z\ \mu(\psi = z) + \sum_{w\in\mathbb{F}} w\ \mu(\phi = w) \\ &= I_\mu(\psi) + I_\mu(\phi).\end{aligned}$$

which proves 2.

For 3. if ϕ and ψ are non-negative simple functions such that $\phi \leq \psi$

$$\begin{aligned}I_\mu(\phi) &= \sum_{a\geq 0} a\mu(\phi = a) = \sum_{a,b\geq 0} a\mu(\phi = a, \psi = b) \\ &\leq \sum_{a,b\geq 0} b\mu(\phi = a, \psi = b) = \sum_{b\geq 0} b\mu(\psi = b) = I_\mu(\psi),\end{aligned}$$

wherein the third inequality we have used $\{\phi = a, \psi = b\} = \emptyset$ if $a > b$. ■

5.3. **Integrals of positive functions.**

Definition 5.15. Let $L^+ = \{f : X \to [0, \infty] : f \text{ is measurable}\}$. Define

$$\int_X f d\mu = \sup \{I_\mu(\phi) : \phi \text{ is simple and } \phi \leq f\}.$$

Because of item 3. of Proposition 5.14, if ϕ is a non-negative simple function, $\int_X \phi d\mu = I_\mu(\phi)$ so that $\int_X$ is an extension of I_μ. We say the $f \in L^+$ is **integrable** if $\int_X f d\mu < \infty$.

Remark 5.16. Notice that we still have the monotonicity property: $0 \leq f \leq g$ then

$$\begin{aligned}\int_X f d\mu &= \sup \{I_\mu(\phi) : \phi \text{ is simple and } \phi \leq f\} \\ &\leq \sup \{I_\mu(\phi) : \phi \text{ is simple and } \phi \leq g\} \leq \int_X g.\end{aligned}$$

Similarly if $c > 0$,

$$\int_X c f d\mu = c \int_X f d\mu.$$

Also notice that if f is integrable, then $\mu(\{f=\infty\})=0$.

Lemma 5.17. *Let X be a set and $\rho : X \to [0,\infty]$ be a function, let $\mu = \sum_{x\in X}\rho(x)\delta_x$ on $\mathcal{M}=\mathcal{P}(X)$, i.e.*

$$\mu(A)=\sum_{x\in A}\rho(x).$$

If $f : X\to[0,\infty]$ is a function (which is necessarily measurable), then

$$\int_X f d\mu=\sum_X \rho f.$$

Proof. Suppose that $\phi : X\to[0,\infty]$ is a simple function, then $\phi = \sum_{z\in[0,\infty]} z 1_{\phi^{-1}(\{z\})}$ and

$$\begin{aligned}\sum_X \rho\phi &= \sum_{x\in X}\rho(x)\sum_{z\in[0,\infty]} z1_{\phi^{-1}(\{z\})}(x)=\sum_{z\in[0,\infty]} z\sum_{x\in X}\rho(x)1_{\phi^{-1}(\{z\})}(x)\\ &=\sum_{z\in[0,\infty]} z\mu(\phi^{-1}(\{z\}))=\int_X\phi d\mu.\end{aligned}$$

So if $\phi : X\to[0,\infty)$ is a simple function such that $\phi\le f$, then

$$\int_X \phi d\mu=\sum_X\rho\phi\le\sum_X\rho f.$$

Taking the sup over ϕ in this last equation then shows that

$$\int_X f d\mu\le\sum_X\rho f.$$

For the reverse inequality, let $\Lambda\subset\subset X$ be a finite set and $N\in(0,\infty)$. Set $f^N(x)=\min\{N,f(x)\}$ and let $\phi_{N,\Lambda}$ be the simple function given by $\phi_{N,\Lambda}(x):=1_\Lambda(x)f^N(x)$. Because $\phi_{N,\Lambda}(x)\le f(x)$,

$$\sum_\Lambda\rho f^N=\sum_X\rho\phi_{N,\Lambda}=\int_X\phi_{N,\Lambda}d\mu\le\int_X f d\mu.$$

Since $f^N\uparrow f$ as $N\to\infty$, we may let $N\to\infty$ in this last equation to concluded that

$$\sum_\Lambda\rho f\le\int_X f d\mu$$

and since Λ is arbitrary we learn that

$$\sum_X\rho f\le\int_X f d\mu.$$

■

Theorem 5.18 (Monotone Convergence Theorem)**.** *Suppose $f_n\in L^+$ is a sequence of functions such that $f_n\uparrow f$ (f is necessarily in L^+) then*

$$\int f_n\uparrow\int f \text{ as } n\to\infty.$$

Proof. Since $f_n \le f_m \le f$, for all $n \le m < \infty$,

$$\int f_n \le \int f_m \le \int f$$

from which if follows $\int f_n$ is increasing in n and

$$\lim_{n\to\infty} \int f_n \le \int f. \tag{5.10}$$

For the opposite inequality, let ϕ be a simple function such that $0 \le \phi \le f$ and let $\alpha \in (0,1)$. By Proposition 5.14,

$$\int f_n \ge \int 1_{E_n} f_n \ge \int_{E_n} \alpha\phi = \alpha \int_{E_n} \phi. \tag{5.11}$$

Write $\phi = \sum \lambda_i 1_{B_i}$ with $\lambda_i > 0$ and $B_i \in \mathcal{M}$, then

$$\begin{aligned}\lim_{n\to\infty} \int_{E_n} \phi &= \lim_{n\to\infty} \sum \lambda_i \int_{E_n} 1_{B_i} = \sum \lambda_i \mu(E_n \cap B_i) = \sum \lambda_i \lim_{n\to\infty} \mu(E_n \cap B_i) \\ &= \sum \lambda_i \mu(B_i) = \int \phi.\end{aligned}$$

Using this we may let $n \to \infty$ in Eq. (5.11) to conclude

$$\lim_{n\to\infty} \int f_n \ge \alpha \lim_{n\to\infty} \int_{E_n} \phi = \alpha \int_X \phi.$$

Because this equation holds for all simple functions $0 \le \phi \le f$, form the definition of $\int f$ we have $\lim_{n\to\infty} \int f_n \ge \alpha \int f$. Since $\alpha \in (0,1)$ is arbitrary, $\lim_{n\to\infty} \int f_n \ge \int f$ which combined with Eq. (5.10) proves the theorem. ■

The following simple lemma will be use often in the sequel.

Lemma 5.19 (Chebyshev's Inequality)**.** *Suppose that $f \ge 0$ is a measurable function, then for any $\epsilon > 0$,*

$$\mu(f \ge \epsilon) \le \frac{1}{\epsilon} \int_X f d\mu. \tag{5.12}$$

In particular if $\int_X f d\mu < \infty$ then $\mu(f = \infty) = 0$ (i.e. $f < \infty$ a.e.) and the set $\{f > 0\}$ is σ – finite.

Proof. Since $1_{\{f\ge\epsilon\}} \le 1_{\{f\ge\epsilon\}} \frac{1}{\epsilon} f \le \frac{1}{\epsilon} f$,

$$\mu(f \ge \epsilon) = \int_X 1_{\{f\ge\epsilon\}} d\mu \le \int_X 1_{\{f\ge\epsilon\}} \frac{1}{\epsilon} f d\mu \le \frac{1}{\epsilon} \int_X f d\mu.$$

If $M := \int_X f d\mu < \infty$, then

$$\mu(f = \infty) \le \mu(f \ge n) \le \frac{M}{n} \to 0 \text{ as } n \to \infty$$

and $\{f \ge 1/n\} \uparrow \{f > 0\}$ with $\mu(f \ge 1/n) \le nM < \infty$ for all n. ■

Corollary 5.20. *If $f_n \in L^+$ is a sequence of functions then*

$$\int \sum_n f_n = \sum_n \int f_n.$$

In particular, if $\sum_n \int f_n < \infty$ then $\sum_n f_n < \infty$ a.e.

