Addendum to Chapter 10

Notation 10.44. Let $C^k_{per}(\mathbb{R}^d)$ denote the 2π – periodic functions in $C^k(\mathbb{R}^d)$,

$$C^k_{per}(\mathbb{R}^d) := \left\{ f \in C^k(\mathbb{R}^d) : f(x + 2\pi e_i) = f(x) \text{ for all } x \in \mathbb{R}^d \text{ and } i = 1, 2, \ldots, d \right\}.$$

Also let $\langle \cdot, \cdot \rangle$ denote the inner product on the Hilbert space $H := L^2([-\pi, \pi]^d)$ given by

$$\langle f, g \rangle := \left(\frac{1}{2\pi} \right)^d \int_{[-\pi,\pi]^d} f(x) \bar{g}(x) dx.$$

Recall that $\left\{ \chi_k(x) := e^{ik \cdot x} : k \in \mathbb{Z}^d \right\}$ is an orthonormal basis for H in particular for $f \in H$,

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \chi_k \rangle \chi_k \tag{10.24}$$

where the convergence takes place in $L^2([-\pi, \pi]^d)$. For $f \in L^1([-\pi, \pi]^d)$, we will write $\tilde{f}(k)$ for the **Fourier coefficient**,

$$\tilde{f}(k) := \langle f, \chi_k \rangle = \left(\frac{1}{2\pi} \right)^d \int_{[-\pi,\pi]^d} f(x) e^{-ik \cdot x} dx. \tag{10.25}$$

Lemma 10.45. *Let $s > 0$, then the following are equivalent,*

$$\sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + |k|)^s} < \infty, \quad \sum_{k \in \mathbb{Z}^d} \frac{1}{(1 + |k|^2)^{s/2}} < \infty \text{ and } s > d. \tag{10.26}$$

Proof. Let $Q := (0, 1]^d$ and $k \in \mathbb{Z}^d$. For $x = k + y \in (k + Q)$,

$$\begin{aligned} 2 + |k| &= 2 + |x - y| \leq 2 + |x| + |y| \leq 3 + |x| \text{ and} \\ 2 + |k| &= 2 + |x - y| \geq 2 + |x| - |y| \geq |x| + 1 \end{aligned}$$

and therefore for $s > 0$,

$$\frac{1}{(3 + |x|)^s} \leq \frac{1}{(2 + |k|)^s} \leq \frac{1}{(1 + |x|)^s}.$$

Thus we have shown

$$\frac{1}{(3 + |x|)^s} \leq \sum_{k \in \mathbb{Z}^d} \frac{1}{(2 + |k|)^s} 1_{Q+k}(x) \leq \frac{1}{(1 + |x|)^s} \text{ for all } x \in \mathbb{R}^d.$$

Integrating this equation then shows

$$\int_{\mathbb{R}^d} \frac{1}{(3 + |x|)^s} dx \leq \sum_{k \in \mathbb{Z}^d} \frac{1}{(2 + |k|)^s} \leq \int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^s} dx$$

from which we conclude that

$$\sum_{k \in \mathbb{Z}^d} \frac{1}{(2 + |k|)^s} < \infty \text{ iff } s > d. \tag{10.27}$$

Because the functions $1 + t$, $2 + t$, and $\sqrt{1 + t^2}$ all behave like t as $t \to \infty$, the sums in Eq. (10.26) may be compared with the one in Eq. (10.27) to finish the proof. ■

Exercise 10.22 (Riemann Lebesgue Lemma for Fourier Series)**.** Show for $f \in L^1([-\pi, \pi]^d)$ that $\tilde{f} \in c_0(\mathbb{Z}^d)$, i.e. $\tilde{f} : \mathbb{Z}^d \to \mathbb{C}$ and $\lim_{k \to \infty} \tilde{f}(k) = 0$. **Hint:** If $f \in H$, this follows form Bessel's inequality. Now use a density argument.

Exercise 10.23. Suppose $f \in L^1([-\pi,\pi]^d)$ is a function such that $\tilde{f} \in \ell^1(\mathbb{Z}^d)$ and set

$$g(x) := \sum_{k\in\mathbb{Z}} \tilde{f}(k)e^{ik\cdot x} \text{ (pointwise).}$$

(1) Show $g \in C_{per}(\mathbb{R}^d)$.
(2) Show $g(x) = f(x)$ for m – a.e. x in $[-\pi,\pi]^d$. **Hint**: Show $\tilde{g}(k) = \tilde{f}(k)$ and then use approximation arguments to show

$$\int_{[-\pi,\pi]^d} f(x)h(x)dx = \int_{[-\pi,\pi]^d} g(x)h(x)dx \;\forall\; h \in C([-\pi,\pi]^d).$$

(3) Conclude that $f \in L^1([-\pi,\pi]^d) \cap L^\infty([-\pi,\pi]^d)$ and in particular $f \in L^p([-\pi,\pi]^d)$ for all $p \in [1,\infty]$.

Exercise 10.24. Suppose $m \in \mathbb{N}_0$, α is a multi-index such that $|\alpha| \le m$ and $f \in C^m_{per}(\mathbb{R}^d)$[25].

(1) Using integration by parts, show

$$(ik)^\alpha \tilde{f}(k) = \langle \partial^\alpha f, \chi_k\rangle.$$

Note: This equality implies

$$\left|\tilde{f}(k)\right| \le \frac{1}{k^\alpha}\left\|\partial^\alpha f\right\|_H \le \frac{1}{k^\alpha}\left\|\partial^\alpha f\right\|_u.$$

(2) Now let $\Delta f = \sum_{i=1}^d \partial^2 f/\partial x_i^2$, Working as in part 1) show

$$\langle (1-\Delta)^m f, \chi_k\rangle = (1+|k|^2)^m \tilde{f}(k). \tag{10.28}$$

Remark 10.46. Suppose that m is an even integer, α is a multi-index and $f \in C^{m+|\alpha|}_{per}(\mathbb{R}^d)$, then

$$\begin{aligned}
\left(\sum_{k\in\mathbb{Z}^d} |k^\alpha|\left|\tilde{f}(k)\right|\right)^2 &= \left(\sum_{k\in\mathbb{Z}^d} |\langle \partial^\alpha f, \chi_k\rangle|\,(1+|k|^2)^{m/2}(1+|k|^2)^{-m/2}\right)^2 \\
&= \left(\sum_{k\in\mathbb{Z}^d} \left|\langle (1-\Delta)^{m/2}\partial^\alpha f, \chi_k\rangle\right|(1+|k|^2)^{-m/2}\right)^2 \\
&\le \sum_{k\in\mathbb{Z}^d}\left|\langle (1-\Delta)^{m/2}\partial^\alpha f, \chi_k\rangle\right|^2 \cdot \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-m} \\
&= C_m \left\|(1-\Delta)^{m/2}\partial^\alpha f\right\|_H^2
\end{aligned}$$

where $C_m := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-m} < \infty$ iff $m > d/2$. So the smoother f is the faster $\tilde{f}$ decays at infinity. The next problem is the converse of this assertion and hence smoothness of f corresponds to decay of $\tilde{f}$ at infinity and visa-versa.

Exercise 10.25. Suppose $s \in \mathbb{R}$ and $\left\{c_k \in \mathbb{C} : k \in \mathbb{Z}^d\right\}$ are coefficients such that

$$\sum_{k\in\mathbb{Z}^d} |c_k|^2 (1+|k|^2)^s < \infty.$$

[25] We view $C_{per}(\mathbb{R})$ as a subspace of H by identifying $f \in C_{per}(\mathbb{R})$ with $f|_{[-\pi,\pi]} \in H$.

Show if $s > \frac{d}{2} + m$, the function f defined by

$$f(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ik \cdot x}$$

is in $C^m_{per}(\mathbb{R}^d)$. **Hint**: Work as in the above remark to show

$$\sum_{k \in \mathbb{Z}^d} |c_k| \, |k^\alpha| < \infty \text{ for all } |\alpha| \le m.$$

Exercise 10.26 (Poisson Summation Formula)**.** Let $F \in L^1(\mathbb{R}^d)$,

$$E := \left\{ x \in \mathbb{R}^d : \sum_{k \in \mathbb{Z}^d} |F(x + 2\pi k)| = \infty \right\}$$

and set

$$\hat{F}(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} F(x) e^{-ik \cdot x} dx.$$

Further assume $\hat{F} \in \ell^1(\mathbb{N}^d)$.

(1) Show $m(E) = 0$ and $E + 2\pi k = E$ for all $k \in \mathbb{Z}^d$. **Hint**: Compute $\int_{[-\pi,\pi]^d} \sum_{k \in \mathbb{Z}^d} |F(x + 2\pi k)| \, dx$.

(2) Let

$$f(x) := \begin{cases} \sum_{k \in \mathbb{Z}^d} F(x + 2\pi k) & \text{for} \quad x \notin E \\ 0 & \text{if} \quad x \in E. \end{cases}$$

Show $f \in L^1([-\pi, \pi]^d)$ and $\tilde{f}(k) = (2\pi)^{-d/2} \hat{F}(k)$.

(3) Using item 2) and the assumptions on F, show $f \in L^1([-\pi, \pi]^d) \cap L^\infty([-\pi, \pi]^d)$ and

$$f(x) = \sum_{k \in \mathbb{Z}^d} \tilde{f}(k) e^{ik \cdot x} = \sum_{k \in \mathbb{Z}^d} (2\pi)^{-d/2} \hat{F}(k) e^{ik \cdot x} \text{ for } m-\text{a.e. } x,$$

i.e.

$$\sum_{k \in \mathbb{Z}^d} F(x + 2\pi k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{F}(k) e^{ik \cdot x} \text{ for } m-\text{a.e. } x. \tag{10.29}$$

(4) Suppose we now assume that $F \in C(\mathbb{R}^d)$ and F satisfies 1) $|F(x)| \le C(1 + |x|)^{-s}$ for some $s > d$ and $C < \infty$ and 2) $\hat{F} \in \ell^1(\mathbb{Z}^d)$, then show Eq. (10.29) holds for **all** $x \in \mathbb{R}^d$ and in particular

$$\sum_{k \in \mathbb{Z}^d} F(2\pi k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{F}(k).$$

For simplicity, in the remaining problems we will assume that $d = 1$.

Exercise 10.27 (Heat Equation 1.)**.** Let $(t, x) \in [0, \infty) \times \mathbb{R} \to u(t, x)$ be a continuous function such that $u(t, \cdot) \in C_{per}(\mathbb{R})$ for all $t \ge 0$, $\dot{u} := u_t$, u_x, and u_{xx} exists and are continuous when $t > 0$. Further assume that u satisfies the heat equation $\dot{u} = \frac{1}{2} u_{xx}$. Let $\tilde{u}(t, k) := \langle u(t, \cdot), \chi_k \rangle$ for $k \in \mathbb{Z}$. Show for $t > 0$ and $k \in \mathbb{Z}$ that $\tilde{u}(t, k)$ is differentiable in t and $\frac{d}{dt} \tilde{u}(t, k) = -k^2 \tilde{u}(t, k)/2$. Use this result to show

$$u(t, x) = \sum_{k \in \mathbb{Z}} e^{-\frac{t}{2} k^2} \tilde{f}(k) e^{ikx} \tag{10.30}$$

where $f(x) := u(0,x)$ and as above

$$\tilde{f}(k) = \langle f, \chi_k \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}dy.$$

Notice from Eq. (10.30) that $(t,x) \to u(t,x)$ is C^∞ for $t > 0$.

Exercise 10.28 (Heat Equation 2.)**.** Let $q_t(x) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-\frac{t}{2}k^2}e^{ikx}$. Show that Eq. (10.30) may be rewritten as

$$u(t,x) = \int_{-\pi}^{\pi} q_t(x-y)f(y)dy$$

and

$$q_t(x) = \sum_{k\in\mathbb{Z}} p_t(x + k2\pi)$$

where $p_t(x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}x^2}$. Also show $u(t,x)$ may be written as

$$u(t,x) = p_t * f(x) := \int_{\mathbb{R}^d} p_t(x-y)f(y)dy.$$

Hint: To show $q_t(x) = \sum_{k\in\mathbb{Z}} p_t(x+k2\pi)$, use the Poisson summation formula along with the Gaussian integration formula

$$\hat{p}_t(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} p_t(x)e^{i\omega x}dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{t}{2}\omega^2}.$$

Exercise 10.29 (Wave Equation)**.** Let $u \in C^2(\mathbb{R}\times\mathbb{R})$ such that $u(t,\cdot) \in C_{per}(\mathbb{R})$ for all $t \in \mathbb{R}$. Further assume that u solves the wave equation, $u_{tt} = u_{xx}$. Let $f(x) := u(0,x)$ and $g(x) = \dot{u}(0,x)$. Show $\tilde{u}(t,k) := \langle u(t,\cdot), \chi_k\rangle$ for $k \in \mathbb{Z}$ is twice continuously differentiable in t and $\frac{d^2}{dt^2}\tilde{u}(t,k) = -k^2\tilde{u}(t,k)$. Use this result to show

$$u(t,x) = \sum_{k\in\mathbb{Z}}\left(\tilde{f}(k)\cos(kt) + \tilde{g}(k)\frac{\sin kt}{k}\right)e^{ikx} \tag{10.31}$$

with the sum converging absolutely. Also show that $u(t,x)$ may be written as

$$u(t,x) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{-t}^{t} g(x+\tau)d\tau. \tag{10.32}$$

Hint: To show Eq. (10.31) implies (10.32) use

$$\cos kt = \frac{e^{ikt}+e^{-ikt}}{2}, \text{ and } \sin kt = \frac{e^{ikt}-e^{-ikt}}{2i}$$

and

$$\frac{e^{ik(x+t)} - e^{ik(x-t)}}{ik} = \int_{-t}^{t} e^{ik(x+\tau)}d\tau.$$

Exercise 10.30. (Worked Example.) Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C} \cong \mathbb{R}^2$, where we write $z = x + iy = re^{i\theta}$ in the usual way. Also let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and recall that Δ may be computed in polar coordinates by the formula,

$$\Delta u = r^{-1}\partial_r\left(r^{-1}\partial_r u\right) + \frac{1}{r^2}\partial_\theta^2 u.$$

Suppose that $u \in C(\bar{D}) \cap C^2(D)$ and $\Delta u(z) = 0$ for $z \in D$.Let $g = u|_{\partial D}$ and

$$\tilde{g}(k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{ik\theta})e^{-ik\theta}d\theta.$$

(We are identifying $S^1 = \partial D := \left\{z \in \bar{D} : |z| = 1\right\}$ with $[-\pi, \pi]/\pi\tilde{\ } - \pi$ by the map $\theta \in [-\pi, \pi] \to e^{i\theta} \in S^1$.) Let

$$\tilde{u}(r,k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})e^{-ik\theta}d\theta \tag{10.33}$$

then:

(1) $\tilde{u}(r,k)$ satisfies the ordinary differential equation

$$r^{-1}\partial_r\left(r\partial_r\tilde{u}(r,k)\right) = \frac{1}{r^2}k^2\tilde{u}(r,k) \text{ for } r \in (0,1).$$

(2) Recall the general solution to

$$r\partial_r\left(r\partial_r y(r)\right) = k^2 y(r) \tag{10.34}$$

may be found by trying solutions of the form $y(r) = r^\alpha$ which then implies $\alpha^2 = k^2$ or $\alpha = \pm k$. From this one sees that $\tilde{u}(r,k)$ may be written as $\tilde{u}(r,k) = A_k r^{|k|} + B_k r^{-|k|}$ for some constants A_k and B_k when $k \neq 0$. If $k = 0$, the solution to Eq. (10.34) is gotten by simple integration and the result is $\tilde{u}(r,0) = A_0 + B_0 \ln r$. Since $\tilde{u}(r,k)$ bounded near the origin for each k, it follows that $B_k = 0$ for all $k \in \mathbb{Z}$.

(3) So we have shown

$$A_k r^{|k|} = \tilde{u}(r,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})e^{-ik\theta}d\theta$$

and letting $r \uparrow 1$ in this equation implies

$$A_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})e^{-ik\theta}d\theta = \tilde{g}(k).$$

Therefore,

$$u(re^{i\theta}) = \sum_{k\in\mathbb{Z}} \tilde{g}(k)r^{|k|}e^{ik\theta} \tag{10.35}$$

for $r < 1$ or equivalently,

$$u(z) = \sum_{k\in\mathbb{N}_0} \tilde{g}(k)z^k + \sum_{k\in\mathbb{N}} \tilde{g}(-k)\bar{z}^k.$$

(4) Inserting the formula for $\tilde{g}(k)$ into Eq. (10.35) shows that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} r^{|k|}e^{ik(\theta-\alpha)}\right)u(e^{i\alpha})d\alpha \text{ for all } r < 1.$$

Now by simple geometric series considerations we find, setting $\delta = \theta - \alpha$, that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} r^{|k|}e^{ik\delta} &= \sum_{k=0}^{\infty} r^k e^{ik\delta} + \sum_{k=1}^{\infty} r^k e^{-ik\delta} \\
&= \frac{1}{1-re^{i\delta}} + \frac{re^{-i\delta}}{1-re^{-i\delta}} = \frac{1-re^{-i\delta} + re^{-i\delta}\left(1-re^{i\delta}\right)}{1-2r\cos\delta + r^2} \\
&= \frac{1-r^2}{1-2r\cos\delta + r^2}.
\end{aligned}$$

Putting this altogether we have shown

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\alpha)u(e^{i\alpha})d\alpha$$

where

$$P_r(\delta) := \frac{1-r^2}{1-2r\cos\delta + r^2}$$

is the so called Poisson kernel.

10.8. **Radon-Nikodym Theorem and the Dual of L^p.**

Definition 10.47. A complex measure ν on a measurable space $(X,\mathcal{M})$ is a countably additive set function $\nu:\mathcal{M}\to\mathbb{C}$ such that $\nu(\emptyset)=0$.

Theorem 10.48. *Suppose $(X,\mathcal{M})$ is a measurable space, μ is a positive finite measure on $\mathcal{M}$ and ν is a complex measure on $\mathcal{M}$ such that $|\nu(A)|\le\mu(A)$ for all $A\in\mathcal{M}$. Then $d\nu=\rho d\mu$ where $|\rho|\le 1$. Moreover if ν is a positive measure, then $0\le\rho\le 1$.*

Proof. For a simple function, $f\in\mathcal{S}(X,\mathcal{M})$, let $\nu(f):=\sum_{a\in\mathbb{C}}a\nu(f=a)$. Then

$$|\nu(f)|\le\sum_{a\in\mathbb{C}}|a|\,|\nu(f=a)|\le\sum_{a\in\mathbb{C}}|a|\,\mu(f=a)=\int_X|f|\,d\mu.$$

So, by the B.L.T. theorem, ν extends to a continuous linear functional on $L^1(\mu)$ satisfying the bounds

$$|\nu(f)|\le\int_X|f|\,d\mu\le\sqrt{\mu(X)}\,\|f\|_{L^2(\mu)}\text{ for all }f\in L^1(\mu).$$

The Riesz representation Theorem (Proposition 10.15) then implies there exists a unique $\rho\in L^2(\mu)$ such that

$$\nu(f)=\int_X f\rho d\mu\text{ for all }f\in L^2(\mu).$$

Taking $f=\overline{\operatorname{sgn}(\rho)}1_A$ in this equation shows

$$\int_A|\rho|\,d\mu=\nu(\overline{\operatorname{sgn}(\rho)}1_A)\le\mu(A)=\int_A 1d\mu$$

from which it follows that $|\rho|\le 1$, μ – a.e. If ν is a positive measure, then

$$0=\operatorname{Im}\left[\nu(\operatorname{Im}\rho>0)\right]=\int_{\{\operatorname{Im}\rho>0\}}\operatorname{Im}\rho d\mu$$

which shows $\operatorname{Im}\rho\le 0$, μ – a.e. Similarly,

$$0=\operatorname{Im}\left[\nu(\operatorname{Im}\rho<0)\right]=\int_{\{\operatorname{Im}\rho<0\}}\operatorname{Im}\rho d\mu$$

and hence $\operatorname{Im}\rho\ge 0$, μ – a.e. and we have shown ρ is real a.e. Similarly,

$$0\le\nu(\operatorname{Re}\rho<0)=\int_{\{\operatorname{Re}\rho<0\}}\rho d\mu\le 0,$$

shows $\rho\ge 0$ a.e. ■

Definition 10.49. Let μ and ν be two positive measures on $(X,\mathcal{M})$. Then μ and ν are **mutually singular** (written as $\mu\perp\nu$) if there exists $A\in\mathcal{M}$ such that $\nu(A)=0$ and $\mu(A^c)=0$. The measure ν **is absolutely continuous relative to** μ (written as $\nu\ll\mu$) provided $\nu(A)=0$ whenever $\mu(A)=0$.

Theorem 10.50 (Radon Nikodym Theorem)**.** *Suppose that μ,ν are σ – finite positive measures on $(X,\mathcal{M})$. Then there exists a unique measure ν_s and a unique (modulo sets of μ – measure 0) function $\rho:X\to[0,\infty)$ such that $d\nu=d\nu_s+\rho d\mu$ and $\nu_s\perp\mu$. The measure ν_s in the **Lebesgue decomposition** of ν is unique and ρ is unique modulo sets of μ – measure zero. Moreover, $\nu\ll\mu$ iff $\nu_s=0$.*

Proof. Uniqueness. Suppose that $d\nu = \tilde{\rho} d\mu + d\tilde{\nu}_s$ with $\tilde{\rho} \geq 0$ and $\tilde{\nu}_s \perp \mu$ is another such decomposition. Let $A, \tilde{A} \in \mathcal{M}$ be chosen so that $\mu(A) = 0$, $\nu_s(A^c) = 0$, $\mu(\tilde{A}) = 0$ and $\tilde{\nu}_s(\tilde{A}^c) = 0$. Then for $B \in \mathcal{M}$, using $\mu(A) = 0$ and $\nu_s(A^c) = 0$,

$$\nu(A \cap B) = \nu_s(A \cap B) + \mu(\rho 1_{A \cap B}) = \nu_s(B).$$

Now using $\tilde{\nu}_s(\tilde{A}^c) = 0$ and $\mu(A) = 0$,

$$\begin{aligned} \nu(A \cap B) &= \nu(A \cap \tilde{A} \cap B) + \nu(A \cap \tilde{A}^c \cap B) \\ &= \nu(A \cap \tilde{A} \cap B) + \tilde{\nu}_s(A \cap \tilde{A}^c \cap B) + \mu(\tilde{\rho} 1_{A \cap \tilde{A}^c \cap B}) \\ &= \nu(A \cap \tilde{A} \cap B). \end{aligned}$$

Combining these equations shows

$$\nu_s(B) = \nu(A \cap B) = \nu(A \cap \tilde{A} \cap B).$$

By symmetry (or a similar argument) $\tilde{\nu}_s(B) = \nu(A \cap \tilde{A} \cap B)$ and therefore $\nu_s = \tilde{\nu}_s$. This then implies that $\tilde{\rho} d\mu = \rho d\mu$, i.e. $\mu(1_B \rho) = \mu(1_B \tilde{\rho})$ for all $B \in \mathcal{M}$. Let $X_n \uparrow X$ be chosen in $\mathcal{M}$ so that $\mu(X_n)$ and $\nu(X_n) < \infty$. Since $\nu(X_n) < \infty$, $\rho 1_{X_n} \in L^1(\mu)$ and $\tilde{\rho} 1_{X_n} \in L^1(\mu)$ and

$$\mu(1_B \cdot 1_{X_n} \rho) = \mu(1_B \cdot 1_{X_n} \tilde{\rho}) \text{ for all } B \in \mathcal{M}$$

which implies $1_{X_n} \rho = 1_{X_n} \tilde{\rho}$ for μ– a.e. x. Letting $n \to \infty$ then shows that $\rho = \tilde{\rho}$, μ – a.e.

Existence: (Due to Von-Neumann.) First suppose that μ and ν are finite measures and let $\lambda = \mu + \nu$. By Theorem 10.48, $d\nu = h d\lambda$ with $0 \leq h \leq 1$ and this implies, for all non-negative measurable functions f, that

$$\nu(f) = \lambda(fh) = \mu(fh) + \nu(fh) \tag{10.36}$$

or equivalently

$$\nu(f(1-h)) = \mu(fh). \tag{10.37}$$

Taking $f = 1_{\{h=1\}}$ and $f = g 1_{\{h<1\}} (1-h)^{-1}$ with $g \geq 0$ in Eq. (10.37)

$$\mu(\{h = 1\}) = 0 \text{ and } \nu(g 1_{\{h<1\}}) = \mu(g 1_{\{h<1\}} (1-h)^{-1} h) = \mu(\rho g)$$

where $\rho := 1_{\{h<1\}} \frac{h}{1-h}$ and $\nu_s(g) := \nu(g 1_{\{h=1\}})$. This gives the desired decomposition[26] since

$$\nu(g) = \nu(g 1_{\{h=1\}}) + \nu(g 1_{\{h<1\}}) = \nu_s(g) + \mu(\rho g)$$

and

$$\nu_s(h \neq 1) = 0 \text{ while } \mu(h = 1) = \mu(\{h \neq 1\}^c) = 0.$$

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[26]Here is the motivation for this construction. Suppose that $d\nu = d\nu_s + \rho d\mu$ is the Radon-Nikodym decompostion and $X = A \coprod B$ such that $\nu_s(B) = 0$ and $\mu(A) = 0$. Then we find

$$\nu_s(f) + \mu(\rho f) = \nu(f) = \lambda(fg) = \nu(fg) + \mu(fg).$$

Letting $f \to 1_A f$ then implies that

$$\nu_s(1_A f) = \nu(1_A f g)$$

which show that $g = 1$ ν –a.e. on A. Also letting $f \to 1_B f$ implies that

$$\mu(\rho 1_B f(1-g)) = \nu(1_B f(1-g)) = \mu(1_B f g) = \mu(fg)$$

which shows that

$$\rho(1-g) = \rho 1_B (1-g) = g \ \mu - \text{a.e.}.$$

This shows that $\rho = \frac{g}{1-g}$ μ – a.e.

If $\nu \ll \mu$, then $\mu(h=1)=0$ implies $\nu(h=1)=0$ and hence that $\nu_s=0$. If $\nu_s=0$, then $d\nu=\rho d\mu$ and so if $\mu(A)=0$, then $\nu(A)=\mu(\rho 1_A)=0$ as well.

For the σ − finite case, write $X=\coprod_{n=1}^{\infty} X_n$ where $X_n\in\mathcal{M}$ are chosen so that $\mu(X_n)<\infty$ and $\nu(X_n)<\infty$ for all n. Let $d\mu_n=1_{X_n}d\mu$ and $d\nu_n=1_{X_n}d\nu$. Then by what we have just proved there exists $\rho_n\in L^1(X,\mu_n)$ and measure ν_n^s such that $d\nu_n=\rho_n d\mu_n+d\nu_n^s$ with $\nu_n^s\perp\mu_n$, i.e. there exists $A_n,B_n\in\mathcal{M}_{X_n}$ and $\mu(A_n)=0$ and $\nu_n^s(B_n)=0$. Define $\nu_s:=\sum_{n=1}^{\infty}\nu_n^s$ and $\rho:=\sum_{n=1}^{\infty}1_{X_n}\rho_n$, then

$$\nu=\sum_{n=1}^{\infty}\nu_n=\sum_{n=1}^{\infty}\left(\rho_n\mu_n+\nu_n^s\right)=\sum_{n=1}^{\infty}\left(\rho_n 1_{X_n}\mu+\nu_n^s\right)=\rho\mu+\nu_s$$

and letting $A:=\cup_{n=1}^{\infty}A_n$ and $B:=\cup_{n=1}^{\infty}B_n$, we have $A=B^c$ and

$$\mu(A)=\sum_{n=1}^{\infty}\mu(A_n)=0 \text{ and } \nu(B)=\sum_{n=1}^{\infty}\nu(B_n)=0.$$

■

Theorem 10.51. *Let $(X,\mathcal{M},\mu)$ be a σ – finite measure space and suppose that $p,q\in[1,\infty]$ are conjugate exponents. Then for $p\in[1,\infty)$, the map $g\in L^q\to\phi_g\in(L^p)^*$ is an isometric isomorphism of Banach spaces. (Recall that $\phi_g(f):=\int_X fgd\mu$.) We summarize this by writing $(L^p)^*=L^q$ for all $1\le p<\infty$.*

Proof. The only point that we have not yet proved is the surjectivity of the map $g\in L^q\to\phi_g\in(L^p)^*$. When $p=2$ the result follows directly from the Riesz theorem. We will begin the proof under the extra assumption that $\mu(X)<\infty$ in which cased bounded functions are in $L^p(\mu)$ for all p.

Let $\phi\in(L^p)^*$ and define $\nu(A):=\phi(1_A)$. Suppose that $A=\coprod_{n=1}^{\infty}A_n$ with $A_n\in\mathcal{M}$, then

$$\|1_A-\sum_{n=1}^{N}1_{A_n}\|_{L^p}=\|1_{\cup_{n=N+1}^{\infty}A_n}\|_{L^p}=\left[\mu(\cup_{n=N+1}^{\infty}A_n)\right]^{\frac{1}{p}}\to 0 \text{ as } N\to\infty.$$

Therefore

$$\nu(A)=\phi(1_A)=\sum_1^{\infty}\phi(1_{A_n})=\sum_1^{\infty}\nu(A_n)$$

showing ν is a complex measure.[27]

Let us define

(10.38)
$$|\nu|(A):=\sup\{|\phi(f1_A)|:|f|\le 1\}\le\|\phi\|_{(L^p)^*}\cdot\|1_A\|_{L^p}=\|\phi\|_{(L^p)^*}\cdot\mu(A)^{1/p}.$$

You are asked to show in Exercise 10.31 that $|\nu|$ is a measure on $(X,\mathcal{M})$. (This also can be deduced from Lemma 15.4 and Proposition 15.6 below.) From Eq. (10.38),

$$|\nu(A)|\le|\nu|(A)\le\|\phi\|_{(L^p)^*}\,\mu(A)^{1/p} \text{ for all } A\in\mathcal{M}$$

from which if follows that $|\nu|\ll\mu$ and by Theorem 10.48, $d\nu=hd|\nu|$ for some $|h|\le 1$ and by Theorem 10.50, $d|\nu|=\rho d\mu$ for some $\rho\in L^1(\mu)$. Hence, letting $g=\rho h\in L^1(\mu)$, $d\nu=gd\mu$ or equivalently

$$\phi(1_A)=\int_X g1_A d\mu\ \forall\ A\in\mathcal{M}. \tag{10.39}$$

[27]It is at this point that the proof breaks down when $p=\infty$.

By linearity this equation implies

$$\phi(f)=\int_X gfd\mu \tag{10.40}$$

for all simple functions f on X. Replacing f by $1_{\{|g|\leq M\}}f$ in Eq. (10.40) shows

$$\phi(f1_{\{|g|\leq M\}})=\int_X 1_{\{|g|\leq M\}}gfd\mu$$

holds for all simple functions f and then by continuity for all $f\in L^p(\mu)$. By the converse to Holder's inequality, (Proposition 7.26) we learn that

$$\left\|1_{\{|g|\leq M\}}g\right\|_q=\sup_{\|f\|_p=1}\left|\phi(f1_{\{|g|\leq M\}})\right|\leq\sup_{\|f\|_p=1}\|\phi\|_{(L^p)^*}\left\|f1_{\{|g|\leq M\}}\right\|_p\leq\|\phi\|_{(L^p)^*}.$$

Using the monotone convergence theorem we may let $M\to\infty$ in the previous equation to learn $\|g\|_q\leq\|\phi\|_{(L^p)^*}$.With this result, Eq. (10.40) extends by continuity to hold for all $f\in L^p(\mu)$ and hence we have shown that $\phi=\phi_g$.

Case 2. Now suppose that μ is σ – finite and $X_n\in\mathcal{M}$ are sets such that $0<\mu(X_n)<\infty$ and $X_n\uparrow X$ as $n\to\infty$. Then by Case 1. there exits $g_n\in L^q(X_n,\mu)$ such that

$$\phi(f)=\int_{X_n}g_nfd\mu\text{ for all }f\in L^p(X_n,\mu)$$

and

$$\|g_n\|_q=\sup\left\{|\phi(f)|:f\in L^p(X_n,\mu)\text{ and }\|f\|_{L^p(X_n,\mu)}=1\right\}\leq\|\phi\|_{[L^p(\mu)]^*}.$$

It is easy to see that $g_n=g_m$ a.e. on $X_n\cap X_m$ for all m,n so that $g:=\lim_{n\to\infty}g_n$ exists μ – a.e. By the above inequality and Fatou's lemma, we have $\|g\|_q\leq\|\phi\|_{[L^p(\mu)]^*}<\infty$ and since

$$\phi(f)=\int_{X_n}gfd\mu\text{ for all }f\in L^p(X_n,\mu)\text{ and }n,$$

it follows by continuity that

$$\phi(f)=\int_X gfd\mu\text{ for all }f\in L^p(X,\mu),$$

i.e. $\phi=\phi_g$.

■

Remark 10.52. We will show later that Theorem 10.51 fails in general when $p=\infty$.

10.9. **Exercises.**

Exercise 10.31. Show $|\nu|$ be defined as in Eq. (10.38) is a positive measure. Here is an outline.

(1) Show

$$|\nu|(A)+|\nu|(B)\leq|\nu|(A\cup B). \tag{10.41}$$

when A,B are disjoint sets in $\mathcal{M}$.

(2) If $A=\coprod_{n=1}^\infty A_n$ with $A_n\in\mathcal{M}$ then

$$|\nu|(A)\leq\sum_{n=1}^\infty|\nu|(A_n). \tag{10.42}$$

(3) From Eqs. (10.41) and (10.42) it follows that ν is finitely additive, and hence
$$|\nu|(A) = \sum_{n=1}^{N} |\nu|(A_n) + |\nu|(\cup_{n>N} A_n) \geq \sum_{n=1}^{N} |\nu|(A_n).$$
Letting $N \to \infty$ in this inequality shows $|\nu|(A) \geq \sum_{n=1}^{\infty} |\nu|(A_n)$ which combined with Eq. (10.42) shows $|\nu|$ is countable additive.

Exercise 10.32. Suppose μ_i, ν_i are σ – finite measure on measurable spaces, $(X_i, \mathcal{M}_i)$, for $i = 1, 2$. If $\nu_i \ll \mu_i$ for $i = 1, 2$ then $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$ and in fact $\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)} = \rho_1 \otimes \rho_2$ when $\rho_i := d\nu_i / d\mu_i$ for $i = 1, 2$.

Exercise 10.33. Problem 3.13 from Folland.