2. Since $C_0(X)$ is a closed subspace of $BC(X)$ and $C_c(X) \subset C_0(X)$, we always have $\overline{C_c(X)} \subset C_0(X)$. Now suppose that $f \in C_0(X)$ and let $K_n \equiv \{|f| \geq \frac{1}{n}\} \sqsubset\sqsubset X$. By Lemma 8.15 we may choose $\phi_n \in C_c(X, [0,1])$ such that $\phi_n \equiv 1$ on K_n. Define $f_n \equiv \phi_n f \in C_c(X)$. Then

$$\|f - f_n\|_u = \|(1-\phi_n) f\|_u \leq \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

This shows that $f \in \overline{C_c(X)}$.

■

Proposition 8.29 (Alexanderov Compactification)**.** *Suppose that (X, τ) is a non-compact locally compact Hausdorff space. Let $X^* = X \cup \{\infty\}$, where $\{\infty\}$ is a new symbol not in X. The collection of sets,*

$$\tau^* = \tau \cup \{X^* \setminus K : K \sqsubset\sqsubset X\} \subset \mathcal{P}(X^*),$$

is a topology on X^* and (X^*, τ^*) is a compact Hausdorff space. Moreover $f \in C(X)$ extends continuously to X^* iff $f = g + c$ with $g \in C_0(X)$ and $c \in \mathbb{C}$ in which case the extension is given by $f(\infty) = c$.

Proof. 1. (τ^* is a topology.) Let $\mathcal{F} := \{F \subset X^* : X^* \setminus F \in \tau^*\}$, i.e. $F \in \mathcal{F}$ iff F is a compact subset of X or $F = F_0 \cup \{\infty\}$ with F_0 being a closed subset of X. Since the finite union of compact (closed) subsets is compact (closed), it is easily seen that $\mathcal{F}$ is closed under finite unions. Because arbitrary intersections of closed subsets of X are closed and closed subsets of compact subsets of X are compact, it is also easily checked that $\mathcal{F}$ is closed under arbitrary intersections. Therefore $\mathcal{F}$ satisfies the axioms of the closed subsets associated to a topology and hence τ^* is a topology.

2. ((X^*, τ^*) is Hausdorff space.) It suffices to show any point $x \in X$ can be separated from ∞. To do this use Proposition 8.13 twice to find open precompact subset, U and V, such that

$$x \in U \subset \bar{U} \subset V \subset \bar{V} \subset X.$$

Then U and $X^* \setminus \bar{V}$ are disjoint open subsets of X^* such that $x \in U$ and $\infty \in X^* \setminus \bar{V}$.

3. ((X^*, τ^*) is compact.) Suppose that $\mathcal{U} \subset \tau^*$ is an open cover of X^*. Since $\mathcal{U}$ covers ∞, there exists a compact set $K \subset X$ such that $X^* \setminus K \in \mathcal{U}$. Clearly X is covered by $\mathcal{U}_0 := \{V \setminus \{\infty\} : V \in \mathcal{U}\}$ and by the definition of τ^* (or using (X^*, τ^*) is Hausdorff), $\mathcal{U}_0$ is an open cover of X. In particular $\mathcal{U}_0$ is an open cover of K and since K is compact there exists $\Lambda \subset\subset \mathcal{U}$ such that $K \subset \cup \{V \setminus \{\infty\} : V \in \Lambda\}$. It is now easily checked that $\Lambda \cup \{X^* \setminus K\} \subset \mathcal{U}$ is a finite subcover of X^*.

4. (Continuous functions on $C(X^*)$ statements.) Let $i : X \to X^*$ be the inclusion map. Then i is continuous and open, i.e. $i(V)$ is open in X^* for all V open in X. If $f \in C(X^*)$, then $g = f|_X - f(\infty) = f \circ i - f(\infty)$ is continuous on X. Moreover, for all $\epsilon > 0$ there exists an open neighborhood $V \in \tau^*$ of ∞ such that

$$|g(x)| = |f(x) - f(\infty)| < \epsilon \text{ for all } x \in V.$$

Since V is an open neighborhood of ∞, there exists a compact subset, $K \subset X$, such that $V = X^* \setminus K$. By the previous equation we see that $\{x \in X : |g(x)| \geq \epsilon\} \subset K$, so $\{|g| \geq \epsilon\}$ is compact and we have shown g vanishes at ∞.

Conversely if $g \in C_0(X)$, extend g to X^* by setting $g(\infty) = 0$. Given $\epsilon > 0$, the set $K = \{|g| \geq \epsilon\}$ is compact, hence $X^* \setminus K$ is open in X^*. Since $g(X^* \setminus K) \subset (-\epsilon, \epsilon)$ we have shown that g is continuous at ∞. Since g is also continuous at all points in X it follows that g is continuous on X^*. Now it $f = g + c$ with $c \in \mathbb{C}$ and $g \in C_0(X)$, it follows by what we just proved that defining $f(\infty) = c$ extends f to a continuous function on X^*. ■

8.3. **More on Separation Axioms: Normal Spaces.** (The reader may skip to Definition 8.32 if he/she wishes. The following material will not be used in the rest of the book.)

Definition 8.30 ($T_0 - T_2$ Separation Axioms)**.** Let (X, τ) be a topological space. The topology τ is said to be:

1. T_0 if for $x \neq y$ in X there exists $V \in \tau$ such that $x \in V$ and $y \notin V$ or V such that $y \in V$ but $x \notin V$.
2. T_1 if for every $x, y \in X$ with $x \neq y$ there exists $V \in \tau$ such that $x \in V$ and $y \notin V$. Equivalently, τ is T_1 iff all one point subsets of X are closed.[18]
3. T_2 if it is Hausdorff.

Note T_2 implies T_1 which implies T_0. The topology in Example 8.3 is T_0 but not T_1. If X is a finite set and τ is a T_1 – topology on X then $\tau = 2^X$. To prove this let $x \in X$ be fixed. Then for every $y \neq x$ in X there exists $V_y \in \tau$ such that $x \in V_y$ while $y \notin V_y$. Thus $\{x\} = \cap_{y \neq x} V_y \in \tau$ showing τ contains all one point subsets of X and therefore all subsets of X. So we have to look to infinite sets for an example of T_1 topology which is not T_2.

Example 8.31. Let X be any infinite set and let $\tau = \{A \subset X : \#(A^c) < \infty\} \cup \{\emptyset\}$ – the so called **cofinite** topology. This topology is T_1 because if $x \neq y$ in X, then $V = \{x\}^c \in \tau$ with $x \notin V$ while $y \in V$. This topology however is not T_2. Indeed if $U, V \in \tau$ are open sets such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ then $U \subset V^c$. But this implies $\#(U) < \infty$ which is impossible unless $U = \emptyset$ which is impossible since $x \in U$.

The uniqueness of limits of sequences which occurs for Hausdorff topologies (see Remark 8.5) need not occur for T_1 – spaces. For example, let $X = \mathbb{N}$ and τ be the cofinite topology on X as in Example 8.31. Then $x_n = n$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ **for all** $x \in \mathbb{N}$. For the most part we will avoid these pathologies in the future by only considering Hausdorff topologies.

Definition 8.32 (Normal Spaces: T_4 – Separation Axiom)**.** A topological space (X, τ) is said to be **normal** or T_4 if:

1. X is Hausdorff and
2. if for any two closed disjoint subsets $A, B \subset X$ there exists disjoint open sets $V, W \subset X$ such that $A \subset V$ and $B \subset W$.

Example 8.33. By Lemma 8.1 and Corollary 8.26 it follows that metric space and locally compact and σ – compact Hausdorff space (in particular compact Hausdorff spaces) are normal. Indeed, in each case if A, B are disjoint closed subsets of X, there exists $f \in C(X, [0, 1])$ such that $f = 1$ on A and $f = 0$ on B. Now let $U = \left\{f > \frac{1}{2}\right\}$ and $V = \{f < \frac{1}{2}\}$.

[18]If one point subsets are closed and $x \neq y$ in X then $V := \{x\}^c$ is an open set containing y but not x. Conversely if τ is T_1 and $x \in X$ there exists $V_y \in \tau$ such that $y \in V_y$ and $x \notin V_y$ for all $y \neq x$. Therefore, $\{x\}^c = \cup_{y \neq x} V_y \in \tau$.