Proof. First off we show that

$$\int (f_1 + f_2) = \int f_1 + \int f_2$$

by choosing non-negative simple function ϕ_n and ψ_n such that $\phi_n \uparrow f_1$ and $\psi_n \uparrow f_2$. Then $(\phi_n + \psi_n)$ is simple as well and $(\phi_n + \psi_n) \uparrow (f_1 + f_2)$ so by the monotone convergence theorem,

$$\begin{aligned}\int (f_1 + f_2) &= \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \left(\int \phi_n + \int \psi_n \right) \\ &= \lim_{n\to\infty} \int \phi_n + \lim_{n\to\infty} \int \psi_n = \int f_1 + \int f_2.\end{aligned}$$

Now to the general case. Let $g_N \equiv \sum\limits_{n=1}^{N} f_n$ and $g = \sum\limits_{1}^{\infty} f_n$, then $g_N \uparrow g$ and so again by monotone convergence theorem and the additivity just proved,

$$\begin{aligned}\sum_{n=1}^{\infty} \int f_n &:= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \\ &= \lim_{N\to\infty} \int g_N = \int g = \sum_{n=1}^{\infty} \int f_n.\end{aligned}$$

■

Remark 5.21. It is in the proof of this corollary (i.e. the linearity of the integral) that we really make use of the assumption that all of our functions are measurable. In fact the definition $\int f d\mu$ makes sense for **all** functions $f : X \to [0, \infty]$ not just measurable functions. Moreover the monotone convergence theorem holds in this generality with no change in the proof. However, in the proof of Corollary 5.20, we use the approximation Theorem 5.12 which relies heavily on the measurability of the functions to be approximated.

The following Lemma and the next Corollary are simple applications of Corollary 5.20.

Lemma 5.22 (First Borell-Carnteli- Lemma.). *Let $(X, \mathcal{M}, \mu)$ be a measure space, $A_n \in \mathcal{M}$, and set*

$$\{A_n \text{ i.o.}\} = \{x \in X : x \in A_n \text{ for infinitely many } n\text{'s}\} = \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} A_n.$$

If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu(\{A_n \text{ i.o.}\}) = 0$.

Proof. (First Proof.) Let us first observe that

$$\{A_n \text{ i.o.}\} = \left\{ x \in X : \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right\}.$$

Hence if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then

$$\infty > \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \int_X 1_{A_n}\, d\mu = \int_X \sum_{n=1}^{\infty} 1_{A_n}\, d\mu$$

implies that $\sum\limits_{n=1}^{\infty} 1_{A_n}(x) < \infty$ for μ - a.e. x. That is to say $\mu(\{A_n \text{ i.o.}\}) = 0$.

(Second Proof.) Of course we may give a strictly measure theoretic proof of this fact:

$$\begin{aligned}\mu(A_n \text{ i.o.}) &= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} A_n\right)\\ &\leq \lim_{N\to\infty} \sum_{n\geq N} \mu(A_n)\end{aligned}$$

and the last limit is zero since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. ∎

Corollary 5.23. *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ is a collection of sets such that $\mu(A_i \cap A_j) = 0$ for all $i \neq j$, then*

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Proof. Since

$$\begin{aligned}\mu\left(\cup_{n=1}^{\infty} A_n\right) &= \int_X 1_{\cup_{n=1}^{\infty} A_n} d\mu \text{ and}\\ \sum_{n=1}^{\infty} \mu(A_n) &= \int_X \sum_{n=1}^{\infty} 1_{A_n} d\mu\end{aligned}$$

it suffices to show

$$\sum_{n=1}^{\infty} 1_{A_n} = 1_{\cup_{n=1}^{\infty} A_n} \ \mu - \text{a.e.} \tag{5.13}$$

Now $\sum_{n=1}^{\infty} 1_{A_n} \geq 1_{\cup_{n=1}^{\infty} A_n}$ and $\sum_{n=1}^{\infty} 1_{A_n}(x) \neq 1_{\cup_{n=1}^{\infty} A_n}(x)$ iff $x \in A_i \cap A_j$ for some $i \neq j$, that is

$$\left\{x : \sum_{n=1}^{\infty} 1_{A_n}(x) \neq 1_{\cup_{n=1}^{\infty} A_n}(x)\right\} = \cup_{i<j} A_i \cap A_j$$

and the later set has measure 0 being the countable union of sets of measure zero. This proves Eq. (5.13) and hence the corollary. ∎

Example 5.24. Suppose $-\infty < a < b < \infty$, $f \in C([a,b],[0,\infty))$ and m be Lebesgue measure on $\mathbb{R}$. Also let $\pi_k = \{a = a_0^k < a_1^k < \cdots < a_{n_k}^k = b\}$ be a sequence of refining partitions (i.e. $\pi_k \subset \pi_{k+1}$ for all k) such that

$$\text{mesh}(\pi_k) := \max\{\left|a_j^k - a_{j-1}^{k+1}\right| : j = 1, \ldots, n_k\} \to 0 \text{ as } k \to \infty.$$

For each k, let

$$f_k(x) = f(a) 1_{\{a\}} + \sum_{l=0}^{n_k - 1} \min\left\{f(x) : a_l^k \leq x \leq a_{l+1}^k\right\} 1_{(a_l^k, a_{l+1}^k]}(x)$$

then $f_k \uparrow f$ as $k \to \infty$ and so by the monotone convergence theorem,

$$\int_a^b f dm := \int_{[a,b]} f dm = \lim_{k\to\infty} \int_a^b f_k \, dm$$
$$= \lim_{k\to\infty} \sum_{l=0}^{n_k} \min\left\{f(x) : a_l^k \le x \le a_{l+1}^k\right\} m\left((a_l^k, a_{l+1}^k]\right)$$
$$= \int_a^b f(x)dx.$$

The latter integral being the Riemann integral.

We can use the above result to integrate some non-Riemann integrable functions:

Example 5.25. For all $\lambda > 0$, $\int_0^\infty e^{-\lambda x} dm(x) = \lambda^{-1}$ and $\int_{\mathbb{R}} \frac{1}{1+x^2} dm(x) = \pi$. The proof of these equations are similar. By the monotone convergence theorem, Example 5.24 and the fundamental theorem of calculus for Riemann integrals (or see Theorem 5.40 below),

$$\int_0^\infty e^{-\lambda x} dm(x) = \lim_{N\to\infty} \int_0^N e^{-\lambda x} dm(x) = \lim_{N\to\infty} \int_0^N e^{-\lambda x} dx$$
$$= -\lim_{N\to\infty} \frac{1}{\lambda} e^{-\lambda x}|_0^N = \lambda^{-1}$$

and

$$\int_{\mathbb{R}} \frac{1}{1+x^2} dm(x) = \lim_{N\to\infty} \int_{-N}^N \frac{1}{1+x^2} dm(x) = \lim_{N\to\infty} \int_{-N}^N \frac{1}{1+x^2} dx$$
$$= \tan^{-1}(N) - \tan^{-1}(-N) = \pi.$$

Let us also consider the functions x^{-p},

$$\int_{(0,1]} \frac{1}{x^p} \, dm(x) = \lim_{n\to\infty} \int_0^1 1_{(\frac{1}{n},1]}(x) \frac{1}{x^p} dm(x)$$
$$= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \frac{1}{x^p} dx = \lim_{n\to\infty} \left. \frac{x^{-p+1}}{1-p} \right|_{1/n}^1$$
$$= \begin{cases} \frac{1}{1-p} & \text{if} \quad p < 1 \\ \infty & \text{if} \quad p > 1 \end{cases}$$

If $p = 1$ we find

$$\int_{(0,1]} \frac{1}{x^p} \, dm(x) = \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \frac{1}{x} dx = \lim_{n\to\infty} \ln(x)|_{1/n}^1 = \infty.$$

Example 5.26. Let $\{r_n\}_{n=1}^\infty$ be an enumeration of the points in $\mathbb{Q} \cap [0,1]$ and define

$$f(x) = \sum_{n=1}^\infty 2^{-n} \frac{1}{\sqrt{|x - r_n|}}$$

with the convention that

$$\frac{1}{\sqrt{|x - r_n|}} = 5 \text{ if } x = r_n.$$

Since, By Theorem 5.40,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{|x-r_n|}}\,dx &= \int_{r_n}^1 \frac{1}{\sqrt{x-r_n}}dx + \int_0^{r_n} \frac{1}{\sqrt{r_n-x}}dx \\ &= 2\sqrt{x-r_n}|_{r_n}^1 - 2\sqrt{r_n-x}|_0^{r_n} = 2\left(\sqrt{1-r_n} - \sqrt{r_n}\right) \\ &\leq 4,\end{aligned}$$

we find

$$\int_{[0,1]} f(x)dm(x) = \sum_{n=1}^\infty 2^{-n} \int_{[0,1]} \frac{1}{\sqrt{|x-r_n|}}dx \leq \sum_{n=1}^\infty 2^{-n}4 = 4 < \infty.$$

In particular, $m(f=\infty)=0$, i.e. that $f<\infty$ for almost every $x \in [0,1]$ and this implies that

$$\sum_{n=1}^\infty 2^{-n}\frac{1}{\sqrt{|x-r_n|}} < \infty \text{ for a.e. } x \in [0,1].$$

This result is somewhat surprising since the singularities of the summands form a dense subset of $[0,1]$.

Proposition 5.27. *Suppose that $f \geq 0$ is a measurable function. Then $\int_X f d\mu = 0$ iff $f=0$ a.e. Also if $f, g \geq 0$ are measurable functions such that $f \leq g$ a.e. then $\int f d\mu \leq \int g d\mu$. In particular if $f = g$ a.e. then $\int f d\mu = \int g d\mu$.*

Proof. If $f=0$ a.e. and $\phi \leq f$ is a simple function then $\phi = 0$ a.e. This implies that $\mu(\phi^{-1}(\{y\})) = 0$ for all $y>0$ and hence $\int_X \phi d\mu = 0$ and therefore $\int_X f d\mu = 0$.

Conversely, if $\int f d\mu = 0$, then by Chebyshev's Inequality (Lemma 5.19),

$$\mu(f \geq 1/n) \leq n\int f d\mu = 0 \text{ for all } n.$$

Therefore, $\mu(f>0) \leq \sum_{n=1}^\infty \mu(f \geq 1/n) = 0$, i.e. $f=0$ a.e.

For the second assertion let E be the exceptional set where $g>f$, i.e. $E := \{x \in X : g(x) > f(x)\}$. By assumption E is a null set and $1_{E^c}f \leq 1_{E^c}g$ everywhere. Because $g = 1_{E^c}g + 1_E g$ and $1_E g = 0$ a.e.,

$$\int g d\mu = \int 1_{E^c} g d\mu + \int 1_E g d\mu = \int 1_{E^c} g d\mu$$

and similarly $\int f d\mu = \int 1_{E^c} f d\mu$. Since $1_{E^c}f \leq 1_{E^c}g$ everywhere,

$$\int f d\mu = \int 1_{E^c} f d\mu \leq \int 1_{E^c} g d\mu = \int g d\mu.$$

■

Corollary 5.28. *Suppose that $\{f_n\}$ is a sequence of non-negative functions and f is a measurable function such that $f_n \uparrow f$ off a null set, then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

Proof. Let $E \subset X$ be a null set such that $f_n 1_{E^c} \uparrow f 1_{E^c}$ as $n \to \infty$. Then by the monotone convergence theorem and Proposition 5.27,

$$\int f_n = \int f_n 1_{E^c} \uparrow \int f 1_{E^c} = \int f \text{ as } n \to \infty.$$

■

Lemma 5.29 (Fatou's Lemma). *If $f_n : X \to [0,\infty]$ is a sequence of measurable functions then*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

Proof. Define $g_k \equiv \inf_{n\ge k} f_n$ so that $g_k \uparrow \liminf_{n\to\infty} f_n$ as $k \to \infty$. Since $g_k \le f_n$ for all $k \le n$,

$$\int g_k \le \int f_n \text{ for all } n \ge k$$

and therefore

$$\int g_k \le \lim\inf_{n\to\infty} \int f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k \to \infty$ to find

$$\int \lim\inf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \int g_k \le \lim\inf_{n\to\infty} \int f_n.$$

■

5.4. Integrals of Complex Valued Functions.

Definition 5.30. A measurable function $f : X \to \bar{\mathbb{R}}$ is **integrable** if $f_+ \equiv f 1_{\{f\ge 0\}}$ and $f_- = -f\ 1_{\{f\le 0\}}$ are **integrable**. We write L^1 for the space of integrable functions. For $f \in L^1$, let

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

Convention: If $f, g : X \to \bar{\mathbb{R}}$ are two measurable functions, let $f + g$ denote the collection of measurable functions $h : X \to \bar{\mathbb{R}}$ such that $h(x) = f(x) + g(x)$ whenever $f(x) + g(x)$ is well defined, i.e. is not of the form $\infty - \infty$ or $-\infty + \infty$. We use a similar convention for $f - g$. Notice that if $f, g \in L^1$ and $h_1, h_2 \in f + g$, then $h_1 = h_2$ a.e. because $|f| < \infty$ and $|g| < \infty$ a.e.

Remark 5.31. Since

$$f_\pm \le |f| \le f_+ + f_-,$$

a measurable function f is integrable iff $\int |f|\, d\mu < \infty$. If $f, g \in L^1$ and $f = g$ a.e. then $f_\pm = g_\pm$ a.e. and so it follows from Proposition 5.27 that $\int f d\mu = \int g d\mu$. In particular if $f, g \in L^1$ we may define

$$\int_X (f + g)\, d\mu = \int_X h d\mu$$

where h is any element of $f + g$.

Proposition 5.32. *The map*

$$f \in L^1 \to \int_X f d\mu \in \mathbb{R}$$

is linear and has the monotonicity property: $\int f d\mu \le \int g d\mu$ for all $f, g \in L^1$ such that $f \le g$ a.e.

Proof. Let $f, g \in L^1$ and $a, b \in \mathbb{R}$. By modifying f and g on a null set, we may assume that f, g are real valued functions. We have $af + bg \in L^1$ because

$$|af + bg| \leq |a||f| + |b|\,|g| \in L^1.$$

If $a < 0$, then

$$(af)_+ = -af_- \text{ and } (af)_- = -af_+$$

so that

$$\int af = -a \int f_- + a \int f_+ = a(\int f_+ - \int f_-) = a \int f.$$

A similar calculation works for $a > 0$ and the case $a = 0$ is trivial so we have shown that

$$\int af = a \int f.$$

Now set $h = f + g$. Since $h = h_+ - h_-$,

$$h_+ - h_- = f_+ - f_- + g_+ - g_-$$

or

$$h_+ + f_- + g_- = h_- + f_+ + g_+.$$

Therefore,

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+$$

and hence

$$\int h = \int h_+ - \int h_- = \int f_+ + \int g_+ - \int f_- - \int g_- = \int f + \int g.$$

Finally if $f_+ - f_- = f \leq g = g_+ - g_-$ then $f_+ + g_- \leq g_+ + f_-$ which implies that

$$\int f_+ + \int g_- \leq \int g_+ + \int f_-$$

or equivalently that

$$\int f = \int f_+ - \int f_- \leq \int g_+ - \int g_- = \int g.$$

The monotonicity property is also a consequence of the linearity of the integral, the fact that $f \leq g$ a.e. implies $0 \leq g - f$ a.e. and Proposition 5.27. ∎

Definition 5.33. A measurable function $f : X \to \mathbb{C}$ is integrable if $\int_X |f|\, d\mu < \infty$, again we write $f \in L^1$. Because, $\max(|\mathrm{Re}\, f|, |\mathrm{Im}\, f|) \leq |f| \leq \sqrt{2} \max(|\mathrm{Re}\, f|, |\mathrm{Im}\, f|)$, $\int |f|\, d\mu < \infty$ iff

$$\int |\mathrm{Re}\, f|\, d\mu + \int |\mathrm{Im}\, f|\, d\mu < \infty.$$

For $f \in L^1$ define

$$\int f\ d\mu = \int \mathrm{Re}\, f\ d\mu + i \int \mathrm{Im}\, f\ d\mu.$$

It is routine to show the integral is still linear on the complex L^1 (prove!).

Proposition 5.34. *Suppose that $f \in L^1$, then*

$$\left|\int_X f d\mu\right| \leq \int_X |f| d\mu.$$

Proof. Start by writing $\int_X f \, d\mu = Re^{i\theta}$. Then using the monotonicity in Proposition 5.27,

$$\begin{aligned}\left|\int_X f d\mu\right| &= R = e^{-i\theta}\int_X f d\mu = \int_X e^{-i\theta} f d\mu \\ &= \int_X \operatorname{Re}\left(e^{-i\theta} f\right) d\mu \leq \int_X \left|\operatorname{Re}\left(e^{-i\theta} f\right)\right| d\mu \leq \int_X |f| \, d\mu.\end{aligned}$$

■

Proposition 5.35. *$f, g \in L^1$, then*

1. *The set $\{f \neq 0\}$ is σ-finite, in fact $\{|f| \geq \frac{1}{n}\} \uparrow \{f \neq 0\}$ and $\mu(|f| \geq \frac{1}{n}) < \infty$ for all n.*
2. *The following are equivalent*
   (a) $\int_E f = \int_E g$ *for all* $E \in \mathcal{M}$
   (b) $\int\limits_X |f - g| = 0$
   (c) $f = g$ *a.e.*

Proof. 1. By Chebyshev's inequality, Lemma 5.19,

$$\mu(|f| \geq \frac{1}{n}) \leq n \int_X |f| d\mu < \infty$$

for all n.

2. (a) $\Longrightarrow$ (c) Notice that

$$\int_E f = \int_E g \Leftrightarrow \int_E (f - g) = 0$$

for all $E \in \mathcal{M}$. Taking $E = \{\operatorname{Re}(f - g) > 0\}$ and using $1_E \operatorname{Re}(f - g) \geq 0$, we learn that

$$0 = \operatorname{Re}\int_E (f - g) d\mu = \int 1_E \operatorname{Re}(f - g) \Longrightarrow 1_E \operatorname{Re}(f - g) = 0 \text{ a.e.}$$

This implies that $1_E = 0$ a.e. which happens iff

$$\mu\left(\{\operatorname{Re}(f - g) > 0\}\right) = \mu(E) = 0.$$

Similar $\mu(\operatorname{Re}(f - g) < 0) = 0$ so that $\operatorname{Re}(f - g) = 0$ a.e. Similarly, $\operatorname{Im}(f - g) = 0$ a.e and hence $f - g = 0$ a.e., i.e. $f = g$ a.e.

(c) $\Longrightarrow$ (b) is clear and so is (b) $\Longrightarrow$ (a) since

$$\left|\int_E f - \int_E g\right| \leq \int |f - g| = 0.$$

■

Definition 5.36. Let $(X, \mathcal{M}, \mu)$ be a measure space and $L^1(\mu) = L^1(X, \mathcal{M}, \mu)$ denote the set of L^1 functions modulo the equivalence relation; $f \sim g$ iff $f = g$ a.e. We make this into a normed space using the norm

$$\|f - g\|_{L^1} = \int |f - g| \, d\mu$$

and into a metric space using $\rho_1(f, g) = \|f - g\|_{L^1}$.

Remark 5.37. More generally we may define $L^p(\mu) = L^p(X, \mathcal{M}, \mu)$ for $p \in [1, \infty)$ as the set of measurable functions f such that

$$\int_X |f|^p \, d\mu < \infty$$

modulo the equivalence relation; $f \sim g$ iff $f = g$ a.e.

We will see in Section 7 that

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu \right)^{1/p} \text{ for } f \in L^p(\mu)$$

is a norm and $(L^p(\mu), \|\cdot\|_{L^p})$ is a Banach space in this norm.

Theorem 5.38 (Dominated Convergence Theorem)**.** *Suppose $f_n, g_n, g \in L^1$, $f_n \to f$ a.e., $|f_n| \le g_n \in L^1$, $g_n \to g$ a.e. and $\int_X g_n d\mu \to \int_X g d\mu$. Then $f \in L^1$ and*

$$\int_X f d\mu = \lim_{h \to \infty} \int_X f_n d\mu.$$

(In most typical applications of this theorem $g_n = g \in L^1$ for all n.)

Proof. Notice that $|f| = \lim_{n\to\infty} |f_n| \le \lim_{n\to\infty} |g_n| \le g$ a.e. so that $f \in L^1$. By considering the real and imaginary parts of f separately, it suffices to prove the theorem in the case where f is real. By Fatou's Lemma,

$$\begin{aligned} \int_X (g \pm f) d\mu &= \int_X \liminf_{n\to\infty} (g_n \pm f_n) \, d\mu \le \liminf_{n\to\infty} \int_X (g_n \pm f_n) \, d\mu \\ &= \lim_{n\to\infty} \int_X g_n d\mu + \liminf_{n\to\infty} \left(\pm \int_X f_n d\mu \right) \\ &= \int_X g d\mu + \liminf_{n\to\infty} \left(\pm \int_X f_n d\mu \right) \end{aligned}$$

Since $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$, we have shown,

$$\int_X g d\mu \pm \int_X f d\mu \le \int_X g d\mu + \begin{cases} \liminf_{n\to\infty} \int_X f_n d\mu \\ -\limsup_{n\to\infty} \int_X f_n d\mu \end{cases}$$

and therefore

$$\limsup_{n\to\infty} \int_X f_n d\mu \le \int_X f d\mu \le \liminf_{n\to\infty} \int_X f_n d\mu.$$

This shows that $\lim\limits_{n\to\infty} \int_X f_n d\mu$ exists and is equal to $\int_X f d\mu$. ∎

Corollary 5.39. *Let $\{f_n\}_{n=1}^\infty \subset L^1$ be a sequence such that $\sum_{n=1}^\infty \|f_n\|_{L^1} < \infty$, then $\sum_{n=1}^\infty f_n$ is convergent a.e. and*

$$\int_X \left(\sum_{n=1}^\infty f_n \right) d\mu = \sum_{n=1}^\infty \int_X f_n d\mu.$$

Proof. The condition $\sum_{n=1}^\infty \|f_n\|_{L^1} < \infty$ is equivalent to $\sum_{n=1}^\infty |f_n| \in L^1$. Hence $\sum_{n=1}^\infty f_n$ is almost everywhere convergent and if $S_N := \sum_{n=1}^N f_n$, then

$$|S_N| \le \sum_{n=1}^N |f_n| \le \sum_{n=1}^\infty |f_n| \in L^1.$$

So by the dominated convergence theorem,

$$\int_X \left(\sum_{n=1}^\infty f_n\right) d\mu = \int_X \lim_{N\to\infty} S_N d\mu = \lim_{N\to\infty} \int_X S_N d\mu$$
$$= \lim_{N\to\infty} \sum_{n=1}^N \int_X f_n d\mu = \sum_{n=1}^\infty \int_X f_n d\mu.$$

■

Theorem 5.40 (The Fundamental Theorem of Calculus)**.** *Suppose* $-\infty < a < b < \infty$*,* $f \in C((a,b),\mathbb{R}) \cap L^1((a,b),m)$ *and* $F(x) := \int_a^x f(y)dm(y)$*. Then*

1. $F \in C([a,b],\mathbb{R}) \cap C^1((a,b),\mathbb{R})$.
2. $F'(x) = f(x)$ *for all* $x \in (a,b)$.
3. *If* $G \in C([a,b],\mathbb{R}) \cap C^1((a,b),\mathbb{R})$ *is an anti-derivative of* f *on* (a,b) *(i.e.* $f = G'|_{(a,b)}$*) then*

$$\int_a^b f(x)dm(x) = G(b) - G(a).$$

Proof. Since $F(x) := \int_{\mathbb{R}} 1_{(a,x)}(y) f(y) dm(y)$, $\lim_{x\to z} 1_{(a,x)}(y) = 1_{(a,z)}(y)$ for m – a.e. y and $\left|1_{(a,x)}(y)f(y)\right| \le 1_{(a,b)}(y)\left|f(y)\right|$ is an L^1 – function, it follows from the dominated convergence Theorem 5.38 that F is continuous on $[a,b]$. Simple manipulations show,

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \frac{1}{|h|}\begin{cases} \left|\int_x^{x+h}\left[f(y)-f(x)\right]dm(y)\right| & \text{if } h>0 \\ \left|\int_{x+h}^{x}\left[f(y)-f(x)\right]dm(y)\right| & \text{if } h<0 \end{cases}$$
$$\le \frac{1}{|h|}\begin{cases} \int_x^{x+h}\left|f(y)-f(x)\right|dm(y) & \text{if } h>0 \\ \int_{x+h}^{x}\left|f(y)-f(x)\right|dm(y) & \text{if } h<0 \end{cases}$$
$$\le \sup\left\{\left|f(y)-f(x)\right| : y \in \left[x-|h|, x+|h|\right]\right\}$$

and the latter expression, by the continuity of f, goes to zero as $h \to 0$. This shows $F' = f$ on (a,b).

For the converse direction, we have by assumption that $G'(x) = F'(x)$ for $x \in (a,b)$. Therefore by the mean value theorem, $F - G = C$ for some constant C. Hence

$$\int_a^b f(x)dm(x) = F(b) = F(b) - F(a) = (G(b)+C) - (G(a)+C) = G(b) - G(a).$$

■

Example 5.41. The following limit holds,

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n dm(x) = 1.$$

Let $f_n(x) = (1-\frac{x}{n})^n 1_{[0,n]}(x)$ and notice that $\lim_{n\to\infty} f_n(x) = e^{-x}$. We will now show

$$0 \le f_n(x) \le e^{-x} \text{ for all } x \ge 0.$$

It suffices to consider $x \in [0,n]$. Let $g(x) = e^x f_n(x)$, then for $x \in (0,n)$,

$$\frac{d}{dx}\ln g(x) = 1 + n\frac{1}{(1-\frac{x}{n})}(-\frac{1}{n}) = 1 - \frac{1}{(1-\frac{x}{n})} \le 0$$

which shows that $\ln g(x)$ and hence $g(x)$ is decreasing on $[0,n]$. Therefore $g(x) \leq g(0) = 1$, i.e.

$$0 \leq f_n(x) \leq e^{-x}.$$

From Example 5.25, we know

$$\int_0^\infty e^{-x} dm(x) = 1 < \infty,$$

so that e^{-x} is an integrable function on $[0,\infty)$. Hence by the dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \int_0^n (1-\frac{x}{n})^n dm(x) &= \lim_{n\to\infty} \int_0^\infty f_n(x) dm(x) \\ &= \int_0^\infty \lim_{n\to\infty} f_n(x) dm(x) = \int_0^\infty e^{-x} dm(x) = 1.\end{aligned}$$

Example 5.42 (Integration of Power Series)**.** Suppose $R > 0$ and $\{a_n\}_{n=0}^\infty$ is a sequence of complex numbers such that $\sum_{n=0}^\infty |a_n| r^n < \infty$ for all $r \in (0,R)$. Then

$$\int_\alpha^\beta \left(\sum_{n=0}^\infty a_n x^n\right) dm(x) = \sum_{n=0}^\infty a_n \int_\alpha^\beta x^n dm(x) = \sum_{n=0}^\infty a_n \frac{\beta^{n+1} - \alpha^{n+1}}{n+1}$$

for all $-R < \alpha < \beta < R$. Indeed this follows from Corollary 5.39 since

$$\begin{aligned}\sum_{n=0}^\infty \int_\alpha^\beta |a_n| |x|^n dm(x) &\leq \sum_{n=0}^\infty \left(\int_0^{|\beta|} |a_n| |x|^n dm(x) + \int_0^{|\alpha|} |a_n| |x|^n dm(x)\right) \\ &\leq \sum_{n=0}^\infty |a_n| \frac{|\beta|^{n+1} + |\alpha|^{n+1}}{n+1} \leq 2r \sum_{n=0}^\infty |a_n| r^n < \infty\end{aligned}$$

where $r = \max(|\beta|, |\alpha|)$.

Corollary 5.43 (Differentiation Under the Integral)**.** *Suppose that $J \subset \mathbb{R}$ is an open interval and $f : J \times X \to \mathbb{C}$ is a function such that*

1. *$x \to f(t,x)$ is measurable for each $t \in J$.*
2. *$f(t_0,\cdot) \in L^1(\mu)$ for some $t_0 \in J$.*
3. *$\frac{\partial f}{\partial t}(t,x)$ exists for all (t,x).*
4. *There is a function $g \in L^1$ such that $\left|\frac{\partial f}{\partial t}(t,\cdot)\right| \leq g \in L^1$ for each $t \in J$.*

   Then $f(t,\cdot) \in L^1(\mu)$ for all $t \in J$ (i.e. $\int |f(t,x)| d\mu(x) < \infty$), $t \to \int_X f(t,x) d\mu(x)$ is a differentiable function on J and

$$\frac{d}{dt} \int_X f(t,x) d\mu(x) = \int_X \frac{\partial f}{\partial t}(t,x) d\mu(x).$$

Proof. (The proof is essentially the same as for sums.) By considering the real and imaginary parts of f separately, we may assume that f is real. Also notice that

$$\frac{\partial f}{\partial t}(t,x) = \lim_{n\to\infty} n(f(t+n^{-1},x) - f(t,x))$$

and therefore, for $x \to \frac{\partial f}{\partial t}(t,x)$ is a sequential limit of measurable functions and hence is measurable for all $t \in J$. By the mean value theorem,

$$|f(t,x) - f(t_0,x)| \leq g(x) |t - t_0| \text{ for all } t \in J \tag{5.14}$$

and hence

$$|f(t,x)| \le |f(t,x) - f(t_0,x)| + |f(t_0,x)| \le g(x)\,|t-t_0| + |f(t_0,x)|\,.$$

This shows $f(t,\cdot) \in L^1(\mu)$ for all $t \in J$. Let $G(t) := \int_X f(t,x)d\mu(x)$, then

$$\frac{G(t)-G(t_0)}{t-t_0} = \int_X \frac{f(t,x)-f(t_0,x)}{t-t_0}d\mu(x).$$

By assumption,

$$\lim_{t\to t_0} \frac{f(t,x)-f(t_0,x)}{t-t_0} = \frac{\partial f}{\partial t}(t,x) \text{ for all } x \in X$$

and by Eq. (5.14),

$$\left|\frac{f(t,x)-f(t_0,x)}{t-t_0}\right| \le g(x) \text{ for all } t \in J \text{ and } x \in X.$$

Therefore, we may apply the dominated convergence theorem to conclude

$$\begin{aligned}\lim_{n\to\infty} \frac{G(t_n)-G(t_0)}{t_n-t_0} &= \lim_{n\to\infty} \int_X \frac{f(t_n,x)-f(t_0,x)}{t_n-t_0}d\mu(x) \\ &= \int_X \lim_{n\to\infty} \frac{f(t_n,x)-f(t_0,x)}{t_n-t_0}d\mu(x) = \int_X \frac{\partial f}{\partial t}(t_0,x)d\mu(x)\end{aligned}$$

for **all** sequences $t_n \in J \setminus \{t_0\}$ such that $t_n \to t_0$. Therefore, $\dot G(t_0) = \lim_{t\to t_0} \frac{G(t)-G(t_0)}{t-t_0}$ exists and

$$\dot G(t_0) = \int_X \frac{\partial f}{\partial t}(t_0,x)d\mu(x).$$

■

Example 5.44. Recall from Example 5.25 that

$$\lambda^{-1} = \int_{[0,\infty)} e^{-\lambda x}dm(x) \text{ for all } \lambda > 0.$$

Let $\epsilon > 0$. For $\lambda \ge 2\epsilon > 0$ and $n \in \mathbb{N}$ there exists $C_n(\epsilon) < \infty$ such that

$$0 \le \left(-\frac{d}{d\lambda}\right)^n e^{-\lambda x} = x^n e^{-\lambda x} \le C(\epsilon)e^{-\epsilon x}.$$

Using this fact, Corollary 5.43 and induction gives

$$n!\lambda^{-n-1} = \left(-\frac{d}{d\lambda}\right)^n \lambda^{-1} = \int_{[0,\infty)} \left(-\frac{d}{d\lambda}\right)^n e^{-\lambda x}dm(x) = \int_{[0,\infty)} x^n e^{-\lambda x}dm(x).$$

That is $n! = \lambda^n \int_{[0,\infty)} x^n e^{-\lambda x}dm(x)$. Recall that

$$\Gamma(t) := \int_{[0,\infty)} x^{t-1}e^{-x}dx \text{ for } t > 0.$$

(The reader should check that $\Gamma(t) < \infty$ for all $t > 0$.) We have just shown that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Remark 5.45. Corollary 5.43 may be generalized by allowing the hypothesis to hold for $x \in X \setminus E$ where $E \in \mathcal{M}$ is a **fixed** null set, i.e. E must be independent of t. Consider what happens if we formally apply Corollary 5.43 to $g(t) := \int_0^\infty 1_{x\le t}dm(x)$,

$$\dot g(t) = \frac{d}{dt}\int_0^\infty 1_{x\le t}dm(x) \overset{?}{=} \int_0^\infty \frac{\partial}{\partial t}1_{x\le t}dm(x).$$

The last integral is zero since $\frac{\partial}{\partial t}1_{x\le t} = 0$ unless $t = x$ in which case it is not defined. On the other hand $g(t) = t$ so that $\dot{g}(t) = 1$. (The reader should decide which hypothesis of Corollary 5.43 has been violated in this example.)

5.5. **Measurability on Complete Measure Spaces.** In this subsection we will discuss a couple of measurability results concerning completions of measure spaces.

Proposition 5.46. *Suppose that $(X, \mathcal{M}, \mu)$ is a complete measure space*[10] *and $f : X \to \mathbb{R}$ is measurable.*

1. *If $g : X \to \mathbb{R}$ is a function such that $f(x) = g(x)$ for μ – a.e. x, then g is measurable.*
2. *If $f_n : X \to \mathbb{R}$ are measurable and $f : X \to \mathbb{R}$ is a function such that $\lim_{n\to\infty} f_n = f$, μ - a.e., then f is measurable as well.*

Proof. 1. Let $E = \{x : f(x) \ne g(x)\}$ which is assumed to be in $\mathcal{M}$ and $\mu(E) = 0$. Then $g = 1_{E^c} f + 1_E g$ since $f = g$ on E^c. Now $1_{E^c} f$ is measurable so g will be measurable if we show $1_E g$ is measurable. For this consider,

$$(1_E g)^{-1}(A) = \begin{cases} E^c \cup (1_E g)^{-1}(A \setminus \{0\}) & \text{if } 0 \in A \\ (1_E g)^{-1}(A) & \text{if } 0 \notin A \end{cases} \tag{5.15}$$

Since $(1_E g)^{-1}(B) \subset E$ if $0 \notin B$ and $\mu(E) = 0$, it follow by completeness of $\mathcal{M}$ that $(1_E g)^{-1}(B) \in \mathcal{M}$ if $0 \notin B$. Therefore Eq. (5.15) shows that $1_E g$ is measurable.

2. Let $E = \{x : \lim_{n\to\infty} f_n(x) \ne f(x)\}$ by assumption $E \in \mathcal{M}$ and $\mu(E) = 0$. Since $g \equiv 1_E f = \lim_{n\to\infty} 1_{E^c} f_n$, g is measurable. Because $f = g$ on E^c and $\mu(E) = 0$, $f = g$ a.e. so by part 1. f is also measurable. ■

The above results are in general false if $(X, \mathcal{M}, \mu)$ is not complete. For example, let $X = \{0, 1, 2\}$ $\mathcal{M} = \{\{0\}, \{1, 2\}, X, \phi\}$ and $\mu = \delta_0$ Take $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, then $g = 0$ a.e. yet g is not measurable.

Lemma 5.47. *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $\bar{\mathcal{M}}$ is the completion of $\mathcal{M}$ relative to μ and $\bar{\mu}$ is the extension of μ to $\bar{\mathcal{M}}$. Then a function $f : X \to \mathbb{R}$ is $(\bar{\mathcal{M}}, \mathcal{B} = \mathcal{B}_{\mathbb{R}})$ – measurable iff there exists a function $g : X \to \mathbb{R}$ that is $(\mathcal{M}, \mathcal{B})$ – measurable such $E = \{x : f(x) \ne g(x)\} \in \bar{\mathcal{M}}$ and $\bar{\mu}(E) = 0$, i.e. $f(x) = g(x)$ for $\bar{\mu}$ – a.e. x. Moreover for such a pair f and g, $f \in L^1(\bar{\mu})$ iff $g \in L^1(\mu)$ and in which case*

$$\int_X f d\bar{\mu} = \int_X g d\mu.$$

Proof. Suppose first that such a function g exists so that $\bar{\mu}(E) = 0$. Since g is also $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable, we see from Proposition 5.46 that f is $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable.

Conversely if f is $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable, by considering $f_\pm$ we may assume that $f \ge 0$. Choose $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable simple function $\phi_n \ge 0$ such that $\phi_n \uparrow f$ as $n \to \infty$. Writing

$$\phi_n = \sum a_k 1_{A_k}$$

with $A_k \in \bar{\mathcal{M}}$, we may choose $B_k \in \mathcal{M}$ such that $B_k \subset A_k$ and $\bar{\mu}(A_k \setminus B_k) = 0$. Letting

$$\tilde{\phi}_n := \sum a_k 1_{B_k}$$

[10] Recall this means that if $N \subset X$ is a set such that $N \subset A \in \mathcal{M}$ and $\mu(A) = 0$, then $N \in \mathcal{M}$ as well.

we have produced a $(\mathcal{M},\mathcal{B})$ – measurable simple function $\tilde{\phi}_n \geq 0$ such that $E_n := \{\phi_n \neq \tilde{\phi}_n\}$ has zero $\bar{\mu}$ – measure. Since $\bar{\mu}(\cup_n E_n) \leq \sum_n \bar{\mu}(E_n)$, there exists $F \in \mathcal{M}$ such that $\cup_n E_n \subset F$ and $\mu(F) = 0$. It now follows that

$$1_F \tilde{\phi}_n = 1_F \phi_n \uparrow g := 1_F f \text{ as } n \to \infty.$$

This shows that $g = 1_F f$ is $(\mathcal{M},\mathcal{B})$ – measurable and that $\{f \neq g\} \subset F$ has $\bar{\mu}$ – measure zero.

Since $f = g$, $\bar{\mu}$ – a.e., $\int_X f d\bar{\mu} = \int_X g d\bar{\mu}$ so to prove Eq. (5.16) it suffices to prove

$$\int_X g d\bar{\mu} = \int_X g d\mu. \tag{5.16}$$

Because $\bar{\mu} = \mu$ on $\mathcal{M}$, Eq. (5.16) is easily verified for non-negative $\mathcal{M}$ – measurable simple functions. Then by the monotone convergence theorem and the approximation Theorem 5.12 it holds for all $\mathcal{M}$ – measurable functions $g : X \to [0,\infty]$. The rest of the assertions follow in the standard way by considering $(\operatorname{Re} g)_{\pm}$ and $(\operatorname{Im} g)_{\pm}$. ■

5.6. **Comparison of the Lebesgue and the Riemann Integral.** For the rest of this chapter, let $-\infty < a < b < \infty$ and $f : [a,b] \to \mathbb{R}$ be a bounded function. A partition of $[a,b]$ is a finite subset $\pi \subset [a,b]$ containing $\{a,b\}$. To each partition

$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\} \tag{5.17}$$

of $[a,b]$ let

$$\operatorname{mesh}(\pi) := \max\{|t_j - t_{j-1}| : j = 1,\ldots,n\},$$

$$M_j = \sup\{f(x) : t_j \leq x \leq t_{j-1}\}, \quad m_j = \inf\{f(x) : t_j \leq x \leq t_{j-1}\}$$
$$G_\pi = f(a)1_{\{a\}} + \sum_1^n M_j 1_{(t_{j-1},t_j]}, \quad g_\pi = f(a)1_{\{a\}} + \sum_1^n m_j 1_{(t_{j-1},t_j]} \text{ and}$$

$$S_\pi f = \sum M_j(t_j - t_{j-1}) \text{ and } s_\pi f = \sum m_j(t_j - t_{j-1}).$$

Notice that

$$S_\pi f = \int_a^b G_\pi dm \text{ and } s_\pi f = \int_a^b g_\pi dm.$$

The upper and lower Riemann integrals are defined respectively by

$$\overline{\int_a^b} f(x)dx = \inf_\pi S_\pi f \text{ and } \underline{\int_b^a} f(x)dx = \sup_\pi s_\pi f.$$

Definition 5.48. The function f is **Riemann integrable** iff $\overline{\int_a^b} f = \underline{\int_a^b} f$ and which case the Riemann integral $\int_a^b f$ is defined to be the common value:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

The proof of the following Lemma is left as an exercise to the reader.

Lemma 5.49. *If π' and π are two partitions of $[a,b]$ and $\pi \subset \pi'$ then*

$$G_\pi \geq G_{\pi'} \geq f \geq g_{\pi'} \geq g_\pi \text{ and}$$
$$S_\pi f \geq S_{\pi'} f \geq s_{\pi'} f \geq s_\pi f.$$

There exists an increasing sequence of partitions $\{\pi_k\}_{k=1}^\infty$ such that $\operatorname{mesh}(\pi_k) \downarrow 0$ *and*

$$S_{\pi_k} f \downarrow \overline{\int_a^b} f \text{ and } s_{\pi_k} f \uparrow \underline{\int_a^b} f \text{ as } k \to \infty.$$

If we let

$$G \equiv \lim_{k\to\infty} G_{\pi_k} \text{ and } g \equiv \lim_{k\to\infty} g_{\pi_k} \tag{5.18}$$

then by the dominated convergence theorem,

$$\int_{[a,b]} g dm = \lim_{k\to\infty} \int_{[a,b]} g_{\pi_k} = \lim_{k\to\infty} s_{\pi_k} f = \underline{\int_a^b} f(x) dx \tag{5.19}$$

and

$$\int_{[a,b]} G dm = \lim_{k\to\infty} \int_{[a,b]} G_{\pi_k} = \lim_{k\to\infty} S_{\pi_k} f = \overline{\int_a^b} f(x) dx. \tag{5.20}$$

Notation 5.50. For $x \in [a,b]$, let

$$H(x) = \limsup_{y\to x} f(y) = \lim_{\epsilon\downarrow 0} \sup\{f(y) : |y-x| \leq \epsilon,\ y \in [a,b]\} \text{ and}$$
$$h(x) \equiv \liminf_{y\to x} f(y) = \lim_{\epsilon\downarrow 0} \inf\ \{f(y) : |y-x| \leq \epsilon,\ y \in [a,b]\}.$$

Lemma 5.51. *The functions $H, h : [a,b] \to \mathbb{R}$ satisfy:*

1. *$h(x) \leq f(x) \leq H(x)$ for all $x \in [a,b]$ and $h(x) = H(x)$ iff f is continuous at x.*
2. *If $\{\pi_k\}_{k=1}^\infty$ is any increasing sequence of partitions such that* $\operatorname{mesh}(\pi_k) \downarrow 0$ *and G and g are defined as in Eq. (5.18), then*

$$G(x) = H(x) \geq f(x) \geq h(x) = g(x) \quad \forall\ x \notin \pi := \cup_{k=1}^\infty \pi_k. \tag{5.21}$$

(Note π is a countable set.)

3. *H and h are Borel measurable.*

Proof. Let $G_k \equiv G_{\pi_k} \downarrow G$ and $g_k \equiv g_{\pi_k} \uparrow g$.

1. It is clear that $h(x) \leq f(x) \leq H(x)$ for all x and $H(x) = h(x)$ iff $\lim_{y\to x} f(y)$ exists and is equal to $f(x)$. That is $H(x) = h(x)$ iff f is continuous at x.
2. For $x \notin \pi$,

$$G_k(x) \geq H(x) \geq f(x) \geq h(x) \geq g_k(x)\ \forall\ k$$

and letting $k \to \infty$ in this equation implies

$$G(x) \geq H(x) \geq f(x) \geq h(x) \geq g(x)\ \forall\ x \notin \pi. \tag{5.22}$$

Moreover, given $\epsilon > 0$ and $x \notin \pi$,

$$\sup\{f(y) : |y-x| \leq \epsilon,\ y \in [a,b]\} \geq G_k(x)$$

for all k large enough, since eventually $G_k(x)$ is the supremum of $f(y)$ over some interval contained in $[x-\epsilon, x+\epsilon]$. Again letting $k \to \infty$ implies $\sup_{|y-x|\leq\epsilon} f(y) \geq G(x)$ and therefore, that

$$H(x) = \limsup_{y\to x} f(y) \geq G(x)$$

for all $x \notin \pi$. Combining this equation with Eq. (5.22) then implies $H(x) = G(x)$ if $x \notin \pi$. A similar argument shows that $h(x) = g(x)$ if $x \notin \pi$ and hence Eq. (5.21) is proved.

3. The functions G and g are limits of measurable functions and hence measurable. Since $H = G$ and $h = g$ except possibly on the countable set π, both H and h are also Borel measurable. (You justify this statement.)

■

Theorem 5.52. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then*

$$\overline{\int_a^b} f = \int_{[a,b]} H dm \text{ and } \underline{\int_a^b} f = \int_{[a,b]} h dm \tag{5.23}$$

and the following statements are equivalent:

1. $H(x) = h(x)$ *for* m *-a.e.* x,
2. *the set*

$$E := \{x \in [a,b] : f \text{ is disconituous at } x\}$$

is an $\bar{m}$ – *null set.*

3. f *is Riemann integrable.*

If f is Riemann integrable then f is Lebesgue measurable[11], *i.e. f is $\mathcal{L}/\mathcal{B}$ – measurable where $\mathcal{L}$ is the Lebesgue σ – algebra and $\mathcal{B}$ is the Borel σ – algebra on $[a,b]$. Moreover if we let $\bar{m}$ denote the completion of m, then*

$$\int_{[a,b]} H dm = \int_a^b f(x)dx = \int_{[a,b]} f d\bar{m} = \int_{[a,b]} h dm. \tag{5.24}$$

Proof. Let $\{\pi_k\}_{k=1}^\infty$ be an increasing sequence of partitions of $[a,b]$ as described in Lemma 5.49 and let G and g be defined as in Lemma 5.51. Since $m(\pi) = 0$, $H = G$ a.e., Eq. (5.23) is a consequence of Eqs. (5.19) and (5.20). From Eq. (5.23), f is Riemann integrable iff

$$\int_{[a,b]} H dm = \int_{[a,b]} h dm$$

and because $h \leq f \leq H$ this happens iff $h(x) = H(x)$ for m - a.e. x. Since $E = \{x : H(x) \neq h(x)\}$, this last condition is equivalent to E being a m – null set. In light of these results and Eq. (5.21), the remaining assertions including Eq. (5.24) are now consequences of Lemma 5.47. ■

Notation 5.53. In view of this theorem we will often write $\int_a^b f(x)dx$ for $\int_a^b f dm$.

[11] f need not be Borel measurable.

5.7. Exercises.

Exercise 5.1. Let μ be a measure on an algebra $\mathcal{A} \subset \mathcal{P}(X)$, then $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$ for all $A, B \in \mathcal{A}$.

Exercise 5.2. Problem 12 on p. 27 of Folland. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and for $A, B \in \mathcal{M}$ let $\rho(A, B) = \mu(A\Delta B)$ where $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Define $A \sim B$ iff $\mu(A\Delta B) = 0$. Show "$\sim$ " is an equivalence relation, ρ is a metric on $\mathcal{M}/\sim$ and $\mu(A) = \mu(B)$ if $A \sim B$. Also show that $\mu : (\mathcal{M}/\sim) \to [0, \infty)$ is a continuous function relative to the metric ρ.

Exercise 5.3. Suppose that $\mu_n : \mathcal{M} \to [0, \infty]$ are measures on $\mathcal{M}$ for $n \in \mathbb{N}$. Also suppose that $\mu_n(A)$ is increasing in n for all $A \in \mathcal{M}$. Prove that $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ is also a measure.

Exercise 5.4. Now suppose that Λ is some index set and for each $\lambda \in \Lambda$, $\mu_\lambda : \mathcal{M} \to [0, \infty]$ is a measure on $\mathcal{M}$. Define $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(A) = \sum_{\lambda\in\Lambda} \mu_\lambda(A)$ for each $A \in \mathcal{M}$. Show that μ is also a measure.

Exercise 5.5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\rho : X \to [0, \infty]$ be a measurable function. For $A \in \mathcal{M}$, set $\nu(A) := \int_A \rho d\mu$.

1. Show $\nu : \mathcal{M} \to [0, \infty]$ is a measure.
2. Let $f : X \to [0, \infty]$ be a measurable function, show

$$\int_X f d\nu = \int_X f \rho d\mu. \tag{5.25}$$

   Hint: first prove the relationship for characteristic functions, then for simple functions, and then for general positive measurable functions.
3. Show that $f \in L^1(\nu)$ iff $f\rho \in L^1(\mu)$ and if $f \in L^1(\nu)$ then Eq. (5.25) still holds.

Notation 5.54. It is customary to informally describe ν defined in Exercise 5.5 by writing $d\nu = \rho d\mu$.

Exercise 5.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{F})$ be a measurable space and $f : X \to Y$ be a measurable map. Define a function $\nu : \mathcal{F} \to [0, \infty]$ by $\nu(A) := \mu(f^{-1}(A))$ for all $A \in \mathcal{F}$.

1. Show ν is a measure. (We will write $\nu = f_*\mu$ or $\nu = \mu \circ f^{-1}$.)
2. Show

$$\int_Y g d\nu = \int_X (g \circ f)\, d\mu \tag{5.26}$$

   for all measurable functions $g : Y \to [0, \infty]$. **Hint:** see the hint from Exercise 5.5.
3. Show $g \in L^1(\nu)$ iff $g \circ f \in L^1(\mu)$ and that Eq. (5.26) holds for all $g \in L^1(\nu)$.

Exercise 5.7. Let $F : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that $F'(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to\pm\infty} F(x) = \pm\infty$. (Notice that F is strictly increasing so that $F^{-1} : \mathbb{R} \to \mathbb{R}$ exists and moreover, by the implicit function theorem that F^{-1} is a C^1 – function.) Let m be Lebesgue measure on $\mathcal{B}_\mathbb{R}$ and

$$\nu(A) = m(F(A)) = m(\left(F^{-1}\right)^{-1}(A)) = \left(F_*^{-1}m\right)(A)$$

for all $A \in \mathcal{B}_{\mathbb{R}}$. Show $d\nu = F'dm$. Use this result to prove the change of variable formula,

$$\int_{\mathbb{R}} h \circ F \cdot F' dm = \int_{\mathbb{R}} h dm \tag{5.27}$$

which is valid for all Borel measurable functions $h : \mathbb{R} \to [0, \infty]$.

Hint: Start by showing $d\nu = F'dm$ on sets of the form $A = (a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. Then use the uniqueness assertions in Theorem 5.8 to conclude $d\nu = F'dm$ on all of $\mathcal{B}_{\mathbb{R}}$. To prove Eq. (5.27) apply Exercise 5.6 with $g = h \circ F$ and $f = F^{-1}$.

Exercise 5.8. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, show

$$\mu(\{A_n \text{ a.a.}\}) \le \liminf_{n\to\infty} \mu(A_n)$$

and if $\mu(\cup_{m\ge n} A_m) < \infty$ for some n, then

$$\mu(\{A_n \text{ i.o.}\}) \ge \limsup_{n\to\infty} \mu(A_n).$$

Exercise 5.9 (Peano's Existence Theorem)**.** Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function. Then for each $T < \infty$[12] there exists a solution to the differential equation

$$\dot{x}(t) = f(t, x(t)) \text{ for } 0 \le t \le T \text{ with } x(0) = x_0. \tag{5.28}$$

Do this by filling in the following outline for the proof.

1. Given $\epsilon > 0$, show there exists a unique function $x_\epsilon \in C([-\epsilon, \infty) \to \mathbb{R}^d)$ such that $x_\epsilon(t) \equiv x_0$ for $-\epsilon \le t \le 0$ and

$$x_\epsilon(t) = x_0 + \int_0^t f(\tau, x_\epsilon(\tau - \epsilon))d\tau \text{ for all } t \ge 0. \tag{5.29}$$

   Here

$$\int_0^t f(\tau, x_\epsilon(\tau - \epsilon))d\tau = \left(\int_0^t f_1(\tau, x_\epsilon(\tau - \epsilon))d\tau, \dots, \int_0^t f_d(\tau, x_\epsilon(\tau - \epsilon))d\tau\right)$$

   where $f = (f_1, \dots, f_d)$ and the integrals are either the Lebesgue or the Riemann integral since they are equal on continuous functions.
2. Then use Exercise 3.39 to show there exists $\{\epsilon_k\}_{k=1}^{\infty} \subset (0, \infty)$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and x_{ϵ_k} converges to some $x \in C([0, T])$ (relative to the sup-norm: $\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|$) as $k \to \infty$.
3. Pass to the limit in Eq. (5.29) with ϵ replaced by ϵ_k to show x satisfies

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))d\tau \;\forall\, t \in [0, T].$$

4. Conclude from this that $\dot{x}(t)$ exists for $t \in (0, T)$ and that x solves Eq. (5.28).

Exercise 5.10. Folland 2.12 on p. 52.

Exercise 5.11. Folland 2.13 on p. 52.

Exercise 5.12. Folland 2.14 on p. 52.

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[12] Using Corollary 8.12 below, we may in fact allow $T = \infty$.

Exercise 5.13. Give examples of measurable functions $\{f_n\}$ on $\mathbb{R}$ such that f_n decreases to 0 uniformly yet $\int f_n dm = \infty$ for all n. Also give an example of a sequence of measurable functions $\{g_n\}$ on $[0,1]$ such that $g_n \to 0$ while $\int g_n dm = 1$ for all n.

Exercise 5.14. Folland 2.19 on p. 59.

Exercise 5.15. Suppose $\{a_n\}_{n=-\infty}^{\infty} \subset \mathbb{C}$ is a summable sequence (i.e. $\sum_{n=-\infty}^{\infty} |a_n| < \infty$), then $f(\theta) := \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ is a continuous function for $\theta \in \mathbb{R}$ and

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

Exercise 5.16. Folland 2.26 on p. 59.

Exercise 5.17. Folland 2.28 on p. 59.

Exercise 5.18. Folland 2.31b on p. 60.