1. Introduction

Not written as of yet. Topics to mention.

1. A better and more general integral.
   (a) Convergence Theorems
   (b) Integration over diverse collection of sets. (See probability theory.)
   (c) Integration relative to different weights or densities including singular weights.
   (d) Characterization of dual spaces.
   (e) Completeness.
2. Infinite dimensional Linear algebra.
3. ODE and PDE.
4. Harmonic and Fourier Analysis.
5. Probability Theory

2. Limits, sums, and other basics

2.1. **Set Operations.** Suppose that X is a set. Let $\mathcal{P}(X)$ or 2^X denote the power set of X, that is elements of $\mathcal{P}(X) = 2^X$ are subsets of A. For $A \in 2^X$ let

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

and more generally if $A, B \subset X$ let

$$B \setminus A = \{x \in B : x \notin A\}.$$

We also define the symmetric difference of A and B by

$$A \triangle B = (B \setminus A) \cup (A \setminus B).$$

As usual if $\{A_\alpha\}_{\alpha \in I}$ is an indexed collection of subsets of X we define the union and the intersection of this collection by

$$\cup_{\alpha \in I} A_\alpha := \{x \in X : \exists\, \alpha \in I \ \ni\ x \in A_\alpha\} \text{ and}$$
$$\cap_{\alpha \in I} A_\alpha := \{x \in X : x \in A_\alpha \,\forall\, \alpha \in I\}.$$

Notation 2.1. We will also write $\coprod_{\alpha \in I} A_\alpha$ for $\cup_{\alpha \in I} A_\alpha$ in the case that $\{A_\alpha\}_{\alpha \in I}$ are pairwise disjoint, i.e. $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$.

Notice that $\cup$ is closely related to $\exists$ and $\cap$ is closely related to $\forall$. For example let $\{A_n\}_{n=1}^\infty$ be a sequence of subsets from X and define

$$\{A_n \text{ i.o.}\} := \{x \in X : \#\{n : x \in A_n\} = \infty\} \text{ and}$$
$$\{A_n \text{ a.a.}\} := \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}.$$

(One should read $\{A_n \text{ i.o.}\}$ as A_n infinitely often and $\{A_n \text{ a.a.}\}$ as A_n almost always.) Then $x \in \{A_n \text{ i.o.}\}$ iff $\forall N \in \mathbb{N}\ \exists n \geq N \ni x \in A_n$ which may be written as

$$\{A_n \text{ i.o.}\} = \cap_{N=1}^\infty \cup_{n \geq N} A_n.$$

Similarly, $x \in \{A_n \text{ a.a.}\}$ iff $\exists\, N \in \mathbb{N} \ni \forall\, n \geq N,\ x \in A_n$ which may be written as

$$\{A_n \text{ a.a.}\} = \cup_{N=1}^\infty \cap_{n \geq N} A_n.$$

2.2. Limits, Limsups, and Liminfs.

Notation 2.2. The Extended real numbers is the set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, i.e. it is $\mathbb{R}$ with two new points called ∞ and $-\infty$. We use the following conventions, $\pm\infty \cdot 0 = 0$, $\pm\infty + a = \pm\infty$ for any $a \in \mathbb{R}$, $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$ while $\infty - \infty$ is not defined.

If $\Lambda \subset \bar{\mathbb{R}}$ we will let $\sup \Lambda$ and $\inf \Lambda$ denote the least upper bound and greatest lower bound of Λ respectively. We will also use the following convention, if $\Lambda = \emptyset$, then $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Notation 2.3. Suppose that $\{x_n\}_{n=1}^\infty \subset \bar{\mathbb{R}}$ is a sequence of numbers. Then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} \text{ and} \tag{2.1}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}. \tag{2.2}$$

We will also write $\underline{\lim}$ for $\liminf$ and $\overline{\lim}$ for $\limsup$.

Remark 2.4. Notice that if $a_k := \inf\{x_k : k \geq n\}$ and $b_k := \sup\{x_k : k \geq n\}$,then $\{a_k\}$ is an increasing sequence while $\{b_k\}$ is a decreasing sequence. Therefore the limits in Eq. (2.1) and Eq. (2.2) always exist and

$$\liminf_{n\to\infty} x_n = \sup_n \inf\{x_k : k \geq n\} \text{ and}$$
$$\limsup_{n\to\infty} x_n = \inf_n \sup\{x_k : k \geq n\}.$$

The following proposition contains some basic properties of liminfs and limsups.

Proposition 2.5. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two sequences of real numbers. Then*

1. $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} a_n$ *exists in* $\bar{\mathbb{R}}$ *iff* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \in \bar{\mathbb{R}}$.
2. *There is a subsequence* $\{a_{n_k}\}_{k=1}^\infty$ *of* $\{a_n\}_{n=1}^\infty$ *such that* $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.
3.
$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{2.3}$$
whenever the right side of this equation is not of the form $\infty - \infty$.
4. *If* $a_n \geq 0$ *and* $b_n \geq 0$ *for all* $n \in \mathbb{N}$, *then*
$$\limsup_{n\to\infty} (a_n b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n, \tag{2.4}$$
provided the right hand side of (2.4) is not of the form $0 \cdot \infty$ *or* $\infty \cdot 0$.

Proof. We will only prove part 1. and leave the rest as an exercise to the reader. We begin by noticing that

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \; \forall n$$

so that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Now suppose that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R}$. Then for all $\epsilon > 0$, there is an integer N such that

$$a - \epsilon \leq \inf\{a_k : k \geq N\} \leq \sup\{a_k : k \geq N\} \leq a + \epsilon,$$

i.e.

$$a - \epsilon \leq a_k \leq a + \epsilon \text{ for all } k \geq N.$$

Hence by the definition of the limit, $\lim_{k\to\infty} a_k = a$.

If $\liminf_{n\to\infty} a_n = \infty$, then we know for all $M \in (0, \infty)$ there is an integer N such that

$$M \leq \inf\{a_k : k \geq N\}$$

and hence $\lim_{n\to\infty} a_n = \infty$. The case where $\limsup_{n\to\infty} a_n = -\infty$ is handled similarly.

Conversely, suppose that $\lim_{n\to\infty} a_n = A \in \bar{\mathbb{R}}$ exists. If $A \in \mathbb{R}$, then for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $|A - a_n| \leq \epsilon$ for all $n \geq N(\epsilon)$, i.e.

$$A - \epsilon \leq a_n \leq A + \epsilon \text{ for all } n \geq N(\epsilon).$$

From this we learn that

$$A - \epsilon \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq A + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$A \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq A,$$

i.e. that $A = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

If $A = \infty$, then for all $M > 0$ there exist $N(M)$ such that $a_n \geq M$ for all $n \geq N(M)$. This show that

$$\liminf_{n\to\infty} a_n \geq M$$

and since M is arbitrary it follows that

$$\infty \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

The proof is similar if $A = -\infty$ as well. ■

2.3. **Sums of positive functions.** In this and the next few sections, let X and Y be two sets. We will write $\alpha \subset\subset X$ to denote that α is a **finite** subset of X.

Definition 2.6. Suppose that $a : X \to [0, \infty]$ is a function and $F \subset X$ is a subset, then

$$\sum_F a = \sum_{x\in F} a(x) = \sup\left\{\sum_{x\in\alpha} a(x) : \alpha \subset\subset F\right\}.$$

Remark 2.7. Suppose that $X = \mathbb{N} = \{1, 2, 3, \dots\}$, then

$$\sum_{\mathbb{N}} a = \sum_{n=1}^{\infty} a(n) := \lim_{N\to\infty} \sum_{n=1}^{N} a(n).$$

Indeed for all N, $\sum_{n=1}^N a(n) \leq \sum_{\mathbb{N}} a$, and thus passing to the limit we learn that

$$\sum_{n=1}^{\infty} a(n) \leq \sum_{\mathbb{N}} a.$$

Conversely, if $\alpha \subset\subset \mathbb{N}$, then for all N large enough so that $\alpha \subset \{1, 2, \ldots, N\}$, we have $\sum_\alpha a \leq \sum_{n=1}^N a(n)$ which upon passing to the limit implies that

$$\sum_\alpha a \leq \sum_{n=1}^\infty a(n)$$

and hence by taking the supremum over α we learn that

$$\sum_{\mathbb{N}} a \leq \sum_{n=1}^\infty a(n).$$

Remark 2.8. Suppose that $\sum_X a < \infty$, then $\{x \in X : a(x) > 0\}$ is at most countable. To see this first notice that for any $\epsilon > 0$, the set $\{x : a(x) \geq \epsilon\}$ must be finite for otherwise $\sum_X a = \infty$. Thus

$$\{x \in X : a(x) > 0\} = \bigcup\nolimits_{k=1}^\infty \{x : a(x) \geq 1/k\}$$

which shows that $\{x \in X : a(x) > 0\}$ is a countable union of finite sets and thus countable.

Lemma 2.9. *Suppose that $a, b : X \to [0, \infty]$ are two functions, then*

$$\sum_X (a + b) = \sum_X a + \sum_X b \text{ and}$$
$$\sum_X \lambda a = \lambda \sum_X a$$

for all $\lambda \geq 0$.

I will only prove the first assertion, the second being easy. Let $\alpha \subset\subset X$ be a finite set, then

$$\sum_\alpha (a + b) = \sum_\alpha a + \sum_\alpha b \leq \sum_X a + \sum_X b$$

which after taking sups over α shows that

$$\sum_X (a + b) \leq \sum_X a + \sum_X b.$$

Similarly, if $\alpha, \beta \subset\subset X$, then

$$\sum_\alpha a + \sum_\beta b \leq \sum_{\alpha\cup\beta} a + \sum_{\alpha\cup\beta} b = \sum_{\alpha\cup\beta} (a + b) \leq \sum_X (a + b).$$

Taking sups over α and β then shows that

$$\sum_X a + \sum_X b \leq \sum_X (a + b).$$

Lemma 2.10. *Let X and Y be sets, $R \subset X \times Y$ and suppose that $a : R \to \bar{\mathbb{R}}$ is a function. Let ${}_xR := \{y \in Y : (x, y) \in R\}$ and $R_y := \{x \in X : (x, y) \in R\}$. Then*

$$\sup_{(x,y)\in R} a(x, y) = \sup_{x\in X} \sup_{y\in {}_xR} a(x, y) = \sup_{y\in Y} \sup_{x\in R_y} a(x, y) \text{ and}$$
$$\inf_{(x,y)\in R} a(x, y) = \inf_{x\in X} \inf_{y\in {}_xR} a(x, y) = \inf_{y\in Y} \inf_{x\in R_y} a(x, y).$$

(Recall the conventions: $\sup \emptyset = -\infty$ *and* $\inf \emptyset = +\infty$.*)*

Proof. Let $M = \sup_{(x,y)\in R} a(x,y)$, $N_x := \sup_{y\in_x R} a(x,y)$. Then $a(x,y) \le M$ for all $(x,y)\in R$ implies $N_x = \sup_{y\in_x R} a(x,y) \le M$ and therefore that

$$\sup_{x\in X}\sup_{y\in_x R} a(x,y) = \sup_{x\in X} N_x \le M. \tag{2.5}$$

Similarly for any $(x,y)\in R$,

$$a(x,y) \le N_x \le \sup_{x\in X} N_x = \sup_{x\in X}\sup_{y\in_x R} a(x,y)$$

and therefore

$$\sup_{(x,y)\in R} a(x,y) \le \sup_{x\in X}\sup_{y\in_x R} a(x,y) = M \tag{2.6}$$

Equations (2.5) and (2.6) show that

$$\sup_{(x,y)\in R} a(x,y) = \sup_{x\in X}\sup_{y\in_x R} a(x,y).$$

The assertions involving infinums are proved analogously or follow from what we have just proved applied to the function $-a$. ■

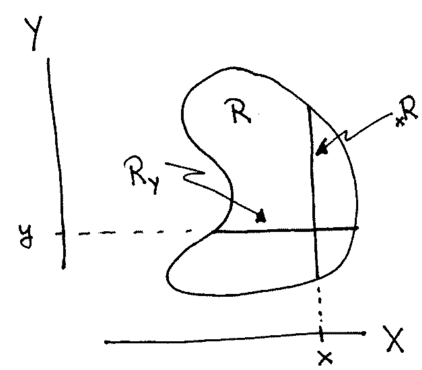

FIGURE 1. The x and y – slices of a set $R \subset X \times Y$.

Theorem 2.11 (Monotone Convergence Theorem for Sums)**.** *Suppose that* $f_n : X \to [0,\infty]$ *is an increasing sequence of functions and*

$$f(x) := \lim_{n\to\infty} f_n(x) = \sup_n f_n(x).$$

Then

$$\lim_{n\to\infty}\sum_X f_n = \sum_X f$$

Proof. We will give two proves. For the first proof, let $\mathcal{P}_f(X) = \{A \subset X : A \subset\subset X\}$. Then

$$\lim_{n\to\infty}\sum_X f_n = \sup_n \sum_X f_n = \sup_n \sup_{\alpha\in\mathcal{P}_f(X)} \sum_\alpha f_n = \sup_{\alpha\in\mathcal{P}_f(X)} \sup_n \sum_\alpha f_n$$
$$= \sup_{\alpha\in\mathcal{P}_f(X)} \lim_{n\to\infty}\sum_\alpha f_n = \sup_{\alpha\in\mathcal{P}_f(X)} \sum_\alpha \lim_{n\to\infty} f_n = \sup_{\alpha\in\mathcal{P}_f(X)} \sum_\alpha f = \sum_X f.$$

(Second Proof.) Let $S_n = \sum_X f_n$ and $S = \sum_X f$. Since $f_n \le f_m \le f$ for all $n \le m$, it follows that

$$S_n \le S_m \le S$$

which shows that $\lim_{n\to\infty} S_n$ exists and is less that S, i.e.

$$A := \lim_{n\to\infty} \sum_X f_n \le \sum_X f. \tag{2.7}$$

Noting that $\sum_\alpha f_n \le \sum_X f_n = S_n \le A$ for all $\alpha \subset\subset X$ and in particular,

$$\sum_\alpha f_n \le A \text{ for all } n \text{ and } \alpha \subset\subset X.$$

Letting n tend to infinity in this equation shows that

$$\sum_\alpha f \le A \text{ for all } \alpha \subset\subset X$$

and then taking the sup over all $\alpha \subset\subset X$ gives

$$\sum_X f \le A = \lim_{n\to\infty} \sum_X f_n \tag{2.8}$$

which combined with Eq. (2.7) proves the theorem. ■

Lemma 2.12 (Fatou's Lemma for Sums)**.** *Suppose that $f_n : X \to [0,\infty]$ is a sequence of functions, then*

$$\sum_X \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \sum_X f_n.$$

Proof. Define $g_k \equiv \inf_{n\ge k} f_n$ so that $g_k \uparrow \liminf_{n\to\infty} f_n$ as $k\to\infty$. Since $g_k \le f_n$ for all $k \le n$,

$$\sum_X g_k \le \sum_X f_n \text{ for all } n \ge k$$

and therefore

$$\sum_X g_k \le \liminf_{n\to\infty} \sum_X f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k \to \infty$ to find

$$\sum_X \liminf_{n\to\infty} f_n = \sum_X \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \sum_X g_k \le \liminf_{n\to\infty} \sum_X f_n.$$

■

Remark 2.13*.* If $A = \sum_X a < \infty$, then for all $\epsilon > 0$ there exists $\alpha_\epsilon \subset\subset X$ such that

$$A \ge \sum_\alpha a \ge A - \epsilon$$

for all $\alpha \subset\subset X$ containing α_ϵ or equivalently,

$$\left| A - \sum_\alpha a \right| \le \epsilon \tag{2.9}$$

for all $\alpha \subset\subset X$ containing α_ϵ. Indeed, choose α_ϵ so that $\sum_{\alpha_\epsilon} a \ge A - \epsilon$.

2.4. **Sums of complex functions.**

Definition 2.14. Suppose that $a : X \to \mathbb{C}$ is a function, we say that

$$\sum_X a = \sum_{x \in X} a(x)$$

exists and is equal to $A \in \mathbb{C}$, if for all $\epsilon > 0$ there is a finite subset $\alpha_\epsilon \subset X$ such that for all $\alpha \subset\subset X$ containing α_ϵ we have

$$\left| A - \sum_\alpha a \right| \leq \epsilon.$$

The following lemma is left as an exercise to the reader.

Lemma 2.15. *Suppose that $a, b : X \to \mathbb{C}$ are two functions such that $\sum_X a$ and $\sum_X b$ exist, then $\sum_X (a + \lambda b)$ exists for all $\lambda \in \mathbb{C}$ and*

$$\sum_X (a + \lambda b) = \sum_X a + \lambda \sum_X b.$$

Definition 2.16 (Summable)**.** We call a function $a : X \to \mathbb{C}$ **summable** if

$$\sum_X |a| < \infty.$$

Proposition 2.17. *Let $a : X \to \mathbb{C}$ be a function, then $\sum_X a$ exists iff $\sum_X |a| < \infty$, i.e. iff a is summable.*

Proof. If $\sum_X |a| < \infty$, then $\sum_X (\operatorname{Re} a)^\pm < \infty$ and $\sum_X (\operatorname{Im} a)^\pm < \infty$ and hence by Remark 2.13 these sums exists in the sense of Definition 2.14. Therefore by Lemma 2.15, $\sum_X a$ exists and

$$\sum_X a = \sum_X (\operatorname{Re} a)^+ - \sum_X (\operatorname{Re} a)^- + i \left(\sum_X (\operatorname{Im} a)^+ - \sum_X (\operatorname{Im} a)^- \right).$$

Conversely, if $\sum_X |a| = \infty$ then, because $|a| \leq |\operatorname{Re} a| + |\operatorname{Im} a|$, we must have

$$\sum_X |\operatorname{Re} a| = \infty \text{ or } \sum_X |\operatorname{Im} a| = \infty.$$

Thus it suffices to consider the case where $a : X \to \mathbb{R}$ is a real function. Write $a = a^+ - a^-$ where

$$a^+(x) = \max(a(x), 0) \text{ and } a^-(x) = \max(-a(x), 0). \tag{2.10}$$

Then $|a| = a^+ + a^-$ and

$$\infty = \sum_X |a| = \sum_X a^+ + \sum_X a^-$$

which shows that either $\sum_X a^+ = \infty$ or $\sum_X a^- = \infty$. Suppose, with out loss of generality, that $\sum_X a^+ = \infty$. Let $X' := \{x \in X : a(x) \geq 0\}$, then we know that $\sum_{X'} a = \infty$ which means there are finite subsets $\alpha_n \subset X' \subset X$ such that $\sum_{\alpha_n} a \geq n$ for all n. Thus if $\alpha \subset\subset X$ is any finite set, it follows that $\lim_{n\to\infty} \sum_{\alpha_n \cup \alpha} a = \infty$, and therefore $\sum_X a$ can not exist as a number in $\mathbb{R}$. ■

Remark 2.18. Suppose that $X = \mathbb{N}$ and $a : \mathbb{N} \to \mathbb{C}$ is a sequence, then it is not necessarily true that

$$\sum_{n=1}^{\infty} a(n) = \sum_{n\in\mathbb{N}} a(n). \tag{2.11}$$

This is because

$$\sum_{n=1}^{\infty} a(n) = \lim_{N\to\infty} \sum_{n=1}^{N} a(n)$$

depends on the ordering of the sequence a where as $\sum_{n\in\mathbb{N}} a(n)$ does not. For example, take $a(n) = (-1)^n/n$ then $\sum_{n\in\mathbb{N}} |a(n)| = \infty$ i.e. $\sum_{n\in\mathbb{N}} a(n)$ does **not** exist while $\sum_{n=1}^{\infty} a(n)$ does exist. On the other hand, if

$$\sum_{n\in\mathbb{N}} |a(n)| = \sum_{n=1}^{\infty} |a(n)| < \infty$$

then Eq. (2.11) is valid.

Theorem 2.19 (Dominated Convergence Theorem for Sums)**.** *Suppose that $f_n : X \to \mathbb{C}$ is a sequence of functions on X such that $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbb{C}$ exists for all $x \in X$. Further assume there is a **dominating function** $g : X \to [0,\infty)$ such that*

$$|f_n(x)| \le g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N} \tag{2.12}$$

and that g is summable. Then

$$\lim_{n\to\infty} \sum_{x\in X} f_n(x) = \sum_{x\in X} f(x). \tag{2.13}$$

Proof. Notice that $|f| = \lim |f_n| \le g$ so that f is summable. By considering the real and imaginary parts of f separately, it suffices to prove the theorem in the case where f is real. By Fatou's Lemma,

$$\begin{aligned}\sum_X (g \pm f) &= \sum_X \liminf_{n\to\infty} (g \pm f_n) \le \liminf_{n\to\infty} \sum_X (g \pm f_n) \\ &= \sum_X g + \liminf_{n\to\infty} \left(\pm \sum_X f_n \right).\end{aligned}$$

Since $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$, we have shown,

$$\sum_X g \pm \sum_X f \le \sum_X g + \begin{cases} \liminf_{n\to\infty} \sum_X f_n \\ -\limsup_{n\to\infty} \sum_X f_n \end{cases}$$

and therefore

$$\limsup_{n\to\infty} \sum_X f_n \le \sum_X f \le \liminf_{n\to\infty} \sum_X f_n.$$

This shows that $\lim\limits_{n\to\infty} \sum_X f_n$exists and is equal to $\sum_X f$. ■

Proof. (Second Proof.) Passing to the limit in Eq. (2.12) shows that $|f| \le g$ and in particular that f is summable. Given $\epsilon > 0$, let $\alpha \subset\subset X$ such that

$$\sum_{X\setminus\alpha} g \le \epsilon.$$

Then for $\beta \subset\subset X$ such that $\alpha \subset \beta$,

$$\begin{aligned}\left|\sum_\beta f - \sum_\beta f_n\right| &= \left|\sum_\beta (f - f_n)\right| \\ &\le \sum_\beta |f - f_n| = \sum_\alpha |f - f_n| + \sum_{\beta\setminus\alpha} |f - f_n| \\ &\le \sum_\alpha |f - f_n| + 2\sum_{\beta\setminus\alpha} g \\ &\le \sum_\alpha |f - f_n| + 2\epsilon.\end{aligned}$$

and hence that

$$\left|\sum_\beta f - \sum_\beta f_n\right| \le \sum_\alpha |f - f_n| + 2\epsilon.$$

Since this last equation is true for all such $\beta \subset\subset X$, we learn that

$$\left|\sum_X f - \sum_X f_n\right| \le \sum_\alpha |f - f_n| + 2\epsilon$$

which then implies that

$$\begin{aligned}\limsup_{n\to\infty} \left|\sum_X f - \sum_X f_n\right| &\le \limsup_{n\to\infty} \sum_\alpha |f - f_n| + 2\epsilon \\ &= 2\epsilon.\end{aligned}$$

Because $\epsilon > 0$ is arbitrary we conclude that

$$\limsup_{n\to\infty} \left|\sum_X f - \sum_X f_n\right| = 0.$$

which is the same as Eq. (2.13). ■

2.5. **Iterated sums.** Let X and Y be two sets. The proof of the following lemma is left to the reader.

Lemma 2.20. *Suppose that $a : X \to \mathbb{C}$ is function and $F \subset X$ is a subset such that $a(x) = 0$ for all $x \notin F$. Show that $\sum_F a$ exists iff $\sum_X a$ exists, and if the sums exist then*

$$\sum_X a = \sum_F a.$$

Theorem 2.21 (Tonelli's Theorem for Sums)**.** *Suppose that $a : X \times Y \to [0, \infty]$, then*

$$\sum_{X\times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

Proof. It suffices to show, by symmetry, that

$$\sum_{X\times Y} a = \sum_X \sum_Y a$$

Let $\Lambda \subset\subset X \times Y$. The for any $\alpha \subset\subset X$ and $\beta \subset\subset Y$ such that $\Lambda \subset \alpha \times \beta$, we have

$$\sum_{\Lambda} a \leq \sum_{\alpha \times \beta} a = \sum_{\alpha} \sum_{\beta} a \leq \sum_{\alpha} \sum_{Y} a \leq \sum_{X} \sum_{Y} a,$$

i.e. $\sum_{\Lambda} a \leq \sum_X \sum_Y a$. Taking the sup over Λ in this last equation shows

$$\sum_{X \times Y} a \leq \sum_X \sum_Y a.$$

We must now show the opposite inequality. If $\sum_{X \times Y} a = \infty$ we are done so we now assume that a is summable. By Remark 2.8, there is a countable set $\{(x'_n, y'_n)\}_{n=1}^{\infty} \subset X \times Y$ off of which a is identically 0.

Let $\{y_n\}_{n=1}^{\infty}$ be an enumeration of $\{y'_n\}_{n=1}^{\infty}$, then since $a(x,y) = 0$ if $y \notin \{y_n\}_{n=1}^{\infty}$, $\sum_{y \in Y} a(x,y) = \sum_{n=1}^{\infty} a(x, y_n)$ for all $x \in X$. Hence

$$\sum_{x \in X} \sum_{y \in Y} a(x,y) = \sum_{x \in X} \sum_{n=1}^{\infty} a(x, y_n) = \sum_{x \in X} \lim_{N \to \infty} \sum_{n=1}^{N} a(x, y_n)$$

$$= \lim_{N \to \infty} \sum_{x \in X} \sum_{n=1}^{N} a(x, y_n), \tag{2.14}$$

wherein the last inequality we have used the monotone convergence theorem with $F_N(x) := \sum_{n=1}^{N} a(x, y_n)$. If $\alpha \subset\subset X$, then

$$\sum_{x \in \alpha} \sum_{n=1}^{N} a(x, y_n) = \sum_{\alpha \times \{y_n\}_{n=1}^{N}} a \leq \sum_{X \times Y} a$$

and therefore,

$$\lim_{N \to \infty} \sum_{x \in X} \sum_{n=1}^{N} a(x, y_n) \leq \sum_{X \times Y} a. \tag{2.15}$$

Hence it follows from Eqs. (2.14) and (2.15) that

$$\sum_{x \in X} \sum_{y \in Y} a(x,y) \leq \sum_{X \times Y} a \tag{2.16}$$

as desired.

Alternative proof of Eq. (2.16). Let $A = \{x'_n : n \in \mathbb{N}\}$ and let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of A. Then for $x \notin A$, $a(x,y) = 0$ for all $y \in Y$.

Given $\epsilon > 0$, let $\delta : X \to [0, \infty)$ be the function such that $\sum_X \delta = \epsilon$ and $\delta(x) > 0$ for $x \in A$. (For example we may define δ by $\delta(x_n) = \epsilon/2^n$ for all n and $\delta(x) = 0$ if $x \notin A$.) For each $x \in X$, let $\beta_x \subset\subset X$ be a finite set such that

$$\sum_{y \in Y} a(x,y) \leq \sum_{y \in \beta_x} a(x,y) + \delta(x).$$

Then

$$\sum_X \sum_Y a \le \sum_{x\in X} \sum_{y\in\beta_x} a(x,y) + \sum_{x\in X} \delta(x)$$

$$= \sum_{x\in X} \sum_{y\in\beta_x} a(x,y) + \epsilon = \sup_{\alpha\subset\subset X} \sum_{x\in\alpha} \sum_{y\in\beta_x} a(x,y) + \epsilon$$

$$\le \sum_{X\times Y} a + \epsilon, \tag{2.17}$$

wherein the last inequality we have used

$$\sum_{x\in\alpha} \sum_{y\in\beta_x} a(x,y) = \sum_{\Lambda_\alpha} a \le \sum_{X\times Y} a$$

with

$$\Lambda_\alpha := \{(x,y) \in X \times Y : x \in \alpha \text{ and } y \in \beta_x\} \subset X \times Y.$$

Since $\epsilon > 0$ is arbitrary in Eq. (2.17), the proof is complete. ■

Theorem 2.22 (Fubini's Theorem for Sums)**.** *Now suppose that $a : X \times Y \to \mathbb{C}$ is a summable function, i.e. by Theorem 2.21 any one of the following equivalent conditions hold:*

1. $\sum_{X\times Y} |a| < \infty$,
2. $\sum_X \sum_Y |a| < \infty$ *or*
3. $\sum_Y \sum_X |a| < \infty$.

Then

$$\sum_{X\times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

Proof. If $a : X \to \mathbb{R}$ is real valued the theorem follows by applying Theorem 2.21 to $a^\pm$ – the positive and negative parts of a. The general result holds for complex valued functions a by applying the real version just proved to the real and imaginary parts of a. ■

2.6. ℓ^p – **spaces, Minkowski and Holder Inequalities.** In this subsection, let $\mu : X \to (0,\infty]$ be a given function. Let $\mathbb{F}$ denote either $\mathbb{C}$ or $\mathbb{R}$. For $p \in (0,\infty)$ and $f : X \to \mathbb{F}$, let

$$\|f\|_p \equiv (\sum_{x\in X} |f(x)|^p \mu(x))^{1/p}$$

and for $p = \infty$ let

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

Also, for $p > 0$, let

$$\ell^p(\mu) = \{f : X \to \mathbb{F} : \|f\|_p < \infty\}.$$

In the case where $\mu(x) = 1$ for all $x \in X$ we will simply write $\ell^p(X)$ for $\ell^p(\mu)$.

Definition 2.23. A **norm** on a vector space L is a function $\|\cdot\| : L \to [0,\infty)$ such that

1. (Homogeneity) $\|\lambda f\| = |\lambda| \, \|f\|$ for all $\lambda \in \mathbb{F}$ and $f \in L$.
2. (Triangle inequality) $\|f + g\| \le \|f\| + \|g\|$ for all $f, g \in L$.
3. (Positive definite) $\|f\| = 0$ implies $f = 0$.

A pair $(L, \|\cdot\|)$ where L is a vector space and $\|\cdot\|$ is a norm on L is called a **normed vector space.**

The rest of this section is devoted to the proof of the following theorem.

Theorem 2.24. *For* $p \in [1,\infty]$, $(\ell^p(\mu), \|\cdot\|_p)$ *is a normed vector space.*

Proof. The only difficulty is the proof of the triangle inequality which is the content of Minkowski's Inequality proved in Theorem 2.30 below. ■

2.6.1. *Some inequalities.*

Proposition 2.25. *Let* $f : [0,\infty) \to [0,\infty)$ *be a continuous strictly increasing function such that* $f(0) = 0$ *(for simplicity) and* $\lim_{s\to\infty} f(s) = \infty$. *Let* $g = f^{-1}$ *and for* $s, t \geq 0$ *let*

$$F(s) = \int_0^s f(s')ds' \text{ and } G(t) = \int_0^t g(t')dt'.$$

Then for all $s, t \geq 0$,

$$st \leq F(s) + G(t)$$

and equality holds iff $t = f(s)$.

Proof. Let

$$A_s := \{(\sigma,\tau) : 0 \leq \tau \leq f(\sigma) \text{ for } 0 \leq \sigma \leq s\} \text{ and}$$
$$B_t := \{(\sigma,\tau) : 0 \leq \sigma \leq g(\tau) \text{ for } 0 \leq \tau \leq t\}$$

then as one sees from Figure 2, $[0,s] \times [0,t] \subset A_s \cup B_t$. (In the figure: $s = 3$, $t = 1$, A_3 is the region under $t = f(s)$ for $0 \leq s \leq 3$ and B_1 is the region to the left of the curve $s = g(t)$ for $0 \leq t \leq 1$.) Hence if m denotes the area of a region in the plane, then

$$st = m([0,s] \times [0,t]) \leq m(A_s) + m(B_t) = F(s) + G(t).$$

As it stands, this proof is a bit on the intuitive side. However, it will become rigorous if one takes m to be Lebesgue measure on the plane which will be introduced later.

We can also give a calculus proof of this theorem under the additional assumption that f is C^1. (This restricted version of the theorem is all we need in this section.) To do this fix $t \geq 0$ and let

$$h(s) = st - F(s) = \int_0^s (t - f(\sigma))d\sigma.$$

If $\sigma > g(t) = f^{-1}(t)$, then $t - f(\sigma) < 0$ and hence if $s > g(t)$, we have

$$\begin{aligned} h(s) = \int_0^s (t - f(\sigma))d\sigma &= \int_0^{g(t)} (t - f(\sigma))d\sigma + \int_{g(t)}^s (t - f(\sigma))d\sigma \\ &\leq \int_0^{g(t)} (t - f(\sigma))d\sigma = h(g(t)). \end{aligned}$$

Combining this with $h(0) = 0$ we see that $h(s)$ takes its maximum at some point $s \in (0,t]$ and hence at a point where $0 = h'(s) = t - f(s)$. The only solution to this equation is $s = g(t)$ and we have thus shown

$$st - F(s) = h(s) \leq \int_0^{g(t)} (t - f(\sigma))d\sigma = h(g(t))$$

with equality when $s = g(t)$. To finish the proof we must show $\int_0^{g(t)} (t - f(\sigma))d\sigma = G(t)$. This is verified by making the change of variables $\sigma = g(\tau)$ and then integrating by parts as follows:

$$\int_0^{g(t)} (t - f(\sigma))d\sigma = \int_0^t (t - f(g(\tau)))g'(\tau)d\tau = \int_0^t (t - \tau)g'(\tau)d\tau$$
$$= \int_0^t g(\tau)d\tau = G(t).$$

■

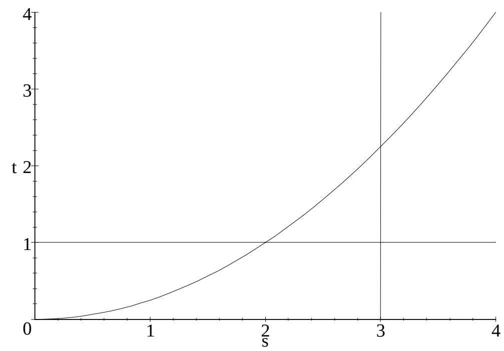

FIGURE 2. A picture proof of Proposition 2.25.

Definition 2.26. The conjugate exponent $q \in [1, \infty]$ to $p \in [1, \infty]$ is $q := \frac{p}{p-1}$ with the convention that $q = \infty$ if $p = 1$. Notice that q is characterized by any of the following identities:

$$\frac{1}{p} + \frac{1}{q} = 1, \ 1 + \frac{q}{p} = q, \ p - \frac{p}{q} = 1 \text{ and } q(p-1) = p. \tag{2.18}$$

Lemma 2.27. *Let $p \in (1, \infty)$ and $q := \frac{p}{p-1} \in (1, \infty)$ be the conjugate exponent. Then*

$$st \le \frac{s^q}{q} + \frac{t^p}{p} \text{ for all } s, t \ge 0$$

with equality if and only if $s^q = t^p$.

Proof. Let $F(s) = \frac{s^p}{p}$ for $p > 1$. Then $f(s) = s^{p-1} = t$ and $g(t) = t^{\frac{1}{p-1}} = t^{q-1}$, wherein we have used $q - 1 = p/(p-1) - 1 = 1/(p-1)$. Therefore $G(t) = t^q/q$ and hence by Proposition 2.25,

$$st \le \frac{s^p}{p} + \frac{t^q}{q}$$

with equality iff $t = s^{p-1}$. ■

Theorem 2.28 (Hölder's inequality)**.** *Let $p, q \in [1, \infty]$ be conjugate exponents. For all $f, g : X \to \mathbb{F}$,*

$$\|fg\|_1 \le \|f\|_p \cdot \|g\|_q. \tag{2.19}$$

If $p \in (1, \infty)$*, then equality holds in Eq. (2.19) iff*

$$(\frac{|f|}{\|f\|_p})^p = (\frac{|g|}{\|g\|_q})^q.$$

Proof. The proof of Eq. (2.19) for $p \in \{1, \infty\}$ is easy and will be left to the reader. The cases where $\|f\|_q = 0$ or ∞ or $\|g\|_p = 0$ or ∞ are easily dealt with and are also left to the reader. So we will assume that $p \in (1, \infty)$ and $0 < \|f\|_q, \|g\|_p < \infty$. Letting $s = |f|/\|f\|_p$ and $t = |g|/\|g\|_q$ in Lemma 2.27 implies

$$\frac{|fg|}{\|f\|_p \|g\|_q} \le \frac{1}{p} \frac{|f|^p}{\|f\|_p} + \frac{1}{q} \frac{|g|^q}{\|g\|^q}.$$

Multiplying this equation by μ and then summing gives

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \le \frac{1}{p} + \frac{1}{q} = 1$$

with equality iff

$$\frac{|g|}{\|g\|_q} = \frac{|f|^{p-1}}{\|f\|_p^{(p-1)}} \iff \frac{|g|}{\|g\|_q} = \frac{|f|^{p/q}}{\|f\|_p^{p/q}} \iff |g|^q \|f\|_p^p = \|g\|_q^q |f|^p.$$

■

Definition 2.29. For a complex number $\lambda \in \mathbb{C}$, let

$$\operatorname{sgn}(\lambda) = \begin{cases} \frac{\lambda}{|\lambda|} & \text{if} \quad \lambda \neq 0 \\ 0 & \text{if} \quad \lambda = 0. \end{cases}$$

Theorem 2.30 (Minkowski's Inequality)**.** *If* $1 \le p \le \infty$ *and* $f, g \in \ell^p(\mu)$ *then*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p,$$

with equality iff

$$\operatorname{sgn}(f) = \operatorname{sgn}(g) \text{ when } p = 1 \text{ and}$$
$$f = cg \text{ for some } c > 0 \text{ when } p \in (1, \infty).$$

Proof. For $p = 1$,

$$\|f + g\|_1 = \sum_X |f + g|\mu \le \sum_X (|f|\mu + |g|\mu) = \sum_X |f|\mu + \sum_X |g|\mu$$

with equality iff

$$|f| + |g| = |f + g| \iff \operatorname{sgn}(f) = \operatorname{sgn}(g).$$

For $p = \infty$,

$$\|f + g\|_\infty = \sup_X |f + g| \le \sup_X (|f| + |g|)$$
$$\le \sup_X |f| + \sup_X |g| = \|f\|_\infty + \|g\|_\infty.$$

Now assume that $p \in (1, \infty)$. Since

$$|f + g|^p \le (2 \max(|f|, |g|))^p = 2^p \max(|f|^p, |g|^p) \le 2^p (|f|^p + |g|^p)$$

it follows that

$$\|f + g\|_p^p \le 2^p \left(\|f\|_p^p + \|g\|_p^p\right) < \infty.$$

The theorem is easily verified if $\|f+g\|_p = 0$, so we may assume $\|f+g\|_p > 0$. Now

$$|f+g|^p = |f+g||f+g|^{p-1} \le (|f|+|g|)|f+g|^{p-1} \tag{2.20}$$

with equality iff $\operatorname{sgn}(f) = \operatorname{sgn}(g)$. Multiplying Eq. (2.20) by μ and then summing and applying Holder's inequality gives

$$\begin{aligned}\sum_X |f+g|^p \mu &\le \sum_X |f|\ |f+g|^{p-1}\mu + \sum_X |g|\ |f+g|^{p-1}\mu \\ &\le (\|f\|_p + \|g\|_p)\, \|\, |f+g|^{p-1}\, \|_q \end{aligned} \tag{2.21}$$

with equality iff

$$\left(\frac{|f|}{\|f\|_p}\right)^p = \left(\frac{|f+g|^{p-1}}{\||f+g|^{p-1}\|_q}\right)^q = \left(\frac{|g|}{\|g\|_p}\right)^p$$
$$\text{and } \operatorname{sgn}(f) = \operatorname{sgn}(g).$$

By Eq. (2.18), $q(p-1) = p$, and hence

$$\||f+g|^{p-1}\|_q^q = \sum_X (|f+g|^{p-1})^q \mu = \sum_X |f+g|^p \mu. \tag{2.22}$$

Combining Eqs. (2.21) and (2.22) implies

$$\|f+g\|_p^p \le \|f\|_p \|f+g\|_p^{p/q} + \|g\|_p \|f+g\|_p^{p/q} \tag{2.23}$$

with equality iff

$$\operatorname{sgn}(f) = \operatorname{sgn}(g) \text{ and}$$
$$\left(\frac{|f|}{\|f\|_p}\right)^p = \frac{|f+g|^p}{\|f+g\|_p^p} = \left(\frac{|g|}{\|g\|_p}\right)^p. \tag{2.24}$$

Solving for $\|f+g\|_p$ in Eq. (2.23) with the aid of Eq. (2.18) shows that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ with equality iff Eq. (2.24) holds which happens iff $f = cg$ with $c > 0$. ■

2.7. **Exercises .**

2.7.1. *Set Theory.* Let $f : X \to Y$ be a function and $\{A_i\}_{i\in I}$ be an indexed family of subsets of Y, verify the following assertions.

Exercise 2.1. $(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c$.

Exercise 2.2. Suppose that $B \subset Y$, show that $B \setminus (\cup_{i\in I} A_i) = \cap_{i\in I}(B \setminus A_i)$.

Exercise 2.3. $f^{-1}(\cup_{i\in I} A_i) = \cup_{i\in I} f^{-1}(A_i)$.

Exercise 2.4. $f^{-1}(\cap_{i\in I} A_i) = \cap_{i\in I} f^{-1}(A_i)$.

Exercise 2.5. Find a counter example which shows that $f(C\cap D) = f(C)\cap f(D)$ need not hold.

Exercise 2.6. Now suppose for each $n \in \mathbb{N} \equiv \{1, 2, \dots\}$ that $f_n : X \to \mathbb{R}$ is a function. Let

$$D \equiv \{x \in X : \lim_{n\to\infty} f_n(x) = +\infty\}$$

show that

$$D = \cap_{M=1}^\infty \cup_{N=1}^\infty \cap_{n\ge N} \{x \in X : f_n(x) \ge M\}. \tag{2.25}$$

Exercise 2.7. Let $f_n : X \to \mathbb{R}$ be as in the last problem. Let

$$C \equiv \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}.$$

Find an expression for C similar to the expression for D in (2.25). (Hint: use the Cauchy criteria for convergence.)

2.7.2. *Limit Problems.*

Exercise 2.8. Prove Lemma 2.15.

Exercise 2.9. Prove Lemma 2.20.

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two sequences of real numbers.

Exercise 2.10. Show $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$.

Exercise 2.11. Suppose that $\limsup_{n\to\infty} a_n = M \in \bar{\mathbb{R}}$, show that there is a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} a_{n_k} = M$.

Exercise 2.12. Show that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{2.26}$$

provided that the right side of Eq. (2.26) is well defined, i.e. no $\infty - \infty$ or $-\infty + \infty$ type expressions. (It is OK to have $\infty + \infty = \infty$ or $-\infty - \infty = -\infty$, etc.)

Exercise 2.13. Suppose that $a_n \ge 0$ and $b_n \ge 0$ for all $n \in \mathbb{N}$. Show

$$\limsup_{n\to\infty}(a_n b_n) \le \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n, \tag{2.27}$$

provided the right hand side of (2.27) is not of the form $0 \cdot \infty$ or $\infty \cdot 0$.

2.7.3. *Dominated Convergence Theorem Problems.*

Notation 2.31. For $u_0 \in \mathbb{R}^n$ and $\delta > 0$, let $B_{u_0}(\delta) := \{x \in \mathbb{R}^n : |x - u_0| < \delta\}$ be the ball in $\mathbb{R}^n$ centered at u_0 with radius δ.

Exercise 2.14. Suppose $U \subset \mathbb{R}^n$ is a set and $u_0 \in U$ is a point such that $U \cap (B_{u_0}(\delta) \setminus \{u_0\}) \ne \emptyset$ for all $\delta > 0$. Let $G : U \setminus \{u_0\} \to \mathbb{C}$ be a function on $U \setminus \{u_0\}$. Show that $\lim_{u\to u_0} G(u)$ exists and is equal to $\lambda \in \mathbb{C}$,[1] iff for all sequences $\{u_n\}_{n=1}^\infty \subset U \setminus \{u_0\}$ which converge to u_0 (i.e. $\lim_{n\to\infty} u_n = u_0$) we have $\lim_{n\to\infty} G(u_n) = \lambda$.

Exercise 2.15. Suppose that Y is a set, $U \subset \mathbb{R}^n$ is a set, and $f : U \times Y \to \mathbb{C}$ is a function satisfying:

1. For each $y \in Y$, the function $u \in U \to f(u, y)$ is continuous on U.[2]
2. There is a summable function $g : Y \to [0, \infty)$ such that

$$|f(u, y)| \le g(y) \text{ for all } y \in Y \text{ and } u \in U.$$

Show that

$$F(u) := \sum_{y\in Y} f(u, y) \tag{2.28}$$

is a continuous function for $u \in U$.

[1]More explicitly, $\lim_{u\to u_0} G(u) = \lambda$ means for every every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|G(u) - \lambda| < \epsilon \text{ whenerver } u \in U \cap (B_{u_0}(\delta) \setminus \{u_0\}).$$

[2]To say $g := f(\cdot, y)$ is continuous on U means that $g : U \to \mathbb{C}$ is continuous relative to the metric on $\mathbb{R}^n$ restricted to U.

Exercise 2.16. Suppose that Y is a set, $J = (a, b) \subset \mathbb{R}$ is an interval, and $f : J \times Y \to \mathbb{C}$ is a function satisfying:

1. For each $y \in Y$, the function $u \to f(u, y)$ is differentiable on J,
2. There is a summable function $g : Y \to [0, \infty)$ such that
$$\left|\frac{\partial}{\partial u} f(u, y)\right| \leq g(y) \text{ for all } y \in Y.$$
3. There is a $u_0 \in J$ such that $\sum_{y \in Y} |f(u_0, y)| < \infty$.

Show:

a) for all $u \in J$ that $\sum_{y \in Y} |f(u, y)| < \infty$.
b) Let $F(u) := \sum_{y \in Y} f(u, y)$, show F is differentiable on J and that
$$\dot{F}(u) = \sum_{y \in Y} \frac{\partial}{\partial u} f(u, y).$$

(Hint: Use the mean value theorem.)

Exercise 2.17 (Differentiation of Power Series)**.** Suppose $R > 0$ and $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ for all $r \in (0, R)$. Show, using Exercise 2.16, $f(x) := \sum_{n=0}^{\infty} a_n x^n$ is continuously differentiable for $x \in (-R, R)$ and
$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Exercise 2.18. Let $\{a_n\}_{n=-\infty}^{\infty}$ be a summable sequence of complex numbers, i.e. $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. For $t \geq 0$ and $x \in \mathbb{R}$, define
$$F(t, x) = \sum_{n=-\infty}^{\infty} a_n e^{-tn^2} e^{inx},$$
where as usual $e^{ix} = \cos(x) + i \sin(x)$. Prove the following facts about F :

1. $F(t, x)$ is continuous for $(t, x) \in [0, \infty) \times \mathbb{R}$. **Hint:** Let $Y = \mathbb{Z}$ and $u = (t, x)$ and use Exercise 2.15.
2. $\partial F(t, x)/\partial t$, $\partial F(t, x)/\partial x$ and $\partial^2 F(t, x)/\partial x^2$ exist for $t > 0$ and $x \in \mathbb{R}$. **Hint:** Let $Y = \mathbb{Z}$ and $u = t$ for computing $\partial F(t, x)/\partial t$ and $u = x$ for computing $\partial F(t, x)/\partial x$ and $\partial^2 F(t, x)/\partial x^2$. See Exercise 2.16.
3. F satisfies the heat equation, namely
$$\partial F(t, x)/\partial t = \partial^2 F(t, x)/\partial x^2 \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

2.7.4. *Inequalities.*

Exercise 2.19. Generalize Proposition 2.25 as follows. Let $a \in [-\infty, 0]$ and $f : \mathbb{R} \cap [a, \infty) \to [0, \infty)$ be a continuous strictly increasing function such that $\lim_{s \to \infty} f(s) = \infty$, $f(a) = 0$ if $a > -\infty$ or $\lim_{s \to -\infty} f(s) = 0$ if $a = -\infty$. Also let $g = f^{-1}$, $b = f(0) \geq 0$,
$$F(s) = \int_0^s f(s')ds' \text{ and } G(t) = \int_0^t g(t')dt'.$$
Then for all $s, t \geq 0$,
$$st \leq F(s) + G(t \vee b) \leq F(s) + G(t)$$

and equality holds iff $t = f(s)$. In particular, taking $f(s) = e^s$, prove Young's inequality stating

$$st \le e^s + (t \vee 1)\ln(t \vee 1) - (t \vee 1) \le e^s + t\ln t - t.$$

Hint: Refer to the following pictures.

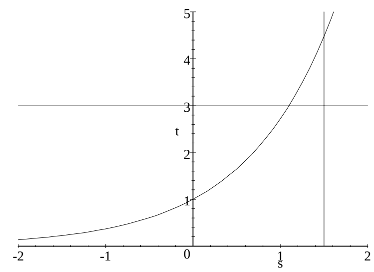

FIGURE 3. Comparing areas when $t \ge b$ goes the same way as in the text.

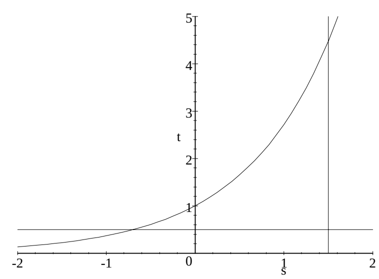

FIGURE 4. When $t \le b$, notice that $g(t) \le 0$ but $G(t) \ge 0$. Also notice that $G(t)$ is no longer needed to estimate st